Notes on gauge dependence of backreaction

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Contents

1 Gauge freedom in the Einstein equations

The Einstein field equations,

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}, \tag{1}$$

express the relation between the curvature of spacetime (represented by the metric tensor g_{ab}) and the distribution of energy and mass (represented by the energy-momentum tensor T_{ab}). Since $1 \leq a, b \leq 4$, (1) describes 16 equations. As the metric, the energy-momentum and the Ricci[1] tensor are symmetric, we see that the Einstein equations describe a system of 10 distinct equations [2, p. 74].

Remark 1.1 (Solving for the metric). Important to note is that if one wants to solve (1), one should solve for g_{ij} and T_{ij} simultaneously [1, Sec. 2.7]. However, given a smooth metric g_{ab}, there exists a smooth symmetric field T_{ab} satisfying the Einstein equations, where one has the possibility to define T_{ab} as the right hand-side of (1).

2 What is a gauge?

We describe what a gauge is in general relativity. We hope to provide two different approaches to describe gauge fixing: a physical and a mathematical approach.

2.1 Gauge fixing: physical approach

We do so with the help of an analogy.

Definition 2.1. A *coordinate system*, or respectively a *gauge*, is a measurement system satisfying the following conditions:

1. It transforms diffeomorphically under a coordinate transformation, or respectively a gauge transformation.

[1] Symmetry of the Ricci tensor R_{ab} is shown in e.g. [1, p. 84].

2. One can derive all previously derived physical quantities after a coordinate transformation, or respectively a gauge transformation.

One can verify that the metric tensor is a potential candidate for a gauge; it measures infinitesimal lengths, and thus is a measuring system, and it transforms diffeomorphically under a gauge transformation [2, p. 74]. Defining a gauge to be a choice of metric, gauge fixing is in line with the principle of General Covariance. This is because the principle states that the metric is the only quantity pertaining to space that can appear in physical laws [3] and a gauge transformation is - and nothing else - a redefinition of the spatial structure of spacetime [reference?] as a gauge transformation is a diffeomorphism $\phi : M \to M$. By exactly the previous argument, it follows from the principle of General Covariance that choosing a metric tensor must be equivalent to fixing a gauge.

Since we derived that choosing a gauge is equivalent to deciding on a metric, one might think that it would be equivalent to choosing any of the tensor fields R_{ab}, g_{ab} or T_{ab}. This because of the Einstein equations $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$, as one might postulate that from any of these fields one can determine the other two. We explain that this cannot be the case for (i) the Ricci tensor R_{ab} nor for (ii) the energy-momentum tensor T_{ab}.

(i) Important to note is that the Christoffel symbol $\Gamma^c{}_{ab}$ is fully determined by the metric g_{ab} as according to Wald [3, p. 36] "we can compute $\Gamma^c{}_{ab}$ (and thence ∇_a) by taking partial derivatives of the coordinate basis components of the metric." It follows that the Riemann tensor,see [4, p. 76] for this expression,

$$R^d{}_{abc} = \partial_b \Gamma^d_{ca} - \partial_c \Gamma^d_{ba} + \Gamma^d_{be}\Gamma^e_{ca} - \Gamma^d_{ce}\Gamma^e_{ba},$$

is completely determined by the metric. By definition of the Ricci tensor,$R_{ab} = R^c{}_{abc}$ for which we, therefore, conclude the same result. In general, one cannot solve for the metric given the Ricci tensor.

(ii) According to [1, Sec. 2.7], in general, "one cannot specify the energy-momentum distribution in the absence of a spacetime metric." Especially, one cannot derive the spacetime metric from the energy-momentum.

It follows that from neither R_{ab} nor T_{ab} one can solve for g_{ab}. However, as showed in (i), one can compute R_{ab} given g_{ab} and by Remark 1.1 one can derive an energy-tensor field T_{ab} if a metric is provided.

2.2 Gauge fixing: mathematical approach

[5, Sec. 1.5] explains that Einstein's theory of general relativity is not a gauge theory. Since GR is the standard in cosmology, we postulate that adopting a gauge theory of gravitation in order to introduce gauges and gauge transformations would be cumbersome for our purposes. Additionally, gauges[2] and gauge transformations[3] are defined on principal bundles. Consequently, we formulate the group of gauge transformations by providing a principal bundle such that this group is the collection of transformations under which the laws of GR should be invariant.

