Differential Equations Worksheet

Accompanies Section 1.5 in ODEP

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1 First-order linear differential equations

In this workshop, you will see how to solve (remember the technical meaning of this word) linear first-order differential equations, that is, equations of the form

$$\frac{dy}{dt} + p(t)y = g(t).$$

Activity 1.1 (Integrating factors)**.**

(a) Multiply this equation by a new unknown function $\mu(t)$.

Answer.

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{y(t)}{2} = \frac{1}{2}\mu(t)e^{t/3}.$$

(b) According to the product rule, what is $(\mu(t)y(t))'$?

$$\frac{d}{dt}\left(\mu(t)y\right) = \underline{\hspace{10cm}}$$

Answer.

$$\frac{d}{dt}(\mu(t)y(t)) = \frac{d\mu}{dt}y(t) + \mu(t)\frac{dy}{dt}.$$

(c) Observe that the expression $\mu(t)dy/dt$ appears in both of the equations you found in Task a and Task b.

Answer. Indeed, it does. See Answer 1 of Task 1.1.a and Answer 1 of Task 1.1.b.

(d) Write down an equation that asserts the equality of the equation sides containing $\mu(t)dy/dt$.

Answer.

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{y(t)}{2} = \frac{d\mu}{dt}y(t) + \mu(t)\frac{dy}{dt}.$$

(e) Simplify the equation you found in Task d, isolate $d\mu/dt$, and integrate to find an expression for $\mu(t)$. (Choose the simplest value for the constant of integration.)

Answer.

$$\mu(t) = e^{t/2}.$$

(f) Now that your **integrating factor** $\mu(t)$ is in hand, substitute it into the equation you found in Task c.

Answer.

$$\begin{aligned} e^{t/2}\frac{dy}{dt} + e^{t/2}\frac{y(t)}{2} &= \frac{1}{2}e^{t/2}e^{t/3} \\ &= \frac{1}{2}e^{5/6}. \end{aligned}$$

(g) Apply the result of Task b to the equation you found in Task f.

Answer.

$$\frac{d}{dt}\left(e^{t/2}y(t)\right) = \frac{1}{2}e^{5t/6}$$

(h) Integrate, using the Fundamental Theorem of Calculus on both sides.

Answer.

$$\begin{aligned} e^{t/2}y(t) &= \frac{1}{2}\frac{6}{5}e^{5t/6} + C \\ &= \frac{3}{5}e^{5t/6} + C \end{aligned}$$

(i) Solve the equation you found in Task h for $y(t)$.

Answer.

$$y(t) = \frac{3}{5}e^{t/3} + Ce^{-t/2}.$$

(j) Check that $y(t)$ as found in Task i is really a solution to the original (**??**).

Solution. Since $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$, we can differentiate to find

$$y' = \frac{1}{5}e^{t/3} - \frac{1}{2}Ce^{-t/2}.$$

Substitution into (**??**) then gives

$$\begin{aligned} \frac{dy}{dt} + \frac{y(t)}{2} &= \frac{1}{5}e^{t/3} - \frac{1}{2}Ce^{-t/2} + \frac{3}{10}e^{t/3} + \frac{1}{2}Ce^{-t/2} \\ &= \frac{1}{2}e^{t/3}, \end{aligned}$$

as required.

Activity 1.2 (Practice integrating factors)**.**

(a) Multiply both sides by $\mu(t)$.

(b) Recall what the product rule tells you about $(\mu(t)y(t))'$.

(c) Write an equation asserting the equality of two different expressions involving $\mu(t)dy/dt$.

(d) Solve your equation for the integrating factor $\mu(t)$.

Hint. Remember, at this stage we can choose any convenient constant of integration: all the possible functions $\mu(t)$ are equally good.

(e) Use $\mu(t)$ and one more integration to solve (**??**).

Hint. Here, the constant of integration is essential. We want to know all of the (infinitely many) functions that satisfy (**??**), not just one of them.

(f) Differentiate the function you found in Task e and substitute into (**??**) to verify that your solution is correct.

The foregoing activities should illustrate the general principle. In ODEP Section 1.5.2 we see that, to solve the equation

$$y' + p(t)y = q(t), \tag{1.1}$$

we can always choose $\mu(t) =$ __________________________________, where $P(t)$ is any antiderivative of $p(t)$.

Warning 1. Note that the formula for the integrating factor assumes that the differential equation has been put into a certain **standard form**. If you memorize the former, be sure to remember the latter as well!