Notes on relativistic gauge theory and backreaction

Dave Verweg
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Abstract. In Section 1 a general treatment on invariance and covariance is provided, which is used to give a precise understanding of invariants in relativity. In Section 2, Bernard's thoughts are commented upon and we conclude in Section 3 with a proof of the backreaction effect not being able to be fully described as a Newtonian phenomenon.

Contents

1 Relativistic gauge theory

1.1 Invariance and covariance of physical systems

—— make sure this section does not include any GR related remarks ——
We will not advance the discussion by trying to define 'symmetries' as these have different meanings in physics than they have in mathematical literature.[1]

> "It is a widely shared opinion that the most outstanding and characteristic feature of General Relativity is its manifest background independence." - Giulini (2007)

To elaborate upon the property of background independence for relativity, it must first be understood what background independence is. Although there are numerous amount of philosophical contributions on the definition and meaning of background independence, we follow Rickles (2008) and Giulini (2007); providing us with a rigorous treatment of the topic. To be precise in our reasoning, we provide a generalization of a topological space.

Definition 1.1 (Lane (1996))**.** A *mathematical structure* is a set S of mathematical objects such that the objects are described axiomatically and the set has at least one relation $R \subset S \times S$ defined on it.

[1] In the literature of physics, an invariant of a set of equations of motion need not be a symmetry, while mathematical contributions most often define a symmetry to be an invariant Giulini (2007).

Example 1.2. Define the equivalence relation $\sim$ on any topological space $(S, \mathfrak{T})$ by letting any two open sets $S_1, S_2 \subset S$ be equivalent if and only if $S_1, S_2 \in \mathfrak{T}$. From this one sees that indeed a topological space is a mathematical structure making Definition 1.1 a generalized of a topological space.

For a given physical theory, its equations of motion take the general form

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0, \tag{1.1}$$

where $\mathcal{D}$ is the collection of *dynamical structures* - i.e. mathematical structures that have no a priori values and must be solved to get assigned values to - and the set $\mathcal{B}$ of *background structures* contains all non-dynamical structures. For example in GR, the metric tensor is a dynamical structure and a coordinate system is a background structure. Let $\mathcal{K}$ be the space of kinematically possible field configurations, also known as field histories, of the theory. Note that the solutions of (1.1) give rise to the subset $\mathcal{P} \subset \mathcal{K}$ of dynamically possible configurations.

Definition 1.3. Let G be a group acting on $\mathcal{K}$ from the left-hand side, i.e. $G \times \mathcal{K} \to \mathcal{K}$. The equations (1.1) of motion are called *covariant* under G if for any $\gamma \in G$,

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \gamma \cdot \mathcal{B}] = 0, \tag{1.2}$$

and *invariant* if

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \mathcal{B}] = 0. \tag{1.3}$$

These concepts allow us to define diffeomorphism covariance and invariance rigorously.

Definition 1.4. Equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ are called *diffeomorphism invariant* if and only if it allows $\mathrm{Diff}(M)$ as invariance group, i.e. $\mathrm{Diff}(M)$ defines a left-action on $\mathcal{K}$ and for any $\gamma \in \mathrm{Diff}(M)$ statement (1.3) is satisfied.

Definition 1.5. Two tensor fields $\tau_1, \tau_2 \in \mathcal{T}(M)$ are *locally diffeomorphism equivalent* if and only if for any point $p \in M$ there exists a neighbourhood U of p and a diffeomorphism $\phi : U \to U$ such that

$$\phi^* \left(\tau_1|_U\right) = \tau_2|_U.$$

One can readily verify that the equivalence defined in Definition 1.5 is an equivalence relation on $\mathcal{T}(M)$.

Definition 1.6. Any field which is either not dynamical or whose solutions are all locally diffeomorphism equivalent is called an *absolute structure*.

Definition 1.7. A theory is called *background independent* if and only if its equations are $\mathrm{Diff}(M)$-invariant and its fields do not include absolute structures.

Q: Should then all equations in the theory be Diff-invariant, etc.?

