Laplacian eigenvalues in regular polygons through the calculus of moving surfaces

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1 Introduction

Recent attempts to the solution of the eigenvalue problem [2] [3] of the Laplacian in regular polygons have partially shifted their attention to the calculus of moving surfaces. The calculus of moving surfaces undoubtedly offers a geometric insight to the problem which can be exploited in a straightforward fashion to aide to the solution of the problem. Through this technique a few illuminating results have been proven. It is possible to prove, and a proof will in fact be offered in this paper, that by re scaling the polygons such that their area is the same as the one of the circle, the first term to appear in the $\frac{1}{N}$ expansion of the eigenvalue is that of power three. Moreover it is possible to substantiate analytically the values for the expansion, and to definitively assert their connection to the Riemann zeta function. The calculus of moving surfaces is thus a powerful tool that may be applied to many other boundary perturbation and optimization problems. In the next few sections a breif account of the theory will be given, after which it will be applied to the problem in question.

2 Calculus of moving surfaces

The calculus of moving surfaces is a mathematical theory that sets itself as a sub-branch of differential geometry. It was first developed by french mathematician Jacques Hadamard and subsequently improved upon by many authors. The main object of study of the calculus of moving surfaces are embedded surfaces moving and stretching in time. The tool used to study these structures is the invariant time derivative whose definition will be built in the next few lines. It is worth recalling a few basic definitions of differtial geometry in order to efficiently describe and attack the calculus of moving surfaces. The next few lines will be dedicated to that purpose.

2.1 Basic manifold theory

The following definitions, taken by Warner's textbook on manifolds and Lie groups [6] will be worked upon.

Definition 1 (Locally Euclidean Space)**.** A locally Euclidean space M of dimension d is a Hausdorff topological space M for which each point has a neighborhood homeomorphic to a subset of $\mathbb{R}^d$. If φ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^d$, then φ is called a coordinate map, the functions $x_i := r_i \circ \varphi$ are called coordinate functions and the pair (U, φ) is called a coordinate system.

... add definitions of differentiable structure, atlas, manifold, submanifold, immersion and embedding. Offer some examples of immersion, submanifold and embedding.

2.2 Basic elements of embedded surfaces

Unfortunately not much has been written to date on the calculus of moving surfaces. Most of the following discussion is taken from and can be expanded on by [1]. We wish now to develop the math behind the intuitive concept of an embedding. Let N be an embedded manifold in M and let the dimension of N be equal to that of M less of one. Let $S^\alpha(t)$ be a coordinate system of a subset of N and let ϕ^{-1} be the inverse of the embedding map between M and N: $\phi : M \to N$. Suppose tat the dependence of the coordinate system on t is smooth: i.e. a $S^\alpha(t)$ is a smooth function of t. Further on we shall refer as the set of all $S^\alpha(t)$ as surface coordinates and they will be denoted by a Greek lettered index. Let Z^i be a coordinate system in a subset of M containing a subset of the image of ϕ^{-1}. We shall denote the i-th coordinate of a point $p \in M$ as $Z^i(p)$. Using the Z^i coordinates and the embedding map one can build a coordinate system for N in the following way:

$$S^i(t) := Z^i(\phi^{-1}(p)), \quad p \in N \tag{1}$$

These coordinates will be identified by the same letter S to emphasize that they are to describe the same object, however one must always keep in mind that, as they are coordinates of M they benefit from an extra component. The coordinate system S^i therefore spans N "inside" of M an can be intuitively be imagined as the parametrization on the surface of the embedded manifold. It is a little bit more tricky to export the idea of a tangent space. Intuitively, it is clear that the tangent space of the embedded manifold will be a subspace of the tangent space of the large manifold. Before discussing the tangent plane as a whole, the attention should be shifted to a single vector. There is a natural connection between the surface and the ambient coordinates. Let f be a function on N to $\mathbb{R}$. We can define a "new" function $\tilde{f} : \phi^{-1}(N) \subset M \to \mathbb{R}$ defined by the relation $\tilde{f} = f \circ \phi$. We can view this function as an copy of f on the point which are the "embedded version" in M or, combined with ϕ^{-1} as a function from N to $\mathbb{R}$, also, by its definition we have $\tilde{f} \circ \phi^{-1} = f$. Differentiating with respect to the α-th coordinate we get:

