CHAPTER 9

Problems in General Form

Up until now, we have always considered our problems to be given in standard form. However, for real-world problems it is often convenient to formulate problems in the following form:

$$\begin{array}{lrcl} \text{maximize} & & c^T x & \\ \text{subject to} & a \leq & Ax & \leq b \\ & l \leq & x & \leq u. \end{array} \tag{9.1}$$

Two-sided constraints such as those given here are called constraints with *ranges*. The vector l is called the vector of *lower bounds*, and u is the vector of *upper bounds*. We allow some of the data to take infinite values; that is, for each $i = 1, 2, \ldots, m$,

$$-\infty \leq a_i \leq b_i \leq \infty,$$

and, for each $j = 1, 2, \ldots, n$,

$$-\infty \leq l_j \leq u_j \leq \infty.$$

In this chapter, we shall show how to modify the simplex method to handle problems presented in this form.

1. The Primal Simplex Method

It is easiest to illustrate the ideas with an example:

$$\begin{array}{lrrcl} \text{maximize} & & 3x_1 - & x_2 & \\ \text{subject to} & 1 \leq & -x_1 + & x_2 & \leq 5 \\ & 2 \leq & -3x_1 + & 2x_2 & \leq 10 \\ & & 2x_1 - & x_2 & \leq 0 \\ & -2 \leq & x_1 & & \\ & 0 \leq & & x_2 & \leq 6. \end{array}$$

With this formulation, zero no longer plays the special role it once did. Instead, that role is replaced by the notion of a variable or a constraint being at its upper or lower bound. Therefore, instead of defining slack variables for each constraint, we use w_i simply to denote the value of the ith constraint:

$$\begin{aligned} w_1 &= \;\; -x_1 + \;\; x_2 \\ w_2 &= -3x_1 + 2x_2 \\ w_3 &= \;\;\; 2x_1 - \;\; x_2. \end{aligned}$$

The constraints can then be interpreted as upper and lower bounds on these variables. Now when we record our problem in a dictionary, we will have to keep explicit track of the upper and lower bound on the original x_j variables and the new w_i variables. Also, the value of a nonbasic variable is no longer implicit; it could be at either its upper or its lower bound. Hence, we shall indicate which is the case by putting a box around the relevant bound. Finally, we need to keep track of the values of the basic variables. Hence, we shall write our dictionary as follows:

$$\begin{array}{rr|rcrcrcr} l & & & & \boxed{-2} & & \boxed{0} & & \\ & u & & & \infty & & 6 & & \\ \hline & & \zeta = & & 3x_1 & - & x_2 & = & -6 \\ \hline 1 & 5 & w_1 = & & -x_1 & + & x_2 & = & 2 \\ 2 & 10 & w_2 = & & -3x_1 & + & 2x_2 & = & 6 \\ -\infty & 0 & w_3 = & & 2x_1 & - & x_2 & = & -4\,. \end{array}$$

Since all the w_i's are between their upper and lower bounds, this dictionary is feasible. But it is not optimal, since x_1 could be increased from its present value at the lower bound, thereby increasing the objective function's value. Hence, x_1 shall be the entering variable for the first iteration. Looking at w_1, we see that x_1 can be raised only 1 unit before w_1 hits its lower bound. Similarly, x_1 can be raised by $4/3$ units, at which point w_2 hits its lower bound. Finally, if x_1 were raised 2 units, then w_3 would hit its upper bound. The tightest of these constraints is the one on w_1, and so w_1 becomes the leaving variable—which, in the next iteration, will then be at its lower bound. Performing the usual row operations, we get

$$\begin{array}{rr|rcrcrcr} l & & & & \boxed{1} & & \boxed{0} & & \\ & u & & & 5 & & 6 & & \\ \hline & & \zeta = & & -3w_1 & + & 2x_2 & = & -3 \\ \hline -2 & \infty & x_1 = & & -w_1 & + & x_2 & = & -1 \\ 2 & 10 & w_2 = & & 3w_1 & - & x_2 & = & 3 \\ -\infty & 0 & w_3 = & & -2w_1 & + & x_2 & = & -2\,. \end{array}$$

Note, of course, that the objective function value has increased (from -6 to -3). For the second iteration, raising x_2 from its lower bound will produce an increase in ζ.

