Double integrals

Math 251 Calculus 3

October 28, 2013

Review and setup

Let $f(x, y)$ be continuous on the rectangle $R = (a, b) \times (c, d)$. Then the limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i,$$

representing n box-volumes with bases ΔA_i and heights $f(x_i, y_i)$, exists regardless of how the rectangle is subdivided. The value of this limit is, by definition,

$$\iint_R f(x, y)\, dA.$$

Computing via iterated integrals

We find the value of the double integral $\iint_R f(x,y)\, dA$ using Fubini's theorem.

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

- Observe that the order of integration is different: $dx\, dy$ is not the same as $dy\, dx$. The limits change accordingly.

Depending on the function, one or the other order of integration might be easier to compute, but that is the only novelty.

The secret to thinking about integrals

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- Only enough to check that it is continuous on the domain of integration. Algebra suffices for this.
- It's the same for several variables. What we need to be careful about picturing is the domain of integration itself.

Decoding the iterated integral

In the order $dy\ dx$, we integrate first in the y-direction—that is, along a vertical segment in the (x, y)-plane. The partial integral

$$A(x) = \int_{y=c}^{d} f(x, y)\ dy$$

is still a function of x, hence the notation. What is the significance of some value $A(x_0)$? It measures the "contribution" of the vertical segment $x = x_0$ to the integral.

Thus, integrating once more over x, $\int_{x=a}^{b} A(x)\ dx$ gives the total "contribution" of all the vertical segments between $x = a$ and $x = b$.

Two orders

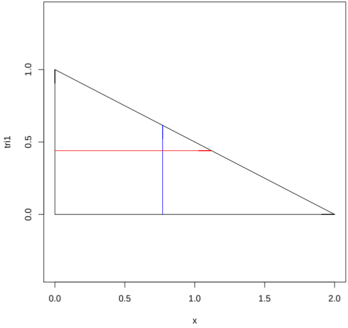
Thus, $\iint_R f(x,y)\,dA = \int_{x=a}^{b} A(x)\,dx = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx.$

Writing the "area element" dA as $dy\,dx$ this way corresponds to this choice of "slicing" the domain: first, find the contribution of a vertical segment, then integrate up the contributions of such segments.

If we write $dA = dx\,dy$ instead, we are integrating first over x—to find the contribution of a horizontal segment—and then over y, to total up the contributions of such segments.

Integrating over a triangle

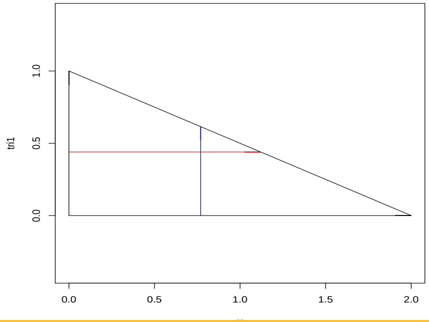
Let R now be the triangular region in the plane bounded by the coordinate axes and the line $x + 2y = 2$.



The first choice is *order of integration*. If we add up the contributions of segments like the blue one, which order of integration is this? Which order adds up the red segments?

The limits are different, depending on the segment

Let's add up the red segments. Since the red segments are indexed by their y-coordinates, this means the corresponding order is $dx\ dy$. In order to find the contribution of a typical segment, we need to know the limits of the inner integral: at which x-coordinate does the typical segment begin, and at which does it end?



Writing down the inner limits

Evidently, all the red segments start at $x = 0$. But they end at a point on the line $x + 2y = 2$. Which point? Well, if the height is y, then x must be $2 - 2y$. So, the contribution of the segment at height y is

$$A(y) = \int_{x=0}^{2-2y} f(x, y)\, dx.$$

Observe that while the upper limit is now a variable expression, *it only involves variables that have not yet been integrated over*.

The outer limits

The last phrase is typical and can help you avoid mistakes if you remember it like a slogan or mantra: *limits only involve variables that have not yet been integrated over*. This means, the last (outermost) integral has ordinary constant limits.

Those limits tell where the red segments of interest live: in this case, between $y = 0$ and $y = 1$.

Our iterated integral looks like this:

$$\int_0^1 \int_0^{2-2y} f(x, y)\, dx\, dy.$$

Work together

Decide with the people around you what the limits should be in the other order for the same triangle.

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- Answer:

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$$\int_{?}^{?} \int_{?}^{?} f(x,y)\, dy\, dx.$$

- Answer:
- $\displaystyle\int_{0}^{2} \int_{0}^{1-x/2} f(x,y)\, dy\, dx$

Workshop 09, coming attractions

In Workshop 09, you will practice drawing domains, choosing integration orders, and finding appropriate limits.

Remember: the inner limits are the ends of a typical segment.

Workshop 09 is due Wednesday, October 30. WeBWorK 7 on optimization is due Friday, November 1.

For tomorrow, read Module 10 and skim section 15.3.