Mathematics 251
Exam 1

September 24, 2013 Name: Answers

1. Let $\vec{v} = \langle 0, -3, 1\rangle$ and $\vec{w} = \langle 2, -3, 4\rangle$.

   (a) (6 points) Find a vector that is orthogonal to the plane containing these vectors.

   Solution: We take the cross product of the vectors. Since it is orthogonal to each vector, it is also orthogonal to the plane containing them.

   $$\begin{aligned}\vec{v} \times \vec{w} &= (-12+3)(\hat{\jmath} \times \hat{k}) + (2-0)(\hat{k} \times \hat{\imath}) + (0-(-6))(\hat{\imath} \times \hat{\jmath}) \\ &= \langle -9, 2, 6\rangle.\end{aligned}$$

   A quick dot product calculation verifies that this vector is indeed orthogonal to $\vec{v}$ and $\vec{w}$—of course we know it *should* be, but perhaps we made an arithmetic mistake.

   (b) (6 points) Find an equation for the plane containing these vectors. (Either a vector equation or a scalar equation is acceptable.)

   Solution: We obtained a normal vector for this plane in the previous part, $\vec{n} = \langle -9, 2, 6\rangle$. Therefore, an equation for the plane containing these vectors is

   $$\begin{gathered}\langle -9, 2, 6\rangle \cdot \langle x, y, z\rangle = 0 \quad \text{or} \\ -9x + 2y + 6z = 0.\end{gathered}$$

2. (a) (12 points) Explain geometrically why the equation

   $$\langle 1, 1, 1\rangle \times \langle x, y, z\rangle = \langle 1, 0, 0\rangle$$

   has no solution, i.e., why the equation is false for every choice of $\langle x, y, z\rangle$.

   Solution: If $\langle 1, 0, 0\rangle$ is the cross product of $\langle 1, 1, 1\rangle$ with anything, then $\langle 1, 0, 0\rangle$ must be orthogonal to $\langle 1, 1, 1\rangle$. But it is clearly not, since the dot product of these vectors is nonzero.

   A more long-winded approach would be to note that the angle between these vectors is, rather than $\pi/2 = \tau/4$,

   $$\cos^{-1} \frac{1}{1+\sqrt{3}}.$$

(b) (12 points) Find a vector $\langle x, y, z\rangle$ that satisfies the equation

$$\langle 1,1,1\rangle \times \langle x, y, z\rangle = \langle 1,-1,0\rangle.$$

Note. There are infinitely many such vectors $\langle x, y, z\rangle$.

Solution: The easiest way to proceed is simply to evaluate the left-hand side.

$$\begin{aligned}\langle 1,1,1\rangle \times \langle x, y, z\rangle &= (z-y)(\hat{\jmath}\times\hat{k}) + (x-z)(\hat{k}\times\hat{\imath}) + (y-x)(\hat{\imath}\times\hat{\jmath}) \\ &= \langle z-y, x-z, y-x\rangle.\end{aligned}$$

By inspection, we see that $x = y$, $z = y+1$, so $\langle 0,0,1\rangle$ is a solution. Any vector $\langle x, x, x+1\rangle$ will do, as is easily checked.

3. Let $P = (1,1,0)$, $Q = (1,-2,1)$, and $R = (3,-2,4)$.

(a) (8 points) Find the cosine of the angle between the line segments $\overline{PQ}$ and $\overline{PR}$.

Solution: We use the dot product–cosine formula, $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$.

First, compute $\vec{v} = \langle 1-1, -2-1, 1-0\rangle = \langle 0,-3,1\rangle$ and $\vec{w} = \langle 3-1, -2-1, 4-1\rangle = \langle 2,-3,4\rangle$. Then $\vec{v}\cdot\vec{w} = 13$. We also find $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{10}$, $\|\vec{w}\| = \sqrt{29}$.

We obtain

$$\cos\theta = \frac{13}{\sqrt{13}\sqrt{29}}.$$

(b) (8 points) Explain why your answer to the previous part means that the points P, Q, and R are the vertices of a triangle (in other words, why the points are not collinear).

Solution: If the points were collinear, the angle between the vectors would have to be either 0 or π. But the cosine of the angle is evidently not 1 or -1.

(c) (8 points) Recall that for *any* two vectors $\vec{u}$, $\vec{v}$, the angle formed by $\vec{u}$ and $\vec{v}$ is acute (resp. obtuse) if $\vec{u}\cdot\vec{v}$ is positive (resp. negative). Are any of the angles of the triangle obtuse? Justify your answer.

