Optimization and the second derivative test

Math 251 Calculus 3

October 11, 2013

Local optima

If $f(x, y)$ is a function of two variables, it probably has maxima and minima.

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- f has a local optimum (or extremum) at (x_0, y_0) if $f(x_0, y_0)$ has either a local max or a local min at (x_0, y_0).

Global extrema

Change the phrase "in some small disk containing (x_0, y_0)" to "in the domain of f".

- f has a global maximum at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for every point (x, y) in some small disk containing (x_0, y_0).
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Stationary points and critical points

If we draw a couple of local optima, we notice something about the tangent planes. They are horizontal, when they exist. This motivates some more definitions.

- When a function has a horizontal tangent plane at a point P, its gradient at P is zero. This is because $\nabla f(P) = \langle f_x(P), f_y(P) \rangle$. We say that P is a *stationary point*.
- When a function is not differentiable at a point, its gradient is typically undefined, although it's possible that the gradient is 0.
- Points at which either of these occur are called *critical points*.

Note that the gradient should be considered to be undefined if *either* of its entries is undefined.

Local optima occur at critical points

If $f(x, y)$ has a local optimum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

> *Take special note of the logical asymmetry of this statement. Its converse is not true!*

A stationary point that is not a local optimum is called a saddle point.

The discriminant

It is impractical to test critical points of $f(x, y)$ for being local optima using the first derivative. But there is a convenient analog of the second derivative test, at least if $f(x, y)$ is smooth enough. Interestingly, all three second derivatives are involved.

- Let $f(x, y)$ be a function with continuous second-order partials. The *Hessian discriminant* of f at (a, b) is defined to be $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$.

Second derivative test

- If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).
- If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).
- If $D(a,b) < 0$, then f has a saddle point at (a,b).
- If $D = 0$, the test is inconclusive.

Inconclusive

Remember, $D = 0$ doesn't mean "saddle point". It means "test fails"!

Global extrema

If f is everywhere smooth (everywhere means, on all of $\mathbf{R}^2$) then its global optima will also be local optima. Of course it may not have global optima.

But, if f has a domain that is a proper subset of $\mathbf{R}^2$, it may have global optima that are not local optima. If the domain is *closed and bounded*, this is guaranteed to be the case.

We have to check the boundary, just like in the one-variable case.

Work together:

- Problem 28 from 14.7

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- Example 5 from 14.7

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- Example 5 from 14.7
- Problem 35 from 14.7 (in groups, with whiteboards)