Mathematics 352
Undetermined Coefficients

April 5, 2013 Name: ______________________________

Due: April 10, 2013

> **Introduction.** In this activity you will get some practice using the method of undetermined coefficients to solve inhomogeneous differential equations $ay'' + by' + cy = g(t)$ in which the inhomogeneous term $g(t)$ is a combination of polynomial, exponential, or trigonometric functions.

Recall that the general solution to the inhomogeneous differential equation $ay''+by'+cy = g(t)$ is y_c+Y, where the *complementary solution* $y_c = c_1y_1 + c_2y_2$ is the general solution of the *associated homogeneous equation* and Y is *any* particular solution of the inhomogeneous equation.

1. Find the *complementary solution* of $y'' - 4y' - 12y = 3e^{5t}$. This is always the first step. You should always do this *before* attempting to find a particular solution Y.

> **Solution:** The complementary solution is given by the roots of the characteristic equation for the associated homogeneous system, $r^2 - 4r - 12 = 0$. The left-hand side factors as $(r-6)(r+2)$, so a fundamental set of solutions is $y_1 = e^{6t}$ and $y_2 = e^{-2t}$, and therefore the complementary solution is
> $$y_c = c_1e^{6t} + c_2e^{-2t}.$$

2. Use the method of undetermined coefficients to find a particular solution Y of $y'' - 4y' - 12y = 3e^{5t}$.

> **Solution:** Since e^{5t} does *not* appear in the complementary solution, we look for a solution of the form Ae^{5t}. Letting Y equal this last expression and differentiating, we find $Y' = 5Y$ and $Y'' = 25Y$. Therefore, on substitution of Y into the left-hand side of the differential equation, we obtain $25Y - 4(5Y) - 12Y = -7Y$. Therefore we must have $-7A = 3$, whence $A = -3/7$. The particular solution so obtained is
> $$Y = -3/7e^{5t}.$$

3. Check, by direct substitution, that your function Y really is a particular solution.

> **Solution:** We have $Y' = -15/7e^{5t}$ and $Y'' = -75/7e^{5t}$. We compute:
> $$\begin{aligned} Y'' - 4Y' - 12Y &= (-75/7 - 4(-15/7) - 12(-3/7))e^{5t} \\ &= (-75/7 + 60/7 + 36/7)e^{5t} \\ &= (21/7)e^{5t} \\ &= 3e^{5t}, \end{aligned}$$

as required.

4. No further calculation is necessary to write down the general solution of the inhomogeneous equation: do so below.

Solution: The general solution is equal to the complementary solution y_c plus the particular solution Y, namely

$$c_1 e^{6t} + c_2 e^{-2t} - \frac{3}{7} e^{5t}.$$

5. Use the method of undetermined coefficients to find a particular solution Y of $y'' - 4y' - 12y = \sin(2t)$.

Solution: Here the method calls for a particular solution of the form $Y = A\cos(2t) + B\sin(2t)$. Differentiating, we find

$$Y' = 2B\cos(2t) - 2A\sin(2t)$$
$$Y'' = -4A\cos(2t) - 4B\sin(2t).$$

Substituting, we observe now that

$$\begin{aligned} Y'' - 4Y' - 12Y &= (-4A - 4(2B) - 12A)\cos(2t) + (-4B - 4(-2A) - 12B)\sin(2t) \\ &= (-16A - 8B)\cos(2t) + (8A - 16B)\sin(2t). \end{aligned}$$

Equating like coefficients, we obtain a 2×2 system of linear equations

$$\begin{aligned} -16A - 8B &= 0 \\ 8A - 16B &= 1. \end{aligned}$$

The first equation entails that $B = -2A$. Substitution into the second yields $8A - 16(-2A) = 1$, whence $A = 1/40$. Hence $B = -1/20$ and the particular solution we are seeking is

$$\frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t).$$

6. Verify, by direct substitution, that Y really is a particular solution.

Solution: Left to the reader.

7. Use the method of undetermined coefficients to find a particular solution Y of $y'' - 4y' - 12y = 2t^3 - t + 3$.

Solution: Here, the particular solution takes the form $Y = At^3 + Bt^2 + Ct + D$. The derivatives are

$$\begin{aligned} Y' &= 3At^2 + 2Bt + C \\ Y'' &= 6At + 2B. \end{aligned}$$

Substitution yields

$$\begin{aligned} Y'' - 4Y' - 12Y &= (6At + 2B) - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) \\ &= -12At^3 + (-12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D). \end{aligned}$$

Equating the coefficients of the various monomials then gives the system of linear equations

$$\begin{aligned} -12A &= 2 \\ -12A - 12B &= 0 \\ 6A - 8B - 12C &= -1 \\ 2B - 4C - 12D &= 3. \end{aligned}$$

This is tedious, but there's no helping it. Evidently, $A = -1/6$ and $B = 1/6$. The third equation then becomes $-1-4/3-12C = -1$, whence $C = -1/9$. Finally, the last equation is $1/3+4/9-12D = 3$, so that $-12D = 3 - 7/9 = 20/9$ and $D = -5/27$. The desired particular solution is

$$-\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}.$$

8. Verify, by direct substitution, that Y really is a particular solution.

Solution: Left, joyfully, to the reader, as are the rest.

9. Write down the form (including the undetermined coefficients) of initial guesses for the particular solutions if $g(t)$ is the indicated function.
   (a) $g(t) = 16e^{7t}\sin(10t)$
   (b) $g(t) = (9t^2 - 103t)\cos(t)$
   (c) $g(t) = -e^{-2t}(3-5t)\cos(9t)$

10. Solve the initial value problem

$$y'' - 4y' - 12y = 3e^{5t}, \quad y(0) = 18/7,\ y'(0) = -1/7$$