Forced oscillations, II

Math 352 Differential Equations

The College of Idaho

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Transient and steady-state decomposition

Our oscillations are controlled by the differential equation

$$mu'' + \gamma u' + ku = F(t).$$

Today we will look briefly at the case $\gamma > 0$. When we think about the general solution to this forced, damped equation, we regard it as usual in two pieces:

$$u_c + U$$

where u_c is the general solution of the associated homogeneous equation (the same system, but unforced) and U is a particular solution.

Why "transient" and "steady-state"?

- Observe that $u_c \to 0$ as $t \to \infty$. Therefore the effects of the initial conditions are ephemeral, which is why we call u_c the *transient* part of the solution, and U the *steady-state* solution. If one returns to the system after time has passed, only the U-behavior of the system is evident.
- The steady-state U is also often called the *forced response* of the system to the forcing function F.
- Observe how the decomposition of the response into $u_c + U$ mirrors the decomposition of the system into internal (m, γ, k, c_1, c_2) and external ($F(t)$) factors. This is what mathematicians and physicists mean by *elegance*.

The solution

- As before, if we assume that $F(t) = F_0 \cos(\omega t)$, we may write $U(t) = R\cos(\omega t - \delta)$ for some amplitude R and phase shift δ. These constants are determined by the formulas

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta},$$

  where $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$.
- We can use these formulas to compare R, the amplitude of the forced response, to F_0/k, the length by which the spring is stretched when subjected to a constant force of magnitude F_0. This depends on ω.

Amplitude from frequency

- It turns out that, if we let $\Gamma = \gamma^2/mk$ (so that it is a constant multiple of Q from last time), we obtain

$$\frac{Rk}{F_0} = \left(\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right)^{-1/2}.$$

- When $\omega \approx 0$, this number is very close to 1, or in other words, $R \approx F_0/k$. This matches with our physical intuition.
- When $\omega \gg 0$, R is small. Extremely high frequency excitation produces a negligible vibration (think of the child kicking his feet in the swings).

Where is the maximum amplitude?

$$\frac{Rk}{F_0} = \left(\left(1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \Gamma \frac{\omega^2}{\omega_0^2} \right)^{-1/2}$$

- The right-hand side can be regarded as a function of ω. It should attain a maximum (the greatest forced amplitude achievable) for some ω_{max}. Some tedious algebra shows that

$$\omega_{\mathrm{max}} = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk} \right).$$

- Observe that $\omega_{\mathrm{max}} < \omega_0$ and that when γ is small, these values are very close.
- The textbook has a more detailed discussion of the variance of Rk/F_0 with ω/ω_0. I encourage you to read it closely.

Wrap-up

In particular, the amplitude achieved when $\omega = \omega_{\max}$ is given by

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-Q}} \approx \frac{F_0}{\gamma\omega_0}\left(1+\frac{Q}{2}\right).$$

Evidently, the approximation fails if $Q > 1$. In fact, if $Q > 1/2$, then $\omega_{\max}$ is a complex number, and R is a monotone decreasing function of ω.

Introduction to higher-order equations

In practice, the linear differential equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(t)$$

is solved by converting it to a system of n first-order differential equations.

Recall that the equation $y' = ay$ has general solution of exponential type:

$$y = ce^{at}.$$

A system of such equations also has a general solution of exponential type, as we shall see.

Matrices and vectors

The solution of such systems is facilitated by a new kind of algebra. An $m \times n$ matrix is an array of numbers (for us, real numbers, also eventually complex numbers inevitably arise) like this:

$$A = \begin{pmatrix} 1 & 4 & 6 & -3 \\ 2 & -2 & 0 & 7 \end{pmatrix}.$$

Here, $m = 2$ and $n = 4$. We always list the row-index first. The same goes if we want to leave the entries (the numbers in the matrix) anonymous:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$