By [8, p. 32], the frame bundle $L(M) = (LM, M, \tau; GL(m))$ is a principal bundle. Following [6, Def 4.2.18], we define a gauge on $L(M)$.

Definition 2.2. A *gauge* for the principal bundle is a global section $s : M \to L(M)$, i.e. a map assigning a tetrad in tangent space TM to every event in spacetime manifold M. A *local gauge* is defined similarly, by taking a local section $s : U \subset M \to L(M)$.

Since local gauge transformations are just coordinate system changes, we see that such transformations are just a special case of gauge transformations.

[2] See e.g. Definition 4.2.18 of [6].

[3] Refer to [6, Sec. 5.3] and [7, Sec. 10.14].

According to [7, Sec. 10.14], there is a group homomorphism $h : \mathrm{Aut}(P) \to \mathrm{Diff}(M)$, where $\mathrm{Aut}(P)$ are all the G-equivariant diffeomorphisms mapping P to itself. Since $\mathrm{Aut}(P) \cong \mathrm{Diff}(M)$, any transformation of tetrad frames can be identified with a global diffeomorphism on M. Since $\mathrm{Aut}(G)$ is the group of gauge transformations, we indeed see that it makes sense to think of a $\phi \in \mathrm{Diff}(M)$ as a gauge transformation. I am left with filling in the details as there are certain conditions on such transformations; see [6, Def. 5.3.1].

Alternatively, I suppose that we can approach this from natural bundles. [8, Sec. 7.1] motivate why it follows from the general covariance principle that GR can be understood as a natural field theory. The principal bundle $\mathrm{Lor}(M) = \text{Metrics}(M; 1, m-1) = L(M) \times_\lambda M(1, m-1)$ can be taken as the configuration bundle as it is a natural bundle [8, p. 124], which relates to the fact that GR is a purely metric theory, i.e. the Levi-Civita connection can be derived from the metric tensor [8, Sec 7.5]. By definition of this natural field formalism, the set of the Lagrangian symmetries is $\mathrm{Diff}(M)$. It would be useful if we could prove $\mathrm{Aut}(\mathrm{Lor}(M)) \cong \mathrm{Diff}(M)$; it would imply that a gauge transformation is equivalent to a change in a metric tensor on M.

Remark 2.1 (Order of the bundle morphism). Within the natural field formalism of GR, one can choose - by definition - a bundle morphism

$$\Gamma : J^h \,\mathrm{Lor}(M) \to L(M)/GL(4),$$

where $L(M)$ is the frame bundle, allowing to define a linear connection on M out of fields and their derivatives up to a finite order h. An interesting direction for further research might be trying to understand whether the order h is in relation with the highest order derivatives of the metric tensor appear in the Einstein field equations.

Remark 2.2 (Imposed topological restrictions of existence of global metrics). [8, p. 124] state that the existence of global metric of signatures, which or not of the form $(m, 0)$ nor $(0, m)$, impose topological restrictions on the spacetime manifold M.

2.3 Example of coordinate and gauge transformations

General gauge transformation.

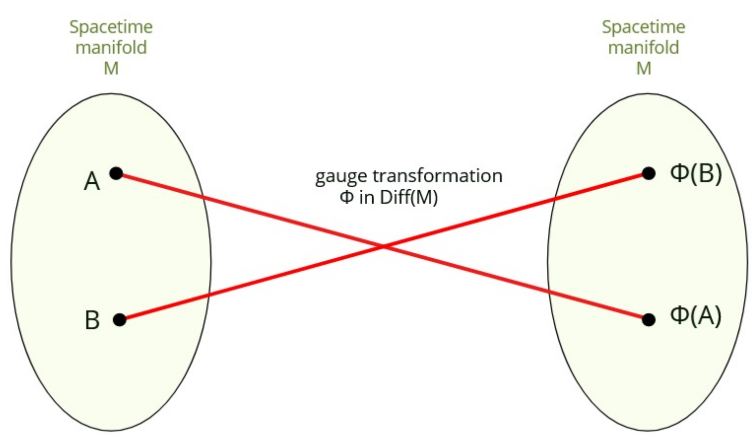

Figure 1: Intuitive depiction of a gauge transformation.

Consider Fig. 2. Take a gauge transformation $\phi : M \to M$ mapping the black dot to the red one. Then, locally, this is equivalent to transforming from the black coordinate system to the red one. We see that a gauge transformation is locally interpretable as a coordinate transformation.