$$\partial_\alpha f = \partial_\alpha(\tilde{f} \circ \phi^{-1}) = \left[d\phi^{-1}\partial_\alpha\right] \tilde{f} \tag{2}$$

to complete our transportation we must now allow for a composition with ϕ^{-1}. In this sense we have transported the vector ∂_α to a vector on the tangent space in M. As all maps used were regular and linear the mapping of the vector is as well. Naively it can be thought of as a rectangular (not necessarily square) matrix. It is thus represented by a tensor-like object Z_i^α which we call the shift tensor. We call the tangent space to the embedding of N in M the span of all vectors $Z_i^\alpha \partial_\alpha{}_{i=1}^d$ and ∂_α are the tangents to the coordinate functions. Furthermore, as the dimension of the tangent space of N is equal to that of M less of one, and each vector in the tangent space to the embedding is determined by one in the tangent space of N we expect there to be another vector in the tangent space of M orthogonal all of the ones in the tangent space of the embedded manifold. We call this normalized vector the normal vector to the surface. There is an ambiguity still which is left, for the direction if the normal vector. For two dimensional closed surfaces embedded in $\mathbb{R}^3$ it is common to take the normal vector "pointing outwards".

A natural question should arise. What is exactly the reason to introduce a new formalism for manifolds and not just add a coordinate to the existing ones. The answer to this question is that, quantities which are more easily thought of as being geometric and invariant of coordinate changes are not always "well behaved" with respect to derivations in time meaning that they do not always evolve in the same way. Consider for example a function $T(t) : N \to \mathbb{R}$ which independent of the choice of coordinates (such as the contraction of two vector fields etc.) and consider also two sets of coordinates: $S^\alpha(t)$ and $S^{\alpha'}(t)$. For any fixed t we have a jacobian $J^\alpha_{\alpha'}(t)$ transforming one set of coordinates into the other. When calculated with respect to the primed coordinates we denote $T(t)$ as $T(t, S^{\alpha'})$ whereas in the other case we omit the prime. Let U and U' also be defined as follows:

$$U := \frac{d}{dt} T(t, S^\alpha) \tag{3}$$

$$U' := \frac{d}{dt} T(t, S^{\alpha'}) \tag{4}$$

Finally, we have the equality:

$$T(t, S^{\alpha'}) = T(t, J^\alpha_{\alpha'} S^{\alpha'}) \tag{5}$$

Differentiating with respect to time we get:

$$U' = U + \partial_\alpha T(t, J^\alpha_{\alpha'} S^{\alpha'}) \partial_t (J^\alpha_{\alpha'} S^{\alpha'}) \tag{6}$$

Because of the last term which in general is nonzero we cannot say that T evolves in the same way in the two reference frames. It is one of the goals of the calculus of moving surfaces to clarify such a divergence from intuition. In the least disruptive way possible we wish to develop a mathematical formalism which provides tools for our basic intuition. This is done not by changing anything already established by differential geometry but rather by introducing a new concept: a new kind of time derivative. In a sense, just like the covariant derivative was built we wish to build a "invariant time derivative". The key to this problem is to let the derivative vary as well as will be shown in the next section.

2.3 Basic elements of calculus of moving surfaces

Before introducing the two main results of the calculus of moving surfaces which will be used for our approach to the solution of the calculus of moving surfaces we must first introduce some kind of velocity for our moving manifold. Our goal will be to give a purely coordinate detached definition, in order to ensure it being a property of the object of study rather than one of the coordinate system of our choosing. Simply taking the time derivative of a coordinate function will not suffice. It is however the first step to the solution of this puzzle, hence consider the "coordinate velocity" defined as:

$$V^i(t,p) := \frac{d}{dt} S^i(t,p) \tag{7}$$

where p is a point in M and $S^i(t,p)$ is the i-th component of a coordinate function at the point p at the time t. We shall prove that as defined in the equation above the coordinate velocity is not invariant to coordinate changes and in fact it does not even transform as a tensor.

