Probability Question - Discontinuities of a Distribution Function

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January 2020 (Updated Nov 2021)

1 Question

Show that a distribution function has at most countably many discontinuities.

2 Answer

Let F be an arbitrary distribution function and $\mathcal{D} = \{x \mid \exists \text{ a discontinuity at } F(x)\}$. Define $F(x-) = \lim_{y \nearrow x} F(y)$ and $F(x+) = \lim_{y \searrow x} F(y)$

Since F is a distribution function it is non-decreasing. For an arbitrary $x_1 \in \mathcal{D}$, since a discontinuity exists at $F(x_1)$, we know that $F(x_1-) < F(x_1+)$. Let A_1 be the interval $\big(F(x_1-), F(x_1+)\big)$. Notice that A_1 is an open interval in $[0, 1]$ therefore it must contain a rational number q_1 (see Section 3 for a proof of this).

Therefore for any discontinuity x_i there $\exists$ an open interval A_i, which is disjoint from all $A_j, j \neq i$, which contains a rational number q_i. Therefore we can create a one-to-one function $f : \mathcal{D} \to \mathbb{Q} \Rightarrow |\mathcal{D}| \leq |\mathbb{Q}| \Rightarrow \mathcal{D}$ is countable $\Rightarrow$ a distribution function has at most countably many discontinuities.

3 Proof - In Any Interval (a, b) where $a < b$ There Exists a Rational Number in (a, b)

This is true because if $a_0.a_1a_2a_3\ldots$ is the infinite decimal expansion of a, and $b_0.b_1b_2b_3\ldots$ is the infinite decimal expansion of b there will exist some smallest index $i \in \mathbb{Z}^+ \cup \{0\}$ such that $10^{-(i)} \geq b - a \geq 10^{-(i+1)} > 0$.

$\Rightarrow a_0.a_1\ldots a_ia_{i+1}a_{i+2} + 10^{-(i+2)} \in \mathbb{Q}$ and $a_0.a_1\ldots a_ia_{i+1}a_{i+2} + 10^{-(i+2)} \in (a, b)$.

4 Supplemental Material

This question is Exercise 1.2.3. in the book Probability: Theory and Examples by Rick Durrett.

I like this problem for its simplicity, but also because the result is very useful. Over the Graduate Probability course I took in the Fall of 2019 it came up again and again.