Math 251
Workshop 09: Integration over planar regions

October 28, 2013 Name: Answers

1 Workshop 09: Integration over planar regions

You saw in the presentation how to decompose double integrals over a triangle into iterated integrals. The same technique works for regions whose boundaries have nice algebraic expressions. We will only concern ourselves with regions whose boundaries are made up of line segments, arcs of circles, and pieces of the graphs of functions. Our regions will also all be "connected". This means, roughly speaking, that they are all one piece.

1.1 More triangles

The simplest regions meeting our descriptions other than rectangles are triangles: specifically triangles with one side parallel to an axis. Give your answers by writing an integral of an anonymous function $f(x,y)$—that is, fill in the limits and the differentials, but don't pick a function to integrate.

1. For each triangle, find the limits of an iterated integral in each of the possible orders of integration. Draw *big, beautiful pictures.*

   (a) The triangle with vertices $(-1,2)$, $(2,2)$, and $(-1,-2)$.

   Solution: The edges of the triangle are segments lying in the lines $x=-1$, $y=2$, and (by a quick calculation) $y-2=(4/3)(x-2)$, or $3y-6=4x-8$, or $4x-3y=2$. If we integrate $dy\,dx$, we are using vertical segments. Evidently, these segments range from $x=-1$ to $x=2$. The segment at x has least height $4x/3-2/3$ and greatest height 2. Thus, our integral would have the form

   $$\int_{-1}^{2}\int_{4x/3-2/3}^{2} f(x,y)\,dy\,dx.$$

   On the other hand, if we integrate using horizontal segments, the order of integration is $dx\,dy$. The segments lie between $y=-2$ and $y=2$, and the segment at height y has left endpoint -1 and right endpoint $(2-3y)/4=1/2-3y/4$. Thus, the integral should be

   $$\int_{-2}^{2}\int_{-1}^{1/2-3y/4} f(x,y)\,dx\,dy.$$

   (b) The triangle with vertices $(-1,2)$, $(-1,-1)$, and $(1,1)$.

   Solution: In this problem, each order of integration presents a different challenge. Let us first identify the bounding curves. These are $x=-1$, $y=x$, and $y-1=(-1/2)(x-1)$. This last is equivalent to the more convenient $x+2y=3$.

   If we use vertical segments, it is clear from a picture that the segments lie between $x=-1$ and $x=1$. The bottom endpoint of the segment at x is thus x, while he top endpoint is $3/2-x/2$. Putting this together, we write

   $$\int_{-1}^{1}\int_{x}^{3/2-x/2} f(x,y)\,dy\,dx.$$

For the other order of integration, we have horizontal segments living between $y = -1$ and $y = 2$. The segment at height y evidently has its left endpoint at $x = -1$, but the right endpoint is troublesome. We will need to use two integrals, because if $y > 1$, the right endpoint lies on the line $x + 2y = 3$, while if $y < 1$, the right endpoint lies on the line $y = x$. We obtain

$$\int_{-1}^{1}\int_{-1}^{x} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{-1}^{3-2y} f(x,y)\,dx\,dy.$$

It is not unusual for one order of integration to be easier. Here, the shape of the boundary gives us a natural preference for the $dy\,dx$ order, because it allows us to use a single iterated integral.

(c) The triangle with verticles $(-1,-1)$, $(3,-1)$, and $(0,2)$.

Solution: This is similar to the previous problem. The bounding curves are $y = -1$, $3x - y = -2$, and $x + y = 2$. With horizontal segments we obtain

$$\int_{-1}^{2}\int_{y/3-2/3}^{2-y} f(x,y)\,dx\,dy,$$

and with vertical ones we get

$$\int_{-1}^{0}\int_{-1}^{3x+2} f(x,y)\,dy\,dx + \int_{0}^{3}\int_{-1}^{2-x} f(x,y)\,dy\,dx.$$

1.2 Regions with curved boundaries

2. Let R be the region whose boundary is the lines $y = 0$ and $x = 2$ and the parabola $x = y^2$. Find limits for iterated integrals over R in both orders.

Solution: This is not very different from the triangle examples. Using vertical segments, we get

$$\int_{0}^{2}\int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx,$$

while horizontal segments yield

$$\int_{0}^{\sqrt{2}}\int_{y^2}^{2} f(x,y)\,dx\,dy.$$

3. Let U and L be the parts of the unit disk that fall in the upper and lower half-planes, respectively. Find limits for iterated integrals over U and L, in both orders.

Solution: In each case the better order is $dy\,dx$. For U we find

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx, \quad \text{or equivalently}$$
$$\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Similarly, for L we find

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dy\,dx, \quad \text{or equivalently}$$
$$\int_{-1}^{0}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

4. Let **D** be the unit disk. Find limits for iterated integrals over **D** in both orders. (Use your answers to the previous part.)

Solution: Here, both orders are equally inconvenient, and not much different. We have

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx,$$
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

1.3 When order matters

5. Let C be the circular cap consisting of those points of **D** whose x-coordinates are at least $-1/2$. Draw C, and try to find limits for an iterated integral over it in one or the other order. Does it make a difference which order you choose?

Solution: Here, the $dy\,dx$ order seems easier, because there is only one type of segment. We modify the integral from the previous part:

$$\int_{-1/2}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

In the $dx\,dy$ order, this is much more difficult, because there are three kinds of segments.

6. Let S be the parabolic sector whose boundary is the parabola $y = x^2$ and the line $y = x+2$.

   (a) Find limits for an iterated integral over S in the order $dy\,dx$.

   Solution: All the segments begin on the parabola and end on the line, so we have

   $$\int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)\,dy\,dx.$$

   (b) Find limits for an iterated integral over S in the order $dx\,dy$.

   Solution: Here, there are two kinds of segments. Segments with $y > 1$ begin on the line and end on the parabola:

   $$\int_{1}^{4}\int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

On the other hand, segments with $y < 1$ both begin and end on the parabola.

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$$