Hence, x_2 is the entering variable. Looking at the basic variables (x_1, w_2, and w_3), we see that w_2 will be the first variable to hit a bound, namely, its lower bound. Hence, w_2 is the leaving variable, which will become nonbasic at its lower bound:

$$
\begin{array}{rr|rrrrr}
l & & & \boxed{1} & & \boxed{2} & \\
 & u & & 5 & & 10 & \\
\hline
 & & \zeta = & 3w_1 & - & 2w_2 = & -1 \\
\hline
-2 & \infty & x_1 = & 2w_1 & - & w_2 = & 0 \\
0 & 6 & x_2 = & 3w_1 & - & w_2 = & 1 \\
-\infty & 0 & w_3 = & w_1 & - & w_2 = & -1\,.
\end{array}
$$

For the third iteration, w_1 is the entering variable, and w_3 is the leaving variable, since it hits its upper bound before any other basic variables hit a bound. The result is

$$
\begin{array}{rr|rrrrr}
l & & & -\infty & & \boxed{2} & \\
 & u & & \boxed{0} & & 10 & \\
\hline
 & & \zeta = & 3w_3 & + & w_2 = & 2 \\
\hline
-2 & \infty & x_1 = & 2w_3 & + & w_2 = & 2 \\
0 & 6 & x_2 = & 3w_3 & + & 2w_2 = & 4 \\
1 & 5 & w_1 = & w_3 & + & w_2 = & 2\,.
\end{array}
$$

Now for the next iteration, note that the coefficients on both w_3 and w_2 are positive. But w_3 is at its upper bound, and so if it were to change, it would have to decrease. However, this would mean a decrease in the objective function. Hence, only w_2 can enter the basis, in which case x_2 is the leaving variable getting set to its upper bound:

$$
\begin{array}{rr|rrrrr}
l & & & -\infty & & 0 & \\
 & u & & \boxed{0} & & \boxed{6} & \\
\hline
 & & \zeta = & 1.5w_3 & + & 0.5x_2 = & 3 \\
\hline
-2 & \infty & x_1 = & 0.5w_3 & + & 0.5x_2 = & 3 \\
2 & 10 & w_2 = & -1.5w_3 & + & 0.5x_2 = & 3 \\
1 & 5 & w_1 = & -0.5w_3 & + & 0.5x_2 = & 3\,.
\end{array}
$$

For this dictionary, both w_3 and x_2 are at their upper bounds and have positive coefficients in the formula for ζ. Hence, neither can be moved off from its bound to increase the objective function. Therefore, the current solution is optimal.

2. The Dual Simplex Method

The problem considered in the previous section had an initial dictionary that was feasible. But as always, we must address the case where the initial dictionary is not feasible. That is, we must define a Phase I algorithm. Following the ideas presented in Chapter 5, we base our Phase I algorithm on a dual simplex method. To this end, we need to introduce the dual of (9.1). So first we rewrite (9.1) as

$$
\begin{array}{ll}
\text{maximize} & c^T x \\
\text{subject to} & \begin{aligned} Ax &\leq b \\ -Ax &\leq -a \\ x &\leq u \\ -x &\leq -l, \end{aligned}
\end{array}
$$

and adding slack variables, we have

$$
\begin{array}{ll}
\text{maximize} & c^T x \\
\text{subject to} & \begin{aligned} Ax + f &= b \\ -Ax + p &= -a \\ x + t &= u \\ -x + g &= -l \\ f,\ p,\ t,\ g &\geq 0. \end{aligned}
\end{array}
$$

We see immediately from the inequality form of the primal that the dual can be written as

$$
\begin{array}{ll}
\text{minimize} & b^T v - a^T q + u^T s - l^T h \\
\text{subject to} & A^T(v - q) - (h - s) = c \\
& \qquad\qquad v,\ q,\ s,\ h \geq 0.
\end{array} \tag{9.2}
$$

Furthermore, at optimality, the dual variables are complementary to the corresponding primal slack variables:

$$
\begin{aligned}
f_i v_i &= 0 \qquad & i &= 1, 2, \ldots, m, \\
p_i q_i &= 0 \qquad & i &= 1, 2, \ldots, m, \\
t_j s_j &= 0 \qquad & j &= 1, 2, \ldots, n, \\
g_j h_j &= 0 \qquad & j &= 1, 2, \ldots, n.
\end{aligned} \tag{9.3}
$$

Note that for each i, if $b_i > a_i$, then at optimality v_i and q_i must be complementary to each other. Indeed, if both were positive, then they could be reduced by an equal amount without destroying feasibility, and the objective function value would strictly decrease, thereby implying that the supposedly optimal solution is not optimal. Similarly, if for some i, $b_i = a_i$, then it is no longer required that v_i and q_i be complementary at optimality; but, given an optimal solution for which both v_i and q_i are positive, we can decrease both these values at the same rate until the smaller of the two reaches zero, all the while preserving feasibility of the solution and not changing the objective function value. Hence, there always exists an optimal solution in which every component of v is complementary to the corresponding component of q. The same argument shows that if there exists an optimal solution, then there exists one in which all the components of h and s are complementary to each other as well.

