Fourier Series Question - $\zeta(4)$

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1 Question

Evaluate $\zeta(4)$ (where ζ is the Riemann Zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s > 1$) using the Fourier Series of x^2 for $-\pi < x < \pi$ and Parseval's Identity.

2 Answer

We first notice that $f(x) = x^2$ is an even function on $[-\pi, \pi]$ therefore its Fourier Series is given by a Fourier Cosine Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\pi}) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{1}$$

The coefficients for the Fourier Cosine Series, a_0 and a_n are given by:

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^\pi x^2 dx = \frac{x^3}{3\pi}\Big|_0^\pi = \frac{\pi^2}{3} \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx \\
a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx \quad \text{(Integrate By Parts)} \\
u = x^2 \quad du = 2x \quad v &= \frac{1}{n}\sin(nx) \quad dv = \cos(nx) \\
a_n &= \frac{2}{\pi}\Bigg(\Big[\frac{x^2}{n}\sin(nx)\Big]\Big|_0^\pi - \int_0^\pi \frac{2x}{n}\sin(nx)\Bigg) \\
a_n &= \frac{2}{\pi}\Bigg(0 - \int_0^\pi \frac{2x}{n}\sin(nx)\Bigg) \\
a_n &= -\frac{4}{\pi n}\Bigg(\int_0^\pi x\sin(nx)\Bigg) \quad \text{(Integrate By Parts)} \\
u = x \quad du = 1 \quad v &= -\frac{1}{n}\cos(nx) \quad dv = \sin(nx) \\
a_n &= -\frac{4}{\pi n}\Bigg(\Big[-\frac{x}{n}\cos(nx)\Big]\Big|_0^\pi + \int_0^\pi \frac{1}{n}\cos(nx)\Bigg) \\
a_n &= -\frac{4}{\pi n}\Bigg(-\frac{\pi}{n}\cos(\pi n) + \Big[\frac{1}{n^2}\sin(nx)\Big]\Big|_0^\pi\Bigg) \\
a_n &= -\frac{4}{\pi n}\Bigg(-\frac{\pi}{n}\cos(\pi n) + 0\Bigg) \\
a_n &= \frac{4}{n^2}\cos(\pi n) \\
a_n &= \frac{4(-1)^n}{n^2} \quad \text{(using fact } \cos(\pi n) = (-1)^n\text{)}
\end{aligned} \tag{2}$$

We can then write the Fourier Series as:

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \quad x \in [-\pi, \pi] \tag{3}$$

Since $f(x) = x^2$ and $f'(x) = 2x$ are piecewise continuous on $[-\pi, \pi]$ we can use Parseval's Equality:

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{L} f(x)^2 dx &= 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\
\frac{1}{\pi}\int_{-\pi}^{\pi} (x^2)^2 dx &= 2(\frac{\pi^2}{3})^2 + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 \\
\frac{x^5}{5\pi}\Big|_{-\pi}^{\pi} &= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16(-1)^{2n}}{n^4} \\
\frac{2\pi^4}{5} - \frac{2\pi^4}{9} &= \sum_{n=1}^{\infty}\frac{16}{n^4} \quad (\text{since } (-1)^{2n} = 1) \\
\frac{\pi^4}{90} &= \sum_{n=1}^{\infty}\frac{1}{n^4}
\end{aligned} \tag{4}$$

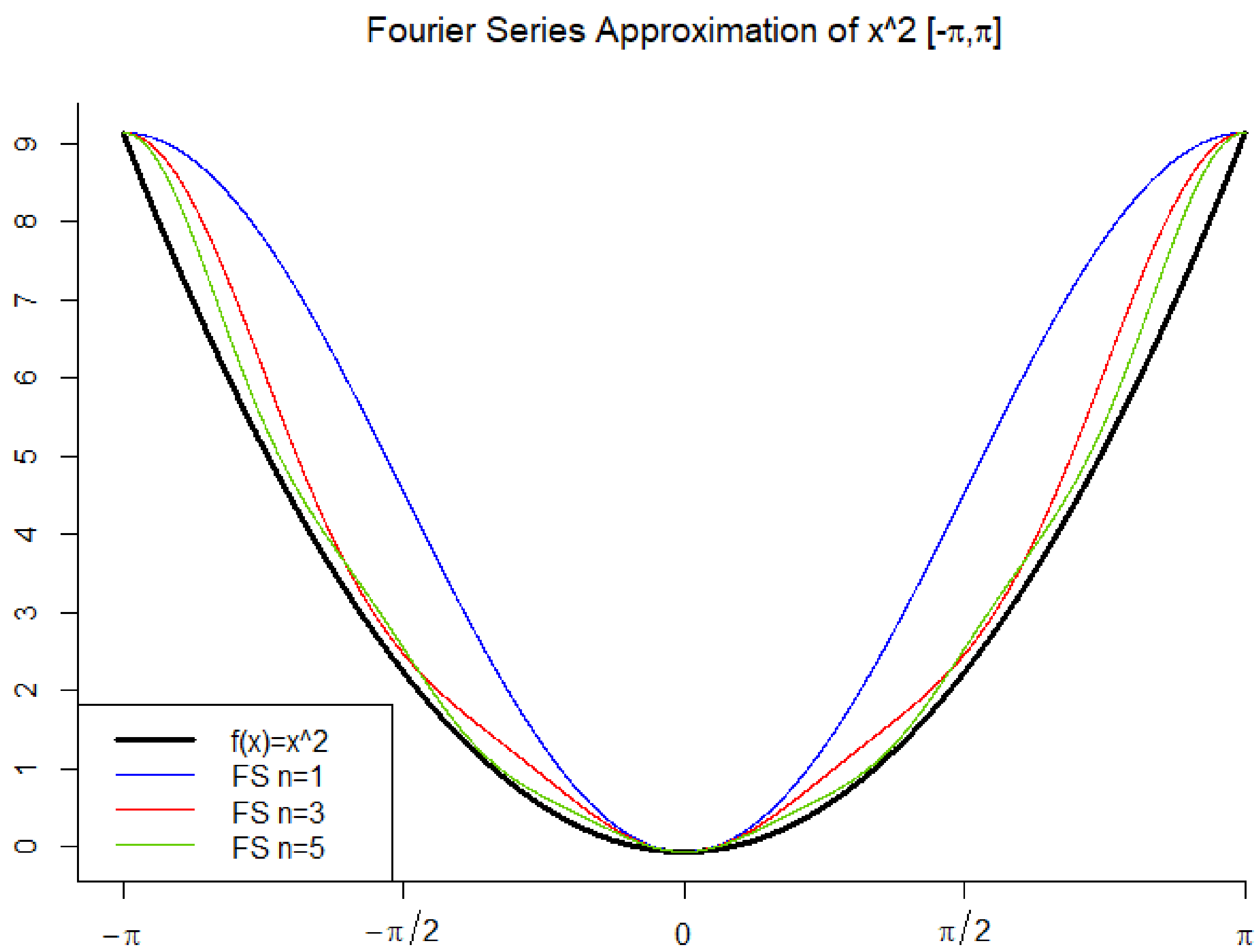
Therefore we conclude that

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \tag{5}$$

3 Supplemental Material

Wow what a result! There is something extraordinary about finding the value of an equation that never terminates. The Riemann Zeta function can be evaluated at other values as well (for example $\zeta(2) = \frac{\pi^2}{6}$).

Graph of the Fourier Series of x^2 for $x \in [-\pi, \pi]$:



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