Linear Systems (Discrete vs. Continuous)

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1 Discrete Dynamical Systems

Let's say there's a matrix A that represents the change in a discrete dynamical system s.t. $\vec{x}(t+1) = A\vec{x}(t)$. Given an initial input vector $\vec{x}(0)$, in order to evaluate the state of the system at $t = 10$ one can either multiply $\vec{x}(0)$ by A ten times ($A^{10}\vec{x}(0)$) or one can come up with a closed equation based on t. The latter, often times, proves much easier.

As is often the case, use eigenvectors. An eigenvector $\vec{u}$ is a vector s.t. $A\vec{u} = \lambda\vec{u}$ for some constant λ. If we can rewrite $\vec{x}(0)$ in eigenvectors, matrix exponentiation simply becomes constant exponentiation.

1.1 Example

Let's say we're trying to model glucose and hormone concentrations in the body. Given t, which is measured in minutes, the amount of glucose decreases by 11 percent of its current level and by 3 percent of the level of hormones in the body. The hormone level, on the other hand, increases by 2 percent based on the amount of glucose, while simultaneously decreasing by 4 percent of its current level.

Given a vector $\vec{x}(0) = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$, if we were to create a matrix A that represented the change in this dynamical system, it might look like:

$$\text{A} = \begin{bmatrix} .89 & -.03 \\ .02 & .96 \end{bmatrix}$$

$$\vec{g}(t+1) = .89\vec{g}(t) - .03\vec{h}(t)$$

$$\vec{h}(t+1) = .02\vec{g}(t) + .96\vec{h}(t)$$

To create a closed equation, find the eigenvectors of the matrix:

$$(.89-\lambda)(.96-\lambda)+.0006$$
$$(\lambda^2-1.85\lambda+.855)$$
$$(\lambda_1,\lambda_2)=(.9,.95)$$
$$e_1=\begin{bmatrix}-3\\1\end{bmatrix}_{.9}\; e_2=\begin{bmatrix}1\\-2\end{bmatrix}_{.95}$$
$$\mathrm{S}=\begin{bmatrix}-3 & 1\\1 & -2\end{bmatrix}$$

To create a closed equation for the system, convert input vector $\vec{x}(0)$ into its constituent eigenvectors. As t changes, these eigenvectors will change according to their eigenvalues, which means they will change by a constant value.

$$\begin{bmatrix}g(t+1)\\h(t+1)\end{bmatrix}=\begin{bmatrix}-3 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}.9^t & 0\\0 & .95^t\end{bmatrix}\begin{bmatrix}-3 & 1\\1 & -2\end{bmatrix}^{-1}\vec{x}(0)$$

multiplying out:

$$\begin{bmatrix}g(t+1)\\h(t+1)\end{bmatrix}=\frac{1}{5}*\begin{bmatrix}-3 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}.9^t & 0\\0 & .95^t\end{bmatrix}\begin{bmatrix}-2g-h\\-g-3h\end{bmatrix}$$
$$\begin{bmatrix}g(t+1)\\h(t+1)\end{bmatrix}=\frac{1}{5}*\begin{bmatrix}-3 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}.9^t(-2g-h)\\.95^t(-g-3h)\end{bmatrix}$$
$$\begin{bmatrix}g(t+1)\\h(t+1)\end{bmatrix}=\frac{1}{5}*\begin{bmatrix}-3*.9^t(-2g-h)+.95^t(-g-3h)\\.9^t(-2g-h)-2*.95^t(-g-3h)\end{bmatrix}$$

So, let's say with intial vector $\vec{x}(0)=\begin{bmatrix}70\\0\end{bmatrix}$ you want to find how much glucose will be in the system after 10 minutes. One way to do this would be to multiply $\vec{x}(0)$ by A ten times. Or, simply plug in the values into the closed equation above s.t.:

$$\vec{g}(10)=\frac{1}{5}*(-3*.9^{10}(-2(70))+.95^{10}(-70))$$
$$\vec{g}(10)=20.91$$

1.2 Trajectories

Beyond finding singular t values, it's often illustrative to graph how a system will change over time. In the above example, since each eigenvalue was greater than 0 but less than 1, as t $\to\infty$ the values will $\to 0$.

