Integrating vector fields along curves

Math 251 Calculus 3

November 20, 2013

Warm-up

Set up the integral of the function $f(x, y) = 2x^2y$ along the arc of the parabola $y = x^2$ connecting $(0, 0)$ to $(2, 4)$.

- ▶ Tip: Recall that you can parametrize the graph of any function $y = g(x)$ as $\vec{r}(t) = \langle t, g(t) \rangle$.

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- Answer: Let $\vec{r}$ as above; then $\|\langle \vec{r}'(t) \rangle\| = \sqrt{1 + t^2}$. We also have $f(x(t), y(t)) = 2(t)^2 \cdot t^2 = 2t^4$ for points on the parabola. The parameter interval is $0 \leq t \leq 2$, so our integral should be. . .

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- This integral is hideously wretched; its value is $\frac{1}{24} \left(266\sqrt{5} + 3 \sinh^{-1}(2) \right) \approx 24.9635$.

Integrating vector fields along paths in the plane

The motivating example is a force field $\vec{F}(x, y)$ that acts throughout some region in the plane. If an object is moved by this force along the path $\mathcal{C}$, we want to call the work done

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}.$$

In the text these vectors are denoted with boldface as opposed to arrows, but the meaning is the same. What is this meaning? On Tuesday (the 19th) we observed that a good definition for ds would be $\|\vec{r}'(t)\|\, dt$.

The scalar line element ds is an incremental bit of arclength along the curve. The vector line element $d\vec{s}$ is an incremental tangent vector to the curve.

Interpreting the differential

We set $d\vec{s} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} ds$.

The tangent vector $\vec{r}(t)$ is the velocity vector for the curve. When we divide it by its length, we get a pure direction vector: the quotient is tangent to the curve and has length 1. When we multiply by the "incremental" scalar ds, we get an "incremental" vector.

But $ds = \|\vec{r}'(t)\| dt$.

Hence $d\vec{s} = \vec{r}'(t) dt$. Observe that there is no magnitude operator and hence no square root. For this reason line integrals of vector fields tend to be more tractable for hand calculation than do line integrals of scalar functions.

Integrals of vector fields along parametrized curves

Putting it all together, if $\vec{r}(t)$ is our parametrized curve with parameter interval $[a, b]$, we can compute the line integral of the vector field $\vec{F}$ by the formula

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt.$$

The geometric meaning of the dot product

Remember that the dot product of two vectors depends on their lengths and the angle θ between them. In particular, if the vectors are orthogonal, the dot product is zero. You might also remember:

- If θ is acute, the dot product is positive, while
- if θ is obtuse, the dot product is negative.

Dotting the vector field with the tangent vector to the curve produces the component of the field in the direction of the curve.

Another convenient notation

Since $d\vec{s}$ is a vector, it has components. We recognize them as the familiar differentials dx and dy as follows. Let $\vec{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ and $\vec{r}(t) = \langle x, y \rangle$. Then as we have seen (note the t-dependence is suppressed for readability)

$$\begin{aligned} \vec{F} \cdot d\vec{s} &= \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \ dt \\ &= \langle F_1(x, y), F_2(x, y) \rangle \cdot \langle x', y' \rangle \ dt \\ &= F_1(x, y) x'(t) \ dt + F_2(x, y) y'(t) \ dt. \end{aligned}$$

We recognize some old friends: $dx = x'(t) \ dt$ and $dy = y'(t) \ dt$. Evidently, we have obtained

$$\vec{F} \cdot d\vec{s} = F_1 \ dx + F_2 \ dy.$$

Properties of line integrals

1. *Linearity*: $\int_{\mathcal{C}} (\vec{F} + \vec{G}) \cdot d\vec{s} = \int_{\mathcal{C}} F \cdot d\vec{s} + \int_{\mathcal{C}} G \cdot d\vec{s}$,

$\int_{\mathcal{C}} (k\vec{F}) \cdot d\vec{s} = k \int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$

2. *Orientation*: $\int_{-\mathcal{C}} \vec{F} \cdot d\vec{s} = - \int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$

3. *Additivity*: If $\mathcal{C}$ is a union of n smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, then

$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{s} + \cdots + \int_{\mathcal{C}_n} \vec{F} \cdot d\vec{s}$

Generalization to three dimensions

It is pretty straightforward. All the two-dimensional vectors become three-dimensional, but the dimension-free formulas stay the same. Refer to the text for the complete details.