Connectivity

1 Acknowledgement

This section is loosely inspired by Chapter 4 of Crossley's *Essential Topology. You can download a PDF print-friendly version.*

2 Introduction

A connected topological space is one that you would say consists of only one "piece". Of course, since we are topologists, we wish to make sense of this idea in terms of open (and closed) sets, but it is perhaps useful to explore a different characterization first.

We often say to beginning calculus students that a continuous function is one whose graph can be drawn without "picking up the pencil". You might try to define a connected space in a similar way.

Definition 2.1. A *path* in a topological space X is a piecewise continuous function $\gamma\colon [0,1] \to X$.

The idea of Definition 2.1 is that the points $\gamma(0)$ and $\gamma(1)$ are the two endpoints. Since the function γ is continuous, the images $\gamma(t)$ for $0 < t < 1$ link the two endpoints without picking up the pencil. Many reasonable people would object to Definition 2.1. After all, it is the *image* of γ that we would typically call the path. However, the definition as given is ubiquitous in topology. When it is convenient we do abuse terminology and refer indirectly to the image of γ as a path joining $\gamma(0)$ to $\gamma(1)$.

Definition 2.2. A topological space X is *path-connected* if, for every pair of points p, $q \in X$, there is a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Exercise 2.3. Definition 2.2 is symmetric in p and q. That is, whenever there is γ connecting p and q as in Definition 2.2, there is a path $-\gamma$ such that $\gamma(0) = q$ and $\gamma(1) = p$.

Hint. Give an explicit formula for $-\gamma$.

The next two exercises suggest that this is a good definition. Remember that connectedness is supposed to mean that the space is "all one piece".

Exercise 2.4. In the topological space $\mathbf{R}_{\text{std}}$, every interval is path-connected.

One generalization of the idea of "interval" is the notion of a convex set.

Definition 2.5. A subset A of $\mathbf{R}^n$ is *convex* if for each pair of points p, $q \in A$, the segment connecting p and q is contained in A.

Recall also that the segment referred to in Definition 2.5 can be parametrized, in the notation of Definition 2.5, as $\gamma(t) = p(1-t) + qt$, for $t \in [0,1]$.

Exercise 2.6. In $\mathbf{R}^2$ with the standard topology, every *convex* set is path-connected.

However, there are spaces that seem to meet our informal criterion that are not path-connected in the sense of Definition 2.2. We need a variant of a definition from elsewhere in the book.

Definition 2.7. The *closed topologists' sine curve* is the set

$$\{(0, y) : -1 \leq y \leq 1\} \cup \left\{(x, \sin \frac{1}{x}) : 0 < x < 1\right\}.$$

Exercise 2.8. The closed topologists' sine curve (Definition 2.7) is not path-connected.

If you disagree that this space is "all one piece", then tell me: when you draw it, where must you pick up the pencil?

Now that you have thought about Exercise 2.8 for a while, let me ask you not to try to continue to prove it. To prove it, we actually will need to use the more fundamental notion of *connectedness*.

Definition 2.9. A topological space X is *connected* if no proper subset of X is both open and closed.

Lemma 2.10. *Let X be a space. Then X is not connected if and only if it is possible to find open subsets U and V of X such that*

1. $U \cap V = \varnothing$
2. $U \cup V = X$
3. *Neither of U and V is empty.*

Such a pair of open sets is sometimes called a separation *of X.*

Definition 2.11. A subset A of the topological space X is connected if A is connected in the sense of Definition 2.9 when given the subspace topology.

Theorem 2.12. *If X is a topological space and A is a connected subset, then $\overline{A}$ is also connected.*

The first definition, Definition 2.2, is more intuitive, but Definition 2.9, is more general. It therefore can capture spaces like the topologists' sine curve (Definition 2.7). Let's worry about this exotic (some would say "pathological") example later, and focus on the familiar for now.

Example 2.13. The empty set is a connected subset of every topological space.

Example 2.14. $\mathbf{R}$ (with the standard topology) is connected.

Example 2.15. Let $a < b$. Then each of the intervals below is a connected subset of $\mathbf{R}$.

1. The open interval (a, b)
2. The closed interval $[a, b]$
3. The half-open interval $(a, b]$
4. The half-open interval $[a, b)$

Example 2.16. Every open ball $B(p, \varepsilon) \subseteq \mathbf{R}^n$ is connected.

The proof of the main theorem of this section, Theorem 2.19, will follow quickly from the conceptually illuminating Lemma 2.18 below. The statement of Lemma 2.18 and the organization of our discussion around it are due to Martin Crossley's book *Essential Topology*. (I will provide a proper citation at a later time.) First we need a short definition.

Definition 2.17. The *0-sphere* S^0 is the subset $\{-1, 1\}$ of $\mathbf{R}$ given the subspace topology.

Lemma 2.18. *Let X be a connected topological space. Then there is no continuous surjective map from X to S^0.*

Theorem 2.19. *Let X and Y be spaces, let X be connected, and let $f\colon X \to Y$ be a continuous function. Then the image of f is connected.*