Notes on relativistic gauge theory and backreaction

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Contents

1 Relativistic gauge theory

(First provide derivation: argumentation for the EH-action - principle of least action - Einstein field equations, i.e. the equations of motion of relativity.)

> "It is a widely shared opinion that the most outstanding and characteristic feature of General Relativity is its manifest background independence." - [1]

To elaborate upon the property of background independence for relativity, it must first be understood what background independence is. Although there are numerous amount of philosophical contributions on the definition and meaning of background independence, we follow [2] and [1]; providing us with a rigorous treatment of the topic. To be precise in our reasoning, we provide a generalization of a topological space.

Definition 1.1 ([3]). A *mathematical structure* is a set S of mathematical objects such that the objects are described axiomatically and the set has at least one relation $R \subset S \times S$ defined on it.

For a given physical theory, its equations of motion take the general form

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0, \tag{1}$$

where $\mathcal{D}$ is the collection of *dynamical structures* - i.e. mathematical structures that have no a priori values and must be solved to get assign values to - and the set $\mathcal{B}$ of *background structures* contains all non-dynamical structures. For example in GR, the metric tensor is a dynamical structure and a coordinate system is a background structure. Let $\mathcal{K}$ be the space of kinematically possible field configurations, also known as field histories, of the theory. Note that the solutions of (1) give rise to the subset $\mathcal{P} \subset \mathcal{K}$ of dynamically possible configurations.

Let $\mathcal{G}$ be a group acting on $\mathcal{K}$ from the left-hand side, i.e. $\mathcal{G} \times \mathcal{K} \to \mathcal{K}$. The equations (1) of motion are called *covariant* under $\mathcal{G}$ if for any $\gamma \in \mathcal{G}$,

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \gamma \cdot \mathcal{B}] = 0, \tag{2}$$

and *invariant* if

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \mathcal{B}] = 0. \tag{3}$$

These concepts allow us to define diffeomorphism covariance and invariance rigorously.

Note that general covariance, independence of coordinates, etc. are covariance groups.

Definition 1.2. Equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ are called *diffeomorphism invariant* if and only if it allows $\mathrm{Diff}(M)$ as invariance group, i.e. for any $\gamma \in \mathrm{Diff}(M)$ statement (3) is satisfied.

======== go on to define background independence and then relate to the gauge transformation group $\mathcal{G}$ ==================

2 Further objectives

We describe several short research questions.

1. Overview of different kinds of transformations
2. Overview of different kinds of invariances
3. Is describing GR with the spinor formalism meaningful to the cosmological backreaction problem?
4. How is the ADM formalism restricting the gauge invariance in an arbitrary spacetime?
5. How is second question related to Weyl's postulate?
6. Prove that the backreaction term is gauge dependent.
7. Prove that there exists a gauge wherein the backreaction vanishes.

3 Transformations

We provide an overview of different kinds of transformations found in the theory of relativity.

1. *Coordinate transformations*.
   Consider a chart (U, φ) on spacetime manifold M with $\varphi = (x^0, \ldots, x^3)$, where the maps $x^0, \ldots, x^3$ are called *(local) coordinates* on U. The *(general) coordinate transformation* between overlapping smooth charts (U, φ) and (V, ψ) with $\psi = (y^0, \ldots, y^3)$ is the transition map .....

   Note: Coordinate transformations are also called gauge transformations of the first kind.

2. *Local Lorentz transformations and local orthogonal rotations*.
   A tetrad $\{e_0, \ldots, e_3\}$ forms the real-valued matrix ${e_i}^\mu$ with $\mu = 1, \ldots, 4$, where the coordinate component ${e_i}^\mu = x^\mu \circ e_i$ for local coordinates (x^μ). The corresponding symmetry is that of the invariance under Lorentz transformations. If ${A_j}^i \in O(1,3)$ such transformation, we have

   $$g({A_i}^a e_a, {A_j}^b e_b) = {A_i}^a {A_j}^b g(e_a, e_b) = {A_i}^a {A_j}^b \eta_{ab} = \eta_{ij},$$

   i.e. the *Lorentz transformation* $e_i \mapsto {A_i}^j(p) e_j$ forms a new tetrad ${A_j}^i(p) e_j$ at every point $p \in M$.

   Carroll [4, p. 44] states that "a Lorentz transformation is a special kind of coordinate transformation, with $x^{\mu'} = {\Lambda^{\mu'}}_\mu x^\mu$."

