Dot and cross product

Math 251 Calculus 3

September 18, 2013

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- Here, we agree to always choose $0 \leq \theta \leq \pi$.
- Suppose $\|\vec{v}\| = 3$ and $\|\vec{w}\| = 2$. What are the maximal and minimal possible values of $\vec{v} \cdot \vec{w}$?

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- Suppose $\vec{v} \cdot \vec{w} = 0$. Now what can you conclude?

Frequent uses of dot product

- Test for *orthogonality*: i.e., whether two vectors are perpendicular
- *Projection*: part 3 of Workshop 02

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  - *Orientation*: The ordered system $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ is right-handed
  - *Cross product–sine formula*: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$.

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  - $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
  - $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

Computing cross products

- There is a method based on the formula for 3×3 determinants outlined in the text.
- I prefer to use bilinearity/distributivity, combined with the fundamental relations
  - $\hat{\imath} \times \hat{\jmath} = \hat{k}$
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  - $\hat{k} \times \hat{\imath} = \hat{\jmath}$
- With anticommutativity, these generate another three relations
  - $\hat{\jmath} \times \hat{\imath} = -\hat{k}$
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