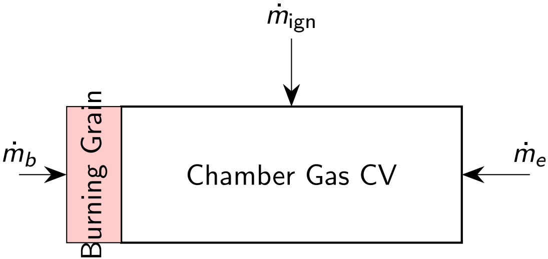
Zero-Dimensional Solid Rocket Motor Internal Ballistics

Mass, Energy, and Pressure Dynamics for AP/HTPB Composite Propellants

Prepared for: Graduate Propulsion Systems Design

Control Volume and Basic Assumptions

- Control volume = gas inside the combustion chamber.
- Gas assumed perfectly mixed and spatially uniform.
- Ideal gas: $pV = mRT$.
- Sources:
  - Propellant–generated mass $\dot{m}_b$
  - Igniter mass $\dot{m}_{\text{ign}}$
  - Heat release from combustion $\dot{Q}_{\text{comb}}$
- Sinks:
  - Nozzle mass flow $\dot{m}_e$
  - Wall heat losses $\dot{Q}_{\text{wall}}$



Mass Balance

Total gas mass:

$$\frac{dm}{dt} = \dot{m}_b + \dot{m}_{\mathrm{ign}} - \dot{m}_e$$

Propellant mass generation:

$$\dot{m}_b = \rho_p \, r_b \, A_b(s)$$

Saint–Robert burn law:

$$r_b = f_{\mathrm{ign}}(t) \, a \, p^n$$

Ignition ramp:

$$f_{\mathrm{ign}}(t) = 1 - \exp\left(-\frac{t - t_{\mathrm{ign,start}}}{\tau_{\mathrm{ign}}}\right)$$

Choked nozzle mass flow:

$$\dot{m}_e = C_d A_t p \sqrt{\frac{\gamma}{RT}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Energy Balance with Heat Release

Internal energy:

$$U = mc_v T$$

Full unsteady first law:

$$\frac{d}{dt}(mc_v T) = \dot{m}_b h_b + \dot{m}_{\text{ign}} h_{\text{ign}} - \dot{m}_e h_e - p\frac{dV}{dt} + \dot{Q}_{\text{wall}} + \dot{Q}_{\text{comb}}$$

Combustion heat release:

$$\dot{Q}_{\text{comb}} = \dot{m}_b \Delta h_c$$

Total heat term:

$$\dot{Q} = \dot{Q}_{\text{wall}} + \dot{Q}_{\text{comb}} + \dot{Q}_{\text{ign}}.$$

Deriving $\frac{dp}{dt}$ (Step 1)

Start with:

$$U = mc_v T$$

$$\frac{d}{dt}(mc_v T) = \dot{m}_b h_b + \dot{m}_{\text{ign}} h_{\text{ign}} - \dot{m}_e h_e - p\frac{dV}{dt} + \dot{Q}$$

Assume:

$$h_b \approx h_e \approx c_p T, \qquad h_{\text{ign}} \approx c_p T_{\text{ign}}$$

Left side:

$$\frac{d}{dt}(mc_v T) = c_v \left(m\frac{dT}{dt} + T\frac{dm}{dt} \right)$$

Right side becomes:

$$c_p T\frac{dm}{dt} - p\frac{dV}{dt} + \dot{Q}$$

Use mass balance:

$$\frac{dm}{dt} = \dot{m}_b + \dot{m}_{\text{ign}} - \dot{m}_e$$

Deriving $\frac{dp}{dt}$ (Step 2)

Rearrange:

$$c_v m \frac{dT}{dt} = (c_p - c_v) T \frac{dm}{dt} - p \frac{dV}{dt} + \dot{Q}$$

Use $c_p - c_v = R$:

$$\frac{dT}{dt} = \frac{RT \frac{dm}{dt} - p \frac{dV}{dt} + \dot{Q}}{c_v m}$$

Differentiate ideal gas:

$$pV = mRT$$

$$\frac{dp}{dt} V + p \frac{dV}{dt} = R \left(T \frac{dm}{dt} + m \frac{dT}{dt} \right)$$

Substitute above dT/dt expression.