Following Nakahara (2003), under a *local orthogonal rotation* (i.e. a local Lorentz transformation or a gauge transformation?) the vierbein $e^{\alpha}{}_{\mu}(p)$ transforms as

$$e^{\alpha}{}_{\mu}(p) \mapsto e^{'\alpha}{}_{\mu}(p) = \Lambda^{\alpha}{}_{\beta}(p) e^{\beta}{}_{\mu}(p).$$

"The indices $\alpha, \beta, \gamma, \ldots$ transform under the local orthogonal rotation and are inert under coordinate changes." The metric tensor is invariant under the rotation and therefore matrix $\{\Lambda^{\alpha}{}_{\beta}(p)\}$ is an element of $SO(3,1)$. The dimension of Lie group $SO(3,1)$ is $m(m-1)/2 = m^2 - m(m+1)/2$, i.e. the difference between the degrees of freedom of $e_{\alpha}{}^{\mu} \in GL(m, \mathbb{R})$ and $g_{\mu\alpha}$ (as g is symmetric).

3. Gauge transformations (of the second kind) [5] makes an error here. Also called local gauge transformations or point dependent gauge transformation [Drechsler 1977]. See p. 102 (v) of Dodson 1988 or *internal automorphism* [6, Sec. 2.2].

4. *Passive and active transformations*.
A passive transformation is a change of coordinates, while an active transformation is simply a diffeomorphism $\phi \in \text{Diff}(M)$ [7, p. 52]. The latter transformation gives rise to a tensor field by consider $\tau \in \mathcal{T}(M)$ via the pullback,

$$\tau \mapsto \phi^* \tau, \qquad (\phi^* \tau)(p) = \tau \circ \phi(p)$$

with $p \in M$, as in general $\phi^* \tau(p) \neq \tau \circ \phi(p)$ for all $p \in M$.

4 Invariances

We provide an overview of invariances relevant to relativity.

Definition 4.1 ([8])**.** A relation R on a set A, i.e. a set $R \subset A \times A$, is *invariant* with respect to a set Φ of one-to-one mappings $\phi : A \to A$ if it is invariant with respect to every element $\phi \in \Phi$.

Definition 4.2 ([9])**.** A ternary relation $R(xyz)$ between points is *invariant* with respect to a given transformation $\phi : p \to p'$ and its inverse $p' \to p$ if $R(abc)$ implies $R(a'b'c')$ and vice versa.

Consider a spacetime manifold $(M, \mathcal{F})$ with $\mathcal{F}$ a collection of tensor fields $\tau \in \mathcal{T}(M)$ defined on M. Take $R \subset \text{Met}(M) \times \text{Met}(M)$ to be the set of all pairs $(g, \hat{g})$ such that there exists a diffeomorphism $f : \text{Met}(M) \to \text{Met}(M)$ mapping $f(g) = \hat{g}$ where both metric tensors $g, \hat{g}$ make the variation of the Einstein-Hilbert action

$$S = \int \left(\frac{1}{2\kappa} R + \mathcal{L}_M \right) \sqrt{-g} d^4 x,$$

with $\kappa = 8\pi G c^{-4}$ the Einstein gravitational constant, vanish, i.e. $\delta S = 0$.

Note:

$$\{\text{coordinate transformations}\} \subset \mathcal{G} \cong \text{Diff}(M).$$

1. *General covariance*.
"All physical laws are independent of the choice of a particular coordinate system" [4, p. 14]. Hence, general covariance is the invariance under coordinate transformations (see Einstein in Baez [7, p. 15]). It follows that the laws of physics should be tensorial.

2. *Invariant functional under field transformation*.
Consider two tensor fields $\tau, \hat{\tau} \in \mathcal{T}(M)$. A functional $F : \mathcal{T}(M) \to \mathbb{R}$ is invariant under the field transformation $\tau \mapsto \hat{\tau}$ if $F(\tau) = F(\hat{\tau})$.

3. *Diffeomorphism invariance*.
   "Passive diff invariance is a property of a formulation of a dynamical theory, while active diff invariance is a property of the dynamical theory itself." and "A theory is invariant under active diffs, when a smooth displacement of the dynamical fields (the dynamical fields alone) over the manifold, sends solutions of the equations of motion into solutions of the equations of motion." [10]. Recall that for GR the equations of motion are the Einstein field equations.

4. *Background independence*.
   Background independent theories only possess dynamical structures and not be affected by geometrical structures. See e.g. Baez or [2].

5 Spinors

Hopefully, the talk with Bernard on April 8 will clarify this question.

6 ADM formalism restricts gauge invariance

Arnowitt, Deser and Misner (ADM) introduce their formalism in [11], i.e. "the slicing of spacetime into a one-parameter family of spacelike hypersurfaces."[1] A more explicit treatment is given in [12, 13].

[1] Cited from [12, p. 506].

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.
2. $\phi \in \mathrm{Diff}(M)$ diffeomorphism.
3. $\gamma \in \mathcal{G}$ gauge transformation.
4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.
5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.

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