Undetermined coefficients: degeneracy and products

Math 352 Differential Equations

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The nondegenerate case

The Method of Undetermined Coefficients finds a particular solution of the inhomogeneous equation

$$ay'' + by' + cy = g(t),$$

when $g(t)$ is a polynomial, an exponential function, or a linear combination of sines and cosines (with like frequencies).

Our findings are summarized in the next table, and should be memorized.

The shape of Y for simple g

$g(t)$	$Y(t)$
t^n	$A_n t^n + \cdots + A_0$
$\exp(rt)$	$A\exp(rt)$
$a\cos(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
$a\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$

- if $g(t)$ is a linear combination of the above forms, guess Y to be a linear combination of the corresponding entries
- $g(t) = 3t^2 + \exp(2t) \implies Y(t) = At^2 + Bt + C + D\exp(2t)$.

Efficiency in choosing the form of Y

- Pay attention to what you're doing:
  - Suppose $g(t) = 3t^3 + t^4$. Two polynomials (?), so a suitable choice for $Y(t)$ could be

$$Y(t) = \underbrace{At^3 + Bt^2 + Ct + D}_{\text{from } 3t^3} + \underbrace{Et^4 + Ft^3 + Gt^2 + Ht + I}_{\text{from } t^4}.$$

- Problem: a large system of equations awaiting you, with many solutions (rather than just one).

How to be efficient

- Previous UC guess is a disguised version of

$$Y(t) = At^4 + Bt^3 + Ct^2 + Dt + E$$

- To get this, regard $g(t) = 3t^3 + t^4$ as a single polynomial (not a sum)
- Always think of $g(t)$ in the way that leads to the fewest terms in the linear combination.

Products

- Similarly when inhomogeneous term is a product of atomic ones, e.g.

$$g(t) = \exp(2t)\sin(t).$$

- Naïve guess: $Y = A\exp(2t)\sin(t)$, but...
- having been burned before, write down the product of *the corresponding table entries*,

$$Y(t) = (A\exp(2t))(B\cos(t) + C\sin(t)).$$

This looks more promising, and an expansion gives

$$Y(t) = AB\exp(2t)\cos(t) + AC\exp(2t)\sin(t).$$

Relabel or reorganize

- Only the *form* of Y matters, not names of coefficients.
- Thus AB and AC should just be relabeled.
- OK to reuse the letters, so write

$$Y(t) = A\exp(2t)\sin(t) + B\exp(2t)\cos(t).$$

  This is more efficient because it is a *smaller* system with *fewer* solutions.
- Another approach is to think about exponential factors separately.
  - Observe that Y is the *original* exponential from g times the *guess* for the trignonometric factor.
  - This "works" with polynomials also.

Try these out

Write down the most efficient form for Y that you can, for each of these instances of $g(t)$.

$$e^{7t}(2\cos(2t) - 8\sin(2t))$$

$$t^2 e^{2t}$$

$$3t^3 \sin 5t$$

Try these out

Write down the most efficient form for Y that you can, for each of these instances of $g(t)$.

$$e^{7t}(2\cos(2t) - 8\sin(2t)) \implies Ae^{7t}\cos(2t) + Be^{7t}\sin(2t)$$

$$t^2e^{2t} \implies At^2e^{2t} + Bte^{2t} + Ce^{2t}$$

$3t^3 \sin 5t$ ...shown on the next slide.

Not too cool

$$\begin{aligned} At^3 \cos(5t) + Bt^3 \sin(5t) + Ct^2 \cos(5t) + Dt^2 \sin(5t) \\ + Et \cos(5t) + Ft \sin(5t) + G \cos(5t) + H \sin(5t) \end{aligned}$$

This results in an 8×8 system of equations, which I wouldn't ask you to solve by hand. We'll see how to solve them using Sage in a couple of weeks.

Degeneracy

- What to do when the complementary solution $c_1 y_1 + c_2 y_2$ appears in $g(t)$?

$$y'' + 5y' + 6y = 6e^{-2t} \tag{1}$$

- No choice of A makes $Y = A\exp(-2t)$ a solution to the above inhomogeneous equation, because such a function is a solution to the *associated homogeneous equation*.

In other words, $Y'' + 5Y' + 6Y = 0$, so there's no way we're getting any exponentials on the right-hand side.

What to do???

In such cases, guided either by past experience or divine inspiration, one uses a higher-degree polynomial. Let the coefficient be At rather than A.

A degenerate solution

- Here, you can check that $6t\exp(-2t)$ is a solution. It is found by the same method as usual. But wait, you say!

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- Here, you can check that $6t \exp(-2t)$ is a solution. It is found by the same method as usual. But wait, you say!
- What if we consider the equation $y'' + 4y' + 4y = 6 \exp(-2t)$?
- If a little medicine is good, then a lot of medicine must be even better. So use a second-degree polynomial, and try to find Y of the form

$$Y(t) = At^2 e^{-2t}.$$

Coming attractions

Undetermined coefficients doesn't work as nicely if the inhomogeneous term $g(t)$ isn't a linear combination of products of polynomials, trig functions, and exponentials. For that reason, we must investigate a second method: VARIATION OF PARAMETERS.