

Differential Equations Worksheet

Accompanies Section 1.5 in ODEP

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1 First-order linear differential equations

In this workshop, you will see how to solve (remember the technical meaning of this word) linear first-order differential equations, that is, equations of the form

$$\frac{dy}{dt} + p(t)y = g(t).$$

Every student of the mathematical sciences should know how to solve first-order linear equations, along with the separable equations we encountered previously. First-order linear equations can always be solved (in principle¹) via the method of integrating factors.

Consider the equation

$$\frac{dy}{dt} + \frac{y(t)}{2} = \frac{1}{2}e^{t/3}. \quad (1.1)$$

Activity 1.1 (Integrating factors).

- (a) Multiply this equation by a new unknown function $\mu(t)$.

Answer.

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{y(t)}{2} = \frac{1}{2}\mu(t)e^{t/3}.$$

- (b) According to the product rule, what is $(\mu(t)y(t))'$?

$$\frac{d}{dt}(\mu(t)y) = \underline{\hspace{10cm}}$$

Answer.

$$\frac{d}{dt}(\mu(t)y(t)) = \frac{d\mu}{dt}y(t) + \mu(t)\frac{dy}{dt}.$$

- (c) Observe that the expression $\mu(t)dy/dt$ appears in both of the equations you found in [Task a](#) and [Task b](#).

Answer. Indeed, it does. See [Answer 1 of Task 1.1.a](#) and [Answer 1 of Task 1.1.b](#).

¹The integrals that arise may be intractable for hand calculation.

- (d) Write down an equation that asserts the equality of the equation sides containing $\mu(t)dy/dt$.

Answer.

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{y(t)}{2} = \frac{d\mu}{dt}y(t) + \mu(t)\frac{dy}{dt}.$$

- (e) Simplify the equation you found in [Task d](#), isolate $d\mu/dt$, and integrate to find an expression for $\mu(t)$. (Choose the simplest value for the constant of integration.)

Answer.

$$\mu(t) = e^{t/2}.$$

- (f) Now that your **integrating factor** $\mu(t)$ is in hand, substitute it into the equation you found in [Task c](#).

Answer.

$$\begin{aligned} e^{t/2}\frac{dy}{dt} + e^{t/2}\frac{y(t)}{2} &= \frac{1}{2}e^{t/2}e^{t/3} \\ &= \frac{1}{2}e^{5t/6}. \end{aligned}$$

- (g) Apply the result of [Task b](#) to the equation you found in [Task f](#).

Answer.

$$\frac{d}{dt}\left(e^{t/2}y(t)\right) = \frac{1}{2}e^{5t/6}$$

- (h) Integrate, using the Fundamental Theorem of Calculus on both sides.

Answer.

$$\begin{aligned} e^{t/2}y(t) &= \frac{1}{2}\frac{6}{5}e^{5t/6} + C \\ &= \frac{3}{5}e^{5t/6} + C \end{aligned}$$

- (i) Solve the equation you found in [Task h](#) for $y(t)$.

Answer.

$$y(t) = \frac{3}{5}e^{t/3} + Ce^{-t/2}.$$

- (j) Check that $y(t)$ as found in [Task i](#) is really a solution to the original [\(1.1\)](#).

Solution. Since $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$, we can differentiate to find

$$y' = \frac{1}{5}e^{t/3} - \frac{1}{2}Ce^{-t/2}.$$

Substitution into [\(1.1\)](#) then gives

$$\begin{aligned} \frac{dy}{dt} + \frac{y(t)}{2} &= \frac{1}{5}e^{t/3} - \frac{1}{2}Ce^{-t/2} + \frac{3}{10}e^{t/3} + \frac{1}{2}Ce^{-t/2} \\ &= \frac{1}{2}e^{t/3}, \end{aligned}$$

as required.

Consider the equation

$$y' - 2ty = 2t. \quad (1.2)$$

Activity 1.2 (Practice integrating factors).

- (a) Multiply both sides by $\mu(t)$.
- (b) Recall what the product rule tells you about $(\mu(t)y(t))'$.
- (c) Write an equation asserting the equality of two different expressions involving $\mu(t)dy/dt$.
- (d) Solve your equation for the integrating factor $\mu(t)$.

Hint. Remember, at this stage we can choose any convenient constant of integration: all the possible functions $\mu(t)$ are equally good.

- (e) Use $\mu(t)$ and one more integration to solve (1.2).

Hint. Here, the constant of integration is essential. We want to know all of the (infinitely many) functions that satisfy (1.2), not just one of them.

- (f) Differentiate the function you found in Task e and substitute into (1.2) to verify that your solution is correct.

The foregoing activities should illustrate the general principle. In ODEP Section 1.5.2 we see that, to solve the equation

$$y' + p(t)y = q(t), \quad (1.3)$$

we can always choose $\mu(t) = \frac{1}{P(t)}$, where $P(t)$ is any antiderivative of $p(t)$.

Warning 1. Note that the formula for the integrating factor assumes that the differential equation has been put into a certain **standard form**. If you memorize the former, be sure to remember the latter as well!