Notes on gauge dependence of backreaction

March 22, 2021
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Abstract. The theory of gauges on spacetimes is formalized by introducing mathematical frame fields, which are generalizations of Einstein's tetrad fields. The frame bundle admits choices of gauge and gauge transformations as it is a principal bundle. Relating the above, we show the following:

(i) Any coordinate system and local Lorentz frame is a gauge, but a gauge need not be a choice of coordinate system nor a local Lorentz frame.

(ii) A tetrad field determines a unique metric tensor on the spacetime manifold.

(iii) The Schwarzschild and Kerr black hole and its effects on the curvature of spacetime can be constructed on manifold $\mathbb{R} \times (0, \infty) \times S^2$ by fixing an appropriate gauge; we provide explicit examples of such gauges.

(iv) We conjecture that there are four degrees of freedom to fix gauges and show that - if this is true - the Schwarzschild black hole does not make any restrictions on this freedom.

Contents

1 Tetrad fields

A tetrad is an orthonormal basis $\{\hat{e}_{(0)}, \dots, \hat{e}_{(3)}\}$, denoted by $\hat{e}_{(a)}$, of the tangent space T_pM at some point $p \in M$. For any coordinate basis $\hat{e}_{(\mu)}$ at $p \in M$, we may write

$$\hat{e}_{(\mu)}(p) = e_\mu{}^a(p)\hat{e}_{(a)},$$

where the components $e_\mu{}^a$ form an 4×4 invertible matrix. It follows that one can construct a section $e_\mu{}^a : M \to TM$, which is called the *tetrad field*, assigning a tetrad to every point in spacetime M. The metric tensor is expressible through

$$g_{\mu\nu}(p) = e_\mu{}^a(p)e_\nu{}^b(p)\eta_{ab},$$

in terms of the frame field and the Minkowski metric η_{ab}.

Remark 1.1 (Tetrads and coordinate systems). Any coordinate system (x^a) can be identified to a tetrad field, since

$$\left.\frac{\partial}{\partial x^a}\right|_p, \qquad \text{for } a = 0, 1, 2, 3, \tag{1}$$

determine a basis for T_pM for any $p \in M$. However, the converse is not true. A tetrad (λ_a) need not be equivalent coordinate system. In particular, we would need

$$\lambda_a = \frac{\partial}{\partial y^a}, \qquad \text{for } a = 0, 1, 2, 3, \tag{2}$$

for some local coordinates (y^i). Hence, (2) would need to be integrated to derive such suitable coordinate system, which cannot be done in general. We conclude that one is not always able to choose a corresponding coordinate system for any tetrad field [1, Sec. 4.2].

Remark 1.2 (Tetrads and metrics). A tetrad field determines a unique metric tensor, while in general one cannot determine the tetrad field if a metric is given. This is shown by Einstein [2] as he states: "*die Metrik ist gemäß $g_{\mu\nu} = h_{\mu a} h_{\nu a} \cdots$, durch das n-Bein-Field, aber nicht umgekehrt letzteres durch erstere bestimmt.*" (See the Appendix of [3].)

2 Relativistic gauge theory on frame fields

Let LM be the set of all (p, e_a) with e_a any basis of tangent space T_pM. By [4, p. 32], the frame bundle $L(M) = (LM, M, \pi; GL(m))$ is a principal bundle, where $\pi : LM \to M$ is the natural projection $(p, e_a) \mapsto p$. Following [5, Def 4.2.18], we define a gauge on $L(M)$.

Definition 2.1. A *gauge* or *frame field* for the principal bundle is a global section $s : M \to LM$, i.e. a map assigning a tetrad in tangent space TM to every event in spacetime manifold M. A *local gauge* or *local frame field*[1] is defined similarly, by taking a local section $s : U \subset M \to LM$.

Consequently, we can define what a change of gauge is.

Definition 2.2. A *gauge transformation* is a diffeomorphism $\phi : LM \to LM$ such that:

(i) it preserves the fibres, i.e. $\pi \circ \phi = \pi$;

(ii) it is $GL(4)$-equivariant, i.e. $\phi(p \cdot g) = \phi(p) \cdot g$ for any frame $p \in LM$ and any $g \in GL(4)$.

We denote the space of all gauge transformations by $\mathcal{G}$.

Since the composition of $GL(4)$-equivariant maps is itself $GL(4)$-equivariant and the fibres are preserved under a composition of fibre preserving maps, $\mathcal{G}$ is a group under composition.

Example 2.3 (Local Lorentz frame is a gauge). If an observer in spacetime (M, g) is freely falling, i.e. following a geodesic, his reference frame is a local Lorentz frame. Considering an event $p \in M$ in the spacetime, any local Lorentz frame at p can be locally described as a coordinate system [6, Sec. 8.6]. By construction, it follows that any local Lorentz frame is just a gauge.

