Introduction to equations with matrices

Math 352 Differential Equations

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The machinery of linear algebra

- Fundamental algebraic construction of linear algebra: *linear combinations*.
- *Vector spaces* are places where it makes sense to form linear combos.
- Coefficients can be real, complex, ...
- The objects being combined are often called vectors, even if they are not vectors like $\langle 2, -3, 4, 12 \rangle$ or $2\vec{i} - 6\vec{j}$.
- Suppose that y_1, y_2, y_3, ... y_n are differentiable and that a_1, ... a_n are real numbers (scalars).
- Then $a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$ is also differentiable.
- So, the set of differentiable functions forms what is called a *real vector space*.

Linear systems

A system of linear (algebraic) equations is a system like this:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & a_{34}x_4 & = & b_3 \end{array}$$

For example, initial value problems give rise to such systems, as did undetermined coefficients:

$$\begin{array}{ccccc} c_1 & + & c_2 & = & 2 \\ -4c_1 & + & 2c_2 & = & -10 \end{array}$$

Matrix form of a system

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

It's cleaner and allows us to focus on the arithmetic.

$$\begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

Solutions of matrix equations

- The vector of x_i is the variable of the matrix equation.
- A *solution* of the matrix equation is a choice of x_i that makes all the individual linear equations true.

Two matrix equations are *equivalent* if they have exactly the same set of solutions.

- Matrices are solved by transforming them into equivalent equations whose solutions are obvious.

An obvious matrix

Lots of other matrix equations' solutions are obvious, but this is what I really meant:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Reduced echelon form

We say a matrix A is in reduced echelon form if:

- the first nonzero entry of each row is a 1
  - such an entry is called a *pivot* or *leading 1*
- each pivot is the only nonzero entry in its column
- each of the pivots after the first one appears to the right of the previous pivot
- each row without a pivot follows all rows with a pivot

Such a matrix certainly has an obvious solution set, and:

- Every matrix is equivalent to exactly one matrix in reduced echelon form.

How do we find a reduced echelon equivalent?

- The same way we solve the equations to which the matrix equation corresponds: by adding and subtracting the rows from one another.
- We'll need to keep track of the RHS too, so add it as the last column of the matrix. We'll manipulate this matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Gaussian elimination

- There are three families of operations on matrices that preserve solution sets.
  - If you do any of these operations to a matrix, the resulting matrix is equivalent.
- Swapping rows
  - this doesn't even change the equations, just their order
- Multiplying a row by a nonzero number
  - this changes the equations, but not the solution set
- Adding a nonzero multiple of a row to another row
  - also doesn't change the solution set.

Using these operations, it's possible to transforms each matrix into its unique reduced echelon equivalent.

In practice: upper-triangular

If you are row-reducing a matrix by hand, reduced echelon form is overkill a lot of the time.

- Reduce to an upper-triangular matrix
- All nonzero entries on or above the main diagonal
    - that is, the upper-left-to-lower-right diagonal
    - with slope -1

Entering vectors in Sage

- Initialize a vector value with `vector()`
- `u = vector(QQ, [1, 3/2, -1])`
- Use `QQ` to display fractions instead of decimals
- `u`
  - `(1, -3/2, 1)`
- `v = vector(RR, [1, 3/2, -1]);` v
  - `(1.00000000000000, 1.50000000000000, -1.00000000000000)`

Entering matrices

- Either enter a matrix as a vector of its rows
  - `A = matrix(QQ, [[1, 2], [3, 4], [5, 6]])`
- or as a list with specification of number of rows
  - `A = matrix(QQ, 2, [1,2,3,4,5,6])`
- Obtain reduced echelon form with `rref`:
  - `B = A.rref()`

Example

- `A = matrix(QQ, 2, [1, 1, 2, -4, 2, -10]); show(A)`
  - $\begin{pmatrix} 1 & 1 & 2 \\ -4 & 2 & -10 \end{pmatrix}$
- `B = A.rref(); show(B)`
  - $\begin{pmatrix} 1 & 0 & 7/3 \\ 0 & 1 & -1/3 \end{pmatrix}$

Row operations

- Want to use Sage to check your work in performing row operations?
  - The matrix methods below may be useful.
  - These methods are *destructive*: they change the entries of the matrix on which they're called.
- `A.rescale_row(i,a)`
  - multiply row i by a
- `A.add_multiple_of_row(i,j,a)`
  - add a times row j to row i
- `A.swap_rows(i,j)`
  - swap rows i and j

Sage lab assignment

Use Sage to solve the systems of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 3y & + & 12z & = & 6 \\ x & + & y & + & 4z & = & 2 \\ 2x & + & 5y & + & 20z & = & 10 \\ -x & + & 2y & + & 8z & = & 4 \end{array} \qquad \begin{array}{rcrcrcr} 2x & + & 10y & + & 2z & = & 6 \\ x & + & 5y & + & 2z & = & 6 \\ x & + & 5y & + & z & = & 3 \\ -3x & - & 15y & + & 3z & = & -9 \end{array}$$

$$\begin{array}{rcrcrcrcr} 2x & + & y & - & z & + & 2w & = & -6 \\ 3x & + & 4y & & & + & w & = & 1 \\ x & + & 5y & + & 2z & + & 6w & = & -3 \\ 5x & + & 2y & - & z & - & w & = & 3 \end{array}$$