Variation of parameters

Math 352 Differential Equations

The College of Idaho

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Introduction: This one goes out to all the gs

The method of undetermined coefficients is very useful when the derivatives of the terms appearing in $g(t)$ have a regular, predictable shape and are themselves the derivatives of similar functions. But it's not too hard to find functions $g(t)$ that would demand excessive ingenuity of us when guessing the form of Y.

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It's hard to imagine rules for undetermined coefficients that are both

- comprehensive enough to include a function like this and
- possible to commit to memory.

Variation of parameters: not a new thought

Recall how we discovered the second fundamental solution te^{rt} in the case of a repeated root r of the characteristic equation. We promoted a constant to a functional coefficient and looked for a solution of the form $v(t) \exp(rt)$.

- In other words, we allowed a parameter to vary.

We can try a similar idea when tackling the inhomogeneous equation $y'' + q(t)y' + r(t)y = g(t)$, provided we have already solved the associated homogeneous equation $y'' + q(t)y' + r(t)y = 0$. Let us write y_1 and y_2 for a fundamental set of solutions as usual.

Variation of parameters: execution

We shall search for a particular solution Y of our inhomogeneous equation of the form $Y = u_1 y_1 + u_2 y_2$: that is to say,

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where u_1 and u_2 are unknown coefficient functions we will have to find. We'll be plugging this expression for Y along with its derivatives back into the inhomogeneous equation, so let's compute the derivatives now.

An unjustified assumption

For no very good reason, let's assume that the fundamental solutions y_1 and y_2 and the derivatives u_1' and u_2' satisfy the following equation:

$$u_1' y_1 + u_2' y_2 = 0.$$

- This is the worst rabbit-out-of-the-hat of the term.
- Sorry.
- The idea is evidently due to Lagrange[1], and it greatly simplifies the computation to come.
- At no point does it cause problems of any kind.

[1] Joseph-Louis Lagrange (25 January 1736–10 April 1813) is most remembered for contributions analysis, number theory, and classical and celestial mechanics. His *Mécanique Analytique* was fundamental for the physicists of the 19th century.

The derivatives of Y

Since

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

we also have

$$\begin{aligned} Y' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2', \\ &= u_1y_1' + u_2y_2', \quad \text{and so} \\ Y'' &= u_1y_1'' + u_1'y_1' + u_2'y_2' + u_2y_2''. \end{aligned}$$

Substitute back in

Now we find that $Y''(t) + q(t)Y'(t) + r(t)Y(t)$ reduces to

$(u_1y_1'' + u_1'y_1' + u_2'y_2' + u_2y_2'') + q(t)(u_1y_1' + u_2y_2') + r(t)(u_1y_1 + u_2y_2)$.

Collecting terms along u_1, u_2, and their derivatives, we get

$$\begin{aligned}(y_1'' + qy_1' + ry_1)u_1 + (y_2'' + qy_2' + ry_2)u_2 + y_1'u_1' + y_2'u_2' \\ = y_1'u_1' + y_2'u_2',\end{aligned}$$

because of the assumption $u_1'y_1 + u_2'y_2 = 0$.

Putting it all together

We have shown that if u_1 and u_2 have the desired properties, then they satisfy

$$\begin{aligned} u_1'y_1' + u_2'y_2' &= g(t) \\ u_1'y_1 + u_2'y_2 &= 0. \end{aligned}$$

Solving this system of equations for u_1 and u_2 (in the usual algebraic way, by either substitution or addition of equations) gives

$$u_1' = -\frac{y_2 g}{W(y_1, y_2)}, u_2' = \frac{y_1 g}{W(y_1, y_2)}.$$

Integrating each of these yields the desired functions u_1 and u_2.

The particular solution

That is, a particular solution Y results from any choice of antiderivatives u_1, u_2. We have

$$Y = -y_1 \int \frac{y_2 g}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g}{W(y_1, y_2)}\, dt.$$

Like undetermined coefficients, this method has its own characteristics.

- It is reassuringly mechanistic. Y is given by a formula.
- It is also very general, since there are no conditions on g.
- On the other hand, the integrals that arise may be intractable.

Generalizations

For us, the coefficient functions $q(t)$ and $r(t)$ are constant. What happens when they are not?

- The methods of Chapter 5 are required to solve the general second-order linear homogeneous equation.
- If fundamental solutions are known, VP works just the same to determine a particular solution.
- It is only the characteristic equation and the exponential trick that fail in this case.