Reduction of order

Math 352 Differential Equations

March 19, 2014

Last time: repeated roots

- We completed the investigation of the equation

$$ay'' + by' + cy = 0.$$

in the last remaining case: $D = b^2 - 4ac = 0$.

- A new technique was necessary
- The exponential trick only gave us half of the general solution
  - the 1-dimensional family $cy_1 = ce^{(-b/2a)t}$.

Last time: guess-and-check

- In Workshop 06, you found that if $v(t)$ is an unknown function, then vy_1 is a solution of the DE—that is,

$$a(vy_1)'' + b(vy_1)' + c(vy_1) = 0,$$

—if and only if $v'' = 0$.
- A bit of calculus and a moment's reflection shows that $v = \alpha t + \beta$ in this case.
- Thus you generated a new class of solutions, the functions

$$Y_2 = (\alpha t + \beta)e^{(-b/2a)t}.$$

- Notice! it's a 2-dimensional family. Maybe it's the general solution?

Last time: the general solution

- This new solution Y_2 is a suitable "other half" of our general solution:
  - in the sense that $W(y_1, Y_2)$ is everywhere nonzero.
  - But it is not optimally efficient, because y_1 appears in the new class!
  - Put $\alpha = 1$ and $\beta = 0$ to obtain it.
- The part of Y_2 that is fundamentally "new" is $y_2 = te^{(-b/2a)t}$.
  - You can check that the Wronskian $W(y_1, y_2)$ is still nonzero
  - Therefore y_1 and y_2 generate all solutions to the differential equation through linear combinations.
  - It is equally true that $\alpha y_2 + \beta y_1$ "is" the general solution. There's no contradiction there.

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.
- If $W \neq 0$, who cares where the solutions came from?

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.
- If $W \neq 0$, who cares where the solutions came from?
  - We can compute them,

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.
- If $W \neq 0$, who cares where the solutions came from?
  - We can compute them,
  - maybe we just guess them!

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.
- If $W \neq 0$, who cares where the solutions came from?
  - We can compute them,
  - maybe we just guess them!
  - or find them scrawled in blood on the walls of an ancient tomb.

The magic secret of Wronskians

- Is our advance knowledge of $y_1(t) = e^{(-b/2a)t}$ enough to replace

$$\beta y_1 + \alpha Y_2 \text{ with } c_1 y_1 + c_2 y_2?$$

- Linear algebra sez: YES, but we haven't the time to go into it.
- If $W \neq 0$, who cares where the solutions came from?
  - We can compute them,
  - maybe we just guess them!
  - or find them scrawled in blood on the walls of an ancient tomb.
- The Wronskian doesn't care, and tells us definitively that we have solved the equation.

The general method

- Once you find (by any means necessary) a single solution to a second-order linear homogeneous ODE the method of "promoting c to a function" can get us the other half.
- This method is called "reduction of order"
  - because we end up solving a *first*-order differential equation to get the unknown coefficient function.
- Let's see how it works on an equation with nonconstant coefficients!

The general method: in general

- Suppose that, in the dusty tomb of a king long dead, we find the following equations scrawled in blood on the wall in a shaky but regal hand:

$$2t^2y'' + 3ty' - y = 0, \quad y_1 = t^{-1}.$$

- y_1 is a solution (easily checked)
- cy_1 is also a solution for all real c (why?)
- We will use the method of reduction of order:
  - promote c to a function $v(t)$
  - get a first-order differential equation for $v(t)$
  - obtain another solution $y_2 = vy_1$ that satisfies $W(y_1, y_2) \neq 0$.

Using the method

We write $y_2 = vy_1$ and substitute in. Observe that $(vy_1)' = v'y_1 + vy_1'$ and that $(vy_1)'' = v''y_1 + 2v'y_1' + vy_1''$. Substitution back into the original differential equation $2t^2y'' + 3ty' - y = 0$ then gives us

$$
\begin{aligned}
0 &= 2t^2(v''y_1 + 2v'y_1' + vy_1'') + 3t(v'y_1 + vy_1') - vy_1 \\
&= (2t^2y_1)v'' + (4t^2y_1' + 3ty_1)v' + (2t^2y_1'' + 3ty_1' - y_1)v \\
&= (2t^2y_1)v'' + (4t^2y_1' + 3ty_1)v'.
\end{aligned}
$$

The coefficient of v is zero, because y_1 is a solution to the ODE!

