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Курсовий проект
на тему:

Self-similar actions of the wallpaper groups

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1. Introduction

This article is focused on the crystallographic groups as discrete groups of motions of an n-dimensional Euclidean space having a bounded fundamental domain. Particularly on the existence of faithful self-similar action in case of $n = 2$ using results of Volodymyr Nekrashevych [1] (proposition 2.9.2). We examined every crystallographic group in R^2 whether they have such action in order to develop the criterion in general case. The results are illustrated in Table 1 and 2. Those, along with code, also can be found on the GitHub repository.

2. Prerequisites

In this section we define all the constructions which we will need for studying the main topic.

2.1. **Rooted trees.** Let X be a finite set, which we call *alphabet*. By X^* we denote the set

$$\{x_1x_2\dots x_n : x_i \in X\}$$

of all finite words over the alphabet X, including an empty word $\emptyset$. Naturally, X^* with the concatenation of words as binary action make the *free monoid* generated by X. By $|v|$ we denote the *lengths* of a word $v = x_1x_2\dots x_n$

It is a convenient way to think about X^* as a rooted tree, where two vertices are connected by the edge iff the associated words have forms v and vx respectively, where $v \in X^*, x \in X$. Obviously, the empty word $\emptyset$ is the root of the tree.

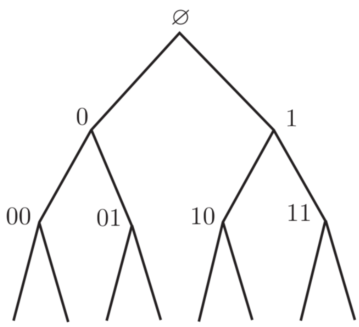

Figure 1. Example of the rooted tree in case of $X = \{0, 1\}$

The set $X^n \subset X$ is called the *n-th level* of the tree X^*. A map $f : X^* \to X^*$ is an *endomorphism* of the tree X^* if for any adjacent vertices u, v their images $f(u)$ and $f(v)$ are also adjacent. Thus, taking into account the exact form of adjacent vertices in X^*, $f(ux) = f(u)y$, where $x, y \in X$.

It is easy to check that $f(X^*) \subset X^*$, which means that we map the entire tree X^* on it's subtree uX^*, where $u \in X^*$ is a *prefix*. If f is bijective then it is called an *automorphism*.

In addition, we consider a X^ω set of all possible infinite-length words over the alphabet X that is the *boundary* of the tree X^*. Naturally, the endomorphism $f(x)$ can be extended uniquely upon the $X^* \sqcup X^\omega$.

Denote by $AutX^*$ the group of all automorphisms of the rooted tree X^*.

2.2. **Groups actions.** A group G is (left) acting on a set X if a map $G \times X \to X, (g, x) \mapsto g(x)$ with following properties is defined:

1. $(gh)(x) = g(h(x))$ for every $g, h \in G, x \in X$
2. $e(x) = x$ for every $x \in X$, where e is a group's identity.

In other words, action of G on X means a homomorphism from G to $Sym(X)$. Recall that $AutX^* \subset Sym(X^*)$ since not every bijection of X^* on itself is a homomorphism.

Group action is

- *transitive* if X is non-empty and for each pair $x, y \in X$ there exists a $g \in G$ such that $g(x) = y$.
- *regular* (or *simply transitive*) if it is transitive and there exists only one g for every pair x, y
- *faithful* if for every $g \neq e \in G$ there exists $x \in X$ such that $g(x) \neq x$
- *free* if for every $g \neq e \in G$, $g(x) \neq x$ for every $x \in X$

Faithful action means that the homomorphism $G \to Sym(X)$ induced by the action *has trivial kernel.*

Proposition 1. *Action is regular iff it is both transitive and free.*

2.3. **Automata.** In general, an *invertible automaton* is a quadruple $A = (S, X, \tau, \pi)$ where S is a finite set of states, X is a finite alphabet, $\tau : S \times X \to S$ a *transition function* and $\pi : S \times X \to X$ an *output function* such that, for each state $s \in S$, the restriction $\pi_s = \pi(s) : X \to X$ is a permutation in S_X (see [3]). If, for instance, the automaton is complete and invertible then its states generate a group, also known as *automaton group.*

