Cross products; classifying planes

Math 251 Calculus 3

September 20, 2013

Distributing and the cyclic relation

Remember: $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$. Plus, scalars operate as expected. So,

$$\begin{aligned}
(\hat{\imath} + \hat{\jmath} + \hat{k}) \times (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \\
&= \hat{\imath} \times (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + \hat{\jmath} \times (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \\
&\quad + \hat{k} \times (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \\
&= 2\hat{\imath} \times \hat{\imath} + 3\hat{\imath} \times \hat{\jmath} + 4\hat{\imath} \times \hat{k} \\
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&+ 2\hat{k} \times \hat{\imath} + 3\hat{k} \times \hat{\jmath} + 4\hat{k} \times \hat{k} \\
&= (3-2)(\hat{\imath} \times \hat{\jmath}) + (2-4)(\hat{k} \times \hat{\imath}) + (4-3)(\hat{\jmath} \times \hat{k}) \\
&= \hat{k} - 2\hat{\jmath} + \hat{\imath} = \langle 1, -2, 1 \rangle.
\end{aligned}$$

Warm-up, II

Find the cross products:

- $\langle 2, 1, 3\rangle \times \langle 4, 2, 1\rangle$
- $\langle -1, 0, 1\rangle \times \langle 2, -1, 5\rangle$

Some setup for classifying planes

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- A vector is *contained* in $\mathbf{P}$ if both its head and its tail (and hence, all the point on the vector's "body") are in $\mathbf{P}$.

Warm-up for Workshop 03

- Choose a pair of orthogonal vectors and draw them in standard position. Your vectors must not be multiples of $\hat{\imath}$ or $\hat{\jmath}$, but I would advise you to choose vectors with z-entry 0—then, you can get away with drawing $\mathbf{R}^3$ as a plane viewed along the positive z-axis. *Hint.* Use the dot product to make sure your vectors really are perpendicular.

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- Because your vectors are orthogonal, there is a rectangle based on these vectors (draw two more sides). Use the cross product–sine formula to verify that the area of the rectangle is equal to the length of the cross product of your vectors.

General parallelogram

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- Check that the length of the cross product yields the area of the parallelogram in this case also.

Wrap-up

Together with the orientation and complementarity properties, the cross product–sine formula *uniquely determines the cross product*. This means that, for any pair of vectors $\vec{v}$ and $\vec{w}$, there is only one vector $\vec{u}$ satifying all three. No matter how it is obtained, it must be equal to the cross product $\vec{v} \times \vec{w}$.

- Use the cross product–sine formula and the orientation property of the cross product to *derive* the rule $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.