Remark 2.4. The Gram-Schmidt procedure allows us to construct an orthonormal basis in any frame. Therefore, we can derive a similar statement for frame fields as Remark 1.2.

Remark 2.5 (Changes under a gauge transformation). How does metric g changes under a gauge transformation? We know how the frame changes. By Remark 2.4, we know how to express such transformed metric.

Conjecture 2.6. There are four degrees of freedom in fixing gauges.

[1] See [4, p. 33].

3 Schwarzschild black hole

We consider the spacetime manifold[2] $M = \mathbb{R}\times(E^3 - O) \cong \mathbb{R}\times(0,\infty)\times S^2$, where O is the the center of a spherical object of mass M and radius r with no electric charge nor angular momentum. Schwarzschild [7] derives an exact solution of the Einstein field equations with a vanishing cosmological constant. The Schwarzschild metric in Schwarzschild coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2, \tag{3}$$

describing the curvature of spacetime outside the spherical mass.

Lemma 3.1. The Schwarzschild metric is obtainable by a suitable choice of gauge.

Proof. Take any coordinate system (x^i), which induces a basis $\{\frac{\partial}{\partial x^i}|_p\}$ for each point p in chart neighborhood U. Then, any orthonormal frame field $e_{(a)} \in \Gamma(LM)$ can locally be expressed at an arbitrary point $p \in U$ by

$$e_{(a)}(p) = e^{\mu}{}_a(p)\frac{\partial}{\partial x^i}\Big|_p, \tag{4}$$

for some map $e^{\mu}{}_a : M \to \mathbb{R}^{4\times 4}$ - called the *vierbein field* - assigning an invertible matrix to every point $p \in M$.

Recall that $e_{(a)}(p)$ is an orthonormal basis spanning the tangent space T_pM, i.e.

$$\big(e_{(a)}(p), e_{(b)}(p)\big) = g_{\mu\nu}(p)e^{\mu}{}_a(p)e^{\nu}{}_b(p) = \eta_{ab} \qquad \forall p \in M, \tag{5}$$

where $\eta_{ab} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric of flat spacetime. We are given the Schwarzschild metric (3), which characterized by $g_{\mu\nu} = 0$ for all $\mu,\nu = 0,1,2,3$ such that $\mu \neq \nu$ and for the diagonal components,

$$g_{00} = -\left(1 - \frac{2M}{r}\right), \qquad g_{11} = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2(\theta).$$

Conveniently writing $g_{\mu\nu}(p) = g_{\mu\nu}$ and $e^{\mu}{}_a(p) = e^{\mu}{}_a$, the diagonal inner-product constraints (5) become

$$\begin{aligned}
-\left(1 - \frac{2M}{r}\right)(e^0{}_0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_0)^2 + r^2(e^2{}_0)^2 + r^2\sin^2(\theta)(e^3{}_0)^2 &= -1, \\
-\left(1 - \frac{2M}{r}\right)(e^0{}_i)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_i)^2 + r^2(e^2{}_i)^2 + r^2\sin^2(\theta)(e^3{}_i)^2 &= 1, \qquad \text{for } i = 1,2,3, \\
-\left(1 - \frac{2M}{r}\right)e^0{}_a e^0{}_b + \left(1 - \frac{2M}{r}\right)^{-1}e^1{}_a e^1{}_b + r^2 e^2{}_a e^2{}_b + r^2\sin^2(\theta)e^3{}_a e^3{}_b &= 0, \qquad \text{for } a \neq b,
\end{aligned} \tag{6}$$

which are 16 equations with 16 unknowns $e^{\mu}{}_a$. One can readily verify that

$$e^0{}_0 = \frac{1}{\sqrt{1 - 2M/r}}, \quad e^1{}_1 = \sqrt{1 - 2M/r}, \quad e^2{}_2 = \frac{1}{r}, \quad e^3{}_3 = \frac{1}{r\sin(\theta)}, \quad e^{\mu}{}_a = 0, \tag{7}$$

for $\mu \neq a$, solves system (6). Notice that it is *not* unique as the solution (7) holds as well if one takes $e^2{}_2 = -\frac{1}{r}$ instead. In vector notation, (7) is described by

$$e_{(0)} = \frac{1}{\sqrt{1 - 2M/r}}\partial_t, \quad e_{(1)} = \sqrt{1 - 2M/r}\partial_r, \quad e_{(2)} = \frac{1}{r}\partial_\theta, \quad e_{(3)} = \frac{1}{r\sin(\theta)}\partial_\phi, \tag{8}$$

as is in accordance with the metric tensor (3). Observe that, by the inner-product signature constraint (5), $\{e_{(a)}\}$ described in (7) is an orthonormal basis by construction. □

[2] We take the Minkowski metric of flat spacetime to be $\eta = \mathrm{diag}(-1,+1,+1,+1)$.

