# Some disambiguation of various notational conventions

Math 352 Differential Equations

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There is a significant danger of confusion that arises when we attempt to port our ideas about second-order linear equations to the case of a $2 \times 2$ system, as in Workshop 11. The danger is that an unwary or reckless reader may conflate the pair of functions from Chapter 3 (an independent set of solutions of a second-order equation) with the pair of functions from Chapter 7 (a single solution of the *system* of differential equations and its derivative).

When we discuss a system of differential equations $\mathbf{u}' = A\mathbf{u}$, the understanding is that the symbols $u_1$ and $u_2$ are reserved for the entries of the vector $\mathbf{u}$. They cannot serve this function and also retain their meaning from Chapter 3. These significations are incompatible.

The remainder of this note will explain the difference between the two pairs and the planes they parameterize. From now on, let us consider the second- order equation

$$u'' + 4u' + 5u = 0. \tag{1}$$

When we choose the fundamental system $u_1 = e^{-2t}\cos t$, $u_2 = e^{-2t}\sin t$, we are also making a specific identification of the *solution set*

$$c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$$

with the $(c_1, c_2)$-coordinate plane. If we were to choose a different fundamental system—say,

$$\begin{aligned} v_1 &= e^{-2t}(\cos t + \sin t) \\ v_2 &= e^{-2t}(\cos t - \sin t) \quad \text{(observe that these functions have a nonzero Wronskian),} \end{aligned}$$

then the *particular* solution $2e^{-2t}\cos t - e^{-2t}\sin t$ is represented by a different coordinate pair. It appears as the point $(2, -1)$ in the $(u_1, u_2)$-coordinate system and as the point $(1/2, 3/2)$ in the $(v_1, v_2)$-coordinate system.

Neither of these is the same thing as the phase plane, and it is essential that you understand the difference well enough to distinguish them.

The phase plane arises when we consider a particular solution $U$ of the differential equation (1) and plot the trajectory of the vector-valued function $\langle U, U' \rangle$. For example, we might have

$$\begin{pmatrix} U \\ U' \end{pmatrix} = \begin{pmatrix} 2e^{-2t}\cos t - e^{-2t}\sin t \\ -4e^{-2t}\cos t - 3e^{-2t}\sin t \end{pmatrix}.$$

The point is, we are evidently not free to choose both entries independently. Our choice of the first entry determines the second entry.

This is because the system of differential equations arising from Equation (1) is

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -5u_1 - 4u_2. \end{aligned}$$

It is not exactly convenient, but the most attractive solution is to write

$$\mathbf{u}^{(1)} = \begin{pmatrix} e^{-2t}\cos t \\ -e^{-2t}\cos t - 2e^{-2t}\sin t \end{pmatrix},$$

$$\mathbf{u}^{(2)} = \begin{pmatrix} e^{-2t}\sin t \\ 2e^{-2t}\cos t - e^{-2t}\sin t \end{pmatrix}.$$

The symbols are pronounced "you-upper-one" and "you-upper-two", respectively. The parentheses in the superscripts are not pronounced and are only there to help avoid confusion with exponents.

This allows us to stick with the irremediable convention that $u_1$ and $u_2$ denote the entries of a vector $\mathbf{u}$ while only slightly changing our previous notation for the elements of a fundamental system of solutions.

That is, with this conventions, the old meaning of $c_1 u_1 + c_2 u_2$ is *the first entry* of the *vector*

$$c_1\mathbf{u}^{(1)} + c_2\mathbf{u}^{(2)}.$$

If this last expression is our particular solution $\mathbf{u}$, then (putting it all together) we have

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \mathbf{u} = c_1\mathbf{u}^{(1)} + c_2\mathbf{u}^{(2)} \\ &= \begin{pmatrix} c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t \\ c_1(-e^{-2t}\cos t - 2e^{-2t}\sin t) + c_2(2e^{-2t}\cos t - e^{-2t}\sin t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t \\ (-c_1 + 2c_2)e^{-2t}\cos t + (-2c_1 - c_2)e^{-2t}\sin t \end{pmatrix}. \end{aligned}$$

Evidently, in the new context we obtain

$$\begin{aligned} u_1 &= c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t, \\ u_2 &= (-c_1 + 2c_2)e^{-2t}\cos t + (-2c_1 - c_2)e^{-2t}\sin t. \end{aligned}$$