Notes on the Global Expansion
An integral formalism

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1 Integral formulation of the isotropic global expansion

Let $t \mapsto \gamma_t$ be a family of geodesics $\gamma_t : \mathbb{R} \to \Sigma_t$ with spacelike hypersurface Σ_t for $t = \text{const}$. Its length is

$$L(t) := \int_{-\infty}^{\infty} \left|\gamma_t'(\tau)\right|_{h(t)} d\tau = \int_{-\infty}^{\infty} \left|h(t)\big(\gamma_t'(\tau), \gamma_t'(\tau)\big)\right| d\tau,$$

which is generally covariant and gauge invariant (Lee, 2018, Prop. 2.47). The *global expansion* of an isotropic cosmology is

$$\mathcal{E}(t) = \frac{d}{dt} L(t).$$

Here the 3-metric on Σ_t is $h(t)$ such that we can decompose the four-metric as

$$ds^2 = -dt^2 + h_{ij}(t) dx^i dx^j.$$

By Sciama et al. (1969), we may write

$$h^{ij}(t) = 2\kappa \int_{\Omega} G^{-ij\nu}{}_{\mu} \big(T^{\mu}{}_{\nu} - \frac{1}{2} T^{\lambda}{}_{\lambda} \delta^{\mu}{}_{\nu}\big) \sqrt{|\det h(t)|} d^4x + \int_{\partial\Omega} G^{-ij\nu}{}_{\mu}{}^{;\sigma} \sqrt{|\det h|} dS_{\sigma}.$$

In general, may decompose the energy-momentum tensor by

$$T_{ab} = \rho u_a u_b + p P_{ab} + 2 q_{(a} u_{b)} + \pi_{ab}$$

with $P_{ab} = g_{ab} + u_a u_b$ the projection tensor and ρ the rest mass density (Tsagas et al., 2008).

References

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