Math 275 **Workshop 01: Slicing spheres**

September 11, 2013

Due: Friday, September 13 Name: ____________________

1 Workshop 01

This workshop will confirm what is, to varying degrees for various people, intuitively clear: that all the nonempty slices of a sphere by planes are circles (we adopt the severely reasonable convention that a circle of radius 0 is a point). Be ready to submit this paper on Friday. Workshop grades are mostly participation: if you appear in class, make an honest effort, and submit on time, you receive all or nearly all of the points.

1.1 Warm-up

Recall that the *unit sphere* in $\mathbf{R}^3$ is the sphere of radius 1 centered at $(0,0,0)$. Its equation is

$$x^2 + y^2 + z^2 = 1.$$

1. Give equations of all the planes parallel to the (y,z)-plane that miss the unit sphere. *Hint.* Use an auxiliary variable.
2. How many planes parallel to the (y,z)-plane meet the sphere at one point? Give equations for all of them.

1.2 Brute force

You read in Module 03 that it is possible to prove algebraically that the intersection of a sphere and a plane is a circle (provided the intersection contains more than one point). After enough mental aerobics, this certainly seems *plausible*. But is it completely, unambiguously obvious that it's impossible to get an elliptical cross-section?

A completely coordinatized approach to this problem involves choosing a sphere with arbitrary center and radius and an arbitrary plane. Using a result about the classification of planes (we shall do this after we have more vector technology ready to deploy), this will involve solving a system of two simultaneous nonlinear equation in eight variables:

$$\begin{aligned}(x-a)^2 + (y-b)^2 + (z-c)^2 &= r^2\\ Ax + By + Cz &= D\end{aligned}$$

This approach is easy to set up, but a pain in the rear to work out the details, and not very illuminating. Instead, we'll unleash the kung fu of symmetry to deal with this complicated geometric situation.

1.3 Symmetry and WLOG

The idea behind a symmetry argument is that a special case might not really be all that special. For us, the special case will be:

The sphere is the unit sphere and the plane is of the form $z = A$.

It seems very special: after all, there are many, many spheres other than the unit sphere, and many planes that are not horizontal. But think of it like a videographer: the right angle and zoom turns any situation into this special one.

That's what we mean by symmetry, or the mathematical abbreviation WLOG. This stands for **W**ithout **L**oss **O**f **G**enerality. The phrase signifies the idea encapsulated above: that what appears to be a special case is in fact sufficiently general.

1.4 The unit sphere and a horizontal plane

1. Draw a big, beautiful picture of the unit sphere sliced by a horizontal plane. Remember the drawing tips:
   - Get the angles right. Make the parallelogram of your plane parallel to the x- and y-axes.
   - Draw a *big* picture. Big pictures have more room for labels.
   - The (x, y)-plane itself does slice the sphere, but it's very special. Pick a more generic slicing plane.
   - Don't be afraid to erase.

2. Now, we're still solving two equations. What are they? (Hint: one from the sphere and one from the slicing plane.)

3. Explain why this situation is algebraically preferable to the brute-force solution outlined above.

4. Write down two equations that all the points in the intersection of the sphere and the plane satisfy. (Hint: one equation involves x and y only, and the other involves z only.) These aren't the same as the equations above, but they aren't too terribly different either.

5. Identify the flaw in this attempted use of WLOG.

   Consider the cylinder of radius 1 with axis the x-axis and the (y,z)-plane. The intersection is evidently the circle of radius 1, centered at $(0,0,0)$, and contained in the (y,z)-plane.

   Therefore, we can say that WLOG, whenever a cylinder is cut by a plane, the intersection is a circle. This is because we can always "rotate and zoom" to place the cylinder so that its radius is 1 and its axis is the x-axis.