2.4. **Self-similar actions.** Let $g : X^* \to X^*$ be an endomorphism of the rooted tree X^*. Consider a vertex $v \in X^*$ and vX^* along with $g(v)X^*$ – subtrees of X^* with v and $g(v)$ as the root respectfully. vX^* represents all the words from X^* that start with v as the prefix. Then, consider a map $g|_v : vX^* \to g(v)X^*$ that is a **restriction** of the g on v. The subtree vX^* is naturally isomorphic to the entire X^* via the map $vw \mapsto w$, as well as $g(v)X^*$. Therefore,

Proposition 2. *$g|_v$ is an endomorphism of vX^* and $g(v)X^*$. It is uniquely determined by the condition*

$$g(vw) = g(v)g|_v(w)$$

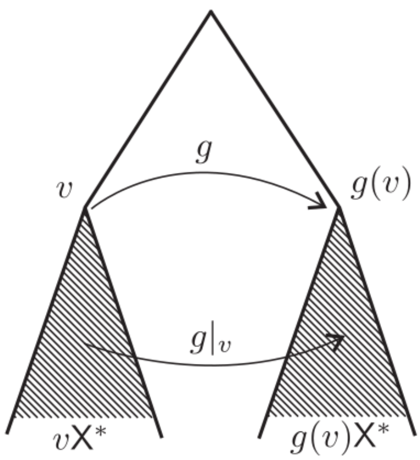

FIGURE 2. Restriction $g|_v$

Here are the obvious properies of the restrictions:

$$g|_{v_1 v_2} = g|_{v_1}|_{v_2}$$
$$(g_1 \cdot g_2)|_v = g_1|_v \cdot g_2|_v$$

Now let's define one of the main properties for this article.

Definition 1. *A faithful action of a group G on X^* (or on X^ω) is said to be* ***self-similar*** *if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that*

$$g(xw) = yh(w)$$

for every $w \in X^*$ ($w \in X^\omega$ resp.)

We will denote self-similar actions as pairs (G, X), where G is the group and X is the alphabet (meaning that G acts on X^* or X^ω).

The pair (h, y) is uniquely determined in the conditions of the Definition 1 by the pair (g, x) since the action is faithful. Hence, we get an automaton with the set of stages G and with the output and transition functions

$$g \cdot x = y \cdot h$$

where $y = g(x)$ and $h = g|_x$. Hereby, any element $g \in G$ can be defined accordingly to it's action on X^*:

$$g = \pi(g|_{x_1} g|_{x_2} \dots g|_{x_n})$$

where $\pi \in Sym(X)$, such that $\pi(x) = y$ from the previous thesis, $X = \{x_1 x_2 \dots x_n\}$. This notation is also called *wreath recursion*, that is a homomorphism

$$\phi : G \to Sym(X) \wr G$$

and the symbol $\wr$ is a *wreath product*, however in this article we will not use any properties of this operation. One can found more details in [1].

The next definition is equivalent to 1 but emphasizes the relation with automata.

Definition 2. *A faithful action of a group G on X^* is said to be* ***self-similar*** *if there exists an automaton (G, X) such that the action of $g \in G$ on X^* coincides with the action of the stage g of the automaton.*

Since G acts on X^* faithfully, then G is isomorphic to a subgroup of the $AutX^*$. Therefore, the definition 1 can also be formulated in term of rooted trees:

Definition 3. *An automorphism group G of the rooted tree X^* is* ***self-similar*** *if for every $g \in G$ and $v \in X^*$ we have $g|_v \in G$.*

An important class of self-similar actions are *contracting* actions, that is there exists a finite set $\mathcal{N}$ such that for every $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $\geq k$. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the self-similar action.

2.5. **Crystallographic groups.** Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the standard scalar product $\langle . \rangle$, euclidean norm $||.||$ and the metric induced by it $d(.,.)$. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ shall be called an *isometry* if for any $x, y \in \mathbb{R}^n$

$$d(x, y) = d(f(x), f(y)).$$

It is easy to prove, that the set of all isometries of $\mathbb{R}^n$ is a group with respect to composition of maps. Hereby $E(n)$ shall denote the group of all isometries of the Euclidean space $\mathbb{R}^n$.