- Values with eigenvalues > 1 will $\to \infty$
- Values with eigenvalues $= 1$ will remain constant.
- Values with $(-1 <$ eigenvalues $< 1)$ will $\to 0$
- Values with eigenvalues < -1 will grow but oscillate about the origin.

Another thing to be cognizant of is, let's say there are two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ with eigenvalues λ_1 and λ_2, respectively. If $\lambda_1 > \lambda_2$, then as t $\to \infty$, trajectories will skew towards v_1 since its eigenvalue will dominate.

2 Continuous Dynamical Systems

Continuous systems are a little different than discrete ones. In a continuous system, instead of matrix A representing the change in state from t to $t+1$, matrix A represents a change of rate matrix s.t. $\frac{d\vec{x}}{dt} = A\vec{x}$.

2.1 Two Methods

2.1.1 Linear Combination

Let's say there's a matrix A $= \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$ s.t. $\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x}$. You want to know how this system changes over time — you want to know, basically, the same thing you wanted to know with the discrete system.

As always, start by finding the eigenvectors of the matrix.

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_6 \vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_3$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \tag{1}$$

This means that when $t = 1$ the eigenvectors change by a factor of 6 and 3, respectively. The goal, then, is to find expressions exp_n s.t. the derivatives of $\frac{dexp_n}{dt} = \lambda_n exp_n$. An obvious solution, then, would be to let $exp_n = e^{\lambda_n t}$.

$$x(t) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \tag{2}$$

$$x(t) = \begin{bmatrix} \frac{e^{3t}}{3} + \frac{2e^{6t}}{3} & \frac{e^{3t}}{3} - \frac{e^{6t}}{3} \\ \frac{2e^{3t}}{3} - \frac{2e^{6t}}{3} & \frac{2e^{3t}}{3} + \frac{e^{6t}}{3} \end{bmatrix} \tag{3}$$

Let's say you started with initial vector x(0) = $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What will this be at $t = 5$?.

$$x(5) \approx \begin{bmatrix} 1.07 * 10^{13} \\ -1.07 * 10^{13} \end{bmatrix}$$

So, as t increases the values of the initial vector are going to get very large. This makes sense, since both eigenvalues are greater than 1.

2.1.2 Matrix Exponentiation

Another method would be to treat the entire matrix A as an exponent s.t. $x(t) = e^{A(t)}$. If this is the case, then $\frac{dx}{dt} = A(t)e^{A(t)} = A(t)x(t)$, which is exactly what we want.

2.1.3 Series of e

The series for $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$ For a matrix, this can be translated s.t. ($e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + ...$). If there is an eigenbasis for A, let S be the change of basis matrix and D the diagonal matrix. Rewrite the above as:

$$e^A = I + SDS^{-1} + \frac{(SDS^{-1})^2}{2} + \frac{(SDS^{-1})^3}{3!} + ...$$

$$e^A = I + SDS^{-1} + \frac{SD^2S^{-1}}{2} + \frac{SD^3S^{-1}}{3!} + ...$$

$$e^A = S^{-1}(I + D + \frac{D^2}{2} + \frac{D^3}{3!} + ...)S$$

$$e^A = S^{-1}(\begin{bmatrix} I + \lambda_1 + \frac{\lambda_1^2}{2} + \frac{\lambda_1^3}{3!} + ... & 0 \\ 0 & I + \lambda_2 + \frac{\lambda_2^2}{2} + \frac{\lambda_2^3}{3!} \end{bmatrix})S$$

$$e^A = S^{-1}e^DS$$

2.2 Trajectories

For continuous dynamical systems:

- For any eigenvalue > 0 the vector will continue to grow. This is because $e^{\lambda(t)}$, as $t \to \infty$, will grow regardless of how small λ is as long as it's positive.
- If an eigenvalue $= 0$ then the vector, regardless of how t changes, will remain unchanged.
- For any eigenvalue < 0 the vector will converge to zero. A system where all eigenvalues are less than zero is considered asymptotically stable.