Math 251 **Workshop 11: integrals in curved coordinates**

November 6, 2013

Due: November 11, 2013 Name: ______________________________

1 Workshop 11: integrals in curved coordinates

This workshop gives you a chance to practice integrals with curved coordinate systems.

1.1 A polar example with an application

1. The region D is that enclosed by the circle of radius 2 centered at the origin, but outside the circle of radius 1 centered at $(1,0)$. Hence the area of D is 3π. Derive this result using an integral. (The polar equation of the small circle is $r = 2\cos\theta$.)

2. The *center of mass* of a lamina is, roughly speaking, the point on which the lamina will balance if placed on a needle. Consider the region D above with uniform density $\rho(x, y) = 1$. Let $(\hat{x}, \hat{y})$ denote its center of mass.

   (a) Argue by symmetry that $\hat{y} = 0$ for the region D. Is this true if the density of the lamina is not required to be uniform?

   (b) The definition of the center of mass is formulated using integrals.

   $$\hat{x} = \frac{1}{M}\iint_D x\rho(x, y)\, dA, \quad \hat{y} = \frac{1}{M}\iint_D y\rho(x, y)\, dA.$$

   Here $M = \iint_D \rho(x, y)\, dA$ is the mass of the solid. Calculate the coordinate $\hat{x}$.

3. Let E be the region of $\mathbf{R}^3$ above the cone $z = \sqrt{x^2 + y^2}$ and below the unit sphere $x^2 + y^2 + z^2 = 1$. Suppose $\rho(x, y, z) = z$ is the density function. Use a curved coordinate system to compute the center of mass of the resulting solid, using symmetry to simplify the argument if appropriate. Consult tables of integrals as necessary. *Hint.* The angle at the cone point is a right angle.