Provide proof

The last equation in the proof provides a useful clue to the next step needed to build our definition of a coordinate free velocity. Namely, being proportional to the shift tensor, it is a linear combination of vectors in the tangent space to the surface under consideration. To get rid of it then we can just contract it with the normal vector, thus obtaining our final, coordinate free definition:

$$C := V^i N_i \tag{8}$$

The proof of this object being coordinate invariant is the same as that of V^i not being so. Geometrically C can be thought of as the velocity of a point on a coordinate function in the normal direction. Because of this it is often called interface velocity.

With this last object we will be able to construct a new type of derivative with which we will be able to operate in a purely coordinate free manner: a derivative not dependant on the coordinate functions we use to describe the manifold, and which preserves the tensor transformation property. One could just state the definition and show that it is well behaved, however due to its intuitive geometrical interpretation before doing so it is worth building in a heuristic way. Consider a point P and an (ambient) coordinate function $S(t,A)$ passing through it on the a subset of $\phi^{-1}(N)$. Let T be a function on that subset. In a small time h the point P as well as the whole coordinate function will move a bit, to a new position on the manifold which we will identify as B. Heuristically (and erroneously if one should be rigorous), one could parametrize the point B with the following equation:

$$T(t,S(t,B)) = T(t,S(t,P)) + h\frac{d}{dt}T(t,S(t,P)) \tag{9}$$

Now let D be a point on the time translated coordinate function "close" to B. Following the coordinate function one might write

$$T(t,S(t,B)) = T(t,S(t,D)) + hV^j\nabla_j T(t,S(t,D)) \tag{10}$$

Eliminating the point B in the two preceding equations we get:

$$T(t,S(t,P)) - T(t,S(t,D)) = h\left(\frac{d}{dt}T(t,S(t,P)) - V^\alpha\nabla_\alpha T(t,S(t,D))\right) \tag{11}$$

Suggesting as a definition for the invariant derivative:

$$\dot{\nabla} T(t, S(t, B)) = \frac{d}{dt} T(t, S(t, P)) - V^{\alpha} \nabla_{\alpha} T(t, S(t, D)) \tag{12}$$

It can be shown that this definition of an invariant time derivative is coordinate independent and allows for correct tensor transformations. SHOW IT

As a useful example to built intuition for this kind of derivative, let $\mathbf{R}$ be a "position vector", a point in the parametrized image of the embedding function in M. Hence $\mathbf{R}$ can be thought of as not directly time dependant: only dependant on time through its coordinate parametrization. We wish now to express in a geometrically interpretable way the invariant time derivative of $\mathbf{R}$. By its definition:

$$\dot{\nabla} \mathbf{R} = \frac{d}{dt} \mathbf{R} - V^{\alpha} \nabla_{\alpha} \mathbf{R} \tag{13}$$

Let now S^i be a set of ambient coordinate functions on the embedding of the manifold. Deriving through we get:

$$\dot{\nabla} \mathbf{R} = \frac{d}{dS^i} \mathbf{R} \frac{d}{dt} S^i(t, P) - V^{\alpha} \nabla_{\alpha} \mathbf{R} = V^i \nabla_i \mathbf{R} - V^{\alpha} Z_{\alpha}^{\ i} \nabla_i \mathbf{R} \tag{14}$$

$$= \left(V^i - V^{\alpha} Z_{\alpha}^{\ i}\right) \nabla_i \mathbf{R} \tag{15}$$

However:

$$V^{\alpha} Z_{\alpha}^{\ i} = V^j Z_j^{\alpha} Z_{\alpha}^i = V^j \left(\delta_j^i - N^i N_j\right) = V^i - V^j N^i N_j \tag{16}$$

Prove.

