Undetermined Coefficients,I

Guess-and-check: expert level

Math 352 Differential Equations

The College of Idaho

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The homogeneous case

In the last couple weeks, we saw how to use the exponential trick and some inspired guessing to get general solutions of the general second-order linear homogeneous equation with constant coefficients,

$$ay'' + by' + cy = 0.$$

There were three distinct cases, corresponding to the three different possibilities $D = b^2 - 4ac > 0$, $D < 0$, and $D = 0$.

Inhomogeneity

One of the key algebraic coincidences that allowed us to use the exponential trick was the *superposition principle*. This is the principle that if y_1 and y_2 are solutions, then all their linear combinations are too. It fails spectacularly if we try it on an inhomogeneous equation, as illustrated in several weekly homework problems.

Inhomogeneity: the translation principle

But this doesn't mean that our work on the homogeneous equation won't help us solve equations of the form

$$ay'' + by' + cy = g(t). \tag{1}$$

This is because any two solutions of such an equation differ by a solution of the *associated homogeneous equation* $ay'' + by' + cy = 0$. That is to say, if Y_1 and Y_2 are both solutions of Equation 1, then $Y_1 - Y_2$ is a solution of the associated homogeneous equation.

The geometry of linear differential operators: homogeneous equations

It is best to think of the whole collection of solutions of the associated homogeneous equation as a set[1] S. A pair of fundamental solutions y_1, y_2 matches S with the Euclidean plane; namely, the point (c_1, c_2) corresponds to the solution $c_1 y_1 + c_2 y_2$. Practice visualizing the plane as embedded in a bigger-dimensional space (the space of all differentiable functions, maybe).

Observe that the point $(0, 0)$ is an element of our plane S, because the zero function is a solution of every homogeneous differential equation. Thus the plane you are imagining passes through the origin of whatever space it lives in.

[1]Many students have learned to find the word "set" unnerving, but it only serves to bind related objects into a conceptual whole. It is a "one" that embodies or instantiates a "many".

The geometry of linear differential operators: inhomogeneous equations

Since the inhomogeneous equation $ay'' + by' + cy = g(t)$ is a second-order equation, intuition and the theory of Wronskians tell us that there should be a pair of fundamental solutions. This sort of happens, but the details are a little different.

The solutions are still a plane

Solutions of the inhomogeneous equation correspond to points in a plane just like the solutions of the homogeneous equation do. The difference is, it's not the same plane.

The plane for inhomogeneous equations doesn't pass through the origin, because the zero function isn't a solution of any inhomogeneous equation.

The geometry of linear differential operators: particular solutions

Recall the translation principle for the inhomogeneous equation:

Translation principle

If Y_1 and Y_2 are solutions of the inhomogeneous equation, then their difference $Y_1 - Y_2$ is a solution of the associated homogeneous equation.

Inverted, it tells us how to construct new solutions of the inhomogeneous equation from a previously known one: add solutions of the associated homogeneous equation.

The general solution of the inhomogeneous equation

Let us suppose that by some devious method we have constructed a single solution of the inhomogeneous equation, say Y, so that $aY'' + bY' + cY = g(t)$. Let also y_1 and y_2 be a fundamental set of solutions of the associated homogeneous equation.

General solution of the inhomogeneous equation

Every function satisfying $ay'' + by' + cy = g(t)$ is of the form

$$Y + c_1 y_1 + c_2 y_2$$

for some numbers c_1 and c_2.

Thus these solutions also form a plane: a plane passing through the nonzero point Y.

Construction of particular solutions

As usual, this existence theorem doesn't tell us anything about how to construct Y, called a *particular solution* of the equation. We know from the general existence theorem for second-order initial value problems that each has a solution. Methods for finding Y vary and depend very much on the form of the inhomogeneous term $g(t)$. We will investigate two such methods: the first of these is *undetermined coefficients*.

Conclusion: examples good, table bad

By clever guessing with sufficient algebraic stamina, it's possible to build Y in a reasonably systematic and efficient way when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine. But I have one warning to give you.

Conclusion: examples good, table bad

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It is, very probably, a big mistake to try to memorize the table given in the text. I've watched people try and fail for years. It is a much better idea to learn the mental yoga of how the method works, by working through and thinking about lots of examples.