Autonomous equations

Math 352 Differential Equations

February 26, 2014

Warm-up: equilibria

Suppose that $dQ/dt = Q^2 - Q - 2$.

Find all the constant functions $Q(t)$ that satisfy this DE.

These are called equilibrium solutions (equilibria for short) because left undisturbed, they just stay put.

Warm-up: in between equilibria

What is the behavior of $Q(t)$...

- if $-1 < Q_0 < 2$?
- if $Q_0 > 2$?
- if $Q_0 < -1$?

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- if $Q(1000000000) > 2$?
- if $Q(1000000000) < -1$?

Warm-up: stability

Suppose $Q = 2$ for all $t < 0$, but at $t = 0$, the system experiences an abrupt local disruption. The disruption bumps Q discontinuously, so that $Q_0 \neq 2$ (but $|Q_0 - 2|$ is still small).

- What is $\lim_{t\to\infty} Q(t)$ in this case?

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- What is $\lim_{t\to\infty} Q(t)$ in this case?
- What if we ask the same question about the other equilibrium solution?
- If you are stuck, think about the equilibria of a pendulum that can swing freely from its pivot to any angle. There are two! How are they different?

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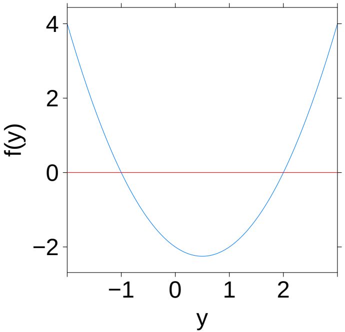
- I don't know why this gets to be called "autonomy", but there you go.

The phase line

Autonomous equations are easy to understand if we graph $f(y)$ against y.

This graph is called the phase line or phase plot.



What is the relationship of the equilibria of

$$y' = y^2 - y - 2$$

to this curve?

Drawing the solution curves

Having drawn the phase plot it's easy to draw a few solutions in the (t, y)-plane.

Carry out the procedure for the following autonomous equations:

- $dy/dt = (y-1)(y-2)$
- $dP/dt = \alpha P(K-P)$, where $\alpha > 0$, $K > 0$
- $dQ/dt = rQ - k$, where $0 < r < 1$, $k > 0$
- $dz/dt = (z+1)(z-3)^2$

Classify the equilibria of each differential equation as stable, unstable, or semistable.