Undetermined coefficients: nondegenerate

Math 352 Differential Equations

March 21, 2014

The translation principle

Last time:

- If Y is a *particular solution* of the equation $ay'' + by' + cy = g(t)$
- and $c_1y_1 + c_2y_2$ is the general solution of the associated homogeneous equation
  - $ay'' + by' + cy = 0$
- then $c_1y_1 + c_2y_2 + Y$ is the general solution of $ay'' + by' + cy = g(t)$.

We apply the method to an example.

An exponential example

$$y'' - 2y' - 3y = 3e^{2t}$$

- Step 1: Find the complementary solution, that is, the general solution of the associated homogeneous equation.

An exponential example

$$y'' - 2y' - 3y = 3e^{2t}$$

- Step 1: Find the complementary solution, that is, the general solution of the associated homogeneous equation.
- The characteristic polynomial is $r^2 - 2r - 3 = (r-3)(r+1)$.

An exponential example

$$y'' - 2y' - 3y = 3e^{2t}$$

- Step 1: Find the complementary solution, that is, the general solution of the associated homogeneous equation.
- The characteristic polynomial is $r^2 - 2r - 3 = (r-3)(r+1)$.
- Therefore, the CS is $y_c = c_1 e^{3t} + c_2 e^{-t}$.

An exponential example

$$y'' - 2y' - 3y = 3e^{2t}$$

- Step 1: Find the complementary solution, that is, the general solution of the associated homogeneous equation.
- The characteristic polynomial is $r^2 - 2r - 3 = (r-3)(r+1)$.
- Therefore, the CS is $y_c = c_1 e^{3t} + c_2 e^{-t}$.
- Get in the habit of doing this step first. Trust me.

Apply undetermined coefficients

- Step 2: Look for a solution of the form $Y = Ae^{2t}$.
- Then $Y' = 2Ae^{2t}$ and $Y'' = 4Ae^{2t}$.
- Substitute:

$$\begin{aligned} 3e^{2t} &= Y'' - 2Y' - 3Y \\ &= 4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} \\ &= 3Ae^{2t}. \end{aligned}$$

Evidently we obtain $A = 1$, so $Y = e^{2t}$ is a particular solution.

Write down the general solution

- Step 3: By the discussion surrounding the translation principle, it follows that

$$c_1e^{3t} + c_2e^{-t} + e^{2t}$$

is the general solution of the inhomogeneous differential equation $ay'' + by' + cy = 0$.

A trigonometric example

$$y'' - 2y' - 3y = 2\sin 2t.$$

- Step 1: the complementary solution is $c_1 e^{3t} + c_2 e^{-t}$ as before.

A trigonometric example

$$y'' - 2y' - 3y = 2\sin 2t.$$

- Step 1: the complementary solution is $c_1 e^{3t} + c_2 e^{-t}$ as before.
- Step 2: try $Y = A\sin 2t$.

A trigonometric example

$$y'' - 2y' - 3y = 2\sin 2t.$$

- Step 1: the complementary solution is $c_1 e^{3t} + c_2 e^{-t}$ as before.
- Step 2: try $Y = A\sin 2t$.
- Then $Y' = 2A\cos 2t$ and $Y'' = -4A\sin 2t$.

Solve for the undetermined coefficient

- We found $Y' = 2A\cos 2t$ and $Y'' = -4A\sin 2t$.
- Substitution gives

$$\begin{aligned} Y'' - 2Y' - 3Y &= -4A\sin 2t - 2(2A\cos 2t) - 3(A\sin 2t) \\ &= -7A\sin 2t + 4A\cos 2t. \end{aligned}$$

Here, something is wrong: there is no choice of A that makes the RHS equal to $2\sin 2t$, since $0 \neq -2/7$.

A better guess

The correct guess is $Y = A\cos 2t + B\sin 2t$.

- Work together to determine A and B.

A better guess

The correct guess is $Y = A\cos 2t + B\sin 2t$.

- Work together to determine A and B.
- You should find $A = 8/65$, $B = -14/65$.

Step 3

- The general solution is:

$$c_1 e^{3t} + c_2 e^{-t} + \frac{8}{65}\cos 2t - \frac{14}{65}\sin 2t$$

Degeneracy

- These examples are *nondegenerate*.
- This means the inhomogeneous term $g(t)$ doesn't appear in the complementary solution.
- Read section 3.5 to see how to handle degeneracies.

Table of guesses (nondegenerate only!)

Table: Guess-o-chart

$g(t)$	$Y(t)$
t^n	$A_n t^n + \cdots + A_0$
$\exp(at)$	$A\exp(at)$
$a\cos(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
$a\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$

- If $g(t)$ is a linear combination of entries from the table, let Y be the corresponding linear combination.

SPRIIIING BREAAAK

- After the break:
  - we will discuss the degenerate cases
  - and the case where $g(t)$ is a product of table entries.

HAVE A GREAT BREAK!