Mathematics 251
Exam 1

September 21, 2011 Name: ______________________________

Instructions: This exam is closed book: no electronic aids or printed references are permitted. Justification of all answers is required for partial credit; please $\boxed{\textbf{box}}$ your final answers. Unless specifically directed, leave all answers in **exact form**, e.g. $\sqrt{3}$ instead of 1.732 and $\pi/2$ instead of 1.57.

Show all pertinent work. *Correct answers without accompanying work will receive little or no credit.* Results from class or from homework or from class can be cited freely. It is in your interest to display your solution in a clear, readable fashion.

If your work continues onto the back of another page, please indicate this. Check and make sure you have all of the pages in the exam; there should be 5, including this one. If you have a question, please raise your hand.

Be sure to read all questions carefully and completely.

Question	Points	Score
1	24	
2	20	
3	24	
4	24	
5	8	
Total:	100	

Good luck!

1. The vector-valued function $\mathbf{r}(t) = \langle R\cos\omega t, R\sin\omega t\rangle$ parametrizes the circle of radius R centered at $(0,0) \in \mathbb{R}^2$ (assume that R, $\omega > 0$).

   (a) (6 points) Find a formula for the tangent vector $\mathbf{r}'(t)$ and use it to verify that the speed of a particle whose position is $\mathbf{r}(t)$ is $R\omega$.

   (b) (6 points) The *unit* tangent vector $\mathbf{T}(t)$ is by definition the unique vector of length one that points in the same direction as $\mathbf{r}'(t)$ (not in the opposite direction). In the notation of the textbook, $\mathbf{T}(t) = \mathbf{e}_{\mathbf{r}'(t)}$. Find a formula for $\mathbf{T}(t)$ if $\mathbf{r}(t)$ is as in the previous part.

   (c) (12 points) Use your formula from the previous part to verify that, for this curve, $\mathbf{T}(t)$ is orthogonal to $\mathbf{T}'(t)$ for all times t.

2. Consider a particle moving in the (x, y)-plane whose position at time t is given for $0 \leq t \leq 2$ by the parametric equations

$$x(t) = 3t - 1, \quad y(t) = 4t^2.$$

(a) (8 points) Find the velocity of the particle at $t = 2$ (your answer should be a *vector*).

(b) (12 points) Parametrize the line that is tangent to the curve at $(x(2), y(2)) = (5, 16)$. Please write your answer both as a vector-valued function and as a set of parametric equations.

3. Consider the *limaçon* curve pictured below. Its equation in polar coordinates is

$$r = f(\theta) = \frac{1}{2} + \cos\theta.$$

(*Limaçon* is the French word for "snail".)

(a) (12 points) Use the polar-to-rectangular conversion formulas and $f(\theta)$ to express x and y in terms of θ alone (for points on the curve). Find $dx/d\theta$ and $dy/d\theta$.

(b) (6 points) Your answer to the previous part includes a set of parametric equations for the curve (with parameter the angular coordinate θ). Write down an equation relating the three derivatives dy/dx, $dx/d\theta$, and $dy/d\theta$ (hint: chain rule). You don't need to provide any justification here, just the equation suffices.

(c) (6 points) There is a unique line tangent to the curve at the origin with positive slope. What is this slope? Use the previous part and a carefully chosen θ.

4. (24 points) Let ℓ_1 be the line in $\mathbb{R}^3$ containing the points $(1, 1, 0)$ and $(0, -1, 1)$. Let ℓ_2 be the line containing the points $(2, 1, -2)$ and $(3, 0, -1)$. Find a unit vector that is perpendicular to both ℓ_1 and ℓ_2.

5. (8 points) (Note: In this problem, no justification or explanation is required.) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$. Identify the correct completion(s) of the sentence: The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar (they lie in one plane) if (select one of (a) through (g)):

   I. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$.
   II. One of the three vectors is a linear combination of the others.
   III. One of the three vectors is parallel to the cross product of the others.
   IV. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

   a. I only
   b. II only
   c. III only
   d. I and IV only
   e. II and IV only
   f. I, III, and IV only
   g. I, II, III, and IV