Any isometry of the space $\mathbb{R}^n$ is a composition of an orthogonal linear map and a translation, i.e. if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$, then there exist a translation t_a and an orthogonal map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = A \circ t_a$. That means for every $x \in \mathbb{R}^n$

$$f(x) = Ax + a$$

where A is an orthogonal matrix. The group of all orthogonal operators is denoted by $O(e)$, whilst the group of all translations is isomorphic to the $\mathbb{R}^n$.

Now, let's introduce a useful construction

Definition 4. *Let H and K denote groups with multiplication '$\circ$' and '$\star$' respectively. Moreover, assume that H is a subgroup of the authomorphism group $AutK$ of the abelian group K. The **semi-direct** product $H \ltimes K$ of the groups H and K is the set of pairs (h, k) with the following multiplication*

$$(h_1, k_1)(h_2, k_2) = (h_1 \circ h_2, k_1 \star h_1(k_2))$$

The multiplication from the previous definition is, in fact, multiplication in the affine group $A(n)$

$$A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$$

and, in our case, the group of isometries

$$E(n) = O(n) \ltimes \mathbb{R}^n$$

Proposition 3. *There is following sequence of subgroups:*

$$E(n) \subset A(n) \subset GL(n+1, \mathbb{R})$$

Definition 5. *Let X be a metric space and Γ a subgroup of a group of its isometries. An open, connected subset $F \subset X$ is a **fundamental domain** if*

$$X = \bigcup_{g \in G} g\bar{F}$$

and $gF \cap g'F = \emptyset$, for $g \neq g' \in G$

Definition 6. *A **crystallographic** group of dimension n is a cocompact and discrete subgroup of $E(n)$.*

Theorem 1. *(Bieberbach)*

1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.

3. Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.

Proposition 4. *Properties of crystallographic groups:*

- If Γ is an abelian crystallographic group; then Γ contains only pure translations.

- (Zassenhaus theorem) A group Γ is isomorphic to a crystallographic group of dimension n iff Γ has a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of Γ.

3. Problem

3.1. **Self-similar actions of $\mathbb{Z}^n$.** Consider a self-similar action (G, X).

Definition 7. *The map $\phi_x : G_x \to G$ defined by the formula*

$$\phi_x(g) = g|_x$$

is a virtual endomorphism of G. Here G_x is the stabilizer of the one-letter word x in G. The $\phi : G \dashrightarrow G$ is called the ***endomorphism associated to the self-similar action****.*

Proposition 5. *The kernel of a self-similar action of a group G with an associated virtual endomorphism ϕ is equal to the subgroup*

$$K(\phi) = \bigcap_{n\geq 1} \bigcap_{g\in G} g^{-1} \cdot Dom\phi^n \cdot g$$

and is the maximal one among the normal ϕ-invariant subgroups.

Corollary 1. *If G is an abelian group, then the kernel of a self-similar action is equal to the subgroup*

$$K(\phi) = \bigcap_{n\geq 1} \phi^{-n}(G)$$

Now, consider such an action $G = \mathbb{Z}^n$ and let $\phi : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$ be the associated virtual endomorphism. ϕ can be uniquely extended to a linear map $A : \mathbb{Q}^n \to \mathbb{Q}^n$.

Hereby A shall also denote the matrix of the linear operator A in the standard basis. The matrix A obviously has rational entries and moreover $k \cdot A$ contains only integers, where $k \in \mathbb{N}$ is such that $k\mathbb{Z}^n \leq Dom\phi$.

If ϕ is a surjection and invertible, then the map ϕ^{-1} is injective and defined on the entire $\mathbb{Z}^n$. Thus and so A^{-1} is a matrix of integers.

Let $T = \{g_i,\ i = 1..k\}$ be a coset transversal of the $Dom\phi$ and $X = \{x_i,\ i = 1..k\}$.

