Tangent planes and differentials

Math 251 Calculus 3

October 1, 2013

Recap: slices, partial derivatives, and tangent lines

We saw previously that if $f(x, y)$ is a function of two variables, each of its partial derivatives $f_x(x, y)$ and $f_y(x, y)$ gives the slopes of tangent lines to slice curves.

If (a, b) is a point in the plane, the slice curves through (a, b) are the graphs of $z = f(a, y)$ (in the plane $x = a$) and $z = f(x, b)$ (in the plane $y = b$).

Partial derivatives

If the slice functions $f(a, y)$ and $f(x, b)$ are differentiable (in the one-variable sense), their derivatives are $f_y(a, y)$ and $f_x(x, b)$. These functions are ordinary derivatives, so they compute tangent slopes in the usual way.

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- $z = f(a, b) + f_y(a, b)(y - b)$

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A candidate for the tangent plane

The tangent lines in the (x, y)-direction determine a unique plane. Its equation is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Rearranging this a bit, we can confirm it's a plane:

$$\langle f_x(a,b), f_y(a,b), 1\rangle \cdot \langle x-a, y-b, z-f(a,b)\rangle = 0.$$

By 12.5, this is the equation of a plane passing through the point $(a, b, f(a,b))$ normal to the vector $\langle f_x(a,b), f_y(a,b), 1\rangle$.

We call the RHS $L(x, y)$ the *linearization* of f at the point (a, b).

Local linearity

Having a tangent plane means the graph of the surface is *locally linear*. If we zoom way in, the surface looks flat. The algebraic criterion for this turns out to be

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|\langle x, y\rangle - \langle a, b\rangle\|} = 0$$

and so we adopt this as the definition of *differentiability*.

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- the graph of $L(x, y)$ is tangent to the graph of f at (a, b)

Criteria for differentiability

If *any* plane is tangent to the graph of f at (a, b), it must be the graph of $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. That's the only plane that contains the tangent lines.

> *Observe that $L(a, b) = f(a, b)$ and that the $x = a$ (resp. $y = b$) slice of $L(x, y)$ is the tangent line to the slice curve in the plane $x = a$ (resp. $y = b$).*

How do we know the tangent plane exists?

> *If f_x and f_y exist and are continuous at (a, b), then the three equivalent conditions given above are all true.*

The linearization approximates f

If (x, y) is near (a, b) (in other words, if the magnitude $\|\langle x, y\rangle - \langle a, b\rangle\|$ is small), then $L(x, y) \approx f(x, y)$, because f is locally linear.

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- $\Delta x = x - a$,
- $\Delta y = y - b$, and
- $\Delta f = f(x, y) - f(a, b)$.

Introducing: differentials

Then

$$\begin{aligned} \Delta f &= f(x,y) - f(a,b) \\ &\approx L(x,y) - L(a,b) \\ &\approx f_x(a,b)\Delta x + f_y(a,b)\Delta y, \end{aligned}$$

at least when Δx and Δy are small.

Differentials and linearization

The approximation: $\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$.

It is common to write

- $dx = \Delta x$
- $dy = \Delta y$
- $df = L(x,y) - L(a,b) = f_x(a,b)dx + f_y(a,b)dy$.

Putting it all together, we have $\Delta f \approx df$, a compact representation of the idea of the linearization. The symbols dx, dy, df are often called "differentials".

What are differentials?

It is common to write $dx = \Delta x$, $dy = \Delta y$,
$df = L(x,y) - L(a,b) = f_x(a,b)dx + f_y(a,b)dy$.

For now, it's best to think of them as convenient abbreviations. The convenience lies in that while $dx = x - a$, the a is suppressed from the notation. So we can abbreviate even further

$$df = f_x dx + f_y dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Of course, one has to remember to evaluate the partials at the appropriate point, but that detail aside, this is quite a compact form for the equation of the tangent plane (supposing it exists).

Abstract nonsense concerning derivatives

We've talked a lot by now about the *partial* derivatives of $f(x, y)$. So what about its actual *derivative*? The best answer, for now, is that the derivative of $f(x, y)$ is the differential df. This is a little peculiar, at first. We won't have much occasion to use this particular notion.

Nonsense, II

Another answer might be that the derivative is the vector $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$. This is satisfying, because this vector seems related to the normal to the tangent plane, something like a "tangent slope". This vector is the *gradient vector* of the function f and we'll see a lot more of it.

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- $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the linearization of f at (a,b)
- The graph of L is a plane passing through $(a,b,f(a,b))$
- If f is locally linear, then the graph of L is tangent to the graph of f.