1.2 Equations of motion for relativity

(First provide derivation: argumentation for the EH-action - principle of least action - Einstein field equations, i.e. the equations of motion of relativity.)

1. Principle of least action
2. So, we need the action for GR. Give an argumentation for the EH-action
3. Use variational approach by invoking Hamilton's principle
4. Derive the Euler-Lagrange - or simply Lagrange's - equations of motions for the system
5. Derive Hamilton's equations of motion of GR and prove that they are equivalent to the Lagrange equations.
6. Derive the Einstein field equations from the Lagrange equations

1.3 Invariants in relativity

Discuss here the invariants in relativity in terms of Section 1.

Remark 1.8 (Invariants in general relativity). Consider the equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ for a relativistic system. Observe that the diffeomorphism group $\mathrm{Diff}(M)$ is not the only invariant appearing in relativity; the Lorentz group is an invariant as well.

Prove: (proper, restricted) Lorentz / Poincare group is a

1.4 Mathematical gauge theory

Define general stuff: principal bundles, etc.

1.5 Mathematical gauge theory for relativity

Take frame bundle and derive the group $\mathcal{G}$ of gauge transformations. Then show $\mathcal{G} \cong \mathrm{Diff}(M)$ and that equations of motion for GR (EFE, Lagrangian, Hamiltonian) are invariant under $\mathcal{G}$, i.e. it is a left-action on $\mathcal{K}$ and satisfies (1.3).

For the reason that $\mathrm{Diff}(M) \cong \mathcal{G}$, diffeomorphism invariance can be identified with gauge invariance.

Prove: coordinate transformations are gauge transformations, etc.....

1.6 Gauge theory for cosmological perturbation theory

Consider a spacetime manifold M and its background M_b with a one-to-one correspondence (bijection) $c : M_b \to M$. Do not confuse: background gauges and frame gauges. However, the former can be understood as a special case of the latter.

Proposition 1.9. A transformation $c \mapsto \phi(c)$ of the background correspondence, where $\phi \circ c : M_b \to M$ is bijective, induces a gauge transformation.

Proof. Consider an arbitrary $p \in M$. There exists an $b \in M_b$ such that $p = c(b)$, and so $M \ni c(b) \mapsto \phi \circ c(b) \in M$, i.e. $p \mapsto \phi(p)$ bijectively. □

1.7 Gauge invariance for backreaction

—— Maybe for next chapter ———-

So what is gauge invariance? How can we be *check* whether the equations written down in GR are gauge invariant? (Look at the Hamiltonian?)

By Definition 1.4 we have that the equations of motion of GR are gauge invariant if $\mathcal{G}$ is a left action on $\mathcal{K}$ and for any gauge transformation $\gamma \in \mathcal{G} \cong \mathrm{Diff}(M)$ the

2 Thoughts on Bernard's thoughts - part I

1. *ADM formalism: problem of curvature and time homogeneity*
   I can't seem to wrap my head about the following. Consider a spacetime model with just one star in it and let us look at what the star - i.e. the matter - does to the curvature of spacetime. To do so, foliation of spacetime of spacelike hypersurfaces with each defined by level sets $t =$ constant. We can draw such level sets on each of these hypersurfaces for the spacelike basis vector x^i for $i = 1, 2, 3$, namely by drawing all lines $x^i =$ constant. Intuitively, one sees that there is a higher density of level sets close to the star compared to further away from the star. This observation is simply a resemblance of the star curving spacetime.

   The thing that I don't understand is the following. That higher density in level sets is there in every basis direction, so especially for the level sets $t =$ constant. But the thing is, how is this information of how tightly these time level sets are packed together conserved in the ADM formalism? From what in the theory can we read this density off? It seems to me that this only works if we assume the cosmology to be "homogeneous in the time direction".

