Notes on gauge dependence of backreaction

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Contents

1 Gauge theory for spacetime

It has been pointed out by e.g. Friedman [1, Sec. 5.5] and Ryckman [2] that Einstein [3] had been confused by the difference between global symmetries on the spacetime manifold and local coordinate transformations as he states that "the general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever (generally covariant)." Global symmetries on a spacetime manifold M are to be understood as diffeomorphisms mapping M unto itself, where we denote the set of all such diffeomorphisms $\phi : M \to M$ by $\mathrm{Diff}(M)$ [2]. Friedman [1, p. 56] explains that the symmetry group of general relativity is $\mathrm{Diff}(M)$, which turns out to be a group [4]. We call elements in $\mathrm{Diff}(M)$ *gauge transformations*, active diffeomorphism or point transformations [1, 2, 5, 6]. In general, a gauge transformation $\phi : M \to M$ can be considered to redefine or 'shuffle around' the points in the spacetime manifold M.

Remark 1.1. Coordinate transformations are not gauge transformations as a coordinate transformation on M is a map $\rho : U \cap V \subset \mathbb{R}^4 \to U \cap V \subset \mathbb{R}^4$, where U and V are overlapping chart neighborhoods on M, sending local coordinates of chart U to V. It therefore does not change the spacetime manifold M itself, but rather a relabeling of coordinates assigned to each event in $U \cap V$.

1.1 Gauge choice

Having defined gauge transformations as elements of the group $\mathrm{Diff}(M)$, one naturally raises the question what a specific choice of gauge is. To the best of my knowledge, there is not yet an intuitive answer to this question in the case of a spacetime (M, g), were we do not consider spacetime manifold M as a perturbed version of some background manifold.[1]

To figure out how we can understand fixing a gauge, we emphasize that relativists and cosmologists in general implicitly assume there to be fields defined on M as "there is no such thing as an empty space i.e., a space without field" according to Einstein [7, p. 157]. More precisely, Hawking and Ellis [8, Sec. 3.2] formulate that "there will be various fields on [spacetime] M, such as the electromagnetic field, the neutrino field, etc., which describe the matter content of spacetime." We therefore define the following - more explicit - notion of spacetime.

Definition 1.1. A *non-empty spacetime* is a triple $(M, g, \mathcal{F})$ with a pseudo-Riemannian manifold M, a Lorentz metric g and a set $\mathcal{F}$ containing predefined tensor fields $\tau \in \mathcal{T}(M)$ defined on M.

[1] Note that in this sense an unperturbed spacetime can be inhomogeneous.

A coordinate transformation transforms one coordinate system into another. Analogously, a gauge transformation transforms one gauge into another gauge. That leaves us with the possibility to identify a gauge with the thing which is transformed under gauge transformations. Let us examine what the thing is an arbitrary gauge transformation transforms. Consider a non-empty spacetime $(M, g, \mathcal{F})$ and take an arbitrary $\phi \in \mathrm{Diff}(M)$.

i. *Spacetime manifold*. By definition of the group $\mathrm{Diff}(M)$, $\phi(M) = M$, so nothing changes for the spacetime manifold M itself.

ii. *Tensor fields*. Take any tensor field $\tau \in \mathcal{F}$, which is a map $\tau : M \to T^r_s(M)$ unto the tensor bundle $T^r_s(M)$ for some $r, s \geq 0$. After the gauge transformation invoked through $\phi : M \to M$, we have that the tensor field τ is transformed to $\tau \circ \phi \in \mathcal{T}(M)$. As $\tau \in \mathcal{F}$ was chosen arbitrarily, all the elements of $\mathcal{F}$ transform like $\tau \to \tau \circ \phi$ under gauge transformation ϕ. Important to stress is that in general $\tau \circ \phi \neq \tau$. Denote the resulting collection of tensor fields after the gauge transformation by $\mathcal{F}_\phi$.

iii. *Metric tensor*. Denote the metric tensor after the gauge transformation by $\tilde{g}_{ab}$. As pointed out above, the formulation of general relativity should be invariant under gauge transformations. In particular, the Einstein field equations

$$R_{ab} - \frac{1}{2} R \tilde{g}_{ab} = T_{ab} - \Lambda \tilde{g}_{ab}, \tag{1}$$

should hold under gauge transformation $\phi : M \to M$. In (1), R_{ab} is the Ricci tensor with trace R, T_{ab} the total energy-momentum tensor of the matter fields and Λ the cosmological constant. Rewriting (1) gives

$$\left(\frac{1}{2} R - \Lambda \right) \tilde{g}_{ab} = R_{ab} - T_{ab}, \tag{2}$$

where we note that R and Λ are scalars. Since $\mathbb{R}_{ab}$ and T_{ab} are completely determined by the matter fields defined on M, hence $\tilde{g}_{ab}$ is fully described by $\mathcal{F}_\phi$.

