Ch 4 - Cosmological backreaction

Dave Verweg
April 28, 2021

Abstract. -

Contents

1 Cosmological backreaction: gauges and gravitational waves

1.1 Cosmological backreaction

Intro to the cosmological backreaction - give small history and concrete derivation + explanation. Follow buchert and ellis + Li.

Remark 1.1 (Comment on the claims in Buchert and Räsänen (2012)). The authors do not take diffeomorphism-invariance, i.e. gauge-invariance, into account as they only consider gauge invariance for cosmological perturbation theory and not $\mathrm{Diff}(M)$.

1.2 Averaging procedure

1.2.1 Gauge dependende of the averaging procedure

Buchert (2000); Buchert and Räsänen (2012) define the following averaging procedure.

Definition 1.2. Choose a hypersurface Σ which is orthogonal to the four-velocity u^α of observers comoving with the dust. Let $\psi : M \to \mathbb{R}$ be a function of Lagrangian coordinates x^i and time t. The *spatial averaging* of scalar field ψ on an arbitrary compact domain $\mathcal{D}$ on the hypersurface Σ is defined as the volume integral

$$\langle\psi\rangle_{\mathcal{D}}(t) := \frac{1}{V_{\mathcal{D}}}\int_D \psi(t, x^i)\sqrt{\det(h_{ij})}d^3x,$$

with the *volume* of $\mathcal{D}$ defined as

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{\det(h_{ij})}d^3x,$$

where h_{ij} is the metric on Σ.

Recall that a compact domain is a non-empty connected open subset such that any open covering of the set contains a finite subcovering. We prove that the averaging procedure defined above is gauge dependent. This can be done by using by a property of the pull-back of a smooth map between manifolds on a differential form.

Proposition 1.3. Let $F : M \to N$ be a smooth map between n-dimensional smooth manifolds with (x^i) and (y^i) smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and let $u : V \to \mathbb{R}$ be continuous. Then, on $U \cap F^{-1}(V)$,

$$F^*(udy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det DF)dx^1 \wedge \cdots \wedge dx^n,$$

where DF is the Jacobian matrix of F in these coordinates.

Proof. By the properties of the pull-back on differential forms, see e.g. Lemma 14.16 of Lee (2013), we have

$$F^*(udy^1 \wedge \cdots \wedge dy^n) = (u \circ F)d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) = (u \circ F)dF^1 \wedge \cdots \wedge dF^n,$$

where F^j is the j-th component of F with respect to the local coordinates (y^i), i.e. $F^j = y^j \circ F$. We see that $F^*(udy^1 \wedge \cdots \wedge dy^n) \in \Omega^n(M)$, so it is a n-form on M.

Take any $v_1, \ldots, v_n \in T_pM$ for an arbitrary $p \in U \cap F^{-1}(V)$. We can decompose these tangent vectors into a linear combination of its basis components, i.e. $v_k = v_k^i \frac{\partial}{\partial x^i}\big|_p$ for any $k = 1, \ldots, n$ and some $v_k^i \in \mathbb{R}$. Hence,

$$\begin{aligned}
F^*(udy^1 \wedge \cdots \wedge dy^n)_p(v_1, \ldots, v_n) &= (u \circ F)_p dF_p^1 \wedge \cdots \wedge dF_p^n \left(v_1^i \frac{\partial}{\partial x^i}\Big|_p, \ldots, v_n^i \frac{\partial}{\partial x^i}\Big