Notes on gauges and backreaction

Dave Verweg

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Abstract

The most important of our current findings and its fundamental implications are being discussed here. We cover the current formalism of invariance - in particular, gauge invariance - in relativistic cosmology. By employing this formalism and a gauge invariant averaging procedure, we show explicitly how the cosmological backreaction effect is captured within the Weyl (or conformal) tensor and, therefore, is affecting the propagation of gravitational waves for an irrotational dust universe. We do so in a covariant and gauge covariant way such that the results hold for any gauge and, thus, in any reference frame.

Keywords: Relativistic cosmology, dark energy theory, cosmological backreaction, gauge invariance.

1. Introduction

Our description is general covariant and gauge covariant, and the corresponding quantities are gauge invariant.

2. Formalism of mathematical gauge theory for relativity

We follow the formalism of mathematical gauge theory to make gauge theory precise as how it occurs in relativistic cosmology. Throughout this paper, we consider (M, g) to be a four-dimensional Lorentzian spacetime manifold.

Definition 2.1. The *(orthonormal) frame bundle* of manifold M is the tuple

$$\big(LM, M, \pi, SO(4)\big),$$

denoted by $L(M)$ or $LM \xrightarrow{\pi} M$, where LM is the collection of all oriented orthonormal bases of T_pM at any $p \in M$.

Note that the frame bundle $L(M)$ is a principal $SO(4)$-bundle, which is convenient as we can define gauges on principal bundles as follows.

Definition 2.2. Let $P \xrightarrow{\pi} M$ be a principal bundle. A *local gauge* is a local section $s : U \subset M \to P$ for an open subset $U \subset M$. We call a local gauge a *gauge* if $U = M$.

Therefore, for the frame bundle $LM \xrightarrow{\pi} M$, a gauge $s : M \to LM$ is a function assigning an orthonormal basis of the tangent space T_pM, i.e. a *tetrad* or a *vierbien*, to every point $p \in M$. Having the definition of gauges in place, the notion of gauge transformations follows naturally.

Definition 2.3. Let $P \xrightarrow{\pi} M$ be a principal bundle. A *gauge transformation* is a transformation $\gamma : \Gamma(P) \to \Gamma(P)$ between gauges.

For the frame bundle $L(M)$, we denote the set of all gauge transformations by $\mathcal{G}$. One can readily verify that $\mathcal{G}$ is a group under composition. Similarly, one can see that the set of diffeomorphisms $\mathrm{Diff}(M)$ of M is a group.

Lemma 2.4. On the frame bundle $L(M)$, one has $\mathcal{G} \cong \mathrm{Diff}(M)$. DV

Proof. **Still busy trying to prove this result.** □

Lemma 2.4 shows us that we can identify any gauge transformation $\gamma \in \mathcal{G}$ uniquely with a diffeomorphism $\phi \in \mathrm{Diff}(M)$. This is in line with the practice by relativists to call $\mathrm{Diff}(M)$ the group of gauge transformations or, for short, the gauge group.

Example 2.5. The Schwarzschild and Kerr geometry describing the curvature of spacetime around a black hole are examples of gauges. (See Chapter 3 for the explicit derivation.)

3. Invariance and covariance in relativity

3.1. Invariance

Invariance, in general, means that quantities or objects do not change with respect to transformations [5].

Definition 3.1. Let G be a group with an operation $\cdot : G \times M \to M$. A tensor $\tau \in \mathcal{T}(M)$ is G-invariant if $\tau(g \cdot p) = \tau(p)$ for any $p \in M$.

If $G = \mathrm{Diff}(M)$ in Definition 3.1, one might speak of diffeomorphism invariance and gauge invariance if $G = \mathcal{G}$.

Email address: dave@verweg.net (Dave Verweg)

3.2. Covariance

Covariance is 'form invariance', in the sense that the form of a physical law is unchanged under a transformation [2]. In general, we consider two kinds: general covariance and gauge covariance.

Definition 3.2. Equations $\mathfrak{E}(D) = 0$ are called *(general) covariant* if they are covariant under coordinate transformations and *gauge covariant* if they are covariant under gauge transformations.

Recall that a *coordinate system* is the chart map $\varphi : U \subset M \to \mathbb{R}^4$ of some smooth chart (U, φ) on manifold M. Let (V, ψ) be a smooth chart overlapping with (U, φ). The *coordinate transformation* from coordinates φ to ψ is the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

Remark 3.3. Lemma 2.4 tells us that gauge invariance is equivalent to diffeomorphism invariance. Allowing us to use them interchangeably.

4. Averaging procedure

Discussion upon a gauge invariant averaging procedure is needed to identify the backreaction locally. Gasperini (2009) do so, but only average over the whole space-like hypersurface of a foliation of spacetime. We provide such procedure for specific compact, and possibly finite, domains of any hypersurface in the foliation.

4.1. Previous averaging methods

We discuss previous averaging procedures that have been used throughout the literature of the cosmological backreaction.

