

# Introduction to flux; conservative vector fields

Math 251 Calculus 3

November 22, 2013

# Warm-up

(problem 2 from WeBWorK 13)

# Integration along vs. across

The vector line integral  $\int_C \vec{F} \cdot d\vec{s}$  is the total contribution of the vector field at each point along the curve. The phrase “along the curve” is very important to the interpretation, because we use the tangent vectors to the curve to define the integral. The particulars have been stated here for curves in  $\mathbf{R}^2$ ; see the text for the complete details.

We can also define a vector line integral using *normal vectors to the curve* if we confine ourselves to  $\mathbf{R}^2$ . Thus we're somehow measuring the contribution of the vector field *across* the curve instead of *along* it.

The resulting integral computes what is called the “flux” of the vector field across the curve.

# What is flux?

Loosely speaking it's the “amount” of something flowing across the curve. Its value depends on the strength of the field, the length of the curve, and how they are oriented relative to one another.

A flux integral is similar to the vector line integrals considered previously, which we might call “flow” integrals in comparison. The difference is that we use a unit *normal* vector where in the flow integral we used a unit *tangent* vector.

$$\text{Flux across } \mathcal{C} = \int_{\mathcal{C}} (\vec{F} \cdot \vec{e}_n) \, ds = \int_{\mathcal{C}} \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, dt,$$

where  $\vec{n}(t) = \langle -y'(t), x'(t) \rangle$  and  $\vec{e}_n = \vec{n} / \|\vec{n}\|$ .

## Normal vectors and orientation

The unit tangent vector to a parametrized curve is *canonical*, meaning there is only one sensible choice. This is the normalized derivative  $\vec{r}'(t)/\|\vec{r}'(t)\|$ . However, there are two good choices for the unit normal vector at each point. The choice  $\vec{n} = \langle -y(t), x(t) \rangle$  is not canonical, but a convention.

*If  $\mathcal{C}$  is a closed curve parametrized by  $\vec{r}(t)$  in such a way that the “inside” of  $\mathcal{C}$  is always to the left, then  $\vec{n}$  points outward from the region enclosed by  $\mathcal{C}$ .*

We leave this topic and the idea of flux for now.

# Conservative fields

One source of many examples of vector fields are the *gradient* vector fields. Choose your favorite smooth function  $V: \mathbf{R}^2 \rightarrow \mathbf{R}$  (or  $\mathbf{R}^3 \rightarrow \mathbf{R}$ ). Then  $\nabla V$  is a vector field on  $\mathbf{R}^2$  (or  $\mathbf{R}^3$ ).

Most vector fields can be seen not to be gradients. In fact, gradients have a very special property. Let us call such vector fields “conservative”. If the vector field  $\vec{F}$  is conservative, then there is some function  $V$  such that  $\nabla V = \vec{F}$ . Such a function is called a *potential function* for  $\vec{F}$ .

# Integration of conservative fields

If  $\vec{F}$  admits a potential function, then line integrals of  $\vec{F}$  are easy to compute.

*Theorem (Fundamental theorem for conservative vector fields). Assume that  $\vec{F} = \nabla V$  throughout some domain  $\mathcal{D}$ . Then for any points  $P$  and  $Q$  in  $\mathcal{D}$  and any path  $\vec{r}(t)$  from  $P$  to  $Q$ ,*

$$\int_{\vec{r}} \vec{F} \cdot d\vec{s} = V(Q) - V(P).$$

In particular, the value of the line integral depends only on the pair  $(P, Q)$  and not on the path connecting them. Fields with this property are called “path-independent”. (Easy proof on p. 963.)

## Circulation of conservative fields

Recall that for a closed curve  $\vec{r}(t)$  (that is, a loop; a curve for which  $\vec{r}(a) = \vec{r}(b)$ ), we define the *circulation* of  $\vec{F}$  around  $\vec{r}$  to be the line integral of  $\vec{F}$  around the curve.

Evidently, path-independent fields have zero circulation around any closed curve, since we may choose  $P = Q$  for such a curve and then

$$\oint_C \vec{F} \cdot d\vec{s} = V(Q) - V(Q) = 0.$$