Proposition 6. *The self-similar action (not necessary faithful) of G on X, that is defined in the following way*

$$g \cdot x_i = x_j \cdot \phi(g_j^{-1} g g_i)$$

where j is such that $g_j^{-1} g g_i \in Dom\phi$.

The defined self-similar action is faithful iff its kernel is trivial. Due to [1] we have a criterion

Theorem 2. *(Nekrashevych) The subgroup $K(\phi)$ is trivial if and only if characteristic polynomial of A is not divisible by a monic polynomial with integral coefficients (or, in other words, if and only if no eigenvalue of A is an algebraic integer)*

3.2. **Crystallographic case.** Now instead of $\mathbb{Z}^n$ consider a crystallographic group $\Gamma \subset E(n)$. Recall that, since $E(n) = O(n) \ltimes \mathbb{R}^n$, every element $g \in \Gamma$ has a form

$$g(x) = a(x) + t,$$

where a is a *linear part* and t is a *translation*. Along with Γ we will consider a group G (sometimes referred to as a *Point Group*) of all the linear parts of Γ and the set L of all parallel translations in Γ. Recall that L is in fact a normal subgroup of finite index, isomorphic to $\mathbb{Z}^n$. Since G preserves the lattice L, relative to a basis in L the transformations in G are represented by matrices with integer entries.

However, it is clear that not every pair (g, t), $g \in G$, $t \in L$ can appear in the Γ. Thus, in order to specify the Γ, we further have to provide a map $a(g)$ such that

$$x \mapsto gx + a(g),\ x \in \mathbb{R}^n$$

It should be also mentioned, that a mapping

$$\alpha : g \mapsto a(g) + L$$

is in fact a one-dimensional cohomologous cocycle, and any crystallographic group can be represented as a triplet (G, L, α), but for the sake of sancta simplicitas we won't use it here.

The central problem of this article is the following: ***which crystallographic groups admit the self-similar action?***.

Keeping in mind the afore-described approach for $\mathbb{Z}^n$, firstly we have to build a virtual endomorphism of Γ. Recall that by the Bieberbach theorem two crystallographic groups Γ_1 and Γ_2 are isomorphic if and only if they are conjugate in the group $A(n)$, or in other words there exists $a \in A(n)$ such that

$$\Gamma_1 = a^{-1}\Gamma_2 a$$

which gives us a natural way to define the virtual endomorphism $\phi : \Gamma \dashrightarrow \Gamma$:

$$\phi_a(g) = a^{-1}ga$$

.

Now, rewrite this equation taking into account $a = g_{(A,v)}$, $g = g_{(O,t)}$:

$$\phi_a(g_{(O,t)}) = g^{-1}_{(A,v)} \cdot g_{(O,t)} \cdot g_{(A,v)} = g^{-1}_{(A,v)} \cdot g_{(OA,Ov+t)} = g^{-1}_{(A^{-1},-A^{-1}v)} \cdot g_{(OA,Ov+t)}$$

(1) $$\phi_a(g_{(O,t)}) = g_{(A^{-1}OA,\, A^{-1}(Ov+t-v))}$$

Here we see, that the linear part G should be invariant with respect to conjugation by A in order to make ϕ to be the endomorphism indeed. In other words, A should belong to a normalizer of G.

Considering the case when $O = \mathbb{I}_n$

(2) $$\phi(g_{(\mathbb{I}_n,t)}) = g_{(\mathbb{I}_n,\, A^{-1}t)}$$

it is clear that A is also "accountable" for the endomorphism on transitions L. Hence, we have the following.

Proposition 7. *If ϕ_a is an associated virtual endomorphism of a faithful self-similar action Γ, X, where $a = g_{(A,v)}$, then A should satisfy the Nekrashevych condition from Theorem 2.*

4. Experiments

Having a theoretical toolkit from the previous section, we investigated all the planar (i.e. two-dimensional) crystallographic groups whether they admit the faithful self-similar action or not. Up to equivalence, there are 17 planar crystallographic groups ([4], [5]), yet amount of unique Point Groups is only 11.

On the Table 1 we present all the associated virtual endomorphisms for the planar crystallographic groups, which were found applying the aforementioned approach.

