Practice with parametrizing lines

Math 251 Calculus 3

November 13, 2013

Starting point and direction vector

Just like the standard form of an equation of a plane, the easiest way to express a line involves a vector. Instead of a normal vector (which, in a way, tells which direction the plane *doesn't* go), we'll use a vector $\vec{v}$ *parallel* to the line, called its direction vector.

If (x_0, y_0, z_0) is a point in $\mathbf{R}^3$, there is exactly one line passing through it parallel to $\vec{v}$. Evidently, these points are all obtained by adding multiples of $\vec{v}$ to the vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Parametrizing the line

A multiple of $\vec{v}$ is a vector $t\vec{v}$, where $t \in \mathbf{R}$. We usually think of t as a time parameter and the expression

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} \tag{1}$$

as a moving point. Really it is a moving *vector*; we imagine it in standard position, so that the tail of $\vec{r}(t)$ is fixed at $(0, 0, 0)$ while the head traces the line through $\vec{r}_0$ parallel to $\vec{v}$.

The scalar form of the equations

If we write $\vec{v}$ in coordinates $\vec{v} = \langle a, b, c \rangle$, then we can decompose Equation 1 into a system of scalar equations. Write

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle = \langle x(t), y(t), z(t) \rangle,$$

to obtain

$$\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}$$

These are evidently equivalent to Equation 1 and are called parametric equations of the line.

Work together

Using the whiteboards, find vector equations for all of the following lines and scalar (parametric) equations for at least one of them.

1. The line passing through $(-5, 6, -1)$ parallel to $\langle 9, 0, -6 \rangle$.
2. The line passing through $(3, 2, -3)$ and $(-1, 4, 2)$.
3. The line passing through $(3, 2, -3)$ orthogonal to the plane $2x - y - z = 3$.

Answers

1. $\vec{r}(t) = \langle -5 + 9t, 6, -1 - 6t \rangle$.
2. $\vec{r}(t) = \langle 3, 2, -3 \rangle + t\langle -4, 2, 5 \rangle$.
3. $\vec{r}(t) = \langle 3, 2, -3 \rangle + t\langle 2, -1, -1 \rangle$.

Scalar equations for 3 are

$$\begin{aligned} x(t) &= 3 + 2t \\ y(t) &= 2 - t \\ z(t) &= -3 - t. \end{aligned}$$

Circles

The other kind of curve you should know how to parametrize is a circle. A circle in the plane determined by its center (x_0, y_0) and its radius r. One parametrization is

$$\vec{r}(t) = \langle x_0, y_0 \rangle + r \langle \cos t, \sin t \rangle.$$

Write down parametrizations for the following.

1. Unit circle
2. Radius 1, center $(1, 0)$
3. Radius 10, center $(-3, -2)$

Pasting parametrizations

Often, we will be interested in parametrizing curves made up of line segments and circular arcs. For example, consider the pseudotriangle whose boundary is the segment connecting $(0, 0)$ to $(1, 0)$, the segment from $(0, 1)$ to $(0, 0)$, and the arc of the unit circle lying in the first quadrant (which connects $(0, 1)$ to $(1, 0)$). Try to find a piecewise function $\vec{r}(t)$ that traces once around this path.

Start by parametrizing each boundary segment separately.