Math 251
Quiz 09 (Group quiz: Green's theorem) solutions

December 2, 2013 Name: Answers

See the quiz (with no solutions) for the instructions that originally accompanied it.

1. Verify Green's theorem for the line integral
$$\oint_C xy\,dx + y\,dy,$$
where $\mathcal{C}$ is the unit circle, oriented counterclockwise. This means: compute each side of Equation **??** and check that the two values are equal.

Solution: To evaluate the left-hand side, we use the standard parametrization of the unit circle, $\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, so that

$$\begin{aligned}\oint_C xy\,dx + y\,dy &= \int_0^{2\pi} \left(-\cos t \sin^2 t + \sin t \cos t\right)\,dt \\ &= 0,\end{aligned}$$

by a simple u-substitution (let $u = \sin t$, $du = \cos t\,dt$).

To evaluate the right-hand side of Green's theorem, we compute

$$\frac{\partial}{\partial x} y - \frac{\partial}{\partial y} xy = -x.$$

By symmetry, we expect the integral of the function $-x$ over the unit disk centered at the origin to be zero; we confirm it by writing $-x = -r\cos\theta$ and converting the integral to polar coordinates.

$$\int_0^{2\pi}\int_0^1 -r^2\cos\theta\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{3}\cos\theta\,d\theta = 0,$$

as expected. (Note that some easy integration steps are omitted with the expectation that the reader will fill in the details himself.)

2. Use Green's theorem to evaluate the line integrals.

(a) $\oint_C y^2\,dx + x^2\,dy$, where $\mathcal{C}$ is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$, oriented counterclockwise.

Solution: According to Green's theorem, we may replace the line integral with a suitable double integral. This is particularly appealing since the limits of integration in the iterated integral will all be constants. We find
$$\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2 = 2x - 2y,$$
so the appropriate double integral is

$$\begin{aligned}\int_0^1\int_0^1 (2x-2y)\,dy\,dx &= 2\int_0^1\int_0^1 (x-y)\,dy\,dx \\ &= \int_0^1 (x - \frac{1}{2})\,dx \\ &= 0.\end{aligned}$$

(b) $\oint_C x^2 y\,dx$, where $\mathcal{C}$ is the unit circle with standard orientation.

Solution: We choose the polar integral with integrand

$$\frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}x^2 y = -x^2 = -r^2\cos^2\theta.$$

Remembering that the polar conversion introduces an extra factor of r, the Green's theorem double integral may be written

$$\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}\cos^2\theta\,d\theta$$
$$= \frac{\pi}{4}.$$

3. Let $I = \oint_C \vec{F}\cdot d\vec{s}$, where $\vec{F} = \langle y + \sin x^2, x^2 + e^{y^2}\rangle$ and C is the circle of radius 4 centered at the origin.
   (a) Which is easier? Evaluating I directly via a parametrization or using Green's theorem?

Solution: Since the hideous function e^{y^2} appears in the vector field F, we suspect Green's theorem will in fact be necessary here. The hideous function is well known as an example of a function that possesses no elementary antiderivative.

   (b) Carry out the evaluation, using the easier method.

Solution: The Green's theorem integrand is the blessedly simple

$$\frac{\partial}{\partial x}(x^2 + e^{y^2}) - \frac{\partial}{\partial y}(y + \sin x^2) = 2x - 1.$$

In polar coordinates, the region of integration has bounds $0 \le r \le 4$, $0 \le \theta \le 2\pi$. Hence, the Green's theorem double integral may be expressed

$$\begin{aligned}
\int_0^{2\pi}\int_0^4 (2r\cos\theta - 1)r\,dr\,d\theta &= \int_0^{2\pi}\int_0^4 (2r^2\cos\theta - r)\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{2r^3}{3}\cos\theta - \frac{r^2}{2}\right)\Bigg|_0^4 d\theta \\
&= \int_0^{2\pi}\left(\frac{128}{3}\cos\theta - 8\right)d\theta \\
&= \frac{128}{3}\sin\theta - 8\theta\Big|_0^{2\pi} \\
&= -16\pi.
\end{aligned}$$