Further investigation of harmonic vibration

Math 352 Differential Equations

7 April 2014

Last time: free harmonic oscillations

The equation of motion for an unforced spring-mass system:

$$mu'' + \gamma u' + ku = 0,$$

where m, $k > 0$ and $\gamma \geq 0$.

Here u is the displacement of the mass from equilibrium.

Determining the spring constant

In practice, the spring constant is obtained via the relation

$$mg = kL.$$

Here L is the marginal length added when the mass m is attached.

No damping

- If $\gamma = 0$, our system is a *simple harmonic oscillator*, vibrating subject to the displacement function

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

- Here $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator.
- If either of c_1 or c_2 is nonzero, then either $u(0)$ or $u'(0)$ is nonzero. Thus either the potential energy or kinetic energy of the mass is nonzero.
- Such a system's motion persists indefinitely.
- The energy added by the initial conditions stays in the system forever.

Reduction to standard form

- Every linear combination of sines and cosines with like frequency can be written as a single sinusoidal function.
- A sinusoidal function is one of the form $R\cos(\omega t - \delta)$, where R is the amplitude, ω is the common frequency, and δ is the phase shift.
- If we know c_1 and c_2 from the initial conditions, we can find R and δ with

$$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta).$$

Getting the new parameters

Using the cosine subtraction identity, we find this entails that

$$c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t).$$

Hence $c_1 = R\cos\delta$, $c_2 = R\sin\delta$, and the usual polar-coordinate equations give us

$$R = \sqrt{c_1^2 + c_2^2}, \quad \tan\delta = c_2/c_1.$$

The arctangent function must be used with due care.

Reminder on arctangent; `atan2`

- Recall: $\tan^{-1} x \in (-\pi/2, \pi/2)$ for all x
- You may need to adjust the value from a calculator.
- Adjustments must be made when the desired value of δ is in the left half-plane: namely, when $c_1 < 0$.
- Sage and many other programming languages include the useful function `atan2`.
- This is a function of two variables that makes this adjustment automatically.
- `atan2(y, x)` returns the unique angle δ with $\sin(\delta) = y$ and $\cos(\delta) = x$
  - (and of course $\tan(\delta) = y/x$)

Classification of damping; overdamped

- $\gamma > 0$: the system is "damped".
- The type of damping corresponds to the sign of $D = \gamma^2 - 4km$.
- When $D > 0$, the system is *overdamped*.
  - The displacement function is a linear combination of two exponentials $e^{r_1 t}$ and $e^{r_2 t}$, with $r_1, r_2 < 0$.
  - The vibration decays as t increases—in practice, very quickly.

Underdamped; critically damped

- When $D < 0$, the displacement function is a linear combination of the functions $e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin(\mu t)$.
- The coefficients λ and μ, as always, are determined by the characteristic polynomial: its roots are $\lambda \pm i\mu$ in this case.
- Here $\lambda < 0$, so the oscillation again decays with time.
- This is called *underdamped*.
- When $D = 0$, the system is *critically damped*.
- The displacement function is a linear combination of e^{rt} and te^{rt}. Here r is the unique root of the characteristic polynomial.
- The graphs of critically damped displacement functions look a lot like those of overdamped ones.

The damped cases: three regimes

- If $\gamma > 0$, then the initial energy is dissipated
  - (and in practice, quickly)
  - in resisting the damping force of the surrounding fluid.
- Clearly, greater values of γ mean "more" damping is occurring.
- The "size" of the damping is measured by a dimensionless coefficient involving all three constants m, γ, and k.

Dimensional analysis: work together

- Find a dimensionless combination of m, γ, and k.
- Hint: $mg = kL$.

Damping and the discriminant

- Let $Q = \gamma^2/4km$.
- Q is dimensionless: all the units cancel out of it.
- Dimensionless coefficients are important, because they don't depend on our scale of measurement.
- Trivially, $Q = 0$ when $\gamma = 0$.
- Our system is underdamped when $0 < Q < 1$.
- Critical damping obtains when $Q = 1$.
- Overdamping is the case $Q > 1$.

Quasiperiod and quasifrequency

The parameter μ determines the quasifrequency of a damped oscillation (since it is not periodic, it doesn't have an honest "frequency"). Some algebra shows that

$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km - \gamma^2}}{2m\sqrt{k/m}} = (1 - Q)^{1/2} \approx 1 - \frac{Q}{2}.$$

The last approximation is a tangent line approximation, valid when Q is small.

Thus, small damping slightly reduces the frequency of the oscillation. These calculations will be of great utility for us in the next section, which concerns *forced vibrations*.