# Optimization and the second derivative test

Math 251 Calculus 3

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# Local optima

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# Global extrema

Change the phrase “in some small disk containing $(x_0, y_0)$” to “in the domain of $f$”.

- $f$ has a global maximum at $(x_0, y_0)$ if $f(x_0, y_0) \geq f(x, y)$ for every point $(x, y)$ in some small disk containing $(x_0, y_0)$.
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# Stationary points and critical points

If we draw a couple of local optima, we notice something about the tangent planes. They are horizontal, when they exist. This motivates some more definitions.

- When a function has a horizontal tangent plane at a point $P$, its gradient at $P$ is zero. This is because $\nabla f_P = \langle f_x(P), f_y(P) \rangle$. We say that $P$ is a *stationary point*.
- When a function is not differentiable at a point, its gradient is typically undefined, although it's possible that the gradient is 0.
- Points at which either of these occur are called *critical points*.

Note that the gradient should be considered to be undefined if *either* of its entries is undefined.

# Local optima occur at critical points

If $f(x, y)$ has a local optimum at $(x_0, y_0)$, then $(x_0, y_0)$ is a critical point of $f$.

> *Take special note of the logical asymmetry of this statement. Its converse is not true!*

A stationary point that is not a local optimum is called a saddle point.

## The discriminant

It is impractical to test critical points of $f(x, y)$ for being local optima using the first derivative. But there is a convenient analog of the second derivative test, at least if $f(x, y)$ is smooth enough. Interestingly, all three second derivatives are involved.

- Let $f(x, y)$ be a function with continuous second-order partials. The *Hessian discriminant* of $f$ at $(a, b)$ is defined to be $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$.

## Second derivative test

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local minimum at $(a, b)$.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local maximum at $(a, b)$.
- If $D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.
- If $D = 0$, the test is inconclusive.

# Inconclusive

Remember, $D = 0$ doesn't mean "saddle point". It means "test fails"!

# Global extrema

If $f$ is everywhere smooth (everywhere means, on all of $\mathbf{R}^2$) then its global optima will also be local optima. Of course it may not have global optima.

But, if $f$ has a domain that is a proper subset of $\mathbf{R}^2$, it may have global optima that are not local optima. If the domain is *closed and bounded*, global optima are guaranteed to exist.

# A little topology

Let $S$ be a subset of the plane. It's OK to assume that $S$ has a reasonable shape: that it's possible to draw it, that its edges (if it has any) are smooth, and so on.

Usually $S$ is defined by algebraic conditions on its coordinates. A point qualifies for membership in $S$ if—and only if—its coordinates meet the conditions.

# Set-builder notation

We describe such sets first via their conditions. You are familiar with doing this. If $f$ is some one-variable function, then

$$\{(x, y) \in \mathbf{R}^2 : y = f(x)\}$$

is the graph of the function $f$. It is pronounced "the set of $(x, y)$ in $\mathbf{R}^2$ such that \$y = f(x)".

If we want to discuss 2-variable functions whose domain is smaller than the plane, we describe their domains this way.

# Closed and bounded

A subset $S$ of the plane is called *closed* if it contains all of its edge points: that is, if all of the edge points meet the membership conditions.

We say $S$ is *bounded* if it is contained in a large enough disk; equivalently, if it is contained in a disk centered at $(0, 0)$; equivalently, if it is possible to draw the set on a finite piece of paper.

> *Note: the textbook calls edge points "boundary points". I prefer to avoid this terminology because the presence of "boundary points" has nothing to do with "boundedness."*

# Existence of global extrema

The symbol $\subset$ denotes set containment.

> **Theorem**. *Let $S \subset \mathbf{R}^2$ be closed and bounded, and let $f: S \to \mathbf{R}^2$ be continuous. Then $f$ has a global maximum and a global minimum on $S$.*

This is like the "closed interval method" from one-variable calculus. The closed and bounded subset in that case is a finite closed interval.

The theorem guarantees the existence of global optima, but tells us nothing about how to find them.

# Interior and edge are separate

We can look for critical points in the interior of the set (interior just means the non-edge parts of the set) and make a table of values. There will only be finitely many such points.

But ... we have to check the edge points, just like in the one-variable case.

And there are more than just 2 endpoints, usually.

# Work together:

From section 14.7:

- Problem 28
- Example 5
- Problem 35 (in groups, with whiteboards)
- Problem 36 if time permits