Systems of equations

Math 352 Differential Equations

The College of Idaho

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Converting to a system

We can convert the second-order homogeneous linear equation

$$mu'' + \gamma u' + ku = 0$$

into a 2×2 system by letting $x_1 = u$, $x_2 = u'$. The equation above then becomes

$$mx_2' + \gamma x_2 + kx_1 = 0,$$

so a 2×2 system whose solutions are the same as the second-order equations' solutions is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 \end{aligned}$$

Warm-up

Solve 1, 3, 5 from section 7.1. When you have a choice to make, always write the choice with *fewer primes*. When you are transforming an initial value problem, don't forget to transform the initial conditions!

Algorithm for solving 2×2 homogeneous

- Obtain coefficient matrix A
- Find eigenvalues: the roots of $\det(A - \lambda I)$
- Solve for the coefficient vectors: the eigenvectors of A
- Write down the solution functions.

Example

Consider the system of differential equations

$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= 4x_1 + x_2
\end{aligned}
$$

In matrix form, this becomes

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}.$$

Write A for the coefficient matrix.

- Check that $\det A - \lambda I = \lambda^2 - 2\lambda - 3$, so that the eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -1$.

Getting the eigenvectors

- Solve the modified system for each eigenvalue to obtain eigenvectors $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$.

The modified system is

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- When $\lambda = 3$, we see that the system reduces to the equation $-2\xi_1 + \xi_2 = 0$, so an eigenvector for $\lambda = 3$ is

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The other eigenvector

When $\lambda = -1$, we see that the system becomes

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Therefore an eigenvector for $\lambda = -1$ is $\vec{\xi}^{(2)} = (1, -2)$.

Putting it all together

- The general solution to the system is, in the "nice" case, the general linear combination of the eigenvectors multiplied by the exponentials with growth constants given by their eigenvalues:

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

- Finally, we can write the solutions in scalar form. The functions x_1 and x_2 are the entries of the vector $\vec{x}$.

$$x_1 = c_1 e^{3t} + c_2 e^{-t}, \quad x_2 = 2c_1 e^{3t} - 2c_2 e^{-t}.$$

Nice matrices

What makes a matrix "nice" is too involved for us to describe completely. If the matrix has as many eigenvalues as it does columns, then it is automatically "nice".

- If $\det A - \lambda I$ factors into *distinct* linear factors, then A is "nice".
- In more technical terms A is *diagonalizable* if and only if A is "nice".

Algorithm, again

- Obtain coefficient matrix A.
- Find eigenvalues: the roots of $\det(A - \lambda I)$.
- Solve for the coefficient vectors: the eigenvectors of A.
- Write down the solution functions x_1 and x_2.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \vec{\xi}^{(1)} e^{\lambda_1 t} + c_2 \vec{\xi}^{(2)} e^{\lambda_2 t}.$$

- Solve 1–7 odd, part (a) only, in section 7.5.