Math 4504: Student Packet

Spring 2018

Contents

1 Begin to Count

Activity: Bunt Chapter 1, Part 1

Problem 1 *Write the numbers from 1 to 100 (counting by fives) in hieroglyphic.*

Problem 2 *Compare and contrast the hieroglyphic counting system with our (Hindu-Arabic) system.*

Problem 3 *Perform the following operations in hieroglyphic, using the method that ancient Egyptians would have:*

(a) $2371 + 185$

(b) $3914 - 1609$

Problem 4 *Write* $\frac{4}{17}$ *as a sum of unique unit fractions in both Hindu-Arabic symbols and in hieroglyphic.*

Problem 5 *Solve the following using the "doubling and adding" method of multiplication. You may use Hindu-Arabic numerals if you'd like!*

(a) 13×33

(b) $36 \div 5$

(c) $6 \div 17$

Problem 6 *What is the value of the tables that ancient Egyptians often used for their calculations?*

Author(s):

2 The method of false position

In this activity we will seek to understand the method of false position.

Exercise 1 *Solve the following algebra problem:*

$$\begin{aligned} x^2 + y^2 &= 52 \\ 2x &= 3y \end{aligned}$$

Question 2 *While I am not sure which method you used to solve this problem, ancient Egyptians used the method of false position to solve problems like this. Moreover, such a method was taught in American schools until the mid 1800's. Here is the solution using false position—without any explanation!*

(a) *Set* $x = 3$ *and* $y = 2$.

(b) $3^2 + 2^2 = 13$.

(c) $52/13 = 4$.

(d) $\sqrt{4} = 2$.

(e) $x = 2 \cdot 3$ *and* $y = 2 \cdot 2$.

Explain the algorithm used and give another example to show you know how it is done.

Exploration 3 *Can you explain* ***why*** *the method of false position works?*

Exercise 4 *Solve the following problem: One hundred dollars is to be split among four siblings: Ali, Brad, Cara, and Denise where Brad gets four more dollars than Ali, Cara gets eight more dollars than Brad, and Denise gets twice as much as Cara. How much does each sibling get?*

Author(s):

Question 5 *Here is the solution by double false position:*

(a) *Suppose Ali gets* 6 *dollars.*

(b) *The total now is not* 100*, but* 70*. We are too low by* 30*.*

(c) *Now suppose Ali gets* 8 *dollars.*

(d) *The total now is not* 100*, but* 80*. We are too low by* 20*.*

(e) *Compute*

$$\frac{8 \cdot 30 - 6 \cdot 20}{30 - 20} = 12.$$

This is the correct answer.

Explain the algorithm used and give another example to show you know how it is done.

Exploration 6 *Can you explain **why** the method of double false position works?*

3 Babylonian numbers

In this activity we explore the number system of the ancient Babylonians.

The ancient Babylonians used cuneiform characters to write their numbers.

Exercise 1 *What are the 2 basic ancient Babylonian numerical symbols and what do they mean?*

Exploration 2 *Discuss the limitations of the Babylonian system. Then debate whether these so-called limitations were actually limitations at all.*

Exploration 3 *Is the Babylonian system more of a place-value system or a concatenation system?*

Problem 4 *Fill out the following table, simplifying any calculations.*

$Hindu-Arabic$	$Cuneiform$	$Hindu-Arabic$	$Cuneiform$	$Hindu-Arabic$	$Cuneiform$
5×1		5×2		5×3	
5×4		5×5		5×6	
5×7		5×8		5×9	
5×10		5×20		5×30	
5×40		5×50		$\frac{1}{5}$	
$\frac{1}{4}$		$\frac{1}{9}$		$\frac{1}{10}$	
$\frac{5}{6}$		$\frac{1}{20}$		$\frac{1}{100}$	

Problem 5 *Use your table to make the following calculations. You should work in base sixty, though you may use Hindu-Arabic numerals.*

(a) 34×5

(b) $1, 47 \div 5$

(c) $150 \div 4$

(d) $8, 6, 15 \div 6, 40$

Author(s):

4 Rational numbers and similarity

In this activity we play a game of "what if" and see a reason that the ancient Greeks might have wanted every number to be rational.

Exploration 1 *Think about plain old plane geometry. What are some theorems that you would want to be true?*

Question 2 *What are the basic theorems involving similar triangles?*

OK—now we are going to do something very strange. Let's suppose that every number is rational. In essence, let's put ourselves into the mindset of the ancient Greeks, **before** they knew that irrational numbers existed.

Exploration 3 *Suppose that you have two triangles whose angles are congruent. Can you make a fairly simple argument, using the fact that the sides are rational numbers, that shows that the sides are proportional? Hint: You may need to use ASA.*

Exploration 4 *Suppose that you have two triangles whose sides are proportional. Can you make a fairly simple argument, using the fact that the sides are rational numbers, that shows that the angles are congruent? Hint: You may need to use SSS.*

Author(s):

5 Pythagorean means

In this activity we explore the three different means of the ancient Greeks.

The arithmetic mean

The *arithmetic mean* is the good-old mean that we are all familiar with.

Question 1 *What is the mean that we are all familiar with? Explain how to compute the mean of $a_1, a_2, \ldots, a_n$. Give some examples.*

The geometric mean

The *geometric mean* is a bit different. The geometric mean of $a_1, a_2, \ldots, a_n$ is given by:

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n}$$

Question 2 *Explain an analogy between the arithmetic mean and the geometric mean.*

Question 3 *Can you explain the geometric mean in terms of geometry? First do it for 2 numbers. Next do it for three.*

The harmonic mean

The *harmonic mean* might be the most mysterious of all. The harmonic mean of $a_1, a_2, \ldots, a_n$ is given by:

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$$

Exploration 4 *Can you find a connection between the harmonic mean and music?*

Question 5 *In the United States, the fuel efficiency of a car is usually given in the units:*

$$\frac{\text{miles}}{\text{gallon}}$$

However, in Europe, the fuel efficiency of a car is usually given in the units:

$$\frac{\text{liters}}{100\text{km}}$$

Give some examples of fuel efficiency (both efficient and inefficient) with each set of units.

Author(s):

Question 6 *Now suppose that a car gets* $60 \frac{\text{miles}}{\text{gallon}}$ *and another car gets* $20 \frac{\text{miles}}{\text{gallon}}$*. What is the average fuel efficiency?*

Question 7 *Now suppose that a car gets* $4 \frac{\text{liters}}{100\text{km}}$ *and another car gets* $20 \frac{\text{liters}}{100\text{km}}$*. What is the average fuel efficiency?*

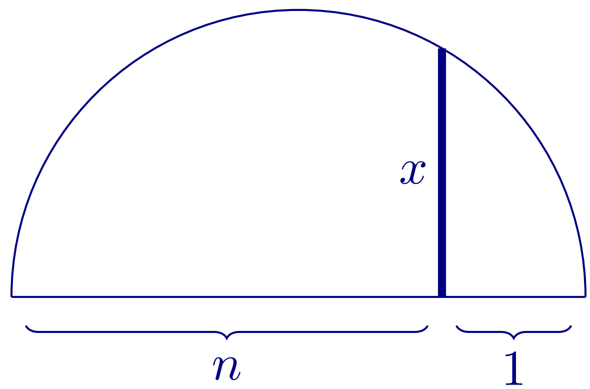
Exploration 8 *Compare your answers to the last two questions. Something fishy is going on, what is it?*

6 Computing quadratures

In this activity we will compute some basic quadratures.