We conclude that under a gauge transformation $\phi \in \mathrm{Diff}(M)$ the collection $\mathcal{F}$ of tensor fields of non-empty spacetime $(M, g, \mathcal{F})$ transforms and its transformed set $\mathcal{F}_\phi$ completely determines the gauge transformed non-empty spacetime $(M, \tilde{g}, \mathcal{F}_\phi)$. By the discussion above, we may identify a gauge with a choice of collection $\mathcal{F}$ of tensor fields on M.

1.2 Gauge dependende of Buchert's averaging procedure

Buchert [9, 10] defines the following averaging procedure.

Definition 1.2. Choose a hypersurface Σ which is orthogonal to the four-velocity u^α of observers comoving with the dust. Let $\psi : M \to \mathbb{R}$ be a function of Lagrangian coordinates x^i and time t. The *spatial averaging* of scalar field ψ on an arbitrary compact domain $\mathcal{D}$ on the hypersurface Σ is defined as the volume integral

$$\langle \psi \rangle_{\mathcal{D}}(t) := \frac{1}{V_{\mathcal{D}}} \int_D \psi(t, x^i) \sqrt{\det(h_{ij})} d^3x,$$

with the *volume* of $\mathcal{D}$ defined as

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{\det(h_{ij})} d^3x,$$

where h_{ij} is the metric on Σ.

Recall that a compact domain is a non-empty connected open subset such that any open covering of the set contains a finite subcovering. We prove that the averaging procedure defined above is gauge dependent. This can be done by using by a property of the pull-back of a smooth map between manifolds on a differential form.

Proposition 1.1. Let $F : M \to N$ be a smooth map between n-dimensional smooth manifolds with (x^i) and (y^i) smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and let $u : V \to \mathbb{R}$ be continuous. Then, on $U \cap F^{-1}(V)$,

$$F^*(udy^1 \wedge \dots \wedge dy^n) = (u \circ F)(\det DF)dx^1 \wedge \dots \wedge dx^n,$$

where DF is the Jacobian matrix of F in these coordinates.

Proof. By the properties of the pull-back on differential forms, see e.g. Lemma 14.16 of Lee [11], we have

$$F^*(udy^1 \wedge \dots \wedge dy^n) = (u \circ F)d(y^1 \circ F) \wedge \dots \wedge d(y^n \circ F) = (u \circ F)dF^1 \wedge \dots \wedge dF^n,$$

where F^j is the j-th component of F with respect to the local coordinates (y^i), i.e. $F^j = y^j \circ F$. We see that $F^*(udy^1 \wedge \dots \wedge dy^n) \in \Omega^n(M)$, so it is a n-form on M.

Take any $v_1, \dots, v_n \in T_pM$ for an arbitrary $p \in U \cap F^{-1}(V)$. We can decompose these tangent vectors into a linear combination of its basis components, i.e. $v_k = v_k^i \frac{\partial}{\partial x^i}\big|_p$ for any $k = 1, \dots, n$ and some $v_k^i \in \mathbb{R}$. Hence,

$$\begin{aligned}
F^*(udy^1 \wedge \dots \wedge dy^n)_p(v_1, \dots, v_n) &= (u \circ F)_p dF_p^1 \wedge \dots \wedge dF_p^n \left(v_1^i \frac{\partial}{\partial x^i}\big|_p, \dots, v_n^i \frac{\partial}{\partial x^i}\big|_p \right) \\
&= (u \circ F)_p \det \left(dF_p^j \left(v_k^i \frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det \left(v_k^i dF_p^j \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det \left(d(y^j \circ F)_p \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det(DF),
\end{aligned}$$