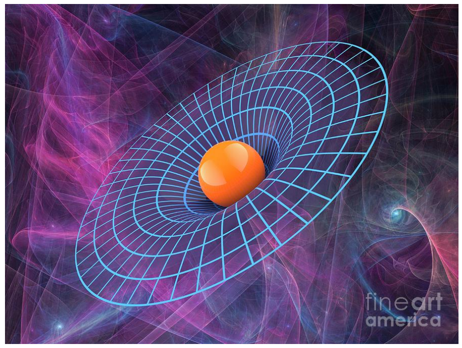

Figure 1: Intuitive depiction of the curvature of 3-dimensional spacetime.

2. *Evolution of the expansion rate of the fluid*
   Consider Equation (7) of Hawking (1966), which is an expression of the covariant derivative θ' of the expansion rate θ in the direction of a worldline. Since the covariant derivate is gauge invariant, is θ' not a possible candidate to consider in order to analyze the backreaction instead of the foliation-depend derivative $\frac{\partial}{\partial x^\mu}\theta$?

3. *Evolution of the (conformal) Weyl tensor*
   The Weyl tensor C_{abcd} is "that part of the curvature that is not determined locally by matter" and can therefore be considered as the free gravitational field Ehlers et al. (1960). The Weyl tensor is the only quantity in relativity not having any Newtonian counterpart. The tensor can be decomposed into its electric and magnetic component, denoted respectively by E_{ab} and H_{ab}. From Equation (15) we see that the rate of change E'_{ab} in the energy component has a source: the shear tensor. Equation (16) shows that H'_{ab} has no source; confirming that it represents gravitational waves without a source.

4. *Perturbations of the Weyl tensor*
   Hawking (1966) states that perturbations of the Weyl tensor do not arise from rotational or density perturbations $\iff$
   $$E_{ab}{}^{;b} = H_{ab}{}^{;b} = 0.$$
   If I understand it correctly, this statement is false.

   Assume that the backreaction is not a Newtonian phenomenon, then it must be captured within the Weyl tensor. The backreaction in itself is not a rotational nor a density perturbation, but causes perturbations in the Weyl tensor. This confirms that Hawking's statement is not correct.

5. *Mistake of Hawking*
   Where does Hawking (1966) you confronted Stephen with? It sounded to me at the moment you were telling me about it that your argument might be useful to understand the underlying assumptions of the ADM formalism.

6. *Backreaction a Newtonian phenomenon*
   Newtonian spacetime is defined on the Euclidean manifold $M = \mathbb{R}^4$ (Barrett, 2015), which is not a curved space. If we prove that
   $$\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}} - \langle\dot{\theta}\rangle_{\mathcal{D}} > 0$$

in some Newtonian setting, then the backreaction is not per se *not* a Newtonian phenomenon. This would imply that the backreaction is cannot be fully captured within the Weyl tensor.

However, it might of course be the opposite. By definition, the backreaction is has an effect on the curvature of spacetime which is *not* determined solely locally by the matter; it is contributed by the deviation from homogeneity or isotropy. There is only one term in which such effect could be captured: the Weyl tensor - having no Newtonian analogue.

7. $\frac{d}{dt}\langle\theta\rangle_V = \langle\dot{\theta}\rangle_V$?
   In my eyes, we can take a cmoving volume V of compact domain $\mathcal{D} \subset M$ independent of the time component t, i.e. $V \equiv V(t)$. Then for $x = (x^1, x^2, x^3)$,

$$\int \langle\dot{\theta}\rangle_V \,\mathrm{d}\,t = \int \frac{\int_V \frac{d}{dt}\theta(t,x^i)\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x}{\int_V \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x}\,\mathrm{d}\,t \tag{2.1}$$

$$= \frac{\int_V \int \frac{d}{dt}\theta(t,x^i)\,\mathrm{d}\,t\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x}{\int_V \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x} \tag{2.2}$$

$$= \frac{\int_V \theta(t,x^i)\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x}{\int_V \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\,x} \tag{2.3}$$

$$= \langle\theta\rangle_V \tag{2.4}$$

$$= \int \frac{d}{dt}\langle\theta\rangle_V, \tag{2.5}$$

   from which we conclude that $\frac{d}{dt}\langle\theta\rangle_V = \langle\dot{\theta}\rangle_V$. What goes wrong here? Or is Buchert not explicit enough in describing the assumptions on choosing domain V?

