Introduction to modeling

Math 352 Differential Equations

February 19, 2014

Modeling is using mathematics to. . .

- explain or describe real phenomena
- investigate questions about the world
- test ideas about the observed reality
- make predictions about reality and its phenomena

Instead of making observations and experiments in the real world, a modeler makes these observations and experiments on *mathematical representations* of the real world.

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- Changing one part of a system may make other parts of it change too.
- Do they change in the same way? at the same rate?
- If not, how are these rates related?

Quantities, rates

- Relationships between quantities: what mathematicians usually call *functions*
- The rates at which quantities change appear as these functions' *derivatives*.

Hence, equations that model relationships among quantities and the rates at which they change are naturally *differential equations*.

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- theoretical coherence
- sounding cool

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The heat equation

Models have to come from somewhere:

- the *one-dimensional heat equation* is derived from physical first principles
  - conservation of energy and
  - Fourier's law of heat transfer, $\mathbf{q} = -k\dfrac{\partial u}{\partial x}$

Here, $\mathbf{q}$ is the heat flux density and u is the temperature in a one-dimensional heated wire. This leads to the *heat equation*,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Logistic growth of population

- Tiny babies know that bacteria populations grow exponentially
- This is because $P' = rP$; the same fraction of the population is always reproducing
- But does this growth continue forever?

Eventually, resource scarcity begins to limit the continued growth.

The logistic differential equation

Let us denote by K the upper limit of the population: that is, if $P > K$ there are too many organisms for the available resources and the population should decrease.

- One differential equation that models this situation is the logistic differential equation, $P' = rP(K - P)$.

Why is the logistic equation good?

In your groups, answer the questions (assuming $P > 0$):

- What if P is much closer to 0 than to K?
- What if P is roughly midway between 0 and K?
- What if P is much closer to K than 0, but still less than K?
- What if $P > K$?
- Can you find any *constant* solutions of the logistic equation?