The method continues

- Now we can see how the order has been reduced. The equation

$$(2t^2y_1)v'' + (4t^2y_1' + 3ty_1)v' = 0$$

  is a first-order equation in v'
  - In fact, it's a separable one.
- The textbook details the solution of this equation—much easier, if we put $y_1 = t^{-1}$ throughout.
- I suggest you check for yourself, without looking in the text, if possible, that in this example we find $v = t^{1/2}$ and therefore $y_2 = t^{-1/2}$.

Inhomogeneity

- Recall the *superposition principle*:
  - if y_1 and y_2 are solutions, then all their linear combinations are too.
- It fails spectacularly if we try it on an inhomogeneous equation, as illustrated in several examples and exercises in the text.

Inhomogeneity: subtracting solutions

- We can use our previous work to solve equations of the form

$$ay'' + by' + cy = g(t). \tag{1}$$

- Suppose that Y_1 and Y_2 are both solutions of Equation (1).
- Behold:

$$\begin{aligned} 0 &= g(t) - g(t) \\ &= (aY_1'' + bY_1' + cY_1) - (aY_2'' + bY_2' + cY_2) \\ &= a(Y_1'' - Y_2'') + b(Y_1' - Y_2') + c(Y_1 - Y_2) \\ &= a(Y_1 - Y_2)'' + b(Y_1 - Y_2)' + c(Y_1 - Y_2) \end{aligned}$$

The translation principle

We just showed:

- any two solutions of Equation (1) differ by a solution of the *associated homogeneous equation*

$$ay'' + by' + cy = 0. \tag{2}$$

Put another way, the difference is a solution of Equation (2).

The geometry of linear differential operators: homogeneous equations

- Think of the set S of solutions of Equation (2)
- A pair of fundamental solutions y_1, y_2 matches S with the plane:
  - namely, via the matching $c_1y_1 + c_2y_2 \longleftrightarrow (c_1, c_2)$.
- We can see S as a plane in a bigger-dimensional space (the space of all differentiable functions, maybe).
- Observe that the point $(0, 0)$ is an element of our plane S, because the zero function is a solution of Equation (2).
- Thus S passes through the origin of whatever space it lives in.

The geometry of linear differential operators: inhomogeneous equations

- Equation (1) is a second-order equation
  - intuition and the theory of Wronskians tell us that there should be a pair of fundamental solutions.
  - This does happen, but the details are a little different.
- Solutions of Equation (1) correspond to points in a plane
- But it's not the same plane
- It's a different plane in the same space.
- It doesn't pass through the origin
  - because the zero function isn't a solution of Equation (1).

The geometry of linear differential operators: particular solutions

Recall the translation principle for Equation (1):

- If Y_1 and Y_2 are solutions of Equation (1), then their difference $Y_1 - Y_2$ is a solution of the associated homogeneous equation.
- Inverted, it tells us how to construct new solutions of Equation (1) from a previously known one:
  - add solutions of Equation (2).

The general solution of the inhomogeneous equation

- Suppose that by some devious method we have constructed a single solution Y of Equation (1).
  - so that $aY'' + bY' + cY = g(t)$.
- Suppose also that y_1 and y_2 are solutions of Equation (2)
  - and that $W(y_1, y_2) \neq 0$.
- Then solution of $ay'' + by' + cy = g(t)$ is of the form

$$Y + c_1 y_1 + c_2 y_2$$

for some numbers c_1 and c_2.

Construction of particular solutions

- This is an existence theorem.
- It doesn't tell us anything about how to construct Y, called a *particular solution* of Equation (1).
- Methods for finding Y vary and depend very much on the form of the inhomogeneous term $g(t)$. We will investigate two such methods:
- *undetermined coefficients* and
- *variation of parameters*.

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.
- That is called the method of undetermined coefficients.

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.
- That is called the method of undetermined coefficients.
  - Expert level guess-and-check.

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.
- That is called the method of undetermined coefficients.
  - Expert level guess-and-check.
- A word to the wise:

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.
- That is called the method of undetermined coefficients.
  - Expert level guess-and-check.
- A word to the wise:
  - Don't try to memorize the table given in the text!

Conclusion: examples good, table bad

- It's possible to build Y by in a reasonably systematic and efficient clever guessing when $g(t)$ is, roughly speaking, built from polynomials, exponential functions, and sine and cosine.
- That is called the method of undetermined coefficients.
  - Expert level guess-and-check.
- A word to the wise:
  - Don't try to memorize the table given in the text!
  - It is a much better idea to learn the mental yoga of the method works, by working through and thinking about lots of examples.