3 Backreaction is not a Newtonian phenomenon

The backreaction is an effect on the curvature of spacetime determined by the deviation from homogeneity or isotropy in some finite domain of spacetime. This effect must, therefore, be containted in "the part of the spacetime curvature not determined locally by the matter at a point, but rather determined by conditions elsewhere."[2] That part of the spacetime curvature is exactly represented by the the Weyl tensor and, thus, we expect that the backreaction is captured within this tensor.

3.1 The conformally flat argument

If the backreaction is not captured entirely within the Weyl tensor, also called the *conformal tensor*, then there must exist a spacetime with a vanishing Weyl tensor $C_{abcd} = 0$, but wherein the backreaction effect is still present. If we find it is not, we can conclude that the backreaction is not a Newtonian phenomenon.

1. *Isolating the backreaction effect*.
   Adopting the definition from Buchert et al. (2015) of the *backreaction* to be

   > "deviation of spatial average properties of an inhomogeneous universe model from the values predicted by a homogeneous-isotropic universe model."

   I don't like the above definition, I prefer something more concrete with backreaction being the effect of the noncommutation $[\partial_t\langle\psi\rangle, \langle\partial_t\psi\rangle] \neq 0$ for any scalar field ψ - and maybe only looking at expansion rate θ.

[2]The quotation is the intuitive definition of Ellis et al. (2012, Sec. 2.7.6) for the Weyl tensor.

So, the backreaction not being a Newtonian phenomenon is equivalent to stating that there do not exist deviations of spatial average properties of an inhomogeneous Newtonian cosmology from the values predicted by a homogeneous-isotropic universe model.

Considering the commutation identity for any scalar tensor field ψ,

$$\partial_t\langle\psi\rangle - \langle\partial_t\psi\rangle = \langle\theta\psi\rangle - \langle\theta\rangle\langle\psi\rangle, \tag{3.1}$$

where "the fluctuation part on the right-hand side of this rule produces the *kinematical backreaction*" Ellis and Buchert (2005).

2. *Newtonian and general relativistic evolution of expansion rate.*
Comparing the modified Raychaudhuri equation for a Newtonian cosmology derived in Buchert and Ehlers (1995, Eq. 8) with its general relativistic representative showed in e.g. Buchert (2000, Eq. 11c), we observe that they are both of the exact same form:

$$\partial_t\langle\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle + \frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle, \tag{3.2}$$

for an irrotational model.

3. *Weyl tensor*.
(**Some intro**) The problem reduces to showing that there is some part of the backreaction effect contained within the Weyl tensor, i.e. by showing C_{abcd} is effected by

Lemma 3.1. Let (M, g) be a spacetime with its matter fields $\{\tau_i : M \to \mathcal{T}(M) \mid i \in I\}$ such that the rotation tensor vanishes, i.e. $\omega^\mu{}_\nu = 0$. The electric component E_{ab} of the Weyl tensor C_{abcd} is depended on the commutator $\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle$ via the relation

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle \tag{3.3}$$

$$- \left(\frac{2}{\mu+p}\right)^2 \Big\langle \Big(\perp E'^a{}_b + E^a{}_b\theta - E^{c(a}\sigma_{b)c} - \eta^a{}_{cde}\eta_{bpqr}u^c u^p \sigma^{dq} E^{er} + 2H^{d(a}\eta_{b)cde}u^c u'^e\Big) \tag{3.4}$$

$$\cdot \Big(\perp E_a'{}^b + E_a{}^b\theta - E_{c(a}\sigma^{b)}{}_c - \eta_{acde}\eta^b_{pqr}u^c u^p \sigma^{dq} E^{er} + 2H^d{}_{(a}\eta^{b)}{}_{cde}u^c u'^e\Big)\Big\rangle_D. \tag{3.5}$$

To the first order, that is,

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - \left(\frac{2}{\mu+p}\right)^2 \Big\langle \left(E'^a{}_b + E^a{}_b\theta\right)\left(E'^b{}_a + E^b{}_a\theta\right)\Big\rangle_D. \tag{3.6}$$