For a real variable ξ, its positive part ξ^+ is defined as

$$\xi^+ = \max\{\xi, 0\}$$

and its negative part ξ^- is defined similarly as

$$\xi^- = \max\{-\xi, 0\}.$$

Clearly, both ξ^+ and ξ^- are nonnegative. Furthermore, they are complementary,

$$\xi^+ = 0 \qquad \text{or} \qquad \xi^- = 0,$$

and their difference represents ξ:

$$\xi = \xi^+ - \xi^-.$$

From the complementarity of the components of v against the components of q, we can think of them as the positive and negative parts of the components of just one vector y. So let us write:

$$v = y^+ \qquad \text{and} \qquad q = y^-.$$

Similarly, let us write

$$h = z^+ \qquad \text{and} \qquad s = z^-.$$

If we impose these complementarity conditions not just at optimality but also from the start, then we can eliminate v, q, s, and h from the dual and write it simply as

$$\begin{array}{ll} \text{minimize} & b^T y^+ - a^T y^- + u^T z^+ - l^T z^- \\ \text{subject to} & A^T y - z = c, \end{array} \tag{9.4}$$

where the notation y^+ denotes the componentwise positive part of y, etc. This problem is an example from the class of problems called piecewise linear programs. Usually, piecewise linear programs are solved by converting them into linear programs. Here, however, we wish to go in the other direction. We shall present an algorithm for (9.4) that will serve as an algorithm for (9.2). We will call this algorithm the *dual simplex method* for problems in general form.

To economize on the presentation, we shall present the dual simplex method in the context of a Phase I algorithm for linear programs in general form. Also, to avoid cumbersome notations, we shall present the algorithm with the following example:

$$
\begin{array}{lrrl}
\text{maximize} & & 2x_1 - x_2 & \\
\text{subject to} & 0 \leq & x_1 + x_2 \leq & 6 \\
& 2 \leq & -x_1 + 2x_2 \leq & 10 \\
& & x_1 - x_2 \leq & 0 \\
& -2 \leq & x_1 & \\
& 1 \leq & x_2 \leq & 5.
\end{array}
\tag{9.5}
$$

The piecewise linear formulation of the dual is

$$
\begin{array}{ll}
\text{minimize} & 6y_1^+ + 10y_2^+ \qquad\qquad + \;2z_1^+ - \;z_2^+ \\
& \qquad\quad - \;2y_2^- + \infty y_3^- + \infty z_1^- + 5z_2^- \\
\text{subject to} & y_1 - \;\; y_2 + \;\; y_3 - \;\; z_1 \qquad\quad = \;\;2 \\
& y_1 + \;2y_2 - \;\; y_3 - \qquad - \;\; z_2 = -1.
\end{array}
$$

Note that the objective function has coefficients that are infinite. The correct convention is that infinity times a variable is plus infinity if the variable is positive, zero if the variable is zero, and minus infinity if the variable is negative.

Since the objective function is nonlinear (taking positive and negative parts of variables is certainly a nonlinear operation), we will not be able to do the usual row operations on the objective function. Therefore, in each iteration, we simply study it as is. But as usual, we prefer to think in terms of maximization, and so we record the negative of the objective function:

$$
\begin{aligned}
-\xi = -6y_1^+ - 10y_2^+ \qquad\qquad & - \;2z_1^+ + \;z_2^+ \\
+ \;2y_2^- - \infty y_3^- & - \infty z_1^- - 5z_2^- .
\end{aligned}
\tag{9.6}
$$

We can of course perform row operations on the two constraints, so we set up the usual sort of dictionary for them:

$$
\begin{aligned}
z_1 &= -2 + y_1 - \;\; y_2 + y_3 \\
z_2 &= \;\;\; 1 + y_1 + 2y_2 - y_3 .
\end{aligned}
\tag{9.7}
$$

For the dual problem, all the action takes place at zero. That is, slopes in the objective function change when a variable goes from negative to positive. Since nonbasic variable are supposed to be set where the action is, we associate a current solution with each dictionary by setting the nonbasic variables to zero. Hence, the solution associated with the initial dictionary is

$$
(y_1, y_2, y_3, z_1, z_2) = (0, 0, 0, -2, 1).
$$

The fact that z_1 is negative implies that z_1^- is a positive number and hence that the objective function value associated with this solution is minus infinity. Whenever the objective function value is minus infinity, we say that the solution is *infeasible*. We also refer to the associated dictionary as infeasible. Hence, the initial dictionary given in (9.7) is infeasible.