When computing a quadrature of a shape in the method of the ancient Greeks, one needs to produce a line segment whose length gives the side of a square of equal area to the original shape.

Question 1 *Consider the figure below. Explain how one could construct it and what segment x represents.*



Question 2 *Construct a rectangle whose side lengths are 8 units and 5 units. Then construct its quadrature. Explain your construction step-by-step, and tell why it works!*

Question 3 *Construct a triangle whose base has length 8 units and whose height has length 5 units. Then construct its quadrature. Explain your construction step-by-step, and tell why it works!*

Question 4 *Suppose you have a square whose side length is 8 units and another square whose side length is 15 units. How would you construct the quadrature of the two areas together? Explain how you know.*

Question 5 *How do you compute the quadrature of a polygon?*

Author(s):

7 It's All Greek To Me!

We investigate solutions to the Problems of Antiquity.

Question 1 *Use your compass and straightedge to double a square with side length s. That is, construct a square whose area is twice that of your original square. Why can't we do the same for the cube?*

Hippocrates used a "continued mean proportional" to double the cube. Let's let a be the side length of the original cube, and x be the side length of the new, larger cube.

Question 2 *Just to check: write an equation relating a and x.*

Here is an outline for how Hippocrates might have gotten his mean proportional:

(a) Start with two cubes of side length a next to each other.

(b) Rearrange the volume of these two cubes into a rectangular prism so that the height is unchanged, but the base rectangle has one side of length x. Call the other side length y.

(c) Rearrange the volume again, but this time into a cube. Use a different side as the base, and leave the "height" x unchanged.

Question 3 *Draw pictures representing the geometry in the method described above.*

Question 4 *Write a proportion corresponding to the first rearrangement using the variables x, y, a, and $2a$. Hint: we know the height of the box stays the same. What does this mean about the area of the base?*

Question 5 *Write a proportion corresponding to the second rearrangement. One of the fractions should be equal to one of the fractions from the previous question! Then write Hippocrates' continued mean proportional by setting three fractions equal to each other.*

Question 6 *How is this continued mean proportional related to the equation you found in Question 2? How might you use this information to duplicate the cube?*

Author(s):

About 420BC, Hippias invented a curve called the "quadratrix". Here is its construction:

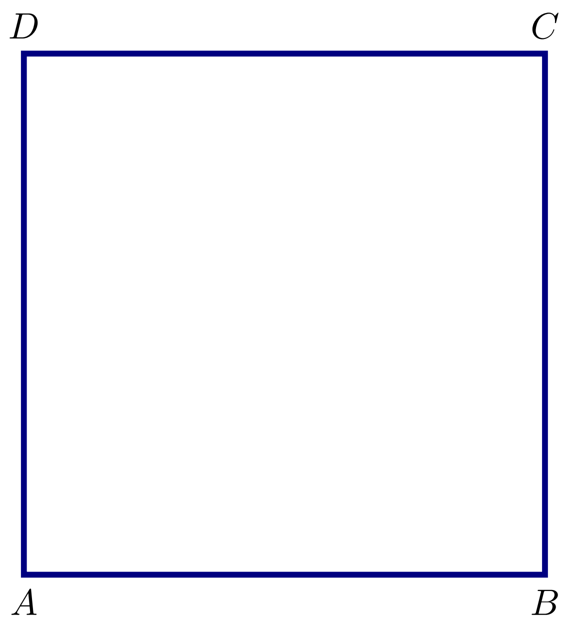
(a) Start with a square $ABCD$.

(b) A line segment congruent with AB and coinciding with AB rotates with center A a quarter turn.

(c) At the same time, and at the same speed, another segment congruent with AB and coinciding with AB moves using straight-line motion through the square until it coincides with CD.

(d) Points on the quadratrix are where the two moving segments intersect.

Question 7 *On the square below, use a ruler and a protractor to construct at least four points on the quadratrix, and then sketch the entire curve.*



Question 8 *Let X be any point on the quadratrix, and X' be the point directly below X on segment AB. If l is the length of segment AB and n is the length of segment XX', explain why it's always true that*

$$\frac{m\angle XAB}{m\angle DAB} = \frac{n}{l}.$$

Question 9 *Use the quadratrix to trisect an angle of $45°$, then an angle of $60°$. Then, explain how the quadratrix can be used to trisect any angle.*

Question 10 *Can you use the quadratrix to square the circle? Explain how!*

8 Proofs of the Pythagorean Theorem

We will study Euclid for two chapters - the first focused on geometry and the second focused on number theory. Euclid's name is worth knowing because of his work called the "Elements", where he attempts to construct all of the mathematics known at the time from basic assumptions he calls "common notions" and "postulates". By the time we are finished with this chapter, you should be able to state Euclid's Fifth Postulate and say something about why it was controversial.

Two important people who influenced Euclid's thinking are Eudoxus, most famous for his "method of exhaustion", and Aristotle who wrote on what proof in mathematics should be, and may have been the first to use the phrase "common notion".

Readings

First reading: Dunham Chapter 2

Second Reading: Proofs of the Pythagorean Theorem

In the second reading, you should read the introduction, and then pick a few of these proofs to study. You do not need to know all of the proofs on this site! You should be able to give, in full detail, the proof from our textbook (which is also Proof #1 on the site) as well as two other proofs of your choice.

Questions

Question 1 *How many proofs are listed on this site?* 118

Question 2 *Which of the following is NOT a category of proofs of the theorem mentioned in the remarks?*

Multiple Choice:

(a) *Proofs by contradiction.*

(b) *Algebraic proofs.*

(c) *Geometric proofs.*

(d) *Trigonometric proofs.*

Question 3 *What are the most important points from this reading?*

Author(s):
See Eudoxus at http://www-groups.dcs.st-and.ac.uk/history/Biographies/Eudoxus.html
See Aristotle at http://www-groups.dcs.st-and.ac.uk/history/Biographies/Aristotle.html
See Proofs of the Pythagorean Theorem at http://www.cut-the-knot.org/pythagoras/

9 Euclid's Elements

We prove some propositions from Euclid's Elements. A full list of the definitions, common notions, postulates, and propositions in Book I is posted on Carmen, downloaded from `http://aleph0.clarku.edu/\protect\unhbox\voidb@x\penalty\@M\{}djoyce/java/elements/bookI/bookI.html`