Proof. Let μ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that the evolution of E_{ab} can be described by

$$-\frac{1}{2}(\mu+p)\sigma_{ab} = \perp E'_{ab} + E_{ab}\theta + h_{(a}{}^f\eta_{b)cde}u^c H_f{}^{d;e} \tag{3.7}$$

$$- E^c{}_{(a}\sigma_{b)c} - E^c{}_{(a}\omega_{b)c} - \eta_{acde}\eta_{bpqr}u^c u^p\sigma^{dq}E^{er} + 2H^d{}_{(a}\eta_{b)cde}u^c u'^e, \tag{3.8}$$

hereby $\perp E'_{ab}$ is the projection of E'_{ab} under the projection tensor $h_a{}^b = g_a{}^b + u_a u^b$, i.e. $\perp E'_{ab} = h_a{}^d h_b{}^e E'_{de}$, and where the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\mu+p)\eta_{abcd}\omega^{cd}u^b, \tag{3.9}$$

of the magnetic component H_{ab} of the Weyl tensor. As we consider an irrotational dust [**define this precisely**], we see that $\omega^{cd} = g^{di}\omega^c{}_i = 0$, implying that $E^c{}_{(a)}\omega_{b)c} = 0$ and that covariant derivative (3.9) also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (3.7) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\mu+p}\left(\perp E'_{ab} + E_{ab}\theta - E^c{}_{(a}\sigma_{b)c} - \eta_{acde}\eta_{bpqr}u^c u^p \sigma^{dq} E^{er} + 2H^d{}_{(a}\eta_{b)cde}u^c u'^e\right), \tag{3.10}$$

Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\mu+p)^2}\left(\perp E'^a{}_b + E^a{}_b\theta - E^{c(a}\sigma_{b)c} - \eta^a{}_{cde}\eta_{bpqr}u^c u^p\sigma^{dq}E^{er} + 2H^{d(a}\eta_{b)cde}u^c u'^e\right) \tag{3.11}$$

$$\cdot\left(\perp E'^{\,b}_a + E_a{}^b\theta - E_{c(a}\sigma^{b)}{}_c - \eta_{acde}\eta^b_{pqr}u^c u^p\sigma^{dq}E^{er} + 2H^d{}_{(a}\eta^{b)}{}_{cde}u^c u'^e\right) \tag{3.12}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{3.13}$$

of Buchert (2000), one sees that (3.3) holds.

To the first order.
Let μ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that to the first order, the evolution of E_{ab} can be described by

$$E'_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)}{}_{cde}u^c H_f{}^{d;e} = -\frac{1}{2}(\mu+p)\sigma_{ab}, \tag{3.14}$$

with the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\mu+p)\eta_{abcd}\omega^{cd}u^b, \tag{3.15}$$

of the magnetic component H_{ab} of the Weyl tensor. Since $\omega^{cd} = g^{di}\omega^c{}_i = 0$, its covariant derivative also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (3.7) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\mu+p}\left(E'_{ab} + E_{ab}\theta\right). \tag{3.16}$$

Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\mu+p)^2}\left(E'^a{}_b + E^a{}_b\theta\right)\left(E'^b{}_a + E^b{}_a\theta\right). \tag{3.17}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{3.18}$$

of Buchert (2000), one sees that **??** holds. □

We want to make Lemma **??** precise. Note that ${h^a}_b = {g^a}_b + u^a u_b$ is a projection tensor projecting into the three-dimensional tangent space orthogonal to u^a (Ellis et al., 2012, Sec. 4.4). For any vector $X^a \in T_pM$ at some point $p \in M$, the projection $X^a_\perp$ orthogonal to u^a is

$$X^a_\perp = {h^a}_b X^b,$$

making that $X^a_\perp u_a = 0$, and

$$\perp E'_{ab} = {h_a}^d {h_b}^e E'_{de}.$$

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.
2. $\phi \in \mathrm{Diff}(M)$ diffeomorphism.
3. $\gamma \in \mathcal{G}$ gauge transformation.
4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.
5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.

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