Central Equilibrium Show Notes

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July 2018

1 Introduction

1.1 Professional History

Bachelors degree in Commerce from Queen's University. Courses in finance, math, statistics, economics.

Interned at BAML 3rd year summer, did a rotational program on trading floor. Joined full-time after school, spent time as repo trader (very short term fixed income) and as Canadian government bond trader.

Returned to university 2017. 1yr of filling in math background, applied to masters programs, now doing my masters statistics graduating spring 2019 at UofT. Always had interest in math/stats which I wanted to pursue further, plus I saw the industry I was in and the world around me being changed by things like 'big data' and 'machine learning', wanted to learn more about those topics.

1.2 Objective

Teach you about the different types of numbers there are, with a particular focus on the rational numbers and irrational numbers. Hope that you can see 'what is a number' in a new way

2 Types of Numbers

There are numerous 'sets' of numbers. What it boils down to is different ways of defining 'what is a number'. Usually when given a problem, the problem specifies what set of numbers certain numbers come from (e.g. $\pi^2 = 2x$ find $x, x \in \mathbb{R}$). Some of the most used/important sets are:

Natural numbers - $\mathbb{N}$ - {1,2,3,4,....}, positive counting numbers (some people include 0 some do not).

Integers - $\mathbb{Z}$ - {...,-3,-2,-1,0,1,2,3,...}, postitive and negative counting numbers.

Rationals - $\mathbb{Q}$ - $\{1,\frac{1}{2},\frac{-5}{12}$,100,...} - numbers expressed as an irreducible fraction ($\frac{p}{q}$ the ratio of two integers with denominator non-zero). These numbers have finite, or repeating/periodic decimal representations.

Irrational - $\mathbb{I}$ - Numbers that cannot be represented as a fraction of two integers $\frac{p}{q}$. These numbers have infinitie non-repeating non-periodic decimal representations.

Reals - $\mathbb{R}$ - All rational and irrational numbers.

Complex - $\mathbb{C}$ - real numbers, imaginary numbers, and sums/differences between them.

Focus of today will mostly be on rational vs. irrational numbers.

3 $\sqrt{2}$ is Irrational

When I returned to school this year this was one of the proofs that jumped out at me. Basically this will be the first example you see in a class on real analysis.

We will use a technique called proof by contradiction. Basically we will assume something is true, and then eventually reach a point where we create a contradiction; this means that what we initially assumed was true is actually false. It is almost like saying you have an animal is either a cat or not a cat, you assume it is a cat but then you make it bark, you then conclude your assumption was false and that the animal is not a cat.

3.1 Proof By Contradiction

Assume $a = \sqrt{2}$ and a is a rational number (can be represented by $a = \frac{p}{q}$, where p and q are in lowest terms).

It then follows that

$$\begin{aligned} a^2 = \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2 \end{aligned} \tag{1}$$

So we see from this that some number squared is an even number. Therefore we can conclude that p is an even number (since if a number multiplied by itself is even the number itself must be even). So p is two times some other number, let's say $p = 2k$. Going back into our equation:

$$\begin{aligned} p^2 &= 2q^2 \\ (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \end{aligned} \tag{2}$$

Using the same logic as before we see q^2 is even, therefore q itself is even.

This is a contradiction, remember that rational numbers are those expressed as an irreducible fraction. But we assumed $\sqrt{2} = \frac{p}{q}$ where p, q in lowest terms, but we just discovered that p, q are both even therefore not in lowest terms...a contradiction! Therefore we conclude $\sqrt{2}$ is not a rational number.

4 Bijections

A concept that will be useful in our discussion on countability.

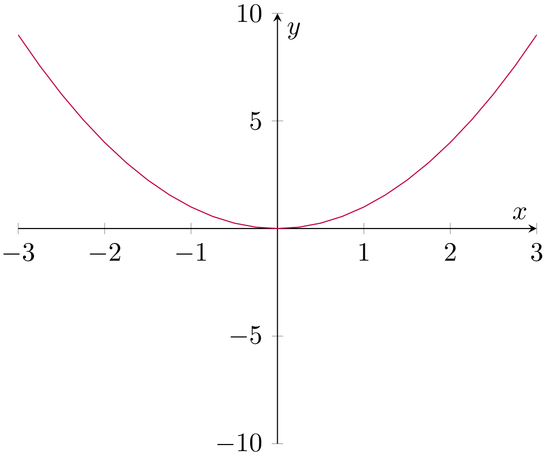
A bijection is function between elements of two sets (call them X, Y) where each element of X is paired with exactly one element of Y and vice-versa.