Proposition 1 (I.5)**.** *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

Question 1 *Prove Proposition I.5.*

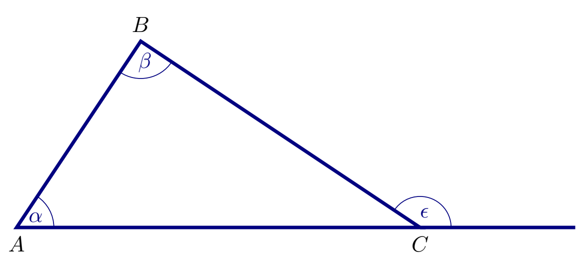
Proposition 2 (I.6)**.** *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

Question 2 *Prove Proposition I.6.*

Proposition 3 (I.15)**.** *If two straight-lines cut one another then they make the vertically opposite angles equal to one another.*

Question 3 *Prove Proposition I.15.*

Proposition 4 (I.16)**.** *For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.*



Question 4 *Prove Proposition I.16.*

Proposition 5 (I.27)**.** *If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.*

Question 5 *Prove Proposition I.27.*

Proposition 6 (I.29)**.** *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.*

Author(s):

Question 6 *Prove Proposition I.29.*

Proposition 7 (I.32)**.** *In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.*

Question 7 *Prove Proposition I.32.*

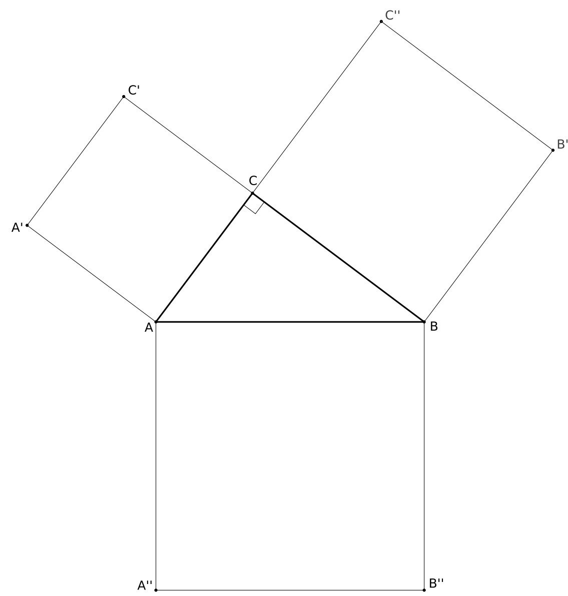
10 The Pythagorean Theorem

In this activity we will prove the most famous theorem of all.

Question 1 *Remind us, what is the most famous theorem of all and what exactly does it assert?*

Euclid's proof

Question 2 *What would one need to prove about the following diagram to prove the Pythagorean Theorem?*



Let's see if we can do this!

Question 3 *Draw a line perpendicular to $\overline{AB}$ that passes though both C and $\overline{A''B''}$. Call the intersection between this line and $\overline{AB}$, point E; call the intersection point between this line and $\overline{A''B''}$, point E'. Explain why $\triangle ACA''$ has half the area of rectangle $AEE'A''$.*

Question 4 *Explain why $\triangle ABA'$ has half the area of square $ACC'A'$.*

Question 5 *Explain why $\triangle ACA''$ is congruent to $\triangle ABA'$.*

Author(s):

Question 6 *Explain why area of square $ACC'A'$ is equal to the area of rectangle $AEE'A''$.*

Question 7 *Use similar ideas to complete a proof the Pythagorean Theorem.*

The converse

Question 8 *What is the converse to the Pythagorean Theorem? Is it true? How do you prove it?*

11 Unsolved Problems

Our next chapter is the first where we'll discuss number theory, a branch of mathematics wherein we study properties of whole numbers and relationships between them. Number theory is sometimes used as an introduction to "higher mathematics", because the definitions are usually easy to grasp. Examples are often easy to come by, and conjectures seem to follow naturally.

Once we begin to ask questions about the relationships between numbers, we quickly realize that many of these questions are incredibly easy to state, but incredibly difficult to prove! In fact, some of the most well-known unsolved problems in mathematics come from the branch of number theory.

In the second reading, we'll be introduced to some famous unsolved problems in mathematics. Some of these problems are mentioned in the first reading, and some of them are not. As you look at this article, make sure to click on some of the links to get more information about topics that look interesting to you. The Collatz problem, for instance, is very easy to try out for yourself. The notion that 10 is a solitary number is related to some things we read about the Pythagoreans, and of course we've already discussed the case of odd perfect numbers.

Readings

First reading: Dunham, Chapter 3, pages 61-73

Second reading: Difficult Problems.

In the second reading, please click on the links on the page to get more information about these problems. You should read at least the following.

(a) The Goldbach Conjecture

(b) Twin Primes

(c) The Twin Prime Conjecture

Questions

Question 1 *How many versions or types of Goldbach's conjecture are listed in the article on that topic?* 7

Question 2 *Which of the following are twin primes?*

Multiple Choice:

(a) 1 *and* 3

(b) 11 *and* 23

Author(s):
See Difficult Problems at `http://mathworld.wolfram.com/UnsolvedProblems.html`
See The Goldbach Conjecture at `http://mathworld.wolfram.com/GoldbachConjecture.html`
See Twin Primes at `http://mathworld.wolfram.com/TwinPrimes.html`
See The Twin Prime Conjecture at `http://mathworld.wolfram.com/TwinPrimeConjecture.html`

(c) 23 *and* 46

(d) 29 *and* 31

Question 3 *What are the most important points from this reading?*

12 The unique factorization theorem

In this activity we investigate unique factorization theorems.

Consider this proposition from Euclid's *Elements*:

Proposition 8 (IX.14). *If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.*

Question 1 *Explain what the proposition above is saying.*

Question 2 *Now consider Euclid's proof:*

> *Let the number A be the least that is measured by the prime numbers B, C, and D. I say that A is not measured by any other prime number except B, C, or D. If possible, let it be measured by the prime number E, and let E not be the same as any one of the numbers B, C, or D.*
>
> *Now, since E measures A, let it measure it according to F, therefore E multiplied by F makes A. And A is measured by the prime numbers B, C, and D. But, if two numbers multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers, therefore each of B, C, and D measures one of the numbers E or F. Now they do not measure E, for E is prime and not the same with any one of the numbers B, C, or D. Therefore they measure F, which is less than A, which is impossible, for A is by hypothesis the least number measured by B, C, and D. Therefore no prime number measures A except B, C, and D. Therefore, if a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.*

Can you explain what this proof is saying?

Now let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

Author(s):

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

Question 3 *Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.*

Question 4 *Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.*

Question 5 *Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.*

Question 6 *Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.*

Definition 1. *Call a positive integer* ***prome*** *in $3\mathbb{Z}$ if it cannot be expressed as the product of two integers both in $3\mathbb{Z}$.*

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

Question 7 *List all the prome numbers less than 297.*

Question 8 *Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?*

Question 9 *Can you find numbers that factor completely into prome numbers in two different ways? How many can you find?*

13 On the Infinitude of Primes

The great theorem of this chapter is, essentially, that there are infinitely many primes. In our readings, we'll see Euclid's proof of this fact as well as another proof by a mathematician named Hillel Furstenberg. Furstenberg is probably most famous for his contributions to an area of mathematics called "ergodic theory", in which we study moving systems.

