Notes on gauge dependence of backreaction

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Abstract. The theory of gauges on spacetimes is formalized by introducing mathematical frame field, which is a generalization of Einstein's tetrad fields. The frame bundle admits choices of gauge and gauge transformations as it is a principal bundle. Relating the above, we show the following:

(i) Any coordinate system and local Lorentz frame is a gauge, but a gauge need not be a choice of coordinate system nor a local Lorentz frame.

(ii) A tetrad field determines a unique metric tensor on the spacetime manifold.

(iii) The Schwarzschild and Kerr black hole can be constructed on manifold $\mathbb{R}\times(0,\infty)\times S^2$ by fixing an appropriate gauge.

Contents

1 Frame fields

A tetrad is an orthonormal basis $\{\hat{e}_{(0)},\dots,\hat{e}_{(3)}\}$, denoted by $\hat{e}_{(a)}$, of the tangent space T_pM at some point $p\in M$. For any coordinate basis $\hat{e}_{(\mu)}$ at $p\in M$, we may write

$$\hat{e}_{(\mu)}(p) = e_\mu{}^a(p)\hat{e}_{(a)},$$

where the components $e_\mu{}^a$ form an 4×4 invertible matrix. It follows that one can construct a section $e_\mu{}^a : M\to TM$, which is called the *tetrad* or *frame field*, assigning a tetrad to every point in spacetime M. The metric tensor is expressible through

$$g_{\mu\nu}(p) = e_\mu{}^a(p)e_\nu{}^b(p)\eta_{ab},$$

in terms of the frame field and the Minkowski metric η_{ab}.

Remark 1.1 (Tetrads and coordinate systems). Any coordinate system (x^a) can be identified to a tetrad field, since

$$\left.\frac{\partial}{\partial x^a}\right|_p, \qquad \text{for } a=0,1,2,3, \tag{1}$$

determine a basis for T_pM for any $p\in M$. However, the converse is not true. A tetrad (λ_a) need not be equivalent coordinate system. In particular, we would need

$$\lambda_a = \frac{\partial}{\partial y^a}, \qquad \text{for } a=0,1,2,3. \tag{2}$$

Hence, (2) would need to be integrated to find a suitable coordinate system, which cannot be done in general. So, one cannot always choose a corresponding coordinate system for any tetrad field [1, Sec. 4.2].

Remark 1.2 (Tetrads and metrics). A tetrad field determines a unique metric tensor, while in general one cannot determine the tetrad field if a metric is given. This is shown by Einstein [2] as he states: "*die Metrik ist gemäß $g_{\mu\nu} = h_{\mu a} h_{\nu a} \cdots$, durch das n-Bein-Field, aber nicht umgekehrt letzteres durch erstere bestimmt.*" (See the Appendix of [3].)

In the subsequent section, we consider frame fields for which we can derive a similar statement as in Remark 1.2, because the Gram-Schmidt procedure allows us to construct an orthonormal basis in any frame.

2 Gauge theory for relativity

Let LM be the set of all (p, e_a) with e_a any basis of tangent space T_pM. By [4, p. 32], the frame bundle $L(M) = (LM, M, \pi; GL(m))$ is a principal bundle, where $\pi : LM \to M$ is the natural projection $(p, e_a) \mapsto p$. Following [5, Def 4.2.18], we define a gauge on $L(M)$.

Definition 2.1. A *gauge* or *frame field* for the principal bundle is a global section $s : M \to LM$, i.e. a map assigning a tetrad in tangent space TM to every event in spacetime manifold M. A *local gauge* or *local frame field*[1] is defined similarly, by taking a local section $s : U \subset M \to LM$.

Consequently, we can define what a change of gauge is.

Definition 2.2. A *gauge transformation* is a diffeomorphism $\phi : LM \to LM$ such that:

(i) it preserves the fibres, i.e. $\pi \circ \phi = \pi$;

(ii) it is $GL(4)$-equivariant, i.e. $\phi(p \cdot g) = \phi(p) \cdot g$ for any frame $p \in LM$ and any $g \in GL(4)$.

We denote the space of all gauge transformations by $\mathcal{G}$.

Since the composition of $GL(4)$-equivariant maps is itself $GL(4)$-equivariant and the fibres are preserved under a composition of fibre preserving maps, $\mathcal{G}$ is a group under composition.

Objectives: how does metric g changes under a gauge transformation? Give example of Schwarzschield-metric, Kerr-black hole, etc.

Example 2.1 (Local Lorentz frame is a gauge). If an observer in spacetime (M, g) is freely falling, i.e. following a geodesic, his reference frame is a local Lorentz frame. Considering an event $p \in M$ in the spacetime, any local Lorentz frame at p can be locally described as a coordinate system [6, Sec. 8.6]. By construction, it follows that any local Lorentz frame is just a gauge.

