Complex eigenvalues and bifurcations

Math 352 Differential Equations

April 30, 2014

The characteristic polynomial

- Matrix-vector differential equations $\vec{x}' = A\vec{x}$ have solutions of exponential type: $\vec{x} = \xi e^{rt}$.
- Here r is an eigenvalue of the coefficient matrix A.
- And ξ is a corresponding eigenvector.
- The eigenvalues are the roots of the characteristic polynomial $\det(A - tI)$, which (for us) is a real polynomial and has degree equal to the size of A.

All about making that FTA

- The characteristic polynomial has n complex roots, counting multiplicities.
- Thus there will be n complex eigenvalues of the matrix if we count them with repeats.

Digression: repeated eigenvalues

- When eigenvalues are repeated, there is a possibility for the matrix to be *deficient*.
- This means there are no corresponding eigenvectors for some occurrences of a repeated eigenvalue.
- This case is complicated, and we don't really have time to discuss it. Some details (though not all) are in Section 7.8.

Our restriction

- For us, eigenvalues will be distinct from now on: this means there really are n of them.
- Complex eigenvalues will only occur in conjugate pairs because the entries of A are real.

Complex eigenvalues

- Combine complex solutions using Euler's formula to obtain real-valued ones.
  - This will work because the eigenvalues occur in conjugate pairs.
- If $r = \lambda + i\mu$ and $\vec{\xi} = \vec{a} + i\vec{b}$, then what is $\vec{\xi}e^{rt}$?
- It is not a surprise to see exponential-trigonometric products as we did previously.

Types of eigenvalues

Mostly, the eigenvalues of a 2×2 real matrix follow one of 3 paradigms:

1. Eigenvalues are real, of opposite sign: $\vec{x} = 0$ is a *saddle point*.
2. Eigenvalues are real, of same sign: $\vec{x} = 0$ is a *node*.
3. Eigenvalues are complex, with nonzero real part: $\vec{x} = 0$ is a *spiral point*.

- Other cases (e.g., a zero eigenvalue, purely imaginary eigenvalues) occur as transition points between these cases.
- Such transitions are of great importance in *bifurcation theory*.

Applications of bifurcation theory

- Biology
  - interactions among a large variety of proteins
  - population ecology
- Physics
  - catastrophe theory
  - tunneling diodes
  - chaos theory