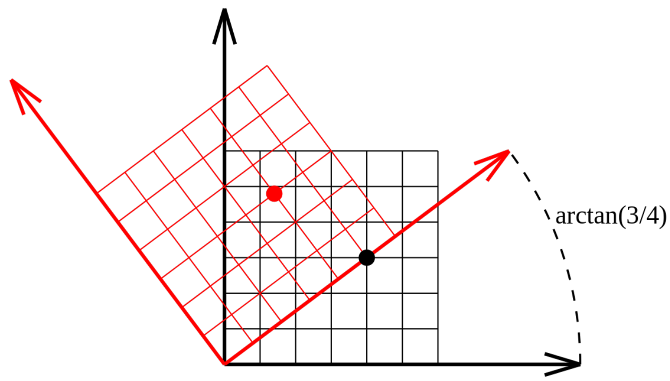

Figure 2: Intuitive depiction of a gauge transformation interpretable as a coordinate transformation.

3 Energy and momentum constraints

We analyze the energy and momentum constraints implied by the Einstein field equations. We postulate that these constraints are solely constraints on the 3-metric when splitting spacetime into spacelike hypersurfaces of constant time, i.e. the ADM formalism.

[ref?] proves that the energy-momentum tensor Tab is divergence-free, i.e.

$$\nabla^a T_{ab} = 0. \tag{2}$$

These four equations can be derived from the fact that the Riemann tensor satisfies the Bianchi identities, see [9, Sec. 1.9].

Baez and Muniain [10, p. 388] state that "T^{00} represents the energy density, the components T^{a0} represent the flow of energy in the ∂_a direction, T^{0b} represents the density of the b-th component of momentum, and T^{ab} for $a \neq 0$" represents the flow of the b-th component of the momentum in the ∂_b direction", therefore (2) describes the conservation of energy if $b = 0$ and for $b \neq 0$ the local momentum conversation.[4]

3.1 Ideas

The 2nd equation under Exercise 45 on p. 423, we see that $G_{00} =$ some scalar. Is this is relation with $\nabla_a G_{a0} = 0$? Maybe we can prove

$$G_{00} = \text{some scalar} \implies \nabla_a G_{a0} = 0 \implies \nabla_a T_{a0} = 0,$$

which is the energy condition. This way we would see that the Einstein equation implies the energy constraint without using the Bianchi idenity of the Riemann tensor. Can we do the same for the momemtum constraints? Can we relate this to freedom of choosing coordinates / gauge?

3.2 Constraints on the 3-metric and extinsic curvature

Four constraints $G^0_b = 8\pi\kappa T^0_b$ for $b = 1, \ldots, 4$ can be derived to imply the following.

$$G^0_0 = -\frac{1}{2}\left({}^3\mathrm{R} + (\operatorname{tr} K)^2 - \operatorname{tr}(K^2)\right), \tag{3}$$

[4] Refer to Wald [3, p. 69] or Malament [1, Sec. 2.5] for a more detailed interpretation of $\nabla^a T_{ab} = 0$.

so $G^0_0 = 8\pi\kappa T^0_0$ is a constraint relating the extrinsic curvature K of any spacelike slice to its scalar curvature ${}^3\mathrm{R}$.

The three Einstein equations $G^0_i = 8\pi\kappa T^0_i$ for $i = 1, 2, 3$ gives

$$G^0_i = {}^3\nabla_j K^j_i - {}^3\nabla_i K^j_j. \tag{4}$$

Remark 3.1 (Lowering and raising indicies). Recall that for a tangent vector v^a at some point $p \in M$, we write

$$v_b = g_{ab} v^a.$$

So, $G_{a0} = g_{c0} G^0_a$. Hence,

$$\nabla_a G_{a0} = \nabla_a g_{c0} G^0_c = g_{c0} \nabla_a G^0_c,$$

where the second step follows as the metric-compatible covariant derivative ∇ commutes with the raising and lowering of indices [4, p. 59]. Since

$$\nabla_0 G_{00} = \nabla_0 g_{c0} G^0_c = g_{c0} \nabla_0 G^0_c$$

1. Check out [11, p. 153]
2. Is it possible that: tangent space of divergence-free velocity fields [12] has some relation to the fact that G_{ab} and T_{ab} are divergence-free?

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