Readings

First reading: Dunham, Chapter 3, pages 73-83

Second reading: On Furstenburg's Proof of the Infinitude of Primes

Questions

Question 1 *What is the example given of an arithmetic progression?* $2 + 7\mathbb{Z}$

Question 2 *To which branch of mathematics is Furstenberg's proof method most related? In other words, what makes his approach different from Euclid's?*

Multiple Choice:

(a) *Number Theory*

(b) *Calculus*

(c) *Topology*

(d) *Ergodic Theory*

Question 3 *What are the most important points from this reading?*

Author(s):
See On Furstenburg's Proof of the Infinitude of Primes at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/40391095

14 Estimating Pi

Question 1 *List as many ways as you can think of for estimating the value of π.*

Draw a (fairly large) circle on a blank sheet of paper. We'll think of this as a unit circle.

Problem 2 *Divide the unit circle into $2^2 = 4$ equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_2. Let $\mathcal{A}_2$ denote the area of the triangle $\triangle OAB_2$ and let θ_2 denote the measure of the angle at O. Explain how to estimate the area of the circle with triangle $\triangle OAB_2$. What is your estimate?*

Problem 3 *Divide the unit circle into $2^3 = 8$ equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_3. Let $\mathcal{A}_3$ denote the area of the triangle $\triangle OAB_3$ and let θ_3 denote the measure of the angle at O. Explain how to estimate the area of the circle with triangle $\triangle OAB_3$. What information do you need to know to actually do this computation?*

Problem 4 *Given an angle θ, explain the relation of $\sin(\theta)$ and $\cos(\theta)$ to the unit circle. How could these values help with the calculation described above?*

Problem 5 *Divide the unit circle into 2^n equal wedges each with its vertex at the center of the circle O. On each wedge, call the two corners of the wedge that lie on the circle A and B_n. Let $\mathcal{A}_n$ denote the area of the triangle $\triangle OAB_n$ and let θ_n denote the measure of the angle at O. Explain why someone would be interested in the value of:*

$$\sin\left(\frac{\theta_n}{2}\right)$$

Problem 6 *Recalling that:*

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}} \qquad \textit{and} \qquad \cos(\theta)^2 + \sin(\theta)^2 = 1$$

Explain why:

$$2\mathcal{A}_{n+1} = \sqrt{\frac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$$

Author(s):

Problem 7 *Let's fill out the following table (a calculator will help!):*

n	$\mathcal{A}_n$	*Approx. Area*	$\sqrt{1-(2\mathcal{A}_n)^2}$	$\dfrac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}$	$2\mathcal{A}_{n+1} = \sqrt{\dfrac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$
2					
3					
4					
5					
6					
7					
8					

What do you notice?

15 Heron's formula

In this activity we will give two proofs of Heron's formula.

We'll start by giving a proof using synthetic geometry.

Part I

Proposition 9. *The bisectors of the angles of a triangle meet at a point that is the center of the triangle's inscribed circle.*

Question 1 *How can we prove this?*

Question 2 *Now draw a triangle with vertices A, B, and C. Draw the incircle. Explain why the radii of the incircle touch the sides of the triangle at right angles.*

Question 3 *Label the intersection of the radii with D between A and B, E between B and C, and F between C and A. Compute the areas of the following triangles:*

$$\triangle AOB, \qquad \triangle BOC, \qquad \triangle COA.$$

Use this to express the area of $\triangle ABC$.

Part II

Question 4 *Explain why*

$$\triangle AOD \cong \triangle AOF, \qquad \triangle BOD \cong \triangle BOE, \qquad \triangle COF \cong \triangle EOF.$$

Question 5 *If $AG \cong CE$, explain why $|BG|$ is the semiperimeter.*

Question 6 *Find segments in your drawing equal to the length of*

$$s - a, \qquad s - b, \qquad s - c.$$

Author(s):

Part III

Proposition 10. *If quadrilateral $AHBO$ has diagonals AB and OH with $\angle HAB$ and $\angle HOB$ being right angles, then $AHOB$ can be inscribed in a circle.*

Question 7 *Can you prove this proposition?*

Proposition 11. *The opposite angles of a cyclic quadrilateral sum to two right angles.*

Question 8 *Can you prove this proposition?*

Question 9 *Now we need to decorate our triangle even more:*

(a) *Draw OL perpendicular to OB cutting AB at K.*

(b) *Draw AM perpendicular to OB.*

(c) *Call the intersection of OL and OM, H.*

(d) *Draw BH.*

Consider quadrilateral $AHBO$, explain why opposite angles sum to two right angles.

Question 10 *Explain why $\triangle COF$ is similar to $\triangle BHA$. Use this to explain why*

$$\frac{|AB|}{|AG|} = \frac{|AH|}{r}.$$

Question 11 *Explain why $\triangle KAH$ is similar to $\triangle KDO$. Use this to explain why*

$$\frac{|AK|}{|KD|} = \frac{|AH|}{r}.$$

Question 12 *Now we see*

$$\frac{|AB|}{|AG|} = \frac{|AK|}{|KD|}.$$

Add 1 to both sides to obtain

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}.$$

Question 13 *Explain why $\triangle KDO$ is similar to $\triangle ODB$. Use this to explain why*

$$|KD| \cdot |BD| = r^2.$$

Question 14 *Multiply both sides of*

$$\frac{|BG|}{|AG|} = \frac{|AD|}{|KD|}$$

by $\dfrac{|BD|}{|BD|}$ *to obtain*

$$r^2|BG|^2 = |AG| \cdot |BG| \cdot |AD| \cdot |BD|.$$

Question 15 *Explain how to deduce Heron's formula.*

A modern proof

Question 16 *Now give a modern proof that a high school student might give.*

Question 17 *Which proof was harder? Why didn't the ancient Greeks use our modern proof?*

16 Solving equations

In this activity we will solve second and third degree equations.

Finding roots of quadratic polynomials is somewhat complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, with the first general solutions appearing in Babylon and China around 2500 years ago. Let's begin with an easier problem: make $a = 1$ and try to solve $x^2 + bx = c$.

Problem 1 *Geometrically, you could visualize $x^2 + bx = c$ as an $x \times x$ square along with a $b \times x$ rectangle. Make a blob for c on the other side. Draw a picture of this!*

Question 2 *What is the total area of the shapes in your picture?*

Problem 3 *Now draw a new picture: take your $b \times x$ rectangle and divide it into two $(b/2) \times x$ rectangles.*

Question 4 *What is the total area of the shapes in your picture?*

Problem 5 *Draw a next picture in your sequence: take both of your $(b/2) \times x$ rectangles and snuggie them next to your $x \times x$ square on adjacent sides. You should now have what looks like an $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ square with a corner cut out of it.*

Question 6 *What is the total area of the shapes in your picture?*

Finally, your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? So if you add that piece in on both sides, the area of both sides of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Question 7 *Can you find x at this point?*

Question 8 *Explain how to solve $ax^2 + bx + c = 0$.*

Author(s):

Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's - and under some pretty strange circumstances! See your text for the low-down on all of the drama.

