Mathematics 352
Exam 3

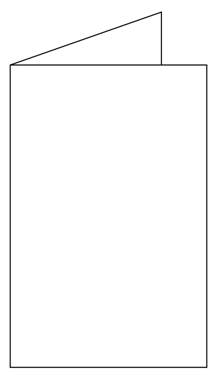
April 29, 2013; 60 minutes   Name: ____________________

This exam is closed book; you can use a calculator (*not* a mobile phone) but no other electronic aids or printed references. *If the wording or intent of any question is unclear, please ask me to clarify.* I am not trying to confuse you with the problem statements.

You can use your own paper or the provided blank copy paper. DO NOT WRITE ANYTHING YOU WANT GRADED ON THE EXAM PAPER. Please write your name on each page you hand in. Show all your reasoning and all pertinent calculations. *Give all answers in exact form. Decimal approximations of any accuracy will not receive full credit.*

When you have finished the exam, place this cover sheet on top of it and fold the packet in half *the long way, with your name facing out.*

YOU ARE STRONGLY ENCOURAGED TO READ ALL THE PROBLEMS BEFORE BEGINNING.



Question	Points	Score
1	20	
2	20	
3	0	
4	0	
Total:	40	

Good luck!

NO WORK ON THIS PAGE
ALL WORK ON SEPARATE PAGES

1. (20 points) Consider a mass-spring system. Suppose it is perturbed and released in such a way that the displacement function for the mass is the solution of the initial value problem

$$2u'' + 0.6u' + 0.45u = 0, \quad u(0) = 3, \quad u'(0) = 0.$$

Solve the initial value problem to find $u(t)$.

Solution: The characteristic equation is $2t^2 + 0.6t + 0.45 = 0$. Multiplying by 20 yields $40t^2 + 12t + 9 = 0$. The roots are therefore

$$r_1, r_2 = -\frac{3}{20} \pm \frac{9}{20} i.$$

The solution we want is therefore of the form

$$u = c_1 e^{-3t/20} \cos\left(\frac{9}{20} t\right) + c_2 e^{-3t/20} \sin\left(\frac{9}{20} t\right).$$

Applying the first initial condition yields $c_1 = 3$. Making this change and computing u', we find

$$u' = \left(\frac{9}{20} c_1 - \frac{9}{20} e^{-3t/20}\right) \cos\left(\frac{9}{20} t\right) + \text{multiple of } \sin\left(\frac{9}{20} t\right),$$

so that $c_1 = 1$. Putting it all together, the desired solution is

$$\left(\sin\left(\frac{9}{20} t\right) + 3\cos\left(\frac{9}{20} t\right)\right) e^{\left(-\frac{3}{20} t\right)}.$$

2. (20 points) *In this problem, assume that the acceleration g due to gravity is exactly* $10\,\mathrm{m/s^2}$.

   (a) Consider a spring-mass system with a mass of 6 kg that stretches the spring a length of 5/3 m when first attached. Suppose this system operates in a viscous medium that provides 60 N of resistive force to a mass moving at 2 m/s.

   Find the position function for the motion that results if the mass is moved 2 m below its equilibrium position and then released. Show all work. Classify the motion as overdamped, underdamped, or critically damped.

Solution: We are given $m = 6$. To find γ, we recall that *viscous* means that the damping force is linearly dependent on the speed (and γ is the proportionality constant). Thus $60 = 2\gamma$, so that $\gamma = 30$. Finally, we obtain k from the relation $mg = kL$, where $L = 5/3$. Evidently $k = mg/L = 36$. Thus, the equation of motion is

$$6u'' + 30u' + 36u = 0,$$

with initial conditions $u(0) = 2$, $u'(0) = 0$.
The characteristic equation is $6t^2+30t+36 = 0$. Dividing by 6, we obtain $t^2+5t+6 = 0$. The roots are evidently -2 and -3, so the displacement function is of the form

$$u = c_1 e^{-2t} + c_2 e^{-3t}.$$

Applying the first initial condition, we see that $c_1 + c_2 = 2$. Therefore

$$u' = -2c_1 e^{-2t} - 3c_2 e^{-3t},$$

and applying the second initial condition, we see that $-2c_1 - 3c_2 = 0$. Solving, one obtains $c_1 = -4$, $c_2 = 6$. The displacement is then $-4e^{(-3t)} + 6e^{(-2t)}$.

(b) How should the damping constant γ be changed to obtain a critically damped system, if the mass and spring constant are left unchanged? (Give an explicit value for γ.)

Solution: We require that $\gamma^2 - 4km = 0$, where $k = 36$, $m = 6$. It appears that we must have $\gamma^2 = 4 \cdot 36 \cdot 6 = 864$, or $\gamma = \sqrt{864} = 12\sqrt{6}$.

3. Use whatever method you like to find the general solution to the equation

$$y'' - 10y' + 25y = e^{5t}.$$

Solution: The characteristic polynomial is $(r-5)^2$, so the general yoga of tells us that the complementary solution is $y_c = c_1 e^{5t} + c_2 t e^{5t}$. Undetermined coefficients then informs us that a particular solution to the equation will be of the form $Y = At^2 e^{5t}$. We compute $Y' = (5At^2 + 2At)e^{5t}$ and $Y'' = (25At^2 + 20At + 2A)e^{5t}$. Substituting, we obtain

$$Y'' - 10Y' + 25Y = 2Ae^{5t}.$$

Therefore $A = 1/2$. The general solution is the sum

$$y_c + Y = c_1 e^{5t} + c_2 t e^{5t} + \frac{1}{2} t^2 e^{5t}.$$

If we foolishly insist on using variation of parameters, then the functions y_1 and y_2 are the fundamental set of solutions from above, $y_1 = e^{5t}$, and $y_2 = te^{5t}$. The Wronskian is

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{10t}.$$

Therefore, the general solution is $u_1 y_1 + u_2 y_2$, where

$$u_1 = \int -\frac{y_2 g}{W}\, dt, \quad u_2 = \int \frac{y_1 g}{W}\, dt.$$

One finds that the integrands are $-t$ and 1 respectively, so that

$$u_1 = \int -t\,dt = -\frac{1}{2}t^2, \quad u_2 = \int 1\,dt = t.$$

The general solution is

$$y = -\frac{t^2}{2}e^{5t} + t(te^{5t}) = \frac{1}{2}t^2e^{5t},$$

as before.

4. Find the general solution of the equation

$$ty'' - (t+1)y' + y = t^2$$

given that $y_1 = e^t$ and $y_2 = t+1$ are solutions of the associated homogeneous equation. *Hint.* Use the method of variation of parameters.

Solution: Let $W = y_1y_2' - y_1'y_2$ as usual. Then $W = e^t - (t+1)e^t = -te^t$. The variations of parameters formulas tell us that a particular solution of the equation will be $u_1y_1 + u_2y_2$, where

$$u_1 = \int \frac{-(t+1)t^2}{te^t}\,dt, \quad u_2 = \int \frac{e^tt^2}{te^t}\,dt.$$

The first integral is annoying. Using integration by parts twice, we find that

$$u_1 = -\frac{t^2+3t+3}{e^t}.$$

For the second, we blessedly obtain

$$u_2 = -\frac{1}{2}t^2$$

almost immediately. Therefore a particular solution is

$$u_1y_1 + u_2y_2 = -(t^2+3t+3) - \frac{1}{2}(t^3+t^2),$$

and the general solution is

$$c_1e^t + c_2(t+1) - t^2 + 3t + 3 - \frac{1}{2}(t^3+t^2).$$