Lemma 3.1, in short, tells us that by choosing an appropriate gauge $s : M \to LM$, e.g. setting $s(t, r, \theta, \phi) = e_{(a)}$ with $\{e_{(a)}\}$ defined as in (5), one constructs a Schwarzschild black hole in M and with it its corresponding metric.

If a gauge describing the Schwarzschild black hole is fixed, it turns out that no degrees of freedom is lost compared to an arbitrary gauge transformation on the spacetime manifold.

Proposition 3.2. There are four degrees of freedom choosing gauges which describe the Schwarzschild black hole.

Proof. Note that the first column of matrix $(e^{\mu}{}_{a})$ cannot be 'altered' with respect to solution described in (5) and at most one variable can be changed per row. This makes that we can set four $e^{\mu}{}_{a} = 0$ to non-zero values. Hence, there is at least four degrees of freedom for fixing an appropriate gauge. By Conjecture 2.6, we find that there is at most four degrees of freedom, which concludes the proof. $\square$

4 Kerr black hole

Kerr [8] was the first to derive the metric tensor describing the curvature of spacetime outside of a rotating black hole of mass[3] M with angular momentum[4] J. In Boyer-Lindquist coordinates, the metric is

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\phi^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2} \sin^2\theta d\phi^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi \tag{9}$$

Lemma 4.1. The Kerr metric is obtainable by a suitable choice of gauge.

Proof. Take any coordinate system (x^i), which induces a basis $\{\frac{\partial}{\partial x^i}|_p\}$ for each point p in chart neighborhood U. Then, any orthonormal frame field $e_{(a)} \in \Gamma(LM)$ can locally be expressed at an arbitrary point $p \in U$ by

$$e_{(a)}(p) = e^{\mu}{}_{a}(p) \frac{\partial}{\partial x^i}\Big|_p, \tag{10}$$

for some vierbein field $e^{\mu}{}_{a} : M \to \mathbb{R}^{4\times4}$.

Recall that $e_{(a)}(p)$ is an orthonormal basis spanning the tangent space T_pM, i.e.

$$g_{\mu\nu}(p) = e_{\mu}{}^{a}(p) e_{\nu}{}^{b}(p) \eta_{ab} \qquad \forall p \in M, \tag{11}$$

where we inverted the inner-product condition (5) using orthonormality $e_{\mu}{}^{a}(p) e_{\nu}{}^{a}(p) = \delta^{\mu}_{\nu}$. If we write $g_{\mu\nu}(p) = g_{\mu\nu}$ and $e_{\mu}{}^{a}(p) = e_{\mu}{}^{a}$ for convenience, system (11) becomes

$$\begin{aligned}
g_{\mu\mu} &= -(e_{\mu}{}^{0})^2 + (e_{\mu}{}^{1})^2 + (e_{\mu}{}^{2})^2 + (e_{\mu}{}^{3})^2, && \text{for } \mu = 0, 1, 2, 3, \\
g_{03} &= -e_0{}^{0} e_3{}^{0} + e_0{}^{1} e_3{}^{1} + e_0{}^{2} e_3{}^{2} + e_0{}^{3} e_3{}^{3}, \\
g_{\mu\nu} &= -e_{\mu}{}^{0} e_{\nu}{}^{0} + e_{\mu}{}^{1} e_{\nu}{}^{1} + e_{\mu}{}^{2} e_{\nu}{}^{2} + e_{\mu}{}^{3} e_{\nu}{}^{3}, && \text{for } \mu \neq \nu \text{ and } (\mu, \nu) \neq (0, 3)\,.
\end{aligned} \tag{12}$$

A solution is described by symmetric $e_{\mu}{}^{a}$ with components

$$\begin{aligned}
e_0{}^{0} &= \frac{1}{\sqrt{1 - 2M/r}}, \quad e_1{}^{1} = \sqrt{1 - 2M/r}, \quad e_2{}^{2} = \frac{1}{r}, \\
e_3{}^{3} &= \sqrt{\frac{\Sigma}{\rho^2}\sin^2\theta + \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1 - 2Mr/\rho^2}}}, \quad e_3{}^{0} = \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1 - 2Mr/\rho^2}}.
\end{aligned} \tag{13}$$

[3] To be specific, M is the Schwarzschild mass [8].

[4] The spheroidal coordinates (r, θ, ϕ) are equivalent to Cartesian coordinates (x, y, z), where one of the transformations is $z = r\cos(\theta)$. The angular momentum is to be considered about this z-axis [8].

In vector notation, that is

$$e_{(0)} = \frac{1}{\sqrt{1-2M/r}}\partial_t + \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1-2Mr/\rho^2}}\partial_\phi, \qquad e_{(1)} = \sqrt{1-2M/r}\partial_r,$$
$$e_{(2)} = \frac{1}{r}\partial_\theta, \qquad e_{(3)} = \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1-2Mr/\rho^2}}\partial_t + \frac{1}{r\sin(\theta)}\partial_\phi. \tag{14}$$

□

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