Mathematics 251
Final Exam

December 5, 2011 Name: ______________________________

Instructions: This exam is closed book: you may refer to one 8.5×11 page of handwritten notes, but no electronic aids or other printed references are permitted. *Justification of all answers is required for partial credit.* Unless specifically directed, leave all answers in **exact form**, e.g. $\sqrt{3}$ instead of 1.732 and $\pi/2$ instead of 1.57.

Show all pertinent work. *Correct answers without accompanying work will receive little or no credit.* Results from homework or from class can be cited freely. It is in your interest to display your solution in a clear, readable fashion. Note that standards of justification are not as high as for homework.

Budget your time wisely. It is a good idea, whenever possible, to look through the entire exam before beginning work on a particular problem.

If your work continues onto the back of another page, please indicate this. Check and make sure you have all of the pages in the exam; there should be 6, including this one. If you have a question, please raise your hand.

Be sure to read all questions carefully and completely.

Question:	1	2	3	4	5	6	Total
Points:	0	24	12	12	0	0	48
Bonus Points:	0	0	0	0	0	0	0
Score:							

Good luck!

1. Consider the helix $\mathcal{C}$ pictured below. It is parametrized by the function $\mathbf{r}\colon \mathbf{R} \to \mathbf{R}^3$, $\mathbf{r} = \langle a\cos t, a\sin t, bt\rangle$.

   (a) Find the unit tangent vector $\mathbf{T}$ to the helix at time t.

   (b) Verify that the curvature of the helix is equal to $|a|/(a^2+b^2)$.

   (c) Find a parametrization of the helix using cylindrical coordinates, i.e., find functions r, θ, $z\colon \mathbf{R} \to \mathbf{R}$ such that for each $t \in \mathbf{R}$, $\mathbf{s}(t) = \langle r(t), \theta(t), z(t)\rangle$ is the cylindrical expression of the point $\mathbf{r}(t)$.

   (d) Write down, but **do not evaluate**, a scalar integral whose value is equal to the value of the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}, \quad \text{where } \mathbf{F}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$$

2. Consider a differentiable function $w = f(x, y, z)$ under the cylindrical coordinate transform $\Phi\colon \mathbf{R}^3 \to \mathbf{R}^3$, $\Phi(\langle x, y, z\rangle) = \langle r\cos\theta, r\sin\theta, z\rangle$.

   (a) (12 points) Use the chain rule to express the partial derivatives $\partial w/\partial r$ and $\partial w/\partial \theta$ in terms of the partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$. This means your answer should *not* contain any terms like $\partial x/\partial r$ or $\partial y/\partial \theta$.

   (b) (12 points) Referring to your answer above, show that

   $$||\nabla f||^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 .$$

   Hint. Expand the left-hand side using the definition of the gradient as a vector and properties of dot products.

3. (12 points) Find the critical points of the function $f(x, y) = \sin x \cos y$ in the region $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and identify the global maximum and minimum of f in the region. Justify your answers for full credit. Elementary properties of the sine and cosine functions may be cited without justification.

4. (12 points) COPY ONE FROM THE BOOK

5. Mark each item TRUE or FALSE. Recall that unless a statement is *always true* (that is, for all permitted values of its variables), then it is false.

Every smooth curve in $\mathbf{R}^3$ admits a parametrization with speed 1.	TRUE FALSE
If the cost function f and constraint function g are differentiable and P is a local minimum of f along the level set $g = 0$, then the level sets $f = 0$ and $g = 0$ are tangent at P provided that $\nabla G_P \neq 0$.	TRUE FALSE
Every vector field whose cross-partials are equal is conservative.	TRUE FALSE
There exist a vector field $\mathbf{F}$ defined on the punctured unit disc $D^* = \{(x, y) \in \mathbf{R}^2 : 0 < x < 1, 0 < y < 1\}$ and a closed curve $\mathcal{C}$ contained in D^* such that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \neq 0$.	TRUE FALSE
A function f is differentiable at the point P if all derivatives $D_{\mathbf{u}} f$ exist at P.	TRUE FALSE
The cross product of two vectors is orthogonal to both vectors.	TRUE FALSE
There is exactly one vector in $\mathbf{R}^n$ of length zero.	TRUE FALSE

6. Green's theorem states that if D is a bounded subset of the plane whose boundary is a smooth simple closed curve $\mathcal{C}$ and $\mathbf{F}$ is a smooth vector field on D, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$