Notes on relativistic gauge theory and backreaction

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Contents

1 Relativistic gauge theory

1.1 Equations of motion for relativity

(First provide derivation: argumentation for the EH-action - principle of least action - Einstein field equations, i.e. the equations of motion of relativity.)

1. Principle of least action
2. So, we need the action for GR. Give an argumentation for the EH-action
3. Use variational approach by invoking Hamilton's principle
4. Derive the Euler-Lagrange - or simply Lagrange's - equations of motions for the system
5. Derive Hamilton's equations of motion of GR and prove that they are equivalent to the Lagrange equations.
6. Derive the Einstein field equations from the Lagrange equations

1.2 Mathematical gauge theory for relativity

Take frame bundle and derive the group $\mathcal{G}$ of gauge transformations. Then show $\mathcal{G} \cong \mathrm{Diff}(M)$ and that equations of motion for GR (EFE, Lagrangian, Hamiltonian) are invariant under $\mathcal{G}$, i.e. it is a left-action on $\mathcal{K}$ and satisfies (3).

1.3 Invariances of a relativistic system

We will not advance the discussion by trying to define 'symmetries' as these have different meanings in physics than they have in mathematical literature.[1]

> "It is a widely shared opinion that the most outstanding and characteristic feature of General Relativity is its manifest background independence." - Giulini (2007)

[1] In the literature of physics, an invariant of a set of equations of motion need not be a symmetry, while mathematical contributions most often define a symmetry to be an invariant Giulini (2007).

To elaborate upon the property of background independence for relativity, it must first be understood what background independence is. Although there are numerous amount of philosophical contributions on the definition and meaning of background independence, we follow Rickles (2008) and Giulini (2007); providing us with a rigorous treatment of the topic. To be precise in our reasoning, we provide a generalization of a topological space.

Definition 1.1 (Lane (1996))**.** A *mathematical structure* is a set S of mathematical objects such that the objects are described axiomatically and the set has at least one relation $R \subset S \times S$ defined on it.

Example 1.2. Define the equivalence relation $\sim$ on any topological space $(S, \mathcal{T})$ by letting any two open sets $S_1, S_2 \subset S$ be equivalent if and only if $S_1, S_2 \in \mathcal{T}$. From this one sees that indeed a topological space is a mathematical structure making Definition 1.1 a generalized of a topological space.

For a given physical theory, its equations of motion take the general form

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0, \tag{1}$$

where $\mathcal{D}$ is the collection of *dynamical structures* - i.e. mathematical structures that have no a priori values and must be solved to get assigned values to - and the set $\mathcal{B}$ of *background structures* contains all non-dynamical structures. For example in GR, the metric tensor is a dynamical structure and a coordinate system is a background structure. Let $\mathcal{K}$ be the space of kinematically possible field configurations, also known as field histories, of the theory. Note that the solutions of (1) give rise to the subset $\mathcal{P} \subset \mathcal{K}$ of dynamically possible configurations.

Definition 1.3. Let G be a group acting on $\mathcal{K}$ from the left-hand side, i.e. $G \times \mathcal{K} \to \mathcal{K}$. The equations (1) of motion are called *covariant* under G if for any $\gamma \in G$,

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \gamma \cdot \mathcal{B}] = 0, \tag{2}$$

and *invariant* if

$$\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{C}[\gamma \cdot \mathcal{D}, \mathcal{B}] = 0. \tag{3}$$

These concepts allow us to define diffeomorphism covariance and invariance rigorously.

Definition 1.4. Equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ are called *diffeomorphism invariant* if and only if it allows $\mathrm{Diff}(M)$ as invariance group, i.e. $\mathrm{Diff}(M)$ defines a left-action on $\mathcal{K}$ and for any $\gamma \in \mathrm{Diff}(M)$ statement (3) is satisfied.

Remark 1.5 (Invariants in general relativity). Consider the equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ for a relativistic system. For the reason that $\mathrm{Diff}(M) \cong \mathcal{G}$, diffeomorphism invariance can be identified with gauge invariance. Observe that the diffeomorphism group $\mathrm{Diff}(M)$ is not the only invariant appearing in relativity; the Lorentz group is an invariant as well.

Definition 1.6. Two tensor fields $\tau_1, \tau_2 \in \mathcal{T}(M)$ are *locally diffeomorphism equivalent* if and only if for any point $p \in M$ there exists a neighbourhood U of p and a diffeomorphism $\phi : U \to U$ such that

$$\phi^* \left(\tau_1|_U\right) = \tau_2|_U.$$

One can readily verify that the equivalence defined in Definition 1.6 is an equivalence relation on $\mathcal{T}(M)$.

Definition 1.7. Any field which is either not dynamical or whose solutions are all locally diffeomorphism equivalent is called an *absolute structure*.

Definition 1.8. A theory is called *background independent* if and only if its equations are $\mathrm{Diff}(M)$-invariant and its fields do not include absolute structures.

Q: Should then all equations in the theory be Diff-invariant, etc.?

1.4 Gauge invariance

So what is gauge invariance? How can we be *check* whether the equations written down in GR are gauge invariant? (Look at the Hamiltonian?)

