Optimization of functions with closed and bounded domains

Math 251 Calculus 3

October 14, 2013

Warm-up

lkl;k;lk;lkj

Setup

An analog of the intermediate value theorem tells us: if f is a continuous real-valued function whose domain is closed and bounded, then

- f has a global maximum and f has a global minimum.

If a global optimum is an *interior point* of the domain, it is also a local optimum. Hence the techniques of calculus will locate it.

If, on the other hand, it is an *edge point* of the domain, we must find it by less direct methods.

Exercise 35

Let $f(x, y) = x + y - x^2 - y^2 - xy$, and suppose f to be defined only on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$. This function is smooth on its domain, so its critical points are stationary points.

A little computation yields $\nabla f = \langle -2x - y + 1, -2y - x + 1\rangle$. The stationary points of f are thus the solutions (in the domain) of the system

$$\begin{aligned} 2x + y - 1 &= 0 \\ x + 2y - 1 &= 0. \end{aligned}$$

Solving for the stationary points

Multiplying the second equation by -2 yields

$$\begin{aligned} 2x + y - 1 &= 0 \\ -2x - 4y + 2 &= 0, \end{aligned}$$

and therefore we find (by adding the equations)

$$-3y + 1 = 0.$$

A unique interior stationary point

Hence, $y = 1/3$, and it is easy to see that $x = 1/3$ as well. We compute:

$$f(1/3, 1/3) = 1/3.$$

Note: this does *not* tell us whether $(1/3, 1/3)$ is a local optimum. We are just interested in the value. We'll have finitely many to compare it to, so we don't bother with the second derivative test.

The edge

We must test the edge separately. In this case, the edge is actually four edges: the segments that make up the edge of the square.

- $\{(x, y) \in \mathbf{R}^2 : x = 0, 0 \leq y \leq 2\}$ (the left)
- $\{(x, y) \in \mathbf{R}^2 : y = 0, 0 \leq x \leq 2\}$ (the bottom)
- $\{(x, y) \in \mathbf{R}^2 : x = 2, 0 \leq y \leq 2\}$ (the right)
- $\{(x, y) \in \mathbf{R}^2 : y = 2, 0 \leq x \leq 2\}$ (the top)

Testing the edges is a routine exercise in slice curves and one-variable calculus.

The left

On the left, our slice curve is $f(0, y) = y - y^2$. Its domain is the interval $0 \leq x \leq 2$. Since the slice curve is continuous on this closed interval, it possesses global optima.

- In your groups, find the global optima of the slice curve. (Remember to check the endpoints of the domain!)

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- $f'(0, y) = 1 - 2y$
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- $f(0, 1/2) = 1/4$; $f(0, 0) = 0$; $f(0, 2) = -2$

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- $f(0, 1/2) = 1/4$; $f(0, 0) = 0$; $f(0, 2) = -2$
- This shows that $f(0, y)$ is maximized at $(0, 1/2)$ with value $1/4$
- and minimized at $(0, 2)$ with value -2

The bottom

On the bottom, our slice curve is $f(x, 0) = x - x^2$. Its domain is the interval $0 \leq y \leq 2$. Since the slice curve is continuous on this closed interval, it possesses global optima.

- In your groups, find the global max and min of this slice curve on the domain $0 \leq y \leq 2$.

The bottom

On the bottom, our slice curve is $f(x,0) = x - x^2$. Its domain is the interval $0 \leq y \leq 2$. Since the slice curve is continuous on this closed interval, it possesses global optima.

- In your groups, find the global max and min of this slice curve on the domain $0 \leq y \leq 2$.
- The function f is symmetric is x and y, so this was really easy. Usually, there will be a separate check for each curve segment of the edge.

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The top

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- In your groups, find the global max and min of this slice curve.
- We have $f'(x,2) = -2x - 1$, so there is a stationary point at $x = -1/2$, not in the domain.
- Thus, the optima must occur at the endpoints: $f(0,2) = -2$, while $f(2,2) = -8$.

The right

A similar symmetry argument disposes of the right edge of the square. Evidently, the maximum value of f on this edge is -2, attained at $(2, 0)$, while the minimum is -8, attained at $(2, 2)$.

Putting it all together

Considering all the edge pieces together, the maximum edge value is $1/4$, attained at $(0, 1/2)$ and $(1/2, 0)$, while the minimum edge value is -8, attained at $(2, 2)$.

The only interior critical point was $(1/3, 1/3)$, with value $1/3$.

This shows that the global maximum of f occurs at $(1/3, 1/3)$, while the global min occurs at $(2, 2)$.

Work together

Try problem 36 from 14.7:

Find the maximum of $f(x, y) = y^2 + xy - x^2$ on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.