Introduction to modeling

Math 352 Differential Equations

February 19, 2014

Modeling is using mathematics to...

- explain or describe real phenomena
- investigate questions about the world
- test ideas about the observed reality
- make predictions about reality and its phenomena

Instead of making observations and experiments in the real world, a modeler makes these observations and experiments on *mathematical representations* of the real world.

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- Do they change in the same way? at the same rate?
- If not, how are these rates related?

Quantities, rates

- Relationships between quantities: what mathematicians usually call *functions*
- The rates at which quantities change appear as these functions' *derivatives*.

Hence, equations that model relationships among quantities and the rates at which they change are naturally *differential equations*.

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- sounding cool

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The heat equation

Models have to come from somewhere:

- the *one-dimensional* heat equation is derived from physical first principles
  - conservation of energy and
  - Fourier's law of heat transfer, $\mathbf{q} = -k\frac{\partial u}{\partial x}$

Here, $\mathbf{q}$ is the heat flux density and u is the temperature in a one-dimensional heated wire. This leads to the *heat equation*,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

Logistic growth of population

- Tiny babies know that bacteria populations grow exponentially
- This is because $P = rP$; the same fraction of the population is always reproducing
- But does this growth continue forever?

Eventually, resource scarcity begins to limit the continued growth.

The logistic differential equation

Let us denote by K the upper limit of the population: that is, if $P > K$ there are too many organisms for the available resources and the population should decrease.

- One differential equation that well models this situation is the logistic differential equation, $P = rP(K-P).

Why is the logistic equation good?

In your groups, answer the questions (assuming $P > 0$):

- What if P is much closer to 0 than to K?
- What if P is roughly midway between 0 and K?
- What if P is much closer to K than 0, but still less than K?
- What if $P > K$?
- Can you find any *constant* solutions of the logistic equation?

Predator–prey equations

If x and y are populations of two different organisms, then one model for their interaction is the *Lotka–Volterra equations*, sometimes known as the *predator–prey* equations. These are:

$$\frac{dx}{dt} = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = -y(\gamma - \delta x)$$

Here, α, β, γ, and δ are parameters describing the nature of the interspecies interaction.

This model was the gold standard in biomathematics for a generation, but may have been supplanted in the late 80s by an alternative model.

Don't jump to the solutions

- Start with differential equations
- Resist the urge to jump straight to solutions

Interpret and predict, even w/o solution

- Logistic direction field
- What will happen if $P(0) > K$? if $P(0) < K$?