1. Find an equation of the tangent plane to the graph of

$$G(u, w) = \sin(uw)$$

at $(\pi/6, 1)$.

Solution: We first compute the partial derivatives of G, finding

$$G_u(u, w) = w\cos(uw),$$
$$G_w(u, w) = u\cos(uw).$$

Then we evaluate each of G, G_u, G_w at the point $(\pi/6, 1)$, obtaining

$$G(\pi/6, 1) = 1/2, \quad G_u(\pi/6, 1) = \frac{\sqrt{3}}{2}, \quad G_w(\pi/6, 1) = \frac{\pi}{4\sqrt{3}}.$$

Hence, the desired equation is

$$z = L(u, w) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) + \frac{\pi}{4\sqrt{3}}\left(y - 1\right).$$

2. Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\mathbf{n} = \langle 3, 2, 2\rangle$ is normal to the tangent plane.

Solution: Evidently, the tangent plane to the graph of any function $g(x, y)$ at the point $(a, b, g(a, b))$ has the normal vector $\langle g_x(a, b), g_y(a, b), -1\rangle$. For us $g(x, y) = 3x^2 - 4y^2$, and so the normal vector becomes $\langle 6a, -8b, -1\rangle$. If $\mathbf{n}$ is to be normal to the tangent plane, it must be parallel (not necessarily equal) to this vector; that is, there must be some nonzero scalar λ satisfying

$$\lambda\langle 3, 2, 2\rangle = \langle 6a, -8b, -1\rangle.$$

Equating entries of this vector we find that

$$3\lambda = 6a,$$
$$2\lambda = -8b,$$
$$2\lambda = -1.$$

We see from the last equation that λ must be equal to $-1/2$. It follows immediately that $a = -1/4$ and $b = 1/8$. Hence, there is only one point on the graph of $g(x, y)$ at which the tangent plane is normal to $\mathbf{n}$: namely, $(-1/4, 1/8, 1/8)$.

3. Use the linear approximation of $f(x,y) = e^{x^2+y}$ at $(0,0)$ to estimate $f(0.01,-0.02)$. Compare with the value obtained using a calculator.

Solution: First, we find the linear approximation $L(x,y)$ of $f(x,y)$ at $(0,0)$. The partial derivatives are

$$f_x(x,y) = 2xe^{x^2+y},$$
$$f_y(x,y) = e^{x^2+y},$$

with values $f_x(0,0) = 0$ and $f_y(0,0) = 1$. Since $f(0,0) = 1$, we have

$$L(x,y) = 1 + y.$$

We conclude that $f(0.01,-0.02) \approx L(0.01,-0.02) = 1 - 0.02 = 0.98$. *Mathematica* gives an approximate numerical value of 0.980297 for $f(0.01,-0.02)$. The approximation is accurate to within 0.04%.

4. Use the linear approximation to $f(x,y) = \sqrt{x/y}$ at $(9,4)$ to estimate $\sqrt{9.1/3.9}$.

Solution: First, we find the linear approximation $L(x,y)$ of $f(x,y)$ at $(9,4)$. The partial derivatives are

$$f_x(x,y) = \frac{1}{2\sqrt{xy}},$$
$$f_y(x,y) = -\frac{\sqrt{x}}{2y\sqrt{y}}.$$

The alert reader will have noticed that these formulas are invalid at $(0,0)$. This is true, but there is no difficulty in using them at other points, in particular $(9,4)$, since f is differentiable in a small enough neighborhood of this point. The values of the partial derivatives are

$$f_x(9,4) = 1/12, \quad f_y(9,4) = -3/16.$$

Since $f(9,4) = 3/2$, we find the linear approximation at $(9,4)$ to be

$$L(x,y) = \frac{3}{2} + \frac{1}{12}(x-9) - \frac{3}{16}(y-4).$$

The fundamental property of the linear approximation is that $L(x,y) \approx f(x,y)$ near $(9,4)$, so it is legit to say that

$$f(9.1,3.9) \approx L(9.1,3.9) = \frac{3}{2} + \frac{1}{120} + \frac{3}{160}.$$

This last value is about 1.52708, whereas $f(9.1,3.9)$ is about 1.52753. This is correct to within 0.03%.