№	**Generators**	ϕ	$\lvert\mathbf{X}\rvert$	Exact form
1	$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 0 & 2 & 0 \\ 1 & 0 & 0 \end{array}\right)$	2	$a = ()(b,b)$ $b = (12)(a,e)$
2	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 0 & 2 & 0 \\ 1 & 0 & 0 \end{array}\right)$	2	$a = ()(b,b)$ $b = (12)(a,e)$ $c = ()(c,ca^{-1})$
3	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	-	-	
4	$\left\langle \left(\begin{array}{cc\|c} -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{array}\right) \right\rangle$	-	-	
5	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle$	-	-	
6	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 0 & 2 & 0 \\ 1 & 0 & 0 \end{array}\right)$	2	$a = ()(a,ac^{-1})$ $b = ()(ba,ba)$ $c = ()(d,d)$ $d = (12)(c,e)$
7	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \left(\begin{array}{cc\|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{array}\right) \right\rangle$	-	-	
8	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 0 & 3 & 0 \\ 1 & 0 & 0 \end{array}\right)$	3	$a = (23)(a,bc^{-1}d^{-1},bc^{-1})$ $b = (132)(bacd,e,e)$ $c = (3)(d,d^{-1},d)$ $d = (123)(bacd,bacd,e)$
9	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 0 & 3 & 0 \\ 1 & 0 & 0 \end{array}\right)$	3	
10	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right)$	2	
11	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right)$	2	
12	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\begin{array}{cc\|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right) \right\rangle$	$\left(\begin{array}{cc\|c} 0 & -3 & 0 \\ 3 & 0 & 0 \end{array}\right)$	9	
13	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 2 & 0 \end{array}\right)$	3	
14	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 2 & 2 & 0 \\ -2 & 0 & 0 \end{array}\right)$	4	
15	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 2 & 2 & 0 \\ -2 & 0 & 0 \end{array}\right)$	4	
16	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 1 & -1 & 0 \\ 1 & 2 & 0 \end{array}\right)$	3	
17	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\left(\begin{array}{cc\|c} 2 & 2 & 0 \\ -2 & 0 & 0 \end{array}\right)$	4	

Table 1. Faithful self-similar actions in the planar crystallographic groups, defined by the associated virtual endomorphism

Some 'notate bene' relating to the table 1: column "generators" skips the default translations (those are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$); zero translations along with unitary matrices are skipped as well; $|X|$ means the size of the alphabet that is the transversal of $\Gamma/\phi(\Gamma)$ (hypothetically, it equals to the determinant of the endomorphism matrix); exact form of the self-similar actions is found in every case, although most of them are huge and ugly.

Table 2 (see appendix) contains normalizers for every point group in $\mathbb{R}$ (i.e. linear part of the given crystallographic group). It was constructed in semi-automatic way using the following technique.

Consider any point group G. By definition, its normalizer $\mathcal{N}$ is the maximal subset of $GL(n, Q)$ that is

$$\mathcal{N}G = G\mathcal{N}$$

.

Thus, let $N = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. For every possible permutation $\pi \in Sym(G)$ we get a system of linear equations:

$$Ng_i = g_{\pi(i)}N \tag{3}$$

that can be solved using Gaussian elimination for example. Then, denoting the set of solutions as $\mathcal{N}_\pi$ we have

$$\mathcal{N} = \bigcup_{\pi \in Sym(G)} \mathcal{N}_\pi$$

.

Fortunately, G are small in the most cases, since for $|G| > 6$ this problem becomes computationally hard. We can reduce the computational space, using an observation that conjugate elements should have the same order.

Proposition 8. *Crystallographic groups 3, 4, 5 and 7 don't admit self-similar action.*

Proof. Due to the Table 2, we see that the normalizer for the groups 3, 4 and 5 contains only diagonal matrices that don't satisfy Nekrashevych condition from theorem 2.

