Self-similar actions of virtually abelian groups

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Self-similar group actions are special actions on the spaces of words that reflect the self-similarity of the space. Self-similar group action naturally arise in many areas of mathematics: dynamical systems, fractal geometry, algebraic topology, automata theory. For the last twenty years self-similar actions were studied for many classes of groups: abelian, nilpotent, solvable, free and linear groups, arithmetic groups.

Self-replicating actions is the special case of self-similar actions. There is a convenient algebraic criterion: a group G admits a self-similar action if and only if there is a virtual homomorphism $\phi : H \to G$ (i.e. $H < G$ is a subgroup of finite index) and the ϕ-core is trivial. Similarly, a group admits a self-replicating action if there is such a surjective ϕ. A self-similar action associated to ϕ is obtained by a certain iterated construction.

Every finitely generated virtually abelian group admits a self-similar action, yet not every one does self-replicating action. Nekrashevych-Sidki [1] showed that if an abelian group has such an action, then it is free. Thus, we consider a generalization of this result to virtually abelian groups.

A crystallographic group of dimension n is a discrete cocompact group of isometries of $\mathbb{R}^n$. Recalling the Bieberbach theorem, every isomorphism of crystallographic group is in fact a conjugation by an element $a = (A, t)$ from the affine group $A(n)$. We have proven the following theorems:

Theorem 1. *Every crystallographic group G admits a self-replicating action, that is generated by a virtual endomorphism $\phi(g) = a^{-1}ga$, where $a = (A, t)$ is an affine transformation with trivial translation $t = 0$ and a scalar matrix A.*

Theorem 2. *A virtually abelian group admits self-replicating action if and only if it is crystallographic.*

Also, we designed and implemented an algorithm to create a self-similar action by the virtual endomorphisms. Using it, we found self-replicating actions with minimal alphabet for every crystallographic group of dimensions 2 and 3.

References

[1] Nekrashevych V. Self-similar groups. – Providence: Mathematical Surveys and Monographs, Vol.117, American Mathematical Society, 2005, 231 pages.