Solution: All of the angles of the triangle have positive cosines, so this triangle is acute.

4. (8 points) (Note: In this problem, no justification or explanation is required.) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be nonzero vectors in $\mathbf{R}^3$. Identify the correct completion(s) of the sentence: The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar (they lie in one plane) if (select one of (a) through (g)):

I. One of the three vectors is parallel to the cross product of the others.
II. There exist scalars a and b with $\vec{w} = a\vec{u} + b\vec{v}$.
III. $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{0}$.
IV. $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

a. I only
b. II only
c. III only
d. II and IV only
e. III and IV only
f. II, III, and IV only
g. I, II, III, and IV

Solution:

I. If, say, $\vec{u}$ is parallel to $\vec{v} \times \vec{w}$, then $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. It is possible for all three to be coplanar, but only when $\vec{v}$ and $\vec{w}$ are parallel to begin with. Usually, $\vec{u}$ will not be coplanar with $\vec{v}$ and $\vec{w}$. This item does not hold.

II. If $\vec{w}$ is a combination of $\vec{u}$ and $\vec{v}$ as indicated, then $\vec{w}$ is indeed coplanar with $\vec{u}$ and $\vec{v}$. This item holds.

III. This is a special case of the first item, so it doesn't hold.

IV. If the three vectors span a box of volume 0, they are coplanar. The box product formula thus tells us that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar in this case. This item holds.

Therefore, the correct answer is d.

5. (12 points) Suppose that $\vec{u}$ and $\vec{v}$ are orthogonal. Use facts about vectors and their dot products to verify the equation

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Do you think this equation can be true for a pair of non-orthogonal vectors? Justify your answer.

Solution: We make repeated use of the formula $\|\vec{w}\|^2 = \vec{w}\cdot\vec{w}$. Applying this to the left-hand side of the equation, we obtain via bilinearity and commutativity

$$(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}.$$

But $\vec{u}$ and $\vec{v}$ are orthogonal, so $\vec{u}\cdot\vec{v} = 0$. Hence, the right-hand side above reduces to $\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v}$. Another application of the formula allows us to rewrite this last as

$$\|\vec{u}\|^2 + \|\vec{v}\|^2.$$

This equation does not hold for non-orthogonal vectors, because such vectors' dot products do not vanish.

6. (8 points) Suppose that $\mathcal{P}$ is a plane in $\mathbf{R}^3$ and that ℓ is a line contained in $\mathcal{P}$. Let Q be a point not on $\mathcal{P}$. Choose the correct relationship between D, the distance from Q to ℓ, and d, the distance from Q to $\mathcal{P}$.

   A. $d \le D$.

   B. $d \ge D$.

   C. $d = D$.

   D. None of the above.

Solution: The answer is B, because the line is constrained to lie in the plane. Thus the distance from ℓ to Q cannot be smaller than the distance from $\mathcal{P}$ to Q. Of course D can be made as large as you please by moving ℓ away from Q within $\mathcal{P}$.

7. Figure 7 shows a contour plot of a function $f(x, y)$.

   (a) (6 points) Starting at $(2,2)$ and moving in the negative x-direction, are the values of $f(x, y)$ decreasing or increasing?

   Solution: The values are increasing.

   (b) (6 points) Starting at point $(2,0)$ and moving in the positive y-direction, are the values of $f(x, y)$ decreasing or increasing?

Solution: The values are increasing.

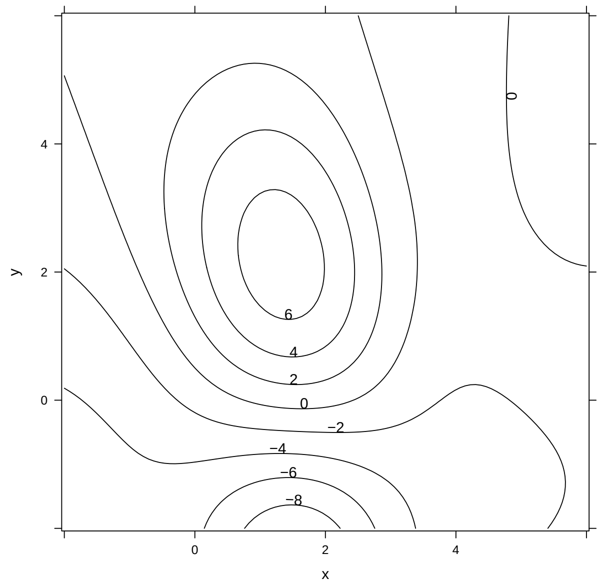

Figure 1: Contour plot for Problem 7