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form

$$x^3 + px + q.$$

All I can tell you are these three steps:

(a) Replace x with $u + v$.

(b) Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms.

(c) Clear denominators and use the quadratic formula.

Question 9 *How many solutions are we supposed to have in total?*

Question 10 *Use the Ferro-Tartaglia method to solve* $x^3 + 9x - 26 = 0$.

Question 11 *How many solutions should our equation above have? Where/what are they? Hint: Make use of an old forgotten foe...*

Question 12 *Is the method described here the same as the one in our text as the proof of the "great theorem"? Explain why or why not.*

Question 13 *Use the Ferro-Tartaglia method to solve* $x^3 = 15x + 4$. *What do you notice?*

Question 14 *How do we do this procedure for other equations of the form*

$$x^3 + px + q = 0?$$

Give an algebraic formula as your solution.

Question 15 *Would Ferro, Tartaglia, Cardano, or Ferrari have answered the previous question differently than you might?*

17 The binomial theorem and π

In this activity we investigate a generalization of the binomial theorem and its connection to an approximation of π.

Question 1 *The binomial theorem states*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Why would we be interested in this?

Question 2 *Newton says*

$$(1+x)^r = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!} \prod_{\ell=0}^{k-1} (r-\ell) \right).$$

How did Newton come up with this? Hint: Calculus!

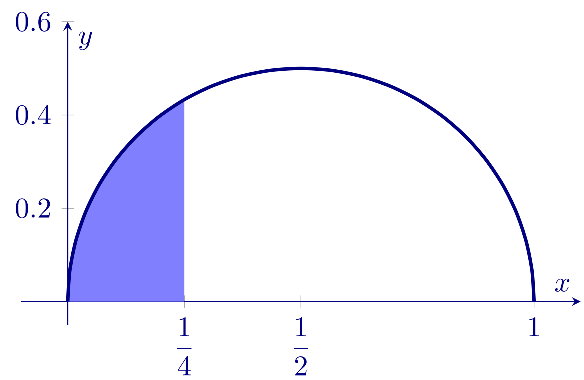
Now we're going to use this to approximate π.

Question 3 *Come up with a function of x for the semicircle of radius $1/2$ centered at $(1/2, 0)$.*

Question 4 *Use Newton's binomial theorem to show your function above is equal to:*

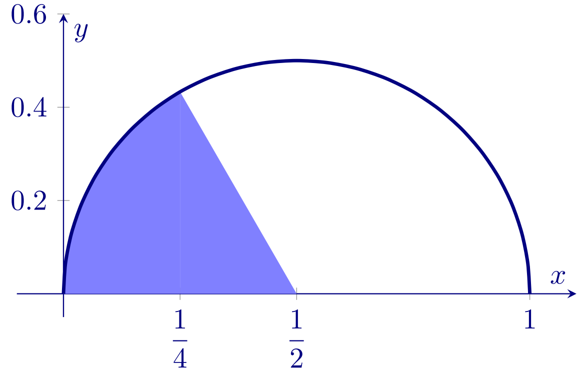
$$x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} - \frac{7x^{11/2}}{256} - \cdots$$

Question 5 *Use calculus to compute the area of the shaded region:*

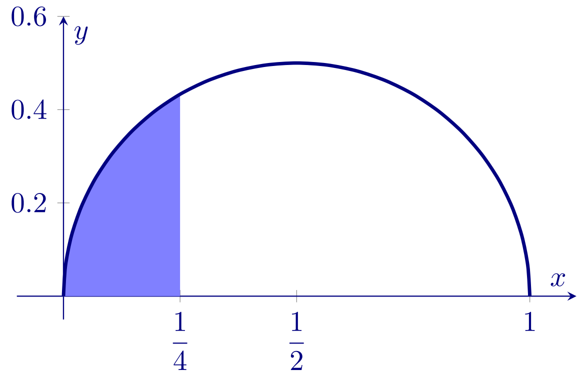


Author(s):

Question 6 *Use proportional reasoning to compute the area of the sector below:*



Question 7 *Use the previous problem, along with the area of a certain 30-60-90 right triangle to give a different computation of the area below.*



Question 8 *Use your work from above to give an approximation of π.*

18 Newton and Leibniz

The first part of Chapter 8 details an unfortunate phase in the history of mathematics: the debate over who actually “invented” calculus. Part of the first reading will tell the story of this debate, while the second reading gives a concise comparison of the works of these two famous mathematicians. After looking through these readings, you should be able to form your own opinion on the matter!

Readings

First reading: Dunham, Chapter 8, pages 184 - 196

Second reading: Newton and Leibniz: The Calculus Controversy.

Questions

Question 1 *In what year (BC) do we find the first evidence of the ideas of calculus?* 1600

Question 2 *In what country did Leibniz study law?*

Multiple Choice:

(a) *England*

(b) *Germany*

(c) *Russia*

(d) *Switzerland*

Question 3 *What are the most important points from this reading?*

Author(s):

See Newton and Leibniz: The Calculus Controversy at https://www.fitchburgstate.edu/uploads/files/Undergraduate_Research_Conference/Sample-Math-Poster.pdf

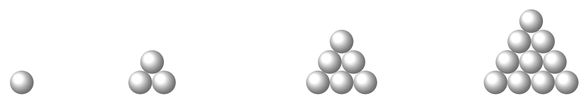
19 Leibniz and series

In this activity we investigate some of the series that Leibniz investigated.

Series pop-up at an early age. I distinctly remember being in fourth grade, sitting at my desk, starring at my ruler, wondering how 1/3 of a foot could simultaneously be 4 inches (clearly a finite number) and $0.333333\ldots$ of a foot (a number that somehow seemed finite and infinite at the same time). I was struggling with the implicit concept that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

Leibniz (and other mathematicians of the era) had similar feelings regarding series. Leibniz's mentor, Christian Huygens, suggested that Leibniz work on computing the sum of the reciprocal of the triangular numbers. Recall that the triangular numbers are the number of dots in discrete equilateral triangles:



Question 1 *Consider Leibniz's "proof."*

$$\begin{aligned}
S &= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots \\
\frac{S}{2} &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots \\
\frac{S}{2} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\
\frac{S}{2} &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \cdots \\
\frac{S}{2} &= 1 \\
S &= 2.
\end{aligned}$$

Can you explain what is going on here? Where might Leibniz want to be a bit more rigorous?

Question 2 *What does*

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

sum to?

Question 3 *Now consider this summation of*

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots$$

Author(s):

Write

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots \\
\sum_{k=1}^{n} \frac{2}{k(k+1)} &= \sum_{k=1}^{n} 2\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
\sum_{k=1}^{n} \frac{2}{k(k+1)} &= 2\left(1 - \frac{1}{n+1}\right) \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{k(k+1)} &= \lim_{n\to\infty} 2\left(1 - \frac{1}{n+1}\right) \\
\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 2.
\end{aligned}
$$

What's going on here? How does this compare to Leibniz's proof?