The gauge dependent, but covariant, averaging of a scalar field over a possible finite domain considered by [1, Sec. 4] is the following. Choose a hypersurface Σ which is orthogonal to the four-velocity u^α of observers comoving with the dust. Let $\psi : M \to \mathbb{R}$ be a function of Lagrangian coordinates x^i and time t. The spatial averaging of scalar field ψ on an arbitrary compact domain $\mathcal{D}$ on the hypersurface Σ is defined as the volume integral

$$\frac{\int_D \psi \sqrt{\det(h_{ij})} d^3x}{\int_{\mathcal{D}} \sqrt{\det(h_{ij})} d^3x}, \tag{4.1}$$

where h_{ij} is the 3-metric on Σ.

Proposition 4.1. The spatial average (4.1) of a scalar field $\psi : M \to \mathbb{R}$ is gauge dependent, i.e. not gauge invariant.

Proof. See Chapter 4 of the more elaborate notes. □

The gauge invariant and covariant averaging of a scalar field proposed by [3] amounts to the following. Consider

> "a cylinder-like domain Ω of M with temporal boundaries determined by the two space-like hypersurfaces on which a suitable scalar field $A(x)$ assumes the constant values A_1 and A_2; the region is bounded in space by the coordinate condition $B(x) < r_0$, where $B(x)$ is a suitable (positive) function of the coordinates with space-like gradient $\partial_\mu B$, and B_0 a positive constant." - [3]

To average over a hypersurface on which $A(x) = A_0$ is constant, [3] define the window function

$$W_\Omega(x) = \delta\big(A(x) - A_0\big)\mathbb{1}\big(B_0 - B(x)\big),$$

where δ and $\mathbb{1}$ are the delta and unit step function, respectively. The average of a scalar field $\psi : M \to \mathbb{R}$ over Σ is defined to be

$$\langle\psi\rangle_\Omega := \frac{\int_M \psi \sqrt{\det(g_{ij})} W_\Omega d^4x}{\int_M \sqrt{\det(g_{ij})} W_\Omega d^4x}, \tag{4.2}$$

which is covariant and gauge invariant if $B(x) \mapsto B\big(\gamma^{-1}(x)\big)$ under any gauge transformation $\gamma \in \mathrm{Diff}(M)$ [3].

Remark 4.2. [3] state

> "that in the case of interest to this paper - i.e. for quasi-homogeneous cosmological backgrounds - a natural candidate for the scalar field $B(x)$ with space-like gradient is missing."

They conclude that one should take $B_0 \to \infty$ in general and thereby pursuing that one should average over all of the hypersurface $A(x) = A_0$. This might not be desirable for the analysis of the cosmological backreaction.

4.2. Gauge invariant averaging over finite domain

We provide a gauge invariant and covariant spatial averaging method for specific, but possibly finite, domains on a hypersurface of the spacetime foliation. We do so by adopting averaging procedure (4.2) and by specifying a scalar field $B(x)$ satisfying the conditions.

Consider a foliation $A(x) = \text{const.}$ of spacetime. Next we construct a compact and possibly finite domain $\mathcal{D}$. Consider the hypersurface $\Sigma \subset M$ specified by $A(x) = A_0$ over which one desire to average over. Note that the metric d of the metric space (M, d) corresponding to our Lorentzian spacetime manifold (M, g) exists. Such metric exists as M is second countable [4, p. 271]. Take a point $p \in \Sigma$ and define $B(x)$ to be

$$d_p : M \to \mathbb{R}, \qquad x \mapsto d_p(x) := d(p, x).$$

The constants A_0, B_0 and the map $B(x)$ define the domain $\mathcal{D}$ as was done for domain Σ by [3].

Throughout the rest op this paper, we assume such choice of domain $\mathcal{D}$ on a hypersurface has been constructed as was done above.

Proposition 4.3. Let $\mathcal{D}$ be any such constructed domain. Average (4.2) is equal to

$$\langle\psi\rangle_{\mathcal{D}}(t) = \frac{\int_\Sigma \psi \sqrt{\det(g_{ij})} W_\Omega(t) d^3x}{\int_\Sigma \sqrt{\det(g_{ij})} W_\Omega d^3x}, \tag{4.3}$$

and is gauge invariant.

Proof. Pick any such domain $\mathcal{D}$. Since $B : M \to \mathbb{R}$ is a scalar field, it transforms as $B(x) \mapsto B\big(\gamma^{-1}(x)\big)$, thus average (4.2) is guage invariant. For the rest of the proof, see Chapter 4 $\square$

5. Backreaction effect on the Weyl tensor

5.1. Backreaction affects the Weyl tensor

Prove: the backreaction affects the evolution of the Weyl tensor.

5.2. Backreaction affects only the Weyl tensor

Prove: the backreaction is a purely general relativistic phenomenon. That is, it is not a Newtonian phenomenon.

6. Quantifying the backreaction effect on gravitational waves

References

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