Nevertheless, the 7-th crystallographic group Γ_7 has the same linear part as the 6-th, where the self-similar action can be constructed. However, considering $A = \left(\begin{array}{cc|c} 0 & a & x\\ b & 0 & y\end{array}\right)$ and $g_2 = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2}\\ 0 & 1 & 0\end{array}\right)$

$$A^{-1}g_2A = \left(\begin{array}{cc|c} 1 & 0 & 0\\ 0 & -1 & \frac{1}{2}a(\frac{2y}{a}+1)+y\end{array}\right) \notin \Gamma_7$$

□

5. Further research

The main next step to be done is to construct a criterion of crystallographic group admitting self-similar action for the general case $\Gamma < E(n)$. As we could see in the proof of proposition 8 for potential A it is not enough to satisfy the Nekrashevych condition and belong to normalizer of G for construction a virtual endomorphism ϕ_A.

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6. Appendix

№	Point Group	Normalizer	Det.	Charpoly
1	$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
2	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
3	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$	ab	$(a-x)(b-x)$
4	same as previous			
5	same as previous			
6	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	ab	$(a-x)(b-x)$
		$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$	$-ab$	x^2-ab
7	same as previous			
8	same as previous			
9	same as previous			
10	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	a^2+b^2	$x^2-2ax+a^2+b^2$
		$\begin{pmatrix} -a & b \\ b & a \end{pmatrix}$	$-a^2-b^2$	$x^2-a^2-b^2$
11	$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
		$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$	a^2	x^2+a^2
		$\begin{pmatrix} a & a \\ -a & a \end{pmatrix}$	$2a^2$	$x^2-2ax+2a^2$
		$\begin{pmatrix} a & -a \\ a & a \end{pmatrix}$	$2a^2$	$x^2-2ax+2a^2$
		$\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$	$-a^2$	$(-a-x)(a-x)$
		$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$	$-a^2$	x^2-a^2
		$\begin{pmatrix} -a & a \\ a & a \end{pmatrix}$	$-2a^2$	x^2-2a^2
		$\begin{pmatrix} a & a \\ a & -a \end{pmatrix}$	$-2a^2$	x^2-2a^2
12	same as previous			
13	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle$	$\begin{pmatrix} b-a & -a \\ a & b \end{pmatrix}$	a^2-ab+b^2	$\dots$
		$\begin{pmatrix} -a & b-a \\ b & a \end{pmatrix}$	$-(a^2-ab+b^2)$	$x^2-a^2+ab-b^2$

Table 2. Point Groups in $\mathbb{R}^n$ (crystallographic groups 1-13 respectfully)

№	Point Group	Normalizer	Det.	Charpoly
14	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
		$\begin{pmatrix} a & a \\ -a & 0 \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} 0 & -a \\ a & a \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$	$-a^2$	$x^2 - a^2$
		$\begin{pmatrix} a & a \\ 0 & -a \end{pmatrix}$	$-a^2$	
		$\begin{pmatrix} -a & 0 \\ a & a \end{pmatrix}$	$-a^2$	
15	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
		$\begin{pmatrix} a & a \\ -a & 0 \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} 0 & -a \\ a & a \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$	$-a^2$	$x^2 - a^2$
		$\begin{pmatrix} a & a \\ 0 & -a \end{pmatrix}$	$-a^2$	
		$\begin{pmatrix} -a & 0 \\ a & a \end{pmatrix}$	$-a^2$	
16	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \right\rangle$	$\begin{pmatrix} b-a & -a \\ a & b \end{pmatrix}$	$a^2 - ab + b^2$	$\dots$
		$\begin{pmatrix} -a & b-a \\ b & a \end{pmatrix}$	$-(a^2 - ab + b^2)$	$x^2 - a^2 + ab - b^2$
17	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	a^2	$(a-x)^2$
		$\begin{pmatrix} a & a \\ -a & 0 \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} 0 & -a \\ a & a \end{pmatrix}$	a^2	$x^2 - ax + a^2$
		$\begin{pmatrix} a & 2a \\ -2a & -a \end{pmatrix}$	$3a^2$	
		$\begin{pmatrix} a & -a \\ a & 2a \end{pmatrix}$	$3a^2$	
		$\begin{pmatrix} 2a & a \\ -a & a \end{pmatrix}$	$3a^2$	

TABLE 3. Point Groups in $\mathbb{R}^n$ (crystallographic groups 14-17 respectfully)