Vectors as displacements; a slicing problem

Math 275 Multivariable Calculus

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- If I move from $(2,3)$ to $(-3,1)$, what is the most natural way to express this change?
- In everyday English, we separate it into two changes: a change in the east–west direction, and a change in the north–south direction.

Two changes in one

"Five blocks south and two blocks west" is a pretty natural way to express the move from $(2, 3)$ to $(-3, 1)$.

- Notice: it's not a sensible *address* in any city or town. Why not?
- It is a *displacement*, not a *location*.
- These are different notions!

Universality of displacement

- If the town is laid out on a square grid (as opposed to rectangles, some other kind of parallelograms, or worse) ...
- ... walking from $(2, 3)$ to $(-3, 1)$ feels the same as walking from $(2013, 2013)$ to $(2008, 2010)$.

We call this displacement the *vector* $\langle -5, -2 \rangle$. Observe the following naïve "equations":

$$(2, 3) + \langle -5, -2 \rangle = (-3, 1)$$
$$(2013, 2013) + \langle -5, -2 \rangle = (2008, 2010)$$

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- But is it completely, unambiguously obvious that it's impossible to get an elliptical cross-section?
- How would we *prove* it?

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- Solve with coordinates?
- A completely coordinatized approach to this problem involves choosing a sphere with arbitrary center and radius and an arbitrary plane.
- One then solves a system of two simultaneous nonlinear equation in eight variables:

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$
$$Ax+By+Cz=D$$

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- Working out the details is horrendous and not very illuminating.
- Instead, we'll unleash the kung fu of symmetry to deal with this complicated geometric situation.

Symmetry arguments

Symmetry argument: a "special" case that isn't really all that special.

For us, the special case will be:

> *The sphere is the unit sphere and the plane is of the form* $z = A$*.*

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- But think of it like a videographer: the right angle and zoom turns any situation into this special one.

WLOG

That's what we mean by symmetry, or the mathematical abbreviation WLOG. This stands for **W**ithout **L**oss **O**f **G**enerality. The phrase signifies the idea encapsulated above: that what appears to be a special case is in fact sufficiently general.