In addition to (co)inventing calculus, Leibniz, dreamed up this beast while trying to solve problems like the one posed to him by Huygens:

$$
\begin{array}{ccccccccc}
 & & & & \frac{1}{1} & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}
$$

Question 4 *What relationships can you find between the entries of the triangle as we move from row to row?*

Question 5 *What are the next two rows? Clearly articulate how to produce more rows of the harmonic triangle.*

Question 6 *Explain how the following expression*

$$\frac{1}{r \cdot \binom{r-1}{c-1}}$$

corresponds to entries of the harmonic triangle. Feel free to draw diagrams and give examples.

Question 7 *Explain how the harmonic triangle is formed. In your explanation, use the notation*

$$\frac{1}{r \cdot \binom{r-1}{c-1}}$$

If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

Question 8 *Can you explain why the numerators of the fractions in the harmonic triangle must always be 1?*

Question 9 *Explain how to use the harmonic triangle to go from:*

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots$$

to

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

Conclude by explaining why Leibniz said:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots=1$$

Question 10 *Can you generalize the results above? Can you give a list of infinite sums and conjecture what they will converge to?*

20 Proofs that the Harmonic Series Diverges

Our Great Theorem of Chapter 8 is Johann Bernoulli's proof that the Harmonic Series diverges. We'll talk about why this is a surprising result, as well as some other attempts that were made at the proof, particularly by Leibniz.

As we saw with the Pythagorean Theorem, mathematicians often like to find several different ways of proving theorems - particularly when the result of the theorem is surprising or important. Sometimes this seems silly to non-mathematicians, since the result of the theorem is already known. But using different methods of proof can help us understand connections we've never before seen between seemingly different areas of mathematics, or shed light on related problems that are not yet solved.

Again, as with the Pythagorean Theorem, you are not expected to know all of the proofs given in the articles below. Choose one or two other than the proof given as the Great Theorem.

Readings

First reading: Dunham, Chapter 8, pages 196-206

Second reading: The Harmonic Series Diverges Again and Again.

Third reading: An Exceedingly Short Proof that the Harmonic Series Diverges

Questions

In the list of proofs that the Harmonic Series diverges, in what year was the earliest one given?

Question 1 1350

Question 2 *The "exceedingly short proof" is done by what method?*

Multiple Choice:

(a) *Guessing.*

(b) *Computing an integral.*

(c) *Evaluating the series directly.*

(d) *Comparison with another series.*

Question 3 *What are the most important points from this reading?*

Author(s):

See The Harmonic Series Diverges Again and Again at http://scipp.ucsc.edu/~haber/archives/physics116A10/harmapa.pdf

See An Exceedingly Short Proof that the Harmonic Series Diverges at http://projecteuclid.org.proxy.lib.ohio-state.edu/download/pdf_1/euclid.mjms/1449161372

21 The Riemann Hypothesis

In Chapter 9, we meet perhaps the most prolific mathematician of all time: Euler. Dunham has written a lot in his career about Euler, and so we have two chapters to study his work. In this first chapter, our Great Theorem will investigate how Euler succeeded where Leibniz and Bernoulli failed to find the sum of $\sum \frac{1}{n^2}$. Euler didn't stop adding up series here, though, and has an impressive list of results similar to this one.

Euler's results, however, quickly bring up another question in the form of "how far can we really go, here?" It's natural at this point to introduce what's known as the Riemann hypothesis. Riemann himself, as well as the hypothesis and Euler's work surrounding it are all described in the second reading. The reading itself appears to be long, but should hopefully be a quick read. Keep in mind, though, that the mathematician who wrote it is joking about just about everything other than the mathematics!

Readings

First reading: Dunham, Chapter 9

Second reading: A Friendly Introduction to the Riemann Hypothesis

Questions

Question 1 *How many problems are in Hilbert's list in total?* 23

Question 2 *What is the value of $\zeta(-2k)$ for k a positive integer?*

Multiple Choice:

(a) *No one knows.*

(b) *The function blows up to infinity there.*

(c) *A purely imaginary number.*

(d) 0

Question 3 *What are the most important points from this reading?*

Author(s):
See A Friendly Introduction to the Riemann Hypothesis at `http://www.math.jhu.edu/~wright/RH2.pdf`

22 Bernoulli, Euler, and series

Here we see some topics that both Bernoulli and Euler were interested in.

Finding the sum of the following series is called "The Basel Problem" as it interested several mathematicians with connections to the city of Basel, Switzerland. (Who were they?)

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} \cdots$$

Notice: we are asking for the sum of the reciprocals of the square numbers.

Question 1 *Consider:*

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \dots$$

Can you explain why

$$f(x) = \frac{\sin(x)}{x} \qquad x \neq 0?$$

Question 2 *Let $g(x)$ be a polynomial with roots $a_1, \dots, a_n$. Suppose also that $g(0) = 0$. What are the factors of $g(x)$?*

Question 3 *Let $g(x)$ be a polynomial with roots $a_1, \dots, a_n$. Suppose also that $g(0) = 1$. What are the factors of $g(x)$?*

Question 4 *What exactly are the roots of $f(x)$? What is $f(0)$? Explain why:*

$$f(x) = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots$$

Question 5 *Explain why:*

$$f(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

Question 6 *Explain why:*

$$f(x) = 1 - x^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + x^4\left(\cdots\right) - x^6\left(\cdots\right) + \cdots$$

Author(s):

Question 7 *Explain why:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exploration 8 *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

23 Very Large Primes

Being one of the most prolific mathematicians in history, Euler considered many subjects. Here, we look at Euler's work with some of Fermat's conjectures in number theory. In an unusual turn of events, we consider the Great Theorem as part of the first reading for this chapter. One of the consequences of this Great Theorem is that people began using Euler's proof method to look for larger and larger primes. This was one of the first ways to look for primes other than simply considering all of their factors. Different methods have since been invented, so Euler's method isn't necessarily used anymore, but our second reading explores "records" for the largest known primes. The article also does a nice job detailing how things changed after the modern computer was invented! You can also click the links in the article if you're interested in learning more about how people search for primes.

Readings

First reading: Dunham, Chapter 10, pages 223 - 235

Second reading: The Largest Known Primes by Year

Questions

Question 1 *How many digits are in the largest prime mentioned in the article?* 22338618

Question 2 *Who has the record for the largest prime found without using a machine?*

Multiple Choice:

(a) *Lucas*

(b) *Euler*

(c) *Ferrier*

(d) *Miller & Wheeler*

Question 3 *What are the most important points from this reading?*

Author(s):
See The Largest Known Primes by Year at http://primes.utm.edu/notes/by_year.html

24 Euler and Fermat

Question 1 *Remind me: what are the necessary ingredients for a proof by induction?*

Our goal is to prove the following theorem:

Theorem 1. *Suppose that a is an even number and p a prime which does not divide a. Suppose that p does divide $a^{2^n}+1$. Then p is of the form $p = 2^{n+1}k+1$ for some positive integer k.*

In order to prove this theorem, we'll need the so-called "Little Fermat Theorem". You can find Euler's proof of this theorem in your text.