where the last step follows from writing out the Jacobian of F in local coordinates (x^i) and (y^i). Notice that (v_k^i) is simply the matrix of all the components of the vectors $v_1, \dots, v_n$ with respect to local coordinates (x^i). As $dx^j \left(\frac{\partial}{\partial x^i}\right) = \delta_i^j$, we find

$$\begin{aligned}
dx_p^1 \wedge \dots \wedge dx^n(v_1, \dots, v_n) &= dx^1 \wedge \dots \wedge dx^n \left(v_1^i \frac{\partial}{\partial x^i}\big|_p, \dots, v_n^i \frac{\partial}{\partial x^i}\big|_p \right) \\
&= \det \left(dx^j \left(v_k^i \frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= \det \left(v_k^i dx^j \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= \det \left(v_k^i \delta_i^j \right) \\
&= \det \left(v_k^j \right).
\end{aligned}$$

Making

$$(u \circ F)_p(\det DF)dx_p^1 \wedge \dots \wedge dx^n(v_1, \dots, v_n) = (u \circ F)_p(\det DF) \det \left(v_k^j \right).$$

Since $p \in U \cap F^{-1}(V)$ was chosen arbitrarily, this concludes the proof.

An alternative proof can be formulated without explicitly evaluating on some collection of vectors $v_1, \dots, v_n$, but rewriting $F^*(udy^1 \wedge \dots dy^n)$ and using Leibniz's formula for the determinant. □

Proposition 1.1 allows us to prove that Buchert's averaging operation depends on the chosen gauge by invoking the theorem on global change of variables.

Theorem 1.1 (Global change of variables)**.** Let M and N be smooth oriented manifolds of dimension n and $F : M \to N$ a diffeomorphism. If the differential form $\omega \in \Omega^n(N)$ has compact support, then the pull-back $F^*\omega$ has compact support and

$$\int_N \omega = \pm \int_M F^*\omega,$$

where we take the plus sign if F is orientation-preserving and the minus sign if F is orientation-reversing.

Proof. See Proposition 16.6 of Lee [11]. □

Theorem 1.2. The spatial averaging of a scalar field defined by Buchert [9] is gauge dependent.

Proof. Let $T : M \to M$ be any gauge transformation and $\mathcal{D}$ an arbitrary compact domain on a hypersurface, which is orthogonal to the four-velocity vector of the observers comoving with the dust. Since T is a diffeomorphism, its inverse $T^{-1} : M \to M$ exists and is a diffeomorphism such that the commutative diagram (3) commutes.

$$\begin{tikzcd} M \arrow[r, "T^{-1}"] & M \\ T(\mathcal{D}) \arrow[u, hook, "\iota_{T(\mathcal{D})}"] \arrow[r, "T^{-1}|_{T(\mathcal{D})}"'] & \mathcal{D} \arrow[u, hook', "\iota_{\mathcal{D}}"'] \end{tikzcd} \tag{3}$$

The result follows by invoking the theorem of global change of variables for the inverse $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ restricted to $T(\mathcal{D})$ on its domain M. For this we need to prove that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth oriented manifolds of the same dimension and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation preserving.

First, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth manifolds of the same dimension. Since the image of a compact set under a continuous map between topological spaces is compact, $T(\mathcal{D})$ is a compact subspace. Since any subspace of a Hausdorff space is Hausdorff itself, cf. [12, p. 110], $\mathcal{D}$ and $T(\mathcal{D})$ are Hausdorff. By definition of compactness of a subspace, any open covering of $\mathcal{D}$ in M has a finite subcover, i.e. $\mathcal{D}$ is compact with the induced topology. Second countability follows by considering some open finite covering $\{B_i\}_{i\in I}$ of $\mathcal{D}$ and noting that $\{B_i \cap \mathcal{D}\}_{i\in I}$ is a countable basis for the topology of $\mathcal{D}$. Since M is a manifold, $\mathcal{D} \subset M$ is also locally Euclidean. We conclude that $\mathcal{D}$ is a smooth manifold, where smoothness follows from inducing a differentiable structure on $\mathcal{D}$ from one on M. Similarly, we can conclude that $T(\mathcal{D})$ is a smooth manifold. Noting that T is a diffeomorphism, we see $\dim \mathcal{D} = \dim T(\mathcal{D})$.