The dual simplex method must start with a dual feasible solution. But since we intend to use the dual simplex method simply to find a feasible solution for (9.5), we are free to change the objective function in (9.5) any way we please. In particular, we can change it from

$$\zeta = 2x_1 - x_2$$

to

$$\eta = -2x_1 - x_2.$$

Making that change to the primal leaves the dual objective function unchanged, but produces a feasible dual dictionary:

$$
\begin{aligned}
z_1 &= 2 + y_1 - y_2 + y_3 \\
z_2 &= 1 + y_1 + 2y_2 - y_3.
\end{aligned}
\tag{9.8}
$$

For comparison purposes, let us also record the corresponding primal dictionary. It is easy to write down the equations defining the w_i's, but how do we know whether the x_j's are supposed to be at their upper or their lower bounds? The answer comes from the requirement that the primal and dual satisfy the complementarity conditions given in (9.3). Indeed, from the dual dictionary we see that $z_1 = 1$. Hence, $z_1^+ = 1$. But since z_1^+ is just a surrogate for h_1, we see that h_1 is positive and hence that g_1 must be zero. This means that x_1 must be at its lower bound. Similarly, for the sake of complementarity, x_2 must also be at its lower bound. Hence, the primal dictionary is

$$
\begin{array}{rr|rrrr}
l & & & \boxed{-2} & \boxed{1} & \\
 & u & & \infty & 5 & \\
\hline
 & & \eta = & -x_1 - & x_2 = & 1 \\
\hline
0 & 6 & w_1 = & x_1 + & x_2 = & -1 \\
2 & 10 & w_2 = & -x_1 + & 2x_2 = & 4 \\
-\infty & 0 & w_3 = & x_1 - & x_2 = & -3\,.
\end{array}
$$

Note that it is infeasible, since w_1 is not between its upper and lower bounds.

We are now ready to describe the first iteration of the dual simplex method. To this end, we ask whether we can improve the dual objective function value by moving one of the nonbasic variables (y_1, y_2, or y_3) away from zero. Of course, each of these three variables can be moved either to the positive or the negative side of zero; we must analyze these six cases individually. First of all, note that since z_1 is positive at the current solution, it follows that $z_1^+ = z_1$ and $z_1^- = 0$ in a neighborhood of the current solution. A similar statement can be made for z_2, and so we can rewrite (9.6) locally around the current solution as

$$\begin{aligned} -\xi = -6y_1^+ - 10y_2^+ \qquad\qquad & - 2z_1 + z_2 \\ + \; 2y_2^- - \infty y_3^- . & \end{aligned}$$

Now, as y_1 is increased from zero, the rate of increase of $-\xi$ is simply the derivative of the right-hand side with respect to y_1, where we must keep in mind that z_1 and z_2 are functions of y_1 via the dictionary (9.8). Hence, the rate of increase is $-6-2+1 = -7$; i.e., the objective function decreases at a rate of 7 units per unit increase of y_1. If, on the other hand, y_2 is decreased from zero into negative territory, then the rate of increase of $-\xi$ is the negative of the derivative of the right-hand side. In this case we get no contribution from y_1^- but we do get something from z_1 and z_2 for a total of $2 - 1 = 1$. Hence, the rate of increase as we move in this direction is one unit increase per unit move. We can analyze changes to y_2 and y_3. The entire situation can be summarized as follows:

$$\begin{aligned} y_1 \nearrow & \quad -6 - 2 + 1 = -7 \\ y_1 \searrow & \qquad 0 + 2 - 1 = 1 \\ y_2 \nearrow & \; -10 + 2 + 2 = -6 \\ y_2 \searrow & \qquad 2 - 2 - 2 = -2 \\ y_3 \nearrow & \qquad 0 - 2 - 1 = -3 \\ y_3 \searrow & \; -\infty + 2 + 1 = -\infty . \end{aligned}$$

