Topology Question - Continuous $f : \mathbb{R}_{\text{co-countable}} \longrightarrow \mathbb{R}_{\text{usual}}$

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1 Question

Prove that there exist no continuous function $f : \mathbb{R}_{\text{co-countable}} \longrightarrow \mathbb{R}_{\text{usual}}$.

Recall:

- $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{usual}}$ are $\mathbb{R}$ with the co-countable $(\mathbb{R}, \mathbb{R}_{\text{co-countable}})$ and usual topologies $(\mathbb{R}, \mathbb{R}_{\text{usual}})$ respectively
- In $(\mathbb{R}, \mathbb{R}_{\text{co-countable}})$, $\mathbb{R}_{\text{co-countable}} \coloneqq \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable}\} \cup \{\emptyset\}$
- In $(\mathbb{R}, \mathbb{R}_{\text{usual}})$, $\mathbb{R}_{\text{usual}}$ is generated by the basis $\mathcal{B} \coloneqq \{(a, b) \mid a, b \in \mathbb{R},\ a < b\}$

2 Answer

Suppose there did exist a continuous function $f : \mathbb{R}_{\text{co-countable}} \longrightarrow \mathbb{R}_{\text{usual}}$. Therefore $\forall\ (a, b) \in \mathbb{R}$, we have by the continuity of f that $f^{-1}\big((a, b)\big) \in \mathbb{R}_{\text{co-countable}}$.

Let D_1 and D_2 be disjoint open intervals in $\mathbb{R}$ (and hence $D_1, D_2 \in \mathbb{R}_{\text{usual}}$).

Since f is continuous, $f^{-1}(D_1), f^{-1}(D_2) \in \mathbb{R}_{\text{co-countable}}$. Therefore by the definition of $\mathbb{R}_{\text{co-countable}}$, $\mathbb{R} \setminus f^{-1}(D_1)$ is countable, and $\mathbb{R} \setminus f^{-1}(D_2)$ is countable.

Also, note that $f^{-1}(\mathbb{R} \setminus D_1) = f^{-1}(\mathbb{R}) \setminus f^{-1}(D_1)$, and $f^{-1}(\mathbb{R} \setminus D_2) = f^{-1}(\mathbb{R}) \setminus f^{-1}(D_2)$.

Also, $D_1 \subseteq \mathbb{R} \setminus D_2 \Rightarrow f^{-1}(D_1) \subseteq f^{-1}(\mathbb{R} \setminus D_2)$. Given $f^{-1}(\mathbb{R} \setminus D_2)$ is countable $\Rightarrow f^{-1}(D_1)$ is countable.

However, we also know that $\mathbb{R} = f^{-1}(D_1) \cup \big(\mathbb{R} \setminus f^{-1}(D_1)\big)$. Therefore $\mathbb{R}$ is a union of two countable sets $\Rightarrow \mathbb{R}$ is countable. This is obviously a contradiction. Therefore there exists no continuous function $f : \mathbb{R}_{\text{co-countable}} \longrightarrow \mathbb{R}_{\text{usual}}$.

3 Supplemental Material

This is not the most difficult problem in topology I came across, however I think it is nice because of the contradiction that is ultimately arrived at, that $\mathbb{R}$ is countable.

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