Oscillatory solutions to second-order homogeneous linear equations

D. Rosoff

College of Idaho

18 March 2013

Warm-up

Recall the formula from last time:

Warm-up

Recall the formula from last time:

Euler's formula

$$e^{\lambda + i\mu} = e^{\lambda}(\cos\mu + i\sin\mu).$$

Warm-up

Recall the formula from last time:

Euler's formula

$$e^{\lambda+i\mu} = e^{\lambda}(\cos\mu + i\sin\mu).$$

Respect. Use Euler's formula to write each of the following complex numbers in the form $a+ib$, called the *rectangular form* of the number.

$$\begin{array}{rl} \exp(1+2i) & \exp(2-3i) \\ \exp(2-(\pi/2)i) & \exp 2\pi i \end{array}$$

Last time

Last time we introduced enough complex algebra to solve the linear homogeneous differential equation

$$ay'' + by' + cy = 0$$

when $D = b^2 - 4ac < 0$.
You used Euler's formula and the standard "exponential trick".

Last time

Specifically, you found two *complex-valued* solutions to $ay'' + by' + cy = 0$. If r_1 and r_2 are the two complex roots of the characteristic equation, we can write $r_j = \lambda \pm i\mu$. Then Euler's formula gives

$$\begin{aligned} y_1 &= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\ y_2 &= e^{\lambda t}(\cos(-\mu t) + i\sin(-\mu t)) \\ &= e^{\lambda t}(\cos(\mu t) - i\sin(\mu t)), \end{aligned}$$

where in the last step we have used the identities

$$\cos(-x) = \cos(x) \text{ and } \sin(-x) = -\sin(x).$$

Getting real solutions

Of course we are most interested in real-valued solutions.
You found that

$$u = \frac{y_1 + y_2}{2},$$
$$v = \frac{y_1 - y_2}{2i}$$

are real-valued (all the i-stuff cancels away). These functions are evidently linear combinations of solutions. So they are solutions to the differential equation: that is, we have

$$au'' + bu' + cu = 0, \quad av'' + bv' + cv = 0.$$

Fundamental solutions when $D < 0$

The method you stepped through in the worksheet is completely general, and yields a fundamental system of solutions (you will check this for yourself in the exercises).

- First, find the roots of the characteristic polynomial. They have the form $\lambda \pm i\mu$, for real numbers λ and μ.
- Write down $u = e^{\lambda t}\cos(\mu t)$ and $v = e^{\lambda t}\sin(\mu t)$.
- The Wronskian of u and v is nonzero, so every solution to $ay'' + by' + cy = 0$ is a linear combination of u and v.
- If necessary, use initial conditions to determine the appropriate coefficients on u and v. Since the Wronskian is nonzero, there is exactly one choice of coefficients.

Exercises

Find the general solution of the differential equation.

- $y'' - 2y' + 6y = 0$
- $y'' + 2y' + 2y = 0$
- $4y'' + 9y = 0$
- $9y'' + 9y' - 4y = 0$
- $y'' + 4y' + (25/4)y = 0$

Solve the initial value problem.

- $y'' + 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0.$
- $y'' + y = 0, \quad y(\pi/3) = 2,\ y'(\pi/3) = -4.$