Matrix-vector equations

Math 352 Differential Equations

The College of Idaho

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Recap

Recall that the equation $y' = ay$ has general solution of exponential type:

$$y = ce^{at}.$$

A system of such equations also has a general solution of exponential type, as we shall see.

Matrices and vectors

We've seen how the $m \times n$ system of equations

$$\begin{aligned} x_1 + 4x_2 + 6x_3 &= -3 \\ 2x_1 - 2x_2 \qquad &= 7 \end{aligned}$$

corresponds to the augmented matrix

$$A = \begin{pmatrix} 1 & 4 & 6 & -3 \\ 2 & -2 & 0 & 7 \end{pmatrix}.$$

This is very convenient if we want to solve for the x_j, but there is another formulation we must also understand.

Matrix-vector form

- Instead of leaving the x_j out entirely, the *matrix-vector form* of the system compresses them into a vector $\vec{x} = (x_1, \ldots, x_n)$. Similarly, we view the constants b_i on the right-hand side as a vector $\vec{b} = (b_1, \ldots, b_m)$.
- According to our convention that vectors are always *column matrices*, $\vec{x}$ is $n \times 1$ while $\vec{b}$ is $m \times 1$.
- This means that, writing A for the $m \times n$ matrix of coefficients (*not* the same as the augmented matrix we used previously) the product Ax is defined, and has the same shape as the constant vector $\vec{b}$.

Matrix-vector form: reloaded

You can check, using the definition of matrix multiplication, that a list of solutions $x_1, \ldots, x_n$ to the $m \times n$ system of equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

is the same thing as a solution to the matrix-vector equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Matrix-vector form: unlocked

Typically, we would suppress the coefficients and just write the last huge mess as $A\vec{x} = \vec{b}$ where A, x, and b are as above. Remember: *solution* means what it always has (at least since week 4): something that, when you plug it in to the equation, makes it true.

- From the matrix-vector point of view, we aren't plugging in a whole list of x_j, but a single vector $\vec{x}$.
- General theory of matrix algebra (which we haven't time to develop) tells us that for a fixed A and $\vec{b}$, there are three possibilities: no solution, one solution, or infinitely many solutions.

Other questions about matrix-vector equations

Instead of thinking of A and $\vec{b}$ as fixed and asking about the set of solution vectors $\vec{x}$, we might ask

- For a fixed A, which $m \times 1$ vectors $\vec{b}$ occur as values of Ax?

This is another question we could answer by row-reduction techniques. But that is beyond the scope of this course.

Unpacking the matrix-vector form

Let's write A_j for the jth column of A regarded as a vector, so that

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Thus A_j is a $m \times 1$ matrix, just like the right-hand side $\vec{b}$ in the equation $A\vec{x} = \vec{b}$.

- You can check, using the definition of matrix multiplication, that $A\vec{x} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$.
- This representation is key in what follows.

Application to ODEs

Let's pass to the application of matrix theory we are interested in: systems of first-order linear ODEs. The algebra is similar to what we have done so far, but the x_j now must be regarded as differentiable *functions* of some usually unwritten variable t. The right-hand side $\vec{b}$ is replaced by the vector of derivatives of the x_j.

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- We usually write this as $\vec{x'} = A\vec{x}$. Observe that $\vec{x}$ is now a variable vector and $\vec{x'}$ stands for the vector of the derivatives.

Assumptions, I

- Like before, the coefficients must be constant for our methods to work. Assume that A has constant entries (they are not functions of t).
- We also assume that A is square, that is, that $m = n$. According to universal mathematical custom we write n for this common value.

Assumptions, II

- Just like we guessed that solutions of the original linear first-order differential equation $y' = ay$ would be of exponential type, we are going to guess the form of the entries of $\vec{x}$.
- In fact, just like we assumed the solutions of $y' = ay$ would be multiples of the exponential e^{at}, we'll assume the solutions of $x' = Ax$ are *vector* multiples of the exponential $e^{\lambda t}$ for suitable λ.
- The λ that "work" are called the *eigenvalues* of the matrix A.

Eigenvalues

- To find the eigenvalues of A, we solve the linear system of ordinary algebraic equations

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$