Self-similar actions of virtually abelian groups

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Theoretical background

Prerequisites. Self-similar actions

Definition. Let X be a finite alphabet. A faithful action of a group G on the set X^* is called *self-similar* if for every $g \in G$ and $x \in X$ there exist $y \in X$ and $h \in G$ such that $g(xw) = yh(w)$ for all $w \in X^*$.
Usually a short notation is used, that originates from another definition using automata:

$$g \cdot x = y \cdot h \tag{1}$$

where h is also noted as $g|_x$ and called a *restriction* of g on x.

Proposition. *Let H be a subgroup of finite index in the group G. If H admits a self-similar action, then G admits a self-similar action.*

Problem

Corrolary. *Every finitely generated virtually abelian group admits a self-similar action. In particular, every crystallographic group admits a self-similar action.*

Question. *Can we* ***describe*** *every self-similar action of finitely generated virtually abelian groups?*

Prerequisites. Virtual endomorphisms

Definition. A *virtual homomorphism* $\phi : G_1 \dashrightarrow G_2$ is a homomorphism of groups $\phi : \text{Dom } \phi \to G_2$, where $\text{Dom } \phi < G_1$ is a subgroup of finite index called the *domain* of the virtual homomorphism.

Proposition. *The subgroup $K(\phi)$, also known as ϕ-**core**, that is defined as*

$$K(\phi) = \bigcap_{n \geq 1} \bigcap_{g \in G} g^{-1} Dom\ \phi^n g$$

*is the maximal one among the **normal ϕ-invariant subgroups** of G.*

Definition. A virtual endomorphism ϕ is said to be *simple* if it's core is trivial, or in other words $K(\phi) = \{e\}$.

Prerequisites. Associative virtual endomorphism

Definition.
The map $\phi_x : G \dashrightarrow G$ defined by the condition

$$g \cdot x = x \cdot \phi_x(g)$$

is called *associative virtual endomorphism* of self-similar action (G, X^*).

Proposition. *Let (G, X^*) be a self-similar action of a group G with an arbitrary associative virtual endomorphism ϕ.*

1. *If N is a normal subgroup of G, and N is ϕ-invariant, then N is contained in the kernel of the self-similar action.*
2. ***The kernel of self-similar action is equal to the ϕ-core.***

Main results

Main results. Free abelian groups

Theorem.

1. *There is a correspondence between virtual endomorphisms $\phi : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$ and pairs of integral $n \times n$ matrices (A_1, A_2) such that $\det(A_1) \neq 0$, given by the rule:*

$$\pi = (A_1, A_2) \mapsto \phi_\pi : A_1(\mathbb{Z}^n) \to \mathbb{Z}^n, \quad \phi_\pi(g) = A_2 A_1^{-1} g$$

   and two pairs (A_1, A_2) and (B_1, B_2) represent the same ϕ if and only if there exists an integral matrix P with $|\det(P)|$ such that $B_1 = A_1 P$, $B_2 = A_2 P$.

2. *The ϕ_π has trivial core if and only if $A = A_2 A_1^{-1}$ is invertible and its characteristic polynomial is not divisible by a monic polynomial with integral coefficients.*
3. *The ϕ_π is surjective iff $|\det(A_2)| = 1$.*

Main results. Finitely generated abelian groups

Every finitely generated abelian group G decomposes into a direct sum $G = \mathbb{Z}^n \oplus F$, where F is a finite abelian group. Virtual endomorphisms consist of triplets:

$$\phi = \begin{pmatrix} A_\phi & 0 \\ B_\phi & C_\phi \end{pmatrix} \tag{2}$$

where $A_\phi \in GL(n, \mathbb{Z})$, $B_\phi \in \mathrm{Hom}(\mathbb{Z}^n, F)$ and C_ϕ is a virtual endomorphism on F.

Theorem. *Let $\phi : G \dashrightarrow G$ be a virtual endomorphism of the finitely generated abelian group $G = \mathbb{Z}^n \oplus F$.*

1. *ϕ is simple if and only if A_ϕ has no eigenvalue that is algebraic integer and C_ϕ is simple as virtual endomorphism on F.*
2. *If ϕ is surjective and simple, then G is free abelian.*

Main results. Crystallographic groups

Definition. A *crystallographic group* of dimension n is a cocompact discrete subgroup in $\mathrm{E}(n)$.

Theorem. *An arbitrary crystallographic group* Γ *that is given by* $(G, \mathbb{Z}^n, \alpha)$ *admits a self-replicating action if and only if there exists* $a = A + t \in \mathrm{A}(n)$ *where* A *is a matrix with rational coefficients such that* A *has no eigenvalue that is algebraic integer and* $a^{-1}\Gamma a \subset \Gamma$.

Corrolary. *Every crystallographic group* Γ *admits a self-replicating action with* $a = A + t$ *being a scalar matrix* A *with trivial translation.*

Combining this result with the results of Bondarenko I. we can formulate the following theorem.

Theorem. *A virtually abelian group* G *admits a self-replicating action if and only if* G *is crystallographic.*

Computational experiments

Computational experiments

We developed some software to work with self-similar actions of crystallographic groups, since they are objects of practical interest. All the needed code one can find on the GitHub repository: https://github.com/davendiy/master_thesis.
Particularly,

1. an algorithm to find the normalizer of a point group;
2. an algorithm to build a self-similar action by an arbitrary conjugation in the Affine group and the respective crystallographic group of any dimension;
3. results of brute-force search of faithful self-similar actions for wallpaper groups and space groups, with alphabet of the minimal possible size.

Thank you for attention!