Hence:

$$\dot{\nabla} \mathbf{R} = V^j N^i N_j \nabla_i \mathbf{R} = C N^i \nabla_i \mathbf{R} \tag{17}$$

Which may be interpreted as the speed of the point in the normal direction. This same result may be used to better express the invariant time derivative of any function of time and coordinates:

$$\dot{\nabla} T(t, S(t, B)) = \frac{d}{dt} T(t, S(t, P)) - V^{\alpha} \nabla_{\alpha} T(t, S(t, D)) = \frac{d}{dt} T(t, S(t, P)) + C N^i \nabla_i T(t, S(t, P)) \tag{18}$$

We now get to the last two results of the calculus of moving surfaces which will be used in the problem. These are the rules for the derivation of integral relations. One refers to volume integration and is completely analogous to the fundamental theorem of calculus while the second does not have a very intuitive meaning. The rules are:

$$\frac{d}{dt} \int_{\Omega} F d\Omega = \int_{\Omega} \frac{d}{dt} F d\Omega + \int_S C F dS \tag{19}$$

Where F is any kind of function on the embedded manifold, Ω is a subset of the manifold and S is the boarder of Ω. This formula can be intuitively interpreted as adding the contribution of a small change in the volume by integrating on its surface multiplied by its velocity. The second equation is:

$$\frac{d}{dt} \int_S F dS = \int_S \dot{\nabla} F dS - \int_S C B_{\alpha}^{\alpha} F dS \tag{20}$$

where everything is as before and B_{α}^{α} is the curvature tensor.

3 Application to the problem

We now turn our attention from the theory of the calculus of moving surfaces to the problem we originally had: that of the solution of the eigenvalue problem for the laplacian in regular polygons. The solution reported here is due to P. Grinfeld, and G. Strang and was originally published in [3]. The goal is to find an analytical expression for the first few terms of the expansion of the eigenvalue in terms of the variable $\frac{1}{N}$, N being the number of sides of the regular polygon. This will be acheived using considerations from the calculus of moving surfaces, and particularly equation 19.

3.1 General strategy

Before attempting a solution we should first provide an outline of the preceidure we will use. First of all we restate the main equation of the problem:

$$\Delta\psi = -\lambda\psi \tag{21}$$

We added a minus sign to make it so that λ the eigenvalue now be positive. This equation we wish to study perturbatively from the solution of a circle, which is known and was reported in a previous section. It is worth noting that we are operating under the assumption that by our problem is well behaved under the idea that by increasing the number of sides of the polygon the solution ψ becomes closer and closer to that of the circle. However founded this assumption may intuitively seem, sometimes in the past it has failed. A notable example that was pointed out by R. Jones [4] comes from thin plate theory and is known as the *polyon-circle paradox*. This paradox is reported in [5]. This being said, we seek an expansion of λ such as:

$$\lambda = \lambda_0 \left(1 + \frac{c_1}{N} + \frac{c_2}{N^2} + \frac{c_3}{N^3} + \dots\right) \tag{22}$$

hence our main objective will be to find an expression for the coefficients c_i. To accomplish this feat we start by looking for a homotopy $\Phi : I \times [0,1] \to \mathbb{R}^2$ suche that: $\Phi(\theta, 0)$ is the parametrization of the circle, and $\Phi(\theta, 1)$ is the parametirzation of the polygon. Different values for the second parameter (which we will henceforth refer to as time) will identify different curves which will be transistioning forms between the polygons and the circle. Further will be said on the choice of homotopy in a following section.

Having found the homotopy expression for the boudary we can incorporate it in the original problem as a boundary condition dependant on t. Our solution ψ will thus depend on this parameter as well. If our choices are sufficiently well behaved we will also be able to apply calculus to ψ also in this new variable, hence, hopefully we are able to construct a Maclaurin expansion of ψ in terms of the variable t. Evaluating it at $t = 1$ we will then have an expression which we will be able to express as a series of $\frac{1}{N}$.

We will show that not everything however is so simple. As it is not always possible to express what we need. In such cases we will Taylor expand over the appropriate variable in order to better arrive at the solution we are seeking.