Deriving $\frac{dp}{dt}$ (Step 3)

Compute:

$$R\left(T\frac{dm}{dt} + m\frac{dT}{dt}\right) = RT\frac{dm}{dt} + \frac{R}{c_v}\left(RT\frac{dm}{dt} - p\frac{dV}{dt} + \dot{Q}\right)$$

Use $\gamma = \frac{c_p}{c_v}$ and $\frac{R}{c_v} = \gamma - 1$:

$$= \gamma RT\frac{dm}{dt} - (\gamma - 1)p\frac{dV}{dt} + (\gamma - 1)\dot{Q}$$

Insert into:

$$\frac{dp}{dt}V + p\frac{dV}{dt} = R\left(T\frac{dm}{dt} + m\frac{dT}{dt}\right)$$

Thus:

$$\frac{dp}{dt}V = \gamma RT\frac{dm}{dt} - \gamma p\frac{dV}{dt} + (\gamma - 1)\dot{Q}$$

Deriving $\frac{dp}{dt}$ (Final Form)

Divide by V:

$$\frac{dp}{dt} = \frac{\gamma R T}{V}\frac{dm}{dt} - \frac{\gamma p}{V}\frac{dV}{dt} + \frac{\gamma - 1}{V}\dot{Q}$$

Use ideal gas $pV = mRT$:

$$\frac{RT}{V} = \frac{p}{m}$$

Therefore:

$$\boxed{\frac{dp}{dt} = \gamma p\left[\frac{1}{m}\frac{dm}{dt} - \frac{1}{V}\frac{dV}{dt}\right] + \frac{\gamma - 1}{V}\dot{Q}}$$

This is the unsteady chamber pressure equation used in 0-D SRM modeling.

Volume Evolution and Web Regression

Web regression:

$$\frac{ds}{dt} = r_b = f_{\mathrm{ign}}(t)\, a\, p^n$$

Grain geometry tables:

$$A_b(s), \qquad V(s), \qquad \frac{dV}{ds}(s)$$

Chamber volume evolution:

$$\frac{dV}{dt} = \frac{dV}{ds}(s)\, \frac{ds}{dt}$$

Propellant Grain

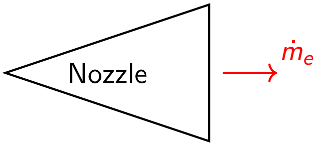
Nozzle Throat Erosion

Throat area:

$$A_t = \pi r_t^2$$

Erosion model:

$$\frac{dr_t}{dt} = k_{\text{eros}} |\dot{m}_e|$$



Final Pressure Equation (SRM Form)

With the SRM source terms:

$$\frac{dm}{dt} = \dot{m}_b + \dot{m}_{\mathrm{ign}} - \dot{m}_e, \qquad \frac{dV}{dt} = \frac{dV}{ds}(s)\frac{ds}{dt},$$

the chamber pressure ODE is:

$$\boxed{\frac{dp}{dt} = \gamma p \left[\frac{\dot{m}_b + \dot{m}_{\mathrm{ign}} - \dot{m}_e}{m} - \frac{1}{V}\frac{dV}{dt}\right] + \frac{\gamma - 1}{V}\dot{Q}}$$

Final ODE System

State vector:

$$y(t) = \begin{bmatrix} p(t) \\ m(t) \\ s(t) \\ r_t(t) \end{bmatrix}$$

ODEs:

$$\frac{dp}{dt} = \gamma p \left[\frac{\dot{m}_b + \dot{m}_{\text{ign}} - \dot{m}_e}{m} - \frac{1}{V}\frac{dV}{dt} \right] + \frac{\gamma - 1}{V}\dot{Q}$$

$$\frac{dm}{dt} = \dot{m}_b + \dot{m}_{\text{ign}} - \dot{m}_e$$

$$\frac{ds}{dt} = f_{\text{ign}}(t)\, a\, p^n$$

$$\frac{dr_t}{dt} = k_{\text{eros}} |\dot{m}_e|$$

Assumptions & Limitations of the 0-D SRM Model

extbfMajor assumptions:

- Chamber gas is **perfectly mixed** (no spatial gradients).
- Propellant gases instantaneously reach chamber temperature.
- Ideal-gas thermodynamics: $pV = mRT$.
- Quasi-steady Saint–Robert burn law: $r_b = ap^n$.
- Choked nozzle flow with fixed C_d.
- Lumped heat loss: $\dot{Q}_{\text{wall}} = -h_w A_w (T - T_w)$.
- Grain geometry encoded through tables $A_b(s)$ and $V(s)$.

extbfLimitations:

- Cannot predict combustion instabilities or pressure oscillations.
- No axial or transverse wave dynamics (1-D/3-D neglected).
- No local flame chemistry or finite-rate kinetics.
- No particle dynamics (Al droplets, slag accumulation).
- No two-phase flow in nozzle.
- No detailed heat transfer to insulation or case.

extitUseful for system-level design and grain optimization, but not for high-fidelity stability or transient ignition modeling.