More practice with parametrizations

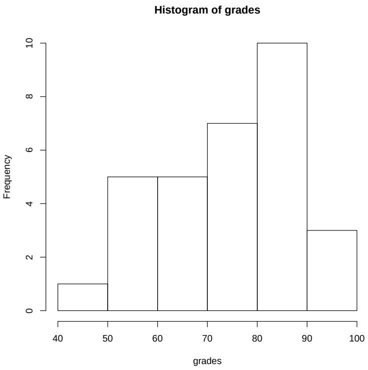
Math 251 Calculus 3

November 18, 2013

Exam data

The average was 74.871 ($n = 31$) with $\sigma = 12.5638$.

min	47
Q1	65.5
Q2	76
Q3	83.5
max	96



Warm-up

Gluing parametrizations. Let P, Q, and R be the points $(1,0)$, $(0,1)$, and $(-1,0)$, respectively, and let γ be the V-shaped path joining P to Q and thence to R via segments.

- Parametrize γ in such a way that $\gamma(0) = P$ and $\gamma(2) = R$.

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- Parametrize γ in such a way that $\gamma(0) = P$ and $\gamma(2) = R$.
- Answer: $\vec{r}(t) = \langle 1 - t, t\rangle$ if $0 \leq t \leq 1$ and $\vec{r}(t) = \langle 1 - t, 1 - t\rangle$ if $1 \leq t \leq 2$.

Tangent vectors, velocity, speed, and arclength

If $\vec{r}(t)$ is a vector function, we say it is continuous if all its entries are continuous functions. Same thing for differentiable. We think of $\vec{r}(t)$ as a position function, because at each time t it shows how to get to our particle from the origin. Then its derivative $\vec{r}'(t)$ turns out to be the instantaneous velocity vector of the moving particle.

- If exactly at time t you detach the moving particle from the function $\vec{r}$ and let it move in the exact direction and speed it had at that moment, then at time $t+1$ it has moved by exactly $\vec{r}'(t)$.

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- How do we differentiate? One entry at a time. $\vec{r}'(t) = \langle x'(t), y'(t) \rangle$ and so on.

Velocity as a vector

Velocity has always been a vector. You know that speed is nonnegative, while in one dimension velocity carries a sign. This is information about the direction of travel, just as meaningful as the speed itself. Since motion in a plane isn't confined to a backward and a forward direction, we need the complete generality of vectors in the plane to describe the possible velocities a moving particle might have.

Usually, when we draw parametrized curves, we don't actually draw the vectors $\vec{r}(t)$. Instead we draw the trajectory: the collection of endpoints of $\vec{r}(t)$. But we frequently will draw the vector $\vec{v}(t) = \vec{r}'(t)$ attached to the curve, with its tail at the endpoint of $\vec{r}(t)$. It's a moving tangent vector!

Speed as the length of velocity

The entries of the velocity vector (tangent vector—same thing) tell you the velocities in each of the coordinate directions. But the speed is something else. It is the magnitude of the velocity vector. In two dimensions,

$$\frac{ds}{dt} = \|\langle x'(t), y'(t)\rangle\| = \sqrt{x'(t)^2 + y'(t)^2}$$

Arc length

We can obtain the length of a parametrized curve (not the same thing as the length of its time domain or parameter interval) by integrating speed over the parameter interval $[a, b]$:

$$s = \int_a^b \frac{ds}{dt}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Practice translating, scaling, gluing

- Circle, center at origin, traced once counterclockwise, starting at $(0, 3)$ with $t \in [0, 1]$.
- Circle, center at (a, b), traced once counterclockwise, starting at $(a + r, b)$ with $t \in [0, 2\pi] = [0, \tau]$.
- Circle, center at origin, traced once *clockwise*, starting at $(1, 0)$.
- Pseudotriangle with boundary the segments from $(0, 1)$ to $(0, 0)$, from $(0, 0)$ to $(1, 0)$, and the arc of the unit circle connecting $(1, 0)$ to $(0, 1)$. Traverse counterclockwise with time interval $[0, 1]$.