3.2 Hadamard's term

Having established how we intend to approach the problem let us start by differentiating with respect to time equation 21. Assuming everything is well behaved enough that we can apply Shwartz's lemma we get:

$$\Delta\partial_t\psi = \lambda'\psi + \lambda\partial_t\psi \tag{23}$$

Where λ' indicates the time derivative of λ. It is a remarkable result of the calculus of moving surfaces that:

$$\lambda' = -\int_{\partial\Omega} C \langle\nabla\psi, \nabla\psi\rangle \, dS \tag{24}$$

C being the interface velocity of the boundary. Because this relation was found by Hadamard we shall refer to λ' as the Hadamard term. As suggested by Strang and Grinfeld to remove this leading term one should keep the area constant. It is intuitive to see why it should be so if only C were inside the integral: the integral acts as a mediator on all of the small displacements of the curve, thus, if the total (signed) area is null, thus should be the integral. This intuition may be corroborated by the following argument: suppose we seek to calculate the area at a time t of the subset Ω enclosed by our curve. This may be expressed as an integral as:

$$A(t) = \int_{\Omega} d\Omega \tag{25}$$

Deriving through by t, we apply equation 19 and find that, because the integrand is 1:

$$A(t)' = \int_{\partial\Omega} C dS \tag{26}$$

It is a little obvious to show that this holds also in the case where we include $|\nabla\psi|^2$. Recall that we are seeking the first term in a perturbation series of the eigenvalue solution for the laplacian in a unit circle. Hence we only only need to evaluate 24 in that case. It was prooven however that:

$$\frac{d}{dr}\psi(1) = \frac{\rho}{\sqrt{\pi}} \tag{27}$$

ρ being such that $\rho^2 = \lambda_0$. Hence, as:

$$\begin{cases} \dfrac{d}{dx} = \cos\theta\dfrac{d}{dr} - \dfrac{\sin\theta}{r}\dfrac{d}{d\theta} \\ \dfrac{d}{dy} = \sin\theta\dfrac{d}{dr} + \dfrac{\cos\theta}{r}\dfrac{d}{d\theta} \end{cases} \tag{28}$$

we can easily find that:

$$\langle\nabla\psi(1),\nabla\psi(1)\rangle = \left(\frac{d}{dx}\psi(1)\right)^2 + \left(\frac{d}{dy}\psi(1)\right)^2 = \left(\frac{d}{dr}\psi(1)\right)^2 + \left(\frac{1}{r}\frac{d}{d\theta}\psi(1)\right)^2 = \left(\frac{d}{dr}\psi(1)\right)^2 = \frac{\lambda_0}{\pi} \tag{29}$$

In the last equality the fact that $\psi(r)$ is independant of the angle θ. As that value is constant it can be brought out of integration the integration and we get yet again the case where only C is to be integrated. We intend now to proove equation 24. Before doing so however it is necessary to prove another simple result which will be used in the proof of 24.

Lemma 1.

$$\int_\Omega \psi\frac{d\psi}{dt}d\Omega = 0 \tag{30}$$

Proof. We start with the normalization condition:

$$\int_\Omega |\psi|\, d\Omega = 1 \tag{31}$$

We wish now derive now by time. As our boundary is dependant on time as well as the function calculating this derivative is not so simple and requires an application of the calculus of moving surfaces. Specifically it is exactly the case of equation 19.

$$\int_\Omega \frac{d}{dt}|\psi|^2\, d\Omega + \int_{\partial\Omega} C\,|\psi|\, dS = 0 \tag{32}$$

Under the Dirichlet boundary conditions, the second term on the left hand side is null ($x \in \partial\Omega \implies \psi(x) = 0$). Expanding the derivative we get that ψ and $\dfrac{d\psi}{dt}$ are orthogonal in $\mathcal{L}^2(\mathbb{R}^2)$. ☐

We now proceed to prove the validity of the formula for Hadamard's term.

Theorem 1. *24 holds.*

Proof. We start by expressing λ as a Rayleigh quotient with unit denominator:

$$\lambda = \int_\Omega \langle\nabla\psi,\nabla\psi\rangle\, d\Omega \tag{33}$$

This formula may be obtained as follows: multiply the eigenvalue equation by ψ and integrate over Ω. As we can take ψ to be unitarely normed the equation reads:

$$\lambda = -\int_\Omega (\Delta\psi)\psi d\Omega \tag{34}$$

By applying Green's first identity we find:

$$\lambda = \int_\Omega \langle \nabla\phi, \nabla\phi \rangle \, d\Omega - \int_{\partial\Omega} \phi \nabla \phi dS \tag{35}$$

