A brief introduction to complex numbers for solving linear equations

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$$ay'' + by' + cy = 0.$$

- Monday, we saw how to find solutions of exponential type $y = ce^{rt}$. The growth constants r that "work" are the roots of the characteristic equation $ar^2 + br + c = 0$.

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- When $b^2 - 4ac > 0$ (the characteristic polynomial has *positive discriminant*), we get two values r_1 and r_2.
- Hence we obtain two solutions, $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$.

Recap

We also know that solutions of *linear homogeneous equations* can be linearly combined to get more solutions. Therefore, we have a 2-dimensional family of solutions

$$c_1 y_1 + c_2 y_2.$$

Exercise

Use the Wronskian to show that when $b^2 - 4ac > 0$, the solutions y_1 and y_2 described above generate all solutions to $ay'' + by' + cy = 0$.

Negative discriminant

None of this tells what happens when $b^2 - 4ac < 0$. The existence theorem still applies, so the solutions have to be out there somewhere. If we expand our thinking to the realm of *complex numbers*, we can get them using the same trick.

Definition

A *complex number* is a symbol $x + iy$, where x and y are ordinary (i.e., real) numbers. We adopt two conventions:
(1) these numbers obey all the rules of arithmetic;
(2) $i^2 + 1 = 0$.

Why bother?

Our polynomial trick relies on getting the roots of $ar^2 + br + c = 0$. They only exist in the extended algebraic system of complex numbers. You are probably aware that polynomials such as $x^2 + 9 = 0$ admit solutions in the complex numbers. There is a somewhat more general statement to make.

Theorem

(Fundamental Theorem of Algebra.) Each polynomial of degree n with complex coefficients has exactly n complex roots, counting multiplicities. Equivalently, each such polynomial splits over the complex numbers into n (not necessarily distinct) linear factors.

This theorem was known to Euler, but it is widely asserted that the first satisfactory proofs were due to Gauss (1777–1855).

Getting solutions

The quadratic formula works in complete generality in the complex context. We will always assume that a, b, and c (the coefficients in our differential equation) are real. This avoids some interesting difficulties with the complex square root. Observe that if $D < 0$, then $D = -|D| = i^2|D|$.

Square roots of negative numbers

We agree to interpret (when $D = b^2 - 4ac < 0$) the symbol $\sqrt{D}$ as:

$$\sqrt{D} = \sqrt{i^2|D|} = i\sqrt{|D|}.$$

The complex exponential

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WTF?

What's e^{rt} when r is complex?

The answer is found in what is called Euler's formula.

Euler's formula

A historical quotation on Euler's formula:

Benjamin Peirce (1809–1880)

"Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth."

Euler's formula

Since r is complex, it is a symbol of the form $x + iy$. Therefore, since t is real, we get

$$e^{rt} = e^{(x+iy)t} = e^{xt}e^{iyt}.$$

This is assuming the addition law for a function we haven't really defined, but if it didn't obey the addition law, we couldn't bear to call it the exponential function.

Euler's formula

This famous and celebrated equation tells us how to interpret the e^{iyt} portion of our exponential expression.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus $\exp(rt) = \exp(xt + iyt) = e^{xt}(\cos(yt) + i\sin(yt))$.

Euler's formula

Putting $\theta = \pi$, we obtain a special case, often referred to as Euler's formula.

Also Euler's formula

$$e^{\pi i} = -1.$$

Respect.