Second, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are orientable and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving. Visser [13, p. 286] shows that non-orientable spacetime manifolds are incompatible with current theories. Therefore, we take M to be orientable. Inducing an atlas on submanifolds $\mathcal{D}$ and $T(\mathcal{D})$ from the oriented atlas of M, makes that both submanifolds are orientable. It follows from Proposition 15.5 of Lee [11] that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving.[2]

[2]For invoking the result [11, Prop. 15.5], note that every diffeomorphism is a local diffeomorphism.

Invoking Thm. 1.1 for $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ to rewrite $\langle\psi\rangle_{\mathcal{D}}(t)$,

$$\begin{aligned}
\langle\psi\rangle_{\mathcal{D}}(t) &= \frac{\int_{\mathcal{D}} \psi(t,x^i)\sqrt{\det(h_{ij})}d^3x}{\int_{\mathcal{D}} \sqrt{\det(h_{ij})}d^3x} \\
&= \frac{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \psi(t,x^i)\sqrt{\det(h_{ij})}dx^1 \wedge dx^2 \wedge dx^3}{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \sqrt{\det(h_{ij})}dx^1 \wedge dx^2 \wedge dx^3} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\sqrt{\det(h_{ij}\circ T^{-1})}d(x^1\circ T^{-1}) \wedge d(x^2\circ T^{-1}) \wedge d(x^3\circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij}\circ T^{-1})}d(x^1\circ T^{-1}) \wedge d(x^2\circ T^{-1}) \wedge d(x^3\circ T^{-1})} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\det T^{-1}\sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})}{\int_{T(\mathcal{D})} \det T^{-1}\sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})},
\end{aligned}$$

where the third equality follows form Prop. 1.1 and noting that $\sqrt{\det h_{ij}} : \mathcal{D} \to \mathbb{R}$, mapping $(t,x^i) \mapsto \sqrt{\det h_{ij}(t,x^i)}$, and ψ are both scalar fields. Seeing that

$$\langle\psi\circ T\rangle_{T(\mathcal{D})}(t) = \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\sqrt{\det(h_{ij})\circ T^{-1}}d^3(x\circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})},$$

we conclude $\langle\psi\rangle_{\mathcal{D}}(t) \neq \langle\psi\circ T\rangle_{T(\mathcal{D})}(t)$ in general as $\det T^{-1} : M \to \mathbb{R}$ may possibly vary of value over M. $\square$

References

[1] M. Friedman, *Foundations of space-time theories: Relativistic physics and philosophy of science*. Princeton university press, 1983.

[2] T. A. Ryckman, “Einstein, cassirer, and general covariance—then and now,” *Science in Context*, vol. 12, no. 4, pp. 585–619, 1999.

[3] A. Einstein, “Grundgedanken und methoden der relativitätstheorie, in ihrer entwicklung dargestellt,” 1920.

[4] M. Göckeler and T. Schücker, *Differential geometry, gauge theories, and gravity*. Cambridge University Press, 1989.

[5] R. M. Wald, *General relativity*. University of Chicago press, 1984.

[6] C. Rovelli and M. Gaul, “Loop quantum gravity and the meaning of diffeomorphism invariance,” in *Towards quantum gravity*, pp. 277–324, Springer, 2000.

[7] A. Einstein, *Relativity: The Special and the General Theory*. Methuen & Co, 1952.

[8] S. Hawking and G. Ellis, *The large scale structure of space-time*, vol. 1. Cambridge university press, 1973.

[9] T. Buchert, “On average properties of inhomogeneous fluids in general relativity: dust cosmologies,” *General Relativity and Gravitation*, vol. 32, no. 1, pp. 105–125, 2000.

[10] T. Buchert and S. Räsänen, “Backreaction in late-time cosmology,” *Annual Review of Nuclear and Particle Science*, vol. 62, pp. 57–79, 2012.

[11] J. M. Lee, “Smooth manifolds,” in *Introduction to Smooth Manifolds*, pp. 1–31, Springer, 2013.

[12] W. A. Sutherland, *Introduction to metric and topological spaces*. Oxford University Press, 2009.

[13] M. Visser, “Lorentzian wormholes. from einstein to hawking,” *Woodbury*, 1995.