Theorem 2 (Little Fermat Theorem)**.** *Let a be a whole number and p a prime which does not divide a. Then p divides $a^{p-1}-1$.*

Question 2 *Remind me: what is the definition of "divides"?*

Question 3 *Prove that Theorem 1 is true in the case that $n = 0$.*

Question 4 *Suppose that A is any whole number, and that you divide A by some number C. What are the possible remainders? What are the possibilities for how you could write A as related to a multiple of C?*

Question 5 *Repeat question 4, but related to Theorem 1 and the case $n = 1$. What are the possibilities for the prime p when you divide by $2^2 = 4$? Eliminate all but two of these cases.*

Question 6 *Use proof by contradiction to eliminate the case you don't want.*

Question 7 *Repeat questions 5 and 6, but for the case $n = 2$. If you're confident you understand what's going on, move to the next question!*

Question 8 *Prove Theorem 1.*

Question 9 *Euler used Theorem 1 to prove that $2^{2^5}+1$ is not prime. How did he do this? Check his work. Could you use his method to prove that $2^{2^6}+1$ is not prime?*

Author(s):

25 Gauss, The Prince of Mathematicians

One of the downsides of using a textbook like Dunham's is that we only have time to talk about a small number of mathematicians. At this point in history, we begin to see an increasing number of names you would recognize from your studies, both in mathematics as well as other subjects. To put Gauss in the epilogue of a chapter about Euler is, in my opinion, a disservice to this great mathematician. Some people even consider him to be the greatest mathematician since antiquity - greater even than Euler. You should, of course, form your own opinions after doing these readings.

We begin with Dunham's biography of Gauss, and then our second reading discusses Gauss's proofs of the Fundamental Theorem of Algebra. You don't need to be able to give these proofs exactly as they are described, but you should be able to talk about the main ideas. Finally, an optional third reading is the Mathematics Genealogy Project, beginning with Gauss. The genealogy project has a page for most modern mathematicians listing both their advisor as well as their advisees. So, you can click through either forward in time or backward in time to see how mathematicians are connected to one another!

Readings

First reading: Dunham, Chapter 10, pages 235 - 244

Second reading: Gauss's Proofs of the Fundamental Theorem of Algebra

Third reading: Mathematics Genealogy Project

Questions

Question 1 *According to Harel Cain, how many proofs of the Fundamental Theorem of Algebra did Gauss Give?* 4

Question 2 *Which of the following did not attempt a proof of the Fundamental Theorem of Algebra?*

Multiple Choice:

(a) *Gauss*

(b) *d'Alembert*

(c) *Leibniz*

(d) *Euler*

Question 3 *What are the most important points from this reading?*

Author(s):
See Gauss's Proofs of the Fundamental Theorem of Algebra at `math.huji.ac.il/~ehud/MH/Gauss-HarelCain.pdf`
See Mathematics Genealogy Project at `https://genealogy.math.ndsu.nodak.edu/id.php?id=18231`

26 Contemplating the Infinite

If you've never thought much about the idea of "infinity" before, then the Great Theorems of Chapters 11 and 12 might be a little surprising. If you've seen them before, it's always good to stop and think a bit about why these theorems are surprising. The historical context in the readings in Dunham should help.

For fun, you might try to explain to a (non-mathematician) friend that there are infinite sets which are the same "size", and infinite sets which are different "sizes". You'll first have to explain what you mean by "size", of course! The second reading might help your explanation, since it can be used effectively in a high school classroom for talking about this topic.

Readings

First reading: Dunham, Chapters 11 and 12

Second reading: The Infinite Hotel.

Questions

Question 1 *After the hotel appears to be full, a family arrives and asks for a room. After moving the patrons around, which room is the new family assigned?* 1

Question 2 *In the story of "Hotel Infinity", what does George build, other than the hotel?*

Multiple Choice:

(a) *A fleet of buses.*

(b) *A courtroom.*

(c) *Billboards.*

(d) *An infinite parking garage.*

Question 3 *What are the most important points from this reading?*

Author(s):
See The Infinite Hotel at http://www.jstor.org.proxy.lib.ohio-state.edu/stable/20876420

27 Cantor Can!

In this activity we look at Cantor's diagonal argument.

It took until the 1700's to get algebra and number systems in place in a workable way. But there was still trouble understanding what infinity was. Was the set of counting numbers really infinite, or was it only as big as the highest number that anyone had ever counted, or as big as the number of atoms in the universe, or. . . ? But even if the set of counting numbers was infinite, then the set of real numbers was also infinite. But then again, were they the same infinity? Some math grad student in Germany around 1850 shocked the math world by saying 'no.'

Question 1 *Here is a table of rational numbers:*

$\cdots$	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
$\cdots$	$\frac{-5}{2}$		$\frac{-3}{2}$		$\frac{-1}{2}$		$\frac{1}{2}$		$\frac{3}{2}$		$\frac{5}{2}$	$\cdots$
$\cdots$	$\frac{-5}{3}$	$\frac{-4}{3}$		$\frac{-2}{3}$	$\frac{-1}{3}$		$\frac{1}{3}$	$\frac{2}{3}$		$\frac{4}{3}$	$\frac{5}{3}$	$\cdots$
$\cdots$	$\frac{-5}{4}$		$\frac{-3}{4}$		$\frac{-1}{4}$		$\frac{1}{4}$		$\frac{3}{4}$		$\frac{5}{4}$	$\cdots$
$\cdots$		$\frac{-4}{5}$	$\frac{-3}{5}$	$\frac{-2}{5}$	$\frac{-1}{5}$		$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$		$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

(a) *What does the 12th row of the table look like?*

(b) *Name three different rational numbers. Will they (eventually) appear on the table?*

(c) *Will every rational number eventually appear in the table above?*

(d) *Can you figure out how to "enumerate" the rationals?*

Question 2 *The question: Are the set of counting numbers and the set of real numbers between* 0 *and* 1 *the same size?*

Author(s):

Cantor's answer: Suppose they were, then you could make a one-to-one, onto match-up:

$$\begin{array}{l} 1 : 0.22343798784\ldots \\ 2 : 0.85984759348\ldots \\ 3 : 0.11290293980\ldots \\ 4 : 0.03432340563\ldots \\ 5 : 0.93928498239\ldots \\ 6 : 0.79788937833\ldots \\ \quad\vdots \end{array}$$

So, you think you did it, eh? I will find a real number between zero and one that is not on your list. How will I do it?

Question 3 *Explain why the same argument does* not *show that the rationals cannot be enumerated.*

Exploration 4 *Is the cardinality of* $\mathbb{R}$ *equal to the cardinality of* $\mathcal{P}(\mathbb{Q})$*?*

28 The Twentieth Century and Beyond

Dunham's story ends with Cantor, but the story of mathematics certainly doesn't. Hopefully, as we've talked about the history of mathematics this semester, you've seen some themes emerging. Mathematics is always growing and changing, and the way mathematicians think about the subject is also growing and changing. The first reading is an attempt to add to these themes. It is written by Michael Atiyah, a famous topologist who won the Fields Medal in 1966. You don't need to read this entire article in detail, but be sure to read enough to get a sense of the themes that Atiyah is describing and how they are similar to (or different from) some of the themes we have discussed previously.