Of these six cases, the only one that brings about an increase in $-\xi$ is the one in which y_1 is sent negative. Hence, y_1 shall be our entering variable, and it will go negative. To find the leaving variable, we must ask: as y_1 goes negative, which of z_1 and z_2 will hit zero first? For the current dictionary, z_2 gets to zero first and so becomes the leaving variable. Performing the usual row operations, the new dictionary for the dual problem is

$$\begin{aligned} z_1 &= \quad 1 + z_2 - 3y_2 + 2y_3 \\ y_1 &= -1 + z_2 - 2y_2 - \; y_3 . \end{aligned}$$

Let us have a look at the new primal dictionary. The fact that y_1 was the entering variable in the dual dictionary implies that w_1 is the leaving variable in the primal. Furthermore, the fact that y_1 has gone negative implies that y_1^- is now positive, and so complementarity then demands that q_1 be zero; i.e., w_1 should go to its lower bound. The fact that z_2 was the leaving variable in the dual dictionary implies that x_2 is the entering variable in the primal. Hence, the new primal dictionary is

$$\begin{array}{rr|rrrcl}
l & & & \boxed{-2} & \boxed{0} & & \\
 & u & & \infty & 6 & & \\
\hline
 & & \eta = & -x_1 - & w_1 = & 2 & \\
\hline
1 & 5 & x_2 = & -x_1 + & w_1 = & 2 & \\
2 & 10 & w_2 = & -3x_1 + & 2w_1 = & 6 & \\
-\infty & 0 & w_3 = & 2x_1 - & w_1 = & -4\,. &
\end{array}$$

We are now ready to begin the second iteration. Therefore, we ask which nonbasic variable should be moved away from zero (and in which direction). As before, we first note that z_1 positive implies that $z_1^+ = z_1$ and $z_1^- = 0$ and that y_1 negative implies that $y_1^+ = 0$ and $y_1^- = -y_1$. Hence, the objective function can be written locally around the current solution as

$$\begin{aligned}
-\xi = &- 10y_2^+ & & - 2z_1 + z_2^+ \\
&+ 2y_2^- - \infty y_3^- & & - 5z_2^-\,.
\end{aligned}$$

We now summarize the possibilities in a small table:

$$\begin{array}{ll}
z_2 \nearrow & 1 - 2 = -1 \\
z_2 \searrow & -5 + 2 = -3 \\
y_2 \nearrow & -10 + 6 = -4 \\
y_2 \searrow & 2 - 6 = -4 \\
y_3 \nearrow & 0 - 4 = -4 \\
y_3 \searrow & -\infty + 4 = -\infty.
\end{array}$$

Note that all the changes are negative, meaning that there are no possibilities to increase the objective function any further. That is, the current dual solution is optimal. Of course, this also could have been deduced by observing that the primal dictionary is feasible (which is what we are looking for, after all).

Even though this example of the dual simplex method has terminated after only one iteration, it should be clear how to proceed had it not terminated.

Now that we have a feasible solution for the primal, we could solve the problem to optimality by simply reinstating the original objective function and proceeding by applying the primal simplex method in a Phase II procedure to find the optimal solution. Since the primal simplex method has already been discussed, we stop here on this problem.

Exercises

Solve the following linear programming problems:

9.1

$$
\begin{array}{lrl}
\text{maximize} & -x_1 + \ \ x_2 & \\
\text{subject to} & -x_1 + \ \ x_2 & \leq 5 \\
& x_1 - 2x_2 & \leq 9 \\
& 0 \leq x_1 & \leq 6 \\
& 0 \leq x_2 & \leq 8.
\end{array}
$$

9.2

$$
\begin{array}{lrl}
\text{maximize} & -3x_1 - x_2 + \ \ x_3 + 2x_4 - \ \ x_5 + \ \ x_6 - \ \ x_7 - 4x_8 & \\
\text{subject to} & x_1 \qquad + 4x_3 + \ \ x_4 - 5x_5 - 2x_6 + 3x_7 - 6x_8 & = \ \ 7 \\
& x_2 - 3x_3 - \ \ x_4 + 4x_5 + \ \ x_6 - 2x_7 + 5x_8 & = -3 \\
& 0 \leq x_1 \leq & \ \ 8 \\
& 0 \leq x_2 \leq & \ \ 6 \\
& 0 \leq x_3 \leq & 10 \\
& 0 \leq x_4 \leq & 15 \\
& 0 \leq x_5 \leq & \ \ 2 \\
& 0 \leq x_6 \leq & 10 \\
& 0 \leq x_7 \leq & \ \ 4 \\
& 0 \leq x_8 \leq & \ \ 3.
\end{array}
$$

Notes

Dantzig (1955) was the first to consider variants of the simplex method that handle bounds and ranges implicitly.