By Definition 1.4 we have that the equations of motion of GR are gauge invariant if $\mathcal{G}$ is a left action on $\mathcal{K}$ and for any gauge transformation $\gamma \in \mathcal{G} \cong \mathrm{Diff}(M)$ the

2 Thoughts on Bernard's thoughts - part I

1. *ADM formalism: problem of curvature and time homogeneity*
   I can't seem to wrap my head about the following. Consider a spacetime model with just one star in it and let us look at what the star - i.e. the matter - does to the curvature of spacetime. To do so, foliation of spacetime of spacelike hypersurfaces with each defined by level sets $t =$ constant. We can draw such level sets on each of these hypersurfaces for the spacelike basis vector x^i for $i = 1, 2, 3$, namely by drawing all lines $x^i =$ constant. Intuitively, one sees that there is a higher density of level sets close to the star compared to further away from the star. This observation is simply a resemblance of the star curving spacetime.

   The thing that I don't understand is the following. That higher density in level sets is there in every basis direction, so especially for the level sets $t =$ constant. But the thing is, how is this information of how tightly these time level sets are packed together conserved in the ADM formalism? From what in the theory can we read this density off? It seems to me that this only works if we assume the cosmology to be "homogeneous in the time direction".

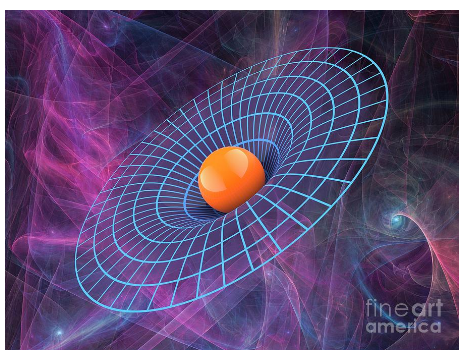

Figure 1: Intuitive depiction of the curvature of 3-dimensional spacetime.

2. *Evolution of the expansion rate of the fluid*
   Consider Equation (7) of Hawking (1966), which is an expression of the covariant derivative θ' of the expansion rate θ in the direction of a worldline. Since the covariant derivate is gauge invariant, is θ' not a possible candidate to consider in order to analyze the backreaction instead of the foliation-depend derivative $\frac{\partial}{\partial x^\mu}\theta$?

3. *Evolution of the (conformal) Weyl tensor*
   The Weyl tensor C_{abcd} is "that part of the curvature that is not determined locally by matter" and can therefore be considered as the free gravitational field Ehlers et al. (1960). The Weyl tensor is the only quantity in relativity not having any Newtonian counterpart. The tensor can be decomposed into its electric and magnetic component, denoted respectively by E_{ab} and H_{ab}. From Equation (15) we see that the rate of change E'_{ab} in the energy component has a source: the shear tensor. Equation (16) shows that H'_{ab} has no source; confirming that it represents gravitational waves without a source.

4. *Perturbations of the Weyl tensor*
   Hawking (1966) states that perturbations of the Weyl tensor do not arise from rotational or

density perturbations $\iff$

$$E_{ab}{}^{;b} = H_{ab}{}^{;b} = 0.$$

If I understand it correctly, this statement is false.

Assume that the backreaction is not a Newtonian phenomenon, then it must be captured within the Weyl tensor. The backreaction in itself is not a rotational nor a density perturbation, but causes perturbations in the Weyl tensor. This confirms that Hawking's statement is not correct.

5. *Mistake of Hawking*
   Where does Hawking (1966) you confronted Stephen with? It sounded to me at the moment you were telling me about it that your argument might be useful to understand the underlying assumptions of the ADM formalism.

6. *Backreaction a Newtonian phenomenon*
   Newtonian spacetime is defined on the Euclidean manifold $M = \mathbb{R}^4$ (Barrett, 2015), which is not a curved space. If we prove that

$$\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}} - \langle\dot{\theta}\rangle_{\mathcal{D}} > 0$$

   in some Newtonian setting, then the backreaction is not per se *not* a Newtonian phenomenon. This would imply that the backreaction is cannot be fully captured within the Weyl tensor.

7. $\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}} = \langle\dot{\theta}\rangle_{\mathcal{D}}$?
   In my eyes, we can take a compact domain $\mathcal{D} \subset M$ independent of the time component t, i.e. $\mathcal{D} \equiv D(t)$. Then for $x = (x^1, x^2, x^3)$,

$$\begin{aligned}
\int \langle\dot{\theta}\rangle_{\mathcal{D}} \,\mathrm{d}\, t &= \int \frac{\int_{\mathcal{D}} \frac{d}{dt}\theta(t, x^i)\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x}{\int_{\mathcal{D}} \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x} \,\mathrm{d}\, t && (4)\\
&= \frac{\int_{\mathcal{D}} \int \frac{d}{dt}\theta(t, x^i)\,\mathrm{d}\, t\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x}{\int_{\mathcal{D}} \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x} && (5)\\
&= \frac{\int_{\mathcal{D}} \theta(t, x^i)\sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x}{\int_{\mathcal{D}} \sqrt{\det(h_{ab}(x^i))}\,\mathrm{d}^3\, x} && (6)\\
&= \langle\theta\rangle_{\mathcal{D}} && (7)\\
&= \int \frac{d}{dt}\langle\theta\rangle_{\mathcal{D}}, && (8)
\end{aligned}$$

   from which we conclude that $\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}} = \langle\dot{\theta}\rangle_{\mathcal{D}}$. What goes wrong here? Or is Buchert not explicit enough in describing the assumptions on choosing domain $\mathcal{D}$?

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.
2. $\phi \in \mathrm{Diff}(M)$ diffeomorphism.
3. $\gamma \in \mathcal{G}$ gauge transformation.
4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.
5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.

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