Euler's formula and sinusoidal solutions

Math 352 Differential Equations

March 12, 2014

Last time: the exponential trick

This week, we've been investigating the second-order homogeneous equation

$$ay'' + by' + cy = 0.$$

- Monday, we saw how to find solutions of exponential type $y = ce^{rt}$. The growth constants r that "work" are the roots of the characteristic equation $ar^2 + br + c = 0$.
- When $b^2 - 4ac > 0$ (the characteristic polynomial has *positive discriminant*), we get two values r_1 and r_2.
- Hence we obtain two solutions, $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$.

Last time: superposition

We also know that solutions of *linear homogeneous equations* can be linearly combined to get more solutions. Therefore, we have a 2-dimensional family of solutions

$$c_1 y_1 + c_2 y_2.$$

Exercise

Use the Wronskian (and Theorem 3.2.4) to show that when $b^2 - 4ac > 0$, the solutions y_1 and y_2 described above generate all solutions to $ay'' + by' + cy = 0$.

Negative discriminant

None of this tells what happens when $b^2 - 4ac < 0$. The existence theorem still applies, so the solutions have to be out there somewhere. If we expand our thinking to the realm of *complex numbers*, we can get them using the same trick.

Definition

A *complex number* is a symbol $x + iy$, where x and y are ordinary (i.e., real) numbers. We adopt two conventions:

i. these numbers obey all the rules of arithmetic;

ii. $i^2 + 1 = 0$.

Why bother?

- We need roots of $ar^2 + br + c = 0$ when $b^2 - 4ac < 0$. > - They are not real, but instead complex numbers.
- Note: a number of the form iy, where y is real, can be called "imaginary". But $x + iy$ is called "complex" when $x \neq 0$.
- But if you refer to $x + iy$ as imaginary and $x \neq 0$, you will sound like the worst sort of ill-mannered oaf.
- Push-ups will be assessed for infractions

Why does it work?

Fundamental Theorem of Algebra

Each polynomial of degree n with complex (coefficients has exactly n complex roots, counting multiplicities. (Equivalently, each such polynomial splits over the complex numbers into n (not necessarily distinct) linear factors.

This theorem was known to Euler, but it is widely asserted that the first satisfactory proofs were due to Gauss (1777–1855).

Getting solutions

Observe that if $D < 0$, then $D = -|D| = i^2|D|$. We use this to generalize the quadratic formula to complex numbers:

Square roots of negative numbers

We agree to interpret (when $D = b^2 - 4ac < 0$) the symbol $\sqrt{D}$ as:

$$\sqrt{D} = \sqrt{i^2|D|} = i\sqrt{|D|}.$$

The complex exponential

Once we have obtained our two complex roots r_1 and r_2, we need to interpret the corresponding exponential solutions.

The complex exponential

Once we have obtained our two complex roots r_1 and r_2, we need to interpret the corresponding exponential solutions.

WTF?

What's e^{rt} when r is complex?

The answer is found in what is called Euler's formula.

Euler's formula

A historical quotation on Euler's formula:

Benjamin Peirce (1809–1880)

"Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth."

Euler's formula

Since r is complex, it is a symbol of the form $x + iy$. Therefore, since t is real, we get

$$e^{rt} = e^{(x+iy)t} = e^{xt}e^{iyt}.$$

This is assuming the addition law for a function we haven't really defined, but if it didn't obey the addition law, we couldn't bear to call it the exponential function.

Euler's formula

This famous and celebrated equation tells us how to interpret the e^{iyt} portion of our exponential expression.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus $\exp(rt) = \exp(xt + iyt) = e^{xt}(\cos(yt) + i\sin(yt))$.

Euler's formula

Putting $\theta = \pi$, we obtain a special case, often referred to as Euler's formula.

Also Euler's formula

$$e^{\pi i} = -1.$$

Respect. Happy Pi Day!