

# Self-similar actions of virtually abelian groups

Зашкольний Давид Олександрович

26 квітня 2023 р.

Науковий керівник: Бондаренко Євген Володимирович

# Contents

## Theoretical background

- Self-similar actions

- Virtual endomorphisms

## Main results

- Self-similar actions of free abelian groups

- Self-similar actions of finitely generated abelian groups

- Self-replicating actions of crystallographic groups

## Computational experiments

# Theoretical background

## Prerequisites. Self-similar actions

**Definition.** Let  $X$  be a finite alphabet. A faithful action of a group  $G$  on the set  $X^*$  is called *self-similar* if for every  $g \in G$  and  $x \in X$  there exist  $y \in X$  and  $h \in G$  such that  $g(xw) = yh(w)$  for all  $w \in X^*$ .

Usually a short notation is used, that originates from another definition using automata:

$$g \cdot x = y \cdot h \tag{1}$$

where  $h$  is also noted as  $g|_x$  and called a *restriction* of  $g$  on  $x$ .

**Proposition.** Let  $H$  be a subgroup of finite index in the group  $G$ . If  $H$  admits a self-similar action, then  $G$  admits a self-similar action.

# Problem

**Corrolary.** *Every finitely generated virtually abelian group admits a self-similar action. In particular, every crystallographic group admits a self-similar action.*

**Question.** *Can we **describe** every self-similar action of finitely generated virtually abelian groups?*

## Prerequisites. Virtual endomorphisms

**Definition.** A *virtual homomorphism*  $\phi : G_1 \dashrightarrow G_2$  is a homomorphism of groups  $\phi : \text{Dom } \phi \rightarrow G_2$ , where  $\text{Dom } \phi < G_1$  is a subgroup of finite index called the *domain* of the virtual homomorphism.

**Proposition.** The subgroup  $K(\phi)$ , also known as  $\phi$ -**core**, that is defined as

$$K(\phi) = \bigcap_{n \geq 1} \bigcap_{g \in G} g^{-1} \text{Dom } \phi^n g$$

is the maximal one among the **normal**  $\phi$ -**invariant subgroups** of  $G$ .

**Definition.** A virtual endomorphism  $\phi$  is said to be *simple* if its core is trivial, or in other words  $K(\phi) = \{e\}$ .

## Prerequisites. Associative virtual endomorphism

### Definition.

The map  $\phi_x : G \dashrightarrow G$  defined by the condition

$$g \cdot x = x \cdot \phi_x(g)$$

is called *associative virtual endomorphism* of self-similar action  $(G, X^*)$ .

**Proposition.** Let  $(G, X^*)$  be a self-similar action of a group  $G$  with an arbitrary associative virtual endomorphism  $\phi$ .

1. If  $N$  is a normal subgroup of  $G$ , and  $N$  is  $\phi$ -invariant, then  $N$  is contained in the kernel of the self-similar action.
2. The kernel of self-similar action is equal to the  $\phi$ -core.

# Main results



# Main results. Free abelian groups

## **Theorem.**

1. *There is a correspondence between virtual endomorphisms  $\phi : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$  and pairs of integral  $n \times n$  matrices  $(A_1, A_2)$  such that  $\det(A_1) \neq 0$ , given by the rule:*

$$\pi = (A_1, A_2) \mapsto \phi_\pi : A_1(\mathbb{Z}^n) \rightarrow \mathbb{Z}^n, \quad \phi_\pi(g) = A_2 A_1^{-1} g$$

*and two pairs  $(A_1, A_2)$  and  $(B_1, B_2)$  represent the same  $\phi$  if and only if there exists an integral matrix  $P$  with  $|\det(P)|$  such that  $B_1 = A_1 P$ ,  $B_2 = A_2 P$ .*

2. *The  $\phi_\pi$  has trivial core if and only if  $A = A_2 A_1^{-1}$  is invertible and its characteristic polynomial is not divisible by a monic polynomial with integral coefficients.*
3. *The  $\phi_\pi$  is surjective iff  $|\det(A_2)| = 1$ .*

## Main results. Finitely generated abelian groups

Every finitely generated abelian group  $G$  decomposes into a direct sum  $G = \mathbb{Z}^n \oplus F$ , where  $F$  is a finite abelian group. Virtual endomorphisms consist of triplets:

$$\phi = \begin{pmatrix} A_\phi & 0 \\ B_\phi & C_\phi \end{pmatrix} \quad (2)$$

where  $A_\phi \in GL(n, \mathbb{Z})$ ,  $B_\phi \in \text{Hom}(\mathbb{Z}^n, F)$  and  $C_\phi$  is a virtual endomorphism on  $F$ .

**Theorem.** Let  $\phi : G \dashrightarrow G$  be a virtual endomorphism of the finitely generated abelian group  $G = \mathbb{Z}^n \oplus F$ .

1.  $\phi$  is simple if and only if  $A_\phi$  has no eigenvalue that is algebraic integer and  $C_\phi$  is simple as virtual endomorphism on  $F$ .
2. If  $\phi$  is surjective and simple, then  $G$  is free abelian.

## Main results. Crystallographic groups

**Definition.** A *crystallographic group* of dimension  $n$  is a cocompact discrete subgroup in  $E(n)$ .

**Theorem.** An arbitrary crystallographic group  $\Gamma$  that is given by  $(G, \mathbb{Z}^n, \alpha)$  admits a self-replicating action if and only if there exists  $a = A + t \in A(n)$  where  $A$  is a matrix with rational coefficients such that  $A$  has no eigenvalue that is algebraic integer and  $a^{-1}\Gamma a \subset \Gamma$ .

**Corrolary.** Every crystallographic group  $\Gamma$  admits a self-replicating action with  $a = A + t$  being a scalar matrix  $A$  with trivial translation.

Combining this result with the results of Bondarenko I. we can formulate the following theorem.

**Theorem.** A virtually abelian group  $G$  admits a self-replicating action if and only if  $G$  is crystallographic.

# Computational experiments

# Computational experiments

We developed some software to work with self-similar actions of crystallographic groups, since they are objects of practical interest. All the needed code one can find on the GitHub repository:

[https://github.com/davendiy/master\\_thesis](https://github.com/davendiy/master_thesis).

Particularly,

1. an algorithm to find the normalizer of a point group;
2. an algorithm to build a self-similar action by an arbitrary conjugation in the Affine group and the respective crystallographic group of any dimension;
3. results of brute-force search of faithful self-similar actions for wallpaper groups and space groups, with alphabet of the minimal possible size.

Thank you for attention!