4) find the explicit form of the solution I_0

$$y' = y^2 - y, \quad y(0) = y_0$$

Let's solve the Cauchy problem.

$$ln\frac{1-y}{y} = x + C$$

$$\frac{1-y}{y} = C_1 e^x$$

$$y = \frac{1}{C_1 e^x + 1}, \quad y = 0$$

$$\frac{1-y_0}{y_0} = C_1 e^0 \quad \Rightarrow \quad y = \frac{1}{\frac{1-y_0}{y_0} e^x + 1}$$

We need to solve the problem of denominator.

$$\frac{1-y_0}{y_0} e^x + 1 \neq 0$$

$$\frac{1-y_0}{y_0} e^x \neq -1$$

$$-\frac{y_0}{1-y_0} \neq e^x$$

Let's consider 2 cases:

1) $-\dfrac{y_0}{1-y_0} < 0 \quad \Rightarrow \quad y_0 \in (-\infty, 0) \cup (1, +\infty) \quad \Rightarrow \quad I_{y_0} = \mathbb{R}$

2) $-\dfrac{y_0}{1-y_0} > 0 \quad \Rightarrow \quad y_0 \in (0,1) \quad \Rightarrow \quad x_0 = \ln -\dfrac{y_0}{1-y_0}$

$$-\frac{y_0}{1-y_0} > 1 \quad \Rightarrow \quad y_0 < y_0 - 1 \quad \Rightarrow \quad 0 < -1 \quad \Rightarrow \quad y_0 \in \emptyset$$

$$-\frac{y_0}{1-y_0} < 1 \quad \Rightarrow \quad y_0 \in (0,1) \quad \Rightarrow \quad x_0 < 0 \quad \Rightarrow \quad I_0 = (-\infty, x_0)$$

Thus,

$$I_{y_0} = \begin{cases} \left(-\infty; \ln -\dfrac{y_0}{1-y_0}\right), \ y_0 \in (0,1) \\ \mathbb{R}, \ y_0 \in (-\infty, 0) \cup (1, +\infty) \end{cases}$$

5) is $g^x : H \subset \mathbb{R} \to \mathbb{R}$, $g^x : y_0 \mapsto y(x, y_0)$ defined on $\mathbb{R}$?

No.

$$\begin{cases} x > \ln -\dfrac{y_0}{1-y_0} \\ y_0 \in (0,1) \end{cases} \Leftrightarrow \quad y_0 < \frac{e^x}{1+e^x} \quad \Rightarrow \quad \forall x > 0 \, g^t \text{ doesn't defined on } y_0 < \frac{e^x}{1+e^x}$$

6) find the $H \subset \mathbb{R}$ such $g^t : H \to \mathbb{R}$ exists for all $x \geq 0$

$H = (-\infty, 0) \cup (1, \infty)$

7) find the $g^1(0)$, $g^1([0,1])$

$$y_0 = 0 \quad \Rightarrow \quad y = \frac{1}{\frac{1-0}{0} e^x + 1} = 0$$

$$y_0 \in [0,1] \quad \Rightarrow \quad \frac{1-y_0}{y_0} \geq 0 \quad \Rightarrow \quad \frac{1-y_0}{y_0} e + 1 \geq 1 \quad \Rightarrow \quad y(1) \in (0,1] = g^1([0,1])$$

8) are there any y_0 such $\forall x \geq 0 \, g^x(y_0) = y_0$

$$g^x(y_0) = y_0 \quad \Rightarrow \quad \frac{1}{\frac{y_0}{1-y_0} + 1} = y_0 \quad \Rightarrow \quad y_0 \left(\frac{y_0}{1-y_0}\right) e^x = 0 \quad \Rightarrow \quad y_0 = 0$$