General solutions to second-order homogeneous equations

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13 March 2013

Introduction

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- We saw how to find solutions of exponential type $y = ce^{rt}$ of the homogeneous equation $ay'' + by' + cy = 0$.
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- Because the equation is linear and homogeneous, *linear combinations of solutions are also solutions*.

Introduction

Therefore, in the case when $\sqrt{b^2 - 4ac}$ is positive, we get two real roots r_1 and r_2. These correspond to solutions

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t},$$

we obtain a 2-dimensional family of solutions

$$c_1 y_1 + c_2 y_2.$$

Question

Are there more? How can we be sure we have found them all?

Introduction

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Theorem 3.2.1.

The initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

admits a unique solution provided that p and q are continuous at t_0. This solution is twice differentiable throughout the largest interval containing t_0 on which both p and q are continuous.

Getting Solutions

The existence and uniqueness theorem says nothing about how to find the solution of an initial value problem. But we have access to what feels like a very large class of solutions to the homogeneous linear equation.

$$c_1 y_1 + c_2 y_2$$

Perhaps it is the case that we can always choose c_1 and c_2 so that the corresponding linear combination is a solution of the *initial value problem*: namely, so that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0'. \end{aligned}$$

Getting Solutions

In solving this system by adding and subtracting equations, one eventually is required to divide by the expression

$$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

It is called the *Wronskian determinant* (*Wronskian* for short) of y_1 and y_2 evaluated at t_0. If this quantity is nonzero, then c_1 and c_2 can indeed be chosen so that the associated linear combination is a solution of the initial value problem, and in fact they can be chosen in exactly one way.

Getting Solutions

If $W(y_1, y_2)(t_0) = 0$, however, then the answer depends on y_0 and y_0'. Most of the time, it will not be possible to choose c_1 and c_2. If y_0 and y_0' are related in exactly the right way, however, then there will be infinitely many choices that "work". We are less interested in this case. It is comparatively rare.

Wronskians and the General Solution

We are only justified in using the term “general solution” if every solution to $ay'' + by' + cy = 0$ is of the form $c_1 y_1 + c_2 y_2$.

Theorem 3.2.4

Suppose that y_1 and y_2 are *any* two solutions of $y'' + p(t)y' + q(t)y = 0$. Then the family of solutions $y = c_1 y_1 + c_2 y_2$ includes every solution of the differential equation if and only if there is at least one point t_0 for which $W(y_1, y_2)(t_0) \neq 0$.

Abel's Theorem

We close with an interesting formula concerning Wronskians. It is due to the Norwegian mathematician Niels Henrik Abel (1802–1829).

Abel's Theorem

If y_1 and y_2 are solutions of the differential equation as before, then

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\, dt\right)$$

where c is independent of t.