Optimization and the second derivative test

Math 251 Calculus 3

October 11, 2013

Local optima

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Global extrema

Change the phrase "in some small disk containing (x_0, y_0)" to "in the domain of f".

- f has a global maximum at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for every point (x, y) in some small disk containing (x_0, y_0).
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Stationary points and critical points

If we draw a couple of local optima, we notice something about the tangent planes. They are horizontal, when they exist. This motivates some more definitions.

- When a function has a horizontal tangent plane at a point P, its gradient at P is zero. This is because $\nabla f(P) = \langle f_x(P), f_y(P) \rangle$. We say that P is a *stationary point*.
- When a function is not differentiable at a point, its gradient is typically undefined, although it's possible that the gradient is 0.
- Points at which either of these occur are called *critical points*.

Note that the gradient should be considered to be undefined if *either* of its entries is undefined.

Local optima occur at critical points

If $f(x, y)$ has a local optimum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

> *Take special note of the logical asymmetry of this statement. Its converse is not true!*

A stationary point that is not a local optimum is called a saddle point.

The discriminant

It is impractical to test critical points of $f(x, y)$ for being local optima using the first derivative. But there is a convenient analog of the second derivative test, at least if $f(x, y)$ is smooth enough. Interestingly, all three second derivatives are involved.

- Let $f(x, y)$ be a function with continuous second-order partials. The *Hessian discriminant* of f at (a, b) is defined to be $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$.

Second derivative test

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D(a, b) < 0$, then f has a saddle point at (a, b).
- If $D = 0$, the test is inconclusive.

Inconclusive

Remember, $D = 0$ doesn't mean "saddle point". It means "test fails"!

Global extrema

If f is everywhere smooth (everywhere means, on all of $\mathbf{R}^2$) then its global optima will also be local optima. Of course it may not have global optima.

But, if f has a domain that is a proper subset of $\mathbf{R}^2$, it may have global optima that are not local optima. If the domain is *closed and bounded*, this is guaranteed to be the case.

A little topology

Let S be a subset of the plane. It's OK to assume that S has a reasonable shape: that it's possible to draw it, that its edges (if it has any) are smooth, and so on.

Usually S is defined by algebraic conditions on its coordinates. A point qualifies for membership in S if—and only if—its coordinates meet the conditions.

Set-builder notation

We describe such sets first via their conditions. You are familiar with doing this. If f is some one-variable function, then

$$\{(x, y) \in \mathbf{R}^2 : y = f(x)\}$$

is the graph of the function f. It is pronounced "the set of (x, y) in $\mathbf{R}^2$ such that \$y = f(x)".

If we want to discuss 2-variable functions whose domain is smaller than the plane, we describe their domains this way.

Closed and bounded

A subset S of the plane is called *closed* if it contains all of its edge points: that is, if all of the edge points meet the membership conditions.

We say S is *bounded* if it is contained in a large enough disk; equivalently, if it is contained in a disk centered at $(0, 0)$; equivalently, if it is possible to draw the set on a finite piece of paper.

> *Note: the textbook calls edge points "boundary points". I prefer to avoid this terminology because the presence of "boundary points" has nothing to do with "boundedness."*

Existence of global extrema

The symbol $\subset$ denotes set containment.

Theorem. *Let $S \subset \mathbf{R}^2$ be closed and bounded, and let $f : S \to \mathbf{R}^2$ be continuous. Then f has a global maximum and a global minimum on S.*

This is like the "closed interval method" from one-variable calculus. The closed and bounded subset in that case is a finite closed interval.

The theorem guarantees the existence of global optima, but tells us nothing about how to find them.

Interior and edge are separate

We can look for critical points in the interior of the set (interior just means the non-edge parts of the set) and make a table of values. There will only be finitely many such points.

But ... we have to check the edge points, just like in the one-variable case.

And there are more than just 2 endpoints, usually.

Work together:

- Problem 28 from 14.7

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- Example 5 from 14.7

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- Problem 35 from 14.7 (in groups, with whiteboards)