

# Problem 12.3.48: a coordinate-free approach

Math 251 Calculus 3

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Assume that  $\vec{u}$  and  $\vec{v}$  are unit vectors with  $\|\vec{u} + \vec{v}\| = \sqrt{3/2}$ . We calculate  $\|2\vec{u} - 3\vec{v}\|$  using some properties of the dot product.

Observe that  $\|2\vec{u} - 3\vec{v}\|^2 = (2\vec{u} - 3\vec{v}) \cdot (2\vec{u} - 3\vec{v})$ . We expand using the bilinearity of dot product:

$$\begin{aligned}\|2\vec{u} - 3\vec{v}\|^2 &= (2\vec{u} - 3\vec{v}) \cdot (2\vec{u} - 3\vec{v}) \\ &= 4\vec{u} \cdot \vec{u} - 6\vec{v} \cdot \vec{u} - 6\vec{u} \cdot \vec{v} + 9\vec{v} \cdot \vec{v} \\ &= 4\|\vec{u}\|^2 - 12\vec{u} \cdot \vec{v} + 9\|\vec{v}\|^2,\end{aligned}$$

where in the last step we use the identity  $\vec{w} \cdot \vec{w} = \|\vec{w}\|^2$  and the commutativity of dot product.

Since  $\|\vec{u}\| = \|\vec{v}\| = 1$ , we have shown that

$$\|2\vec{u} - 3\vec{v}\|^2 = 13 - 12\vec{u} \cdot \vec{v}.$$

Therefore, if we can compute  $\vec{u} \cdot \vec{v}$ , we will be done.

We are given that  $\|\vec{u} + \vec{v}\|^2 = 3/2$ . This means that

$$\begin{aligned}3/2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\ &= 2 + 2\vec{u} \cdot \vec{v},\end{aligned}$$

since  $\vec{u}$  and  $\vec{v}$  are unit vectors. This shows that  $\vec{u} \cdot \vec{v} = -1/4$ .

Putting it all together, we have

$$\|2\vec{u} - 3\vec{v}\| = \sqrt{13 - 12\vec{u} \cdot \vec{v}} = \sqrt{13 + 3} = 4.$$