In more rigorous terms a function $f : X \to Y$ is **bijective** if it is injective and surjective.

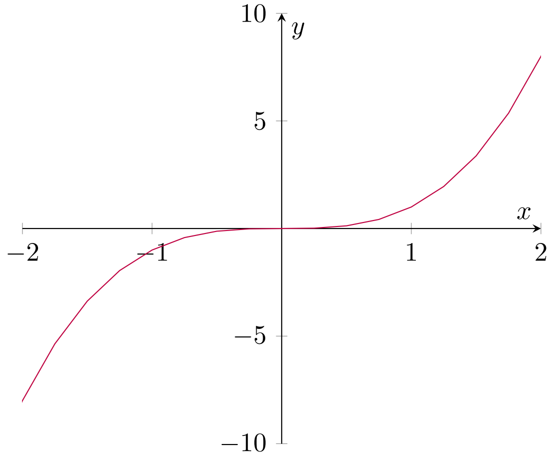
A function is **injective** (one-to-one) if $f(x) = f(y) \Rightarrow x = y$, or $x \neq y \Rightarrow f(x) \neq f(y)$.

A function is **surjective** (onto) if $\forall\, y \in Y\ \exists\, x \in X$ such that $f(x) = y$.

$y = x^2$, $x \in \mathbb{R}$, not bijective
(not injective or surjective)



$y = x^3$, $x \in \mathbb{R}$, is bijective
(both injective and surjective)



5 Countability

A set, call it A, is defined to be countable if there is a bijection that exists from $\mathbb{N}$ to A such that $f(k) = a_k$ with $k \in \mathbb{N}, a_k \in A$. Equivalently A is countable if the elements of A can be listed as $a_1, a_2, a_3, \ldots$. It is worth noting that a set being countable does not mean the set has a finite number of elements. For example N is countable but there are infinitely many elements in the set.

5.1 Proof By Construction That The Rational Numbers Are Countable

In this proof we will construct a bijection $f : \mathbb{N} \to \mathbb{Q}$, therefore demonstrating $\mathbb{Q}$ is countable.

Arrange the rational numbers as follows (assume $\frac{p}{q}$ in lowest terms):

$$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \cancel{\frac{2}{2}}, \frac{-1}{2}, \cancel{\frac{-2}{2}}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \cancel{\frac{3}{3}}, \frac{-1}{3}, \frac{-2}{3}, \cancel{\frac{-3}{3}}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \frac{1}{4}, \cancel{\frac{2}{4}}, \frac{3}{4}, \cancel{\frac{4}{4}}, \frac{-1}{4}, \cancel{\frac{-2}{4}}, \frac{-3}{4}, \cancel{\frac{-4}{4}}, \frac{4}{1}, \cancel{\frac{4}{2}}, \frac{4}{3}, \frac{-4}{1}, \cancel{\frac{-4}{2}}, \frac{-4}{3}, \ldots, \frac{p}{q}, \cdots$$

You are left with:

$$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{4}{1}, \frac{4}{3}, \frac{-4}{1}, \frac{-4}{3}, \ldots, \frac{p}{q}, \cdots$$

Obviously every rational number will appear in this list at some point, therefore it is possible to setup a bijection between every rational number and its list position (i.e. the natural numbers):

N	1	2	3	4	5	6	7	8	...	n	...
$f(\mathbb{N})$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{1}{3}$	...	$\frac{p}{q}$	...

6 Book Recommendation to Audience

Check out the book Prediction Machines, written by three Rotman professors. I recently read it and it gives a pretty good overview of what some of the potential effects on the economy the recent advent of artificial intelligence/machine learning will have (e.g. when the cost of prediction falls dramatically maybe it will make more sense to sell people things before they even know they want it). Clear writing style, short, highly recommend.

Also check out my website at daveveitch.wordpress.com