Math 251 **Workshop 03: Cross products and planes**

September 20, 2013

Due: Monday, September 23 Name: ____________________

1 Workshop 03

In this workshop, you will accomplish the algebraic description of all the planes of $\mathbf{R}^3$ by using the cross product and the rest of the vector geometry we have developed so far.

1.1 Planes and normal vectors

We think of planes as being made up of *points*, but in order to get a classification of planes, it is easier to think in terms of vectors. Like lines, we will use equations to classify planes, but while there are analogous forms to the slope-intercept equation $y = mx + b$, we make use instead of the analog of the more general linear equation $ax + by = c$.

The advantage of this form is that every line admits an equation (vertical lines have no slope-intercept equation). A minor disadvantage is that such equations are not unique: if a line satisfies the equation $ax + by = c$, it also satisfies an equation $\lambda ax + \lambda by = \lambda c$ for each nonzero λ.

1. Let's start with a plane $z = c$. Choose any two points in this plane other than $(0,0,c)$, and write $\vec{v}$ for the vector connecting these two points. Thus the vector $\vec{v}$ lies entirely in the plane. Check that $\vec{v} \cdot \langle 0,0,1\rangle = 0$.

2. Argue that
$$\langle x, y, z\rangle \cdot \langle 0,0,1\rangle = c$$
is an equation for the plane $z = c$ in the sense that the *point* (x, y, z) is on the plane if, and only if, the standard-position *vector* $\langle x, y, z\rangle$'s head is on the plane.

   We frequently will identify points of $\mathbf{R}^3$ with the heads of standard-position vectors.

1.2 Attitude adjustment

Imagine a generic plane through the origin—here, "generic" means only that it is not any of the coordinate planes. So it is tilted with respect to the axes. Planes through the origin are harder to draw, so hold off on the picture for a bit.

3. Explain why, if $P = (x_0, y_0, z_0)$ is a point on this plane, the *vector* $\langle x_0, y_0, z_0\rangle$ is *contained* in the plane. Your explanation should consist of one or more complete and grammatically standard sentences.
4. Add a detail to your mental picture of this tilted plane: a vector, normal to the plane, in standard position. Since the plane passes through the origin, putting the normal vector in standard position makes the vector stick straight out from the plane. Say this normal vector is $\langle a, b, c\rangle$.
5. Without choosing specific values for anything, find the numeric value of

$$\langle x_0, y_0, z_0\rangle \cdot \langle a, b, c\rangle.$$

6. Expand the dot product on the left-hand side to get a single equation for the plane that involves no vector quantities.

1.3 The Classification of Planes

It is a fact of classical geometry that any three points not all on the same line *determine* a unique plane; that is to say, only one plane contains all three points. Let $P = (2,0,1)$, $Q = (1,1,0)$, and $R = (0,0,3)$, and call the plane that contains them **P**. You can draw pictures to accompany this work if you want to.

7. Find two vectors contained in **P** that contains P, Q, and R, and call them $\vec{v}$ and $\vec{w}$.
8. Use the cross product to get a vector normal to **P**. Call this vector $\vec{n} = \langle a, b, c\rangle$.
9. Suppose (x, y, z) is a point in **P**. Use it to construct another vector contained **P**. What is the dot product of this vector with $\vec{n}$?
10. Write down an equation satisfied by all the points of **P** that involves no vector quantities.