3 Schwarzschild black hole

We consider the spacetime manifold[2] $M = \mathbb{R} \times (E^3 - O) \cong \mathbb{R} \times (0, \infty) \times S^2$, where O is the the center of a spherical object of mass M and radius r with no electric charge nor angular momentum. Schwarzschild [7] derives an exact solution of the Einstein field equations with a vanishing cosmological constant. The Schwarzschild metric in Schwarzschild coordinates is

$$g(t, r, \theta, \phi) = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{3}$$

describing the curvature of spacetime outside the spherical mass.

[1] See [4, p. 33].

[2] We take the Minkowski metric of flat spacetime to be $\eta = \mathrm{diag}(-1, +1, +1, +1)$.

Lemma 3.1. The Schwarzschild metric is obtainable by a suitable choice of gauge.

Proof. Take any coordinate system (x^i), which induces a basis $\{\frac{\partial}{\partial x^i}|_p\}$ for each point p in chart neighborhood U. Then, any orthonormal frame field $e_{(a)} \in \Gamma(LM)$ can locally be expressed at an arbitrary point $p \in U$ by

$$e_{(a)}(p) = e^{\mu}{}_{a}(p)\frac{\partial}{\partial x^i}\Big|_p, \tag{4}$$

for some map $e^{\mu}{}_{a} : M \to \mathbb{R}^{4\times 4}$ - called the *vierbein field* - assigning an invertible matrix to every point $p \in M$.

Recall that $e_{(a)}(p)$ is an orthonormal basis spanning the tangent space T_pM, i.e.

$$\big(e_{(a)}(p), e_{(b)}(p)\big) = g_{\mu\nu}(p)e^{\mu}{}_{a}(p)e^{\nu}{}_{b}(p) = \eta_{ab} \qquad \forall p \in M, \tag{5}$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric of flat spacetime. We are given the Schwarzschild metric (3), which characterized by $g_{\mu\nu} = 0$ for all $\mu, \nu = 0, 1, 2, 3$ such that $\mu \neq \nu$ and for the diagonal components,

$$g_{00} = -\left(1 - \frac{2M}{r}\right), \qquad g_{11} = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2(\theta).$$

Conveniently writing $g_{\mu\nu}(p) = g_{\mu\nu}$ and $e^{\mu}{}_{a}(p) = e^{\mu}{}_{a}$, the diagonal inner-product constraints (5) become

$$\begin{aligned}
-\left(1 - \frac{2M}{r}\right)(e^0{}_0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_0)^2 + r^2(e^2{}_0)^2 + r^2\sin^2(\theta)(e^3{}_0)^2 \quad &= -1, \\
-\left(1 - \frac{2M}{r}\right)(e^0{}_i)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_i)^2 + r^2(e^2{}_i)^2 + r^2\sin^2(\theta)(e^3{}_i)^2 \quad &= 1, \qquad \text{for } i = 1, 2, 3, \\
-\left(1 - \frac{2M}{r}\right)e^0{}_a e^0{}_b + \left(1 - \frac{2M}{r}\right)^{-1}e^1{}_a e^1{}_b + r^2 e^2{}_a e^2{}_b + r^2\sin^2(\theta)e^3{}_a e^3{}_b \quad &= 0, \qquad \text{for } a \neq b,
\end{aligned}$$

which are 16 equations with 16 unknowns $e^{\mu}{}_{a}$. One can readily verify that

$$e^0{}_0 = \frac{1}{\sqrt{1 - 2M/r}}, \quad e^1{}_1 = \sqrt{1 - 2M/r}, \quad e^2{}_2 = \frac{1}{r}, \quad e^3{}_3 = \frac{1}{r\sin(\theta)}, \quad e^{\mu}{}_{a} = 0, \tag{6}$$

for $\mu \neq a$, solves the system. Notice that it is *not* unique as the solution (7) holds as well if one takes $e^2{}_2 = -\frac{1}{r}$ instead. In vector notation, (7) is described by

$$e_{(0)} = \frac{1}{\sqrt{1 - 2M/r}}dt, \quad e_{(1)} = \sqrt{1 - 2M/r}dr, \quad e_{(2)} = \frac{1}{r}d\theta, \quad e_{(3)} = \frac{1}{r\sin(\theta)}d\phi, \tag{7}$$

as is in accordance with the metric tensor (3). Observe that, by the inner-product signature constraint (5), $\{e_{(a)}\}$ described in (7) is an orthonormal basis by construction. $\square$

Lemma 3.1, in short, tells us that by choosing an appropriate gauge $s : M \to LM$, e.g. setting $s(t, r, \theta, \phi) = e_{(a)}$ with $\{e_{(a)}\}$ defined as in (5), one constructs a Schwarzschild black hole in M and with it its corresponding metric.

Remark 3.1 (Question). How many degrees of freedom do we have in this solution? Note that the first column of matrix $(e^{\mu}{}_{a})$ cannot be "altered" with respect to solution described in (5).

4 Kerr black hole

We get one extra constraint one the $dtd\omega$ (I think).

References

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