Because of the boundary conditions the second integral on the right side of the equation is zero. Hence, we have proven equation 33. Deriving through by time we get:

$$\lambda' = \frac{d}{dt} \int_\Omega \langle \nabla\psi, \nabla\psi \rangle \, d\Omega \tag{36}$$

which, by application of equation 19, can be expanded as:

$$\lambda' = 2 \int_\Omega \left\langle \nabla\psi, \frac{d}{dt}\nabla\psi \right\rangle d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{37}$$

$$= 2 \int_\Omega \left[\nabla(\psi \nabla \frac{d\psi}{dt}) - \psi \Delta \frac{d\psi}{dt} \right] d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{38}$$

$$= 2 \int_S \psi \nabla \frac{d\psi}{dt} dS - 2 \int_\Omega \psi \Delta \frac{d\psi}{dt} d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{39}$$

The first integral vanishes due to the boundary conditions, and we may express

$$\Delta \frac{d\psi}{dt} = -\frac{d\lambda\psi}{dt} = -\lambda'\psi - \lambda \frac{d\psi}{dt} \tag{40}$$

So, plugging this into equation 39,

$$\lambda' = 2 \int_\Omega \psi \left(\lambda'\psi + \lambda \frac{d\psi}{dt} \right) d\Omega + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{41}$$

$$= 2\lambda' + \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{42}$$

$$\lambda' = - \int_{\partial\Omega} C \langle \nabla\psi, \nabla\psi \rangle \, dS \tag{43}$$

Where we used lemma 1 for the first equality. □

3.3 Homotopy

The choice of a correct homotopy is a very delicate a subtle process. In general, there are a great number of such transformations which might be used, but due to interrelation they might be prohibitive. In the optimum case, one would like to find a homotopy which is area conserving at all times, in order to eliminate Hadamard's term. It is not too hard to write a homotopy mapping the circle to a polygon with equal area, however it is hard to write one conserving the area at all times, i.e. such that:

$$A(t)' = \frac{d}{dt} \int_\Omega d\Omega = 0 \tag{44}$$

Such a homotopy would have some points moving inward and some points moving outward in such a way that the total signed area *averages out*. At the end of the transformation process some of the points of the side of the polygon will be inside the original circle, while the vertices and some other points will be on the outside. Grinfeld and Strang leave this problem to be solved by posterity. In the mean time, they consider a homotopy such that each point moves radially to the inscribed polygon with constant speed. With reference to figure 1 one can write:

$$d(\theta, N) = 1 - \frac{\cos\left(\frac{\pi}{N}\right)}{\cos(\theta)} \tag{45}$$

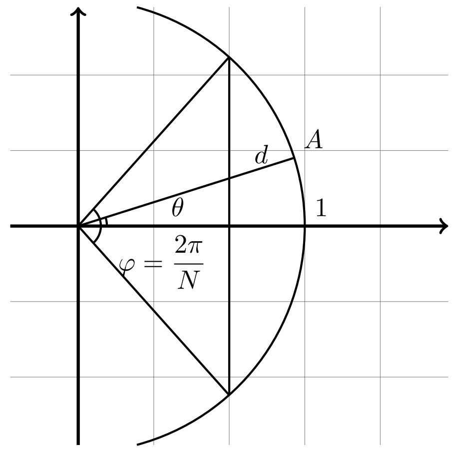

Figure 1: Arc of a circle with side of N-gon. The distance traveled by point A is d.

As each point is moving with constant speed, the speed of each point is proportional to the distance traveled. We set the proportionality constant equal to minus one. Hence, each point moves with speed $V(\theta, N) = -d(\theta, N)$. One can parametrize the homomorphism as:

$$\gamma(\theta, N, t) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} (1 + tV(\theta, N)) \tag{46}$$

This way, for $t = 0$ we return to a parametrization of the circle and for $t = 1$ we get that of the polygon. It should be noted that this function is not valid for all θ; it is only valid for $\theta \in \left[-\frac{pi}{N}, \frac{pi}{N}\right]$. To get a complete curve one must extend this result.

4 Interface velocity

We now seek to calculate C in order to use Hadamard's formula to get the first variation. The tangent to the curve is:

$$T(\theta, N, t) = \frac{d\gamma(\theta, N, t)}{darg2} \tag{47}$$

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