Further investigation of harmonic vibration

Math 352 Differential Equations

11 April 2014

Inhomogeneous equations

- Unforced or *free* vibrations are controlled by the differential equation

$$mu'' + \gamma u' + ku = 0.$$

When external forces are present, we typically add them together and regard them as the inhomogeneous term in the equation

$$mu'' + \gamma u' + ku = F(t).$$

Good news

- We solve this equation for a large class of forcing functions $F(t)$ using the methods of undetermined coefficients and variation of parameters.
- FYI: other methods exist, in particular the method of *Laplace transforms*, that are even better suited to *discontinuous* forcing functions such as square waves.

What else is there to say?

- In the free damped case ($F(t) = 0$, $\gamma > 0$), the displacement decays to zero with increasing time.
- Therefore, we think about the general solution of the forced, damped equation as the sum of two pieces:

$$y_c + Y$$

- Since $y_c \to 0$ as $t \to \infty$, y_c is called the *transient* part of the solution, and Y the *steady-state* solution.

Transient and steady-state

Recall also that in solving an instance of this equation, the coefficients c_1 and c_2 are determined only by the initial conditions, and the steady- state solution is independent of them.

Dual nature of solutions

The forcing function alone determines the steady-state solution; initial conditions only affect the transient.

Beats: when frequencies are near

- Suppose the forcing function is sinusoidal
  - with frequency close to the natural (undamped) frequency ω_0.
- Let ω be the forcing frequency and F_0 the forcing amplitude, so that $F(t) = F_0 \cos(\omega t)$. The equation of motion becomes

$$u'' + \omega_0^2 u = \frac{F_0}{m} \cos(\omega t).$$

- UC: Particular solution should be $Y = A\cos(\omega t) + B\sin(\omega t)$. But in fact one can show that $B = 0$.

Beats: the particular solution

- Furthermore, one can show that $A = \frac{F_0/m}{\omega_0^2 - \omega^2}$, so that

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t).$$

- If the system starts from equilibrium, one can show[1] that

$$c_1 = -\frac{F_0/m}{\omega_0^2 - \omega^2}, \quad c_2 = 0.$$

- The displacement function is thus determined to be

$$u = \frac{F_0/m}{\omega_0^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_0 t) \right).$$

[1] It would be an excellent exercise to carry out the details of these assertions.

We got the beat

We now use the trigonometric identity

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta$$

to rewrite this as

$$u(t) = \left(\frac{2F_0/m}{\omega_0^2 - \omega^2} \sin \frac{\omega_0 - \omega}{2} t \right) \sin \frac{\omega_0 + \omega}{2} t.$$

- We can interpret the equation as a fast oscillation *modulated* by a slow one, since $\omega_0 \approx \omega$.

Resonance

- What happens when $\omega_0 = \omega$?
- Roughly speaking, the period of the slow oscillation becomes infinite.
- We see a steadily increasing amplitude for the fast oscillation.
- The differential equation is degenerate, and the particular solution ends up being

$$Y = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

- If $u(0) = u'(0) = 0$, then $c_1 = c_2 = 0$. Displacement is

$$u = Y = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

- Observe that $\lim_{t\to\infty} u = \infty$, as advertised.

An explanation

"The following argument may shed some light on resonance phenomena. A system has the minimum resistance to oscillations at its natural frequency ω_0, since it will oscillate at ω_0 when it is not disturbed. When an external effect forces the system to oscillate at a different frequency, the system will resist because it does not feel comfortable oscillating at an unnatural frequency. But when the external effect forces the system to oscillate at its natural frequency, the system will gladly cooperate since the request is in perfect compliance with its intrinsic characteristics."

{From Çengel and Palm}