A general method: variation of parameters

Math 352 Differential Equations

April 2, 2014

Introduction: This one goes out to all the gs

The method of undetermined coefficients is very useful when the derivatives of the terms appearing in $g(t)$ have a regular, predictable shape and are themselves the derivatives of similar functions. But it's not too hard to find functions $g(t)$ that would demand excessive ingenuity of us when guessing the form of Y.

Introduction: Too many functions!

Consider $y'' + y' + y = 3\csc(t)$. You can check that a particular solution Y is given by the formula

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Consider $y'' + y' + y = 3\csc(t)$. You can check that a particular solution Y is given by the formula

$$-3\sin(t)\cos(2t) + \frac{3}{2}\ln|\csc(t) - \cot(t)|\sin(2t) + 3\cos(t)\sin(2t).$$

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Variation of parameters: not a new thought

- How did we find te^{rt} in the case of a repeated root r of the characteristic equation?
- We promoted a constant to a function and looked for a solution of the form $v(t)\exp(rt)$.
- In other words, we allowed a parameter to vary.
- VP is the same idea, adapted to equation $y'' + q(t)y' + r(t)y = g(t)$
  - Requires that we have already solved the associated homogeneous equation $y'' + q(t)y' + r(t)y = 0$.
  - So assume we know y_1 and y_2 with $W(y_1, y_2) \neq 0$.

Variation of parameters: execution

- Guess: $Y = u_1 y_1 + u_2 y_2$ where u_1, u_2 are functions of t:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

- Goal: solve for u_1, u_2. We'll be plugging this expression for Y along with its derivatives back into the inhomogeneous equation, so let's compute the derivatives now.

An unjustified assumption

- Assume that the fundamental solutions y_1 and y_2 and the derivatives u_1' and u_2' satisfy the following equation:

$$u_1' y_1 + u_2' y_2 = 0.$$

- This is the worst rabbit-out-of-the-hat of the term.

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- At no point does it cause problems of any kind.

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The derivatives of Y

Since

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

we also have

$$\begin{aligned} Y' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2', \quad \text{blue terms add to 0} \\ &= u_1y_1' + u_2y_2', \quad \text{and so} \\ Y'' &= u_1y_1'' + u_1'y_1' + u_2'y_2' + u_2y_2''. \end{aligned}$$

Substitute back in

Now we find that $Y''(t) + q(t)Y'(t) + r(t)Y(t)$ reduces to

$$(u_1 y_1'' + u_1' y_1' + u_2' y_2' + u_2 y_2'') + q(t)(u_1 y_1' + u_2 y_2') + r(t)(u_1 y_1 + u_2 y_2).$$

Collecting terms along u_1, u_2, and their derivatives, we get

$$\begin{aligned} &(y_1'' + qy_1' + ry_1)u_1 + (y_2'' + qy_2' + ry_2)u_2 + y_1' u_1' + y_2' u_2' \\ &\qquad = y_1' u_1' + y_2' u_2', \end{aligned}$$

because of the assumption $u_1' y_1 + u_2' y_2 = 0$.

Putting it all together

We have shown that if u_1 and u_2 have the desired properties, then they satisfy

$$\begin{aligned} u_1' y_1' + u_2' y_2' &= g(t) \\ u_1' y_1 + u_2' y_2 &= 0. \end{aligned}$$

Solving this system of equations for u_1 and u_2 (in the usual algebraic way, by either substitution or addition of equations) gives

$$u_1' = -\frac{y_2 g}{W(y_1, y_2)}, u_2' = \frac{y_1 g}{W(y_1, y_2)}.$$

Integrating each of these yields the desired functions u_1 and u_2.

The particular solution

That is, a particular solution Y results from any choice of antiderivatives u_1, u_2. We have

$$Y = -y_1 \int \frac{y_2 g}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g}{W(y_1, y_2)}\, dt.$$

Like undetermined coefficients, this method has its own characteristics.

- It is reassuringly mechanistic. Y is given by a formula.
- It is also very general, since there are no conditions on g.
- On the other hand, the integrals that arise may be intractable.

Generalizations

For us, the coefficient functions $q(t)$ and $r(t)$ are constant. What happens when they are not?

- The methods of Chapter 5 are required to solve the general second-order linear homogeneous equation.
- If fundamental solutions are known, VP works just the same to determine a particular solution.
- It is only the characteristic equation and the exponential trick that fail in this case.