Mathematics 352
Variation of Parameters

1. Use variation of parameters to find the general solution of the equation

$$y'' - y' - 2y = e^{3t}.$$

Solution: The characteristic polynomial is $(r-2)(r+1)$, so $y_1 = e^{2t}$ and $y_2 = e^{-t}$ are a fundamental set of solutions to the associated homogeneous equation. We compute $W(y_1, y_2) = (e^{2t})(-e^{-t}) - (2e^{2t})(e^{-t}) = -3e^t$. The variation of parameters formulas yield

$$u_1 = -\int \frac{e^{-t}e^{3t}}{-3e^t}\,dt = \int \frac{e^t}{3}\,dt = \frac{e^t}{3},$$
$$u_2 = \int \frac{e^{2t}e^{3t}}{-3e^t}\,dt = -\frac{1}{3}\int e^{4t}\,dt = -\frac{1}{12}e^{4t}.$$

Therefore a particular solution is

$$u_1y_1 + u_2y_2 = \frac{1}{3}e^{3t} - \frac{1}{12}e^{3t} = \frac{1}{4}e^{3t}.$$

2. Use variation of parameters to find the general solution of the equation

$$y'' - 4y' + 5y = e^{-4x}.$$

3. Verify that $y_1 = t^2$ and $y_2 = t^{-1}$ are solutions of the equation

$$t^2y'' - 2y = 0.$$

4. Use variation of parameters to find the general solution of the equation

$$t^2y'' - 2y = 3t^2 - 1.$$

You can assume $t > 0$ wherever necessary.

Solution: As above we let $y_1 = t^2$ and $y_2 = t^{-1}$. Since their Wronskian is

$$t^2(-t^{-2}) - 2t(t^{-1}) = -1 - 2 = -3,$$

they are an independent set of solutions to the associated homogeneous equation. The variation of parameters equations give

$$u_1 = -\int \frac{t^{-1}(3t^2-1)}{-3}\,dt = \frac{1}{3}\int 3t - t^{-1}\,dt = \frac{1}{3}\left(\frac{3}{2}t^2 - \ln|t|\right) = \frac{1}{2}t^2 - \frac{1}{3}\ln|t|,$$
$$u_2 = \int \frac{t^2(3t^2-1)}{-3}\,dt = \frac{1}{3}\int 3t^4 - t^2\,dt = \frac{1}{3}\left(\frac{3}{5}t^5 - \frac{1}{3}t^3\right) = \frac{1}{5}t^5 - \frac{1}{9}t^3.$$