The Fields Medal is the highest award given to mathematicians, and is sometimes referred to as the "Nobel Prize of Mathematics". The second reading is a bit of history on this prize, and the third reading a list of past winners of the prize. It's worthwhile to note that the first woman to win the Fields Medal was Maryam Mirzakhani, who won the prize in 2014 and passed away in 2017.

Readings

First reading: Mathematics In The Twentieth Century

Second reading: The Fields Medal

Third reading: List of Fields Medalists

Questions

Question 1 *The Fields Medal is awarded every 4 years.*

Question 2 *Which of the following is not a theme discussed by Atiyah?*

Multiple Choice:

(a) *Local to Global*

(b) *Geometry versus Algebra*

(c) *Techniques in Common*

(d) *Increase in Precision*

Question 3 *What are the most important points from this reading?*

Author(s):
See Mathematics In The Twentieth Century at http://www.jstor.org/stable/2695275
See The Fields Medal at http://www.mathunion.org/general/prizes/fields/details/
See List of Fields Medalists at http://www.mathunion.org/general/prizes/fields/prizewinners/

29 Who's Who

You may have heard of many of the mathematicians below! As time has marched forward, more and more mathematicians have made their mark on the subject. So many, in fact, that we hardly have time to talk about them all! Instead, we'll try to build a yearbook of sorts to learn a little bit about the mathematicians whose work you might use!

For each of the mathematicians below, write the years they were born and died (if applicable), and then one reason we could have studied their work. What did they study? What theorems are they famous for? Why are they remembered in the mathematical community? Who did they work with? You'll likely have to look up much of this information online.

Author(s):

Mathematician	Years	One reason you might know the name
Niels Abel		
Maria Agnesi		
Augustin-Louis Cauchy		
Abraham DeMoivre		
Gotthold Eisenstein		
Paul Erdős		
Joseph Fourier		
Évariste Galois		
Joseph-Louis Lagrange		
Adrien-Marie Legendre		
Grigori Perelman		
Henri Poincaré		
Srinivasa Ramanujan		
Karl Weierstrauss		

30 Limits of axioms

In this activity, we discuss how statements can be independent of axioms.

We will motivate our discussion with questions about cardinality.

Question 1 *Given any finite set S, can you prove that the power set of S has a larger cardinality?*

Consider $S = \mathbb{N}$. We wish to show that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. Proceed as follows: Seeking a contradiction, suppose that they are equinumerous and imagine a bijection between every element of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. Call a natural number **selfish** if by your bijection it is pared with a set containing itself. Call a natural number **nonselfish** if it is paired with a set not containing itself.

Question 2 *Give part of an example map between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ and clearly identify selfish numbers and nonselfish numbers based on your map.*

Question 3 *Let B be the set of all nonselfish numbers. Explain why $B \in \mathcal{P}(\mathbb{N})$. Arrive at a contradiction. Hint: Which integers map to B?*

Question 4 *So far we have only shown $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$. How do you conclude that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$?*

Question 5 *Given any set S, can you prove that the power set of S has a larger cardinality? Hint: repeat the argument above.*

We know that $\mathbb{N}$ is countable and that $\mathcal{P}(\mathbb{N})$ is uncountable. Define:

$$\begin{aligned}
\beth_0 &:= |\mathbb{N}| \\
\beth_1 &:= |\mathcal{P}(\mathbb{N})| \\
\beth_2 &:= |\mathcal{P}(\mathcal{P}(\mathbb{N}))| \\
\beth_3 &:= |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| \\
&\vdots
\end{aligned}$$

and so on. These are *beth* numbers. From our work above we see that

$$\beth_0 \leq \beth_1 \leq \beth_2 \leq \beth_3 \leq \cdots$$

Question 6 *Let*

$$\varphi : \mathcal{P}(\mathbb{N}) \to [0, 1]$$

Author(s):

via the map

$$A \mapsto \sum_{x \in A} \frac{1}{2^x}$$

where $A \subset \mathbb{N}$. Explain how every element in the image of φ can be associated to exactly one number $0.a_1a_2a_3\ldots$ where $a_i = 0$ or $a_i = 1$.

Question 7 *Argue that $\beth_1 = \mathcal{P}(\mathbb{N}) = |[0,1]| = |\mathbb{R}|$.*

On the other hand there are also *aleph* numbers. Here

$$\aleph_0 := |\mathbb{N}|$$

but $\aleph_1$ is defined to be the *smallest* infinite cardinal number larger than $\aleph_0$. In general, $\aleph_n$ is the smallest infinite cardinal number larger than $\aleph_{n-1}$. So from this definition we find:

$$\aleph_0 \leq \aleph_1 \leq \aleph_2 \leq \aleph_3 \leq \cdots$$

Question 8 *Say everything you can about the relationship between the aleph numbers and the beth numbers.*

Hilbert's first problem

In 1900 Hilbert made a list of problems to guide the mathematicians of the 20th Century. Here is the first problem on the list:

> Prove the continuum hypothesis.

What is this so-called continuum hypothesis? It states

$$\aleph_1 = \beth_1 = |\mathbb{R}|.$$

Hilbert's second problem

I speculate that finding the "holes" in Euclid's arguments led Hilbert to question the validity of our own proofs. This speculation is supported by the fact that in 1900 the second problem in Hilbert's list of problems was:

> Prove that the axioms of arithmetic are consistent.

Question 9 *What would an counterexample to this claim imply? What would an affirmative proof imply?*

With Hilbert's second problem in mind, in 1901 Bertrand Russell showed that the naive set theory of Cantor cannot be used to answer Hilbert's second problem. Russell proposed that one consider the set of all sets that do not contain themselves.

Question 10 *How does one express this set in "set-builder" notation?*

Question 11 *What is the problem with this set?*

Question 12 *Does this set remind you of anything you've seen before?*

The reader should rest assured that the foundations of mathematics will not come collapsing upon our heads. Russell himself has a resolution based on something called *type theory*, though we cannot discuss this at the moment.

(In)completeness

Now we will turn our attention to Kurt Gödel. In 1931, Gödel proved his (first) incompleteness theorem. To paraphrase:

> Any set of axioms powerful enough to describe "elementary number theory" will have statements that are *true* but *unprovable*, and hence this set of axioms is incomplete.

In 1940, Gödel proved that the continuum hypothesis cannot be disproved using the standard axioms of set theory. Around 1964, Paul Cohen showed that the continuum hypothesis cannot be proved using the standard axioms of set theory. To use the language of vector-spaces,

> The continuum hypothesis is outside the "span" of our standard axioms!

Hence the continuum hypothesis is one of these unprovable statements. There are in fact, many others. Once upon a time, mathematical statements were either true or false. Now we have a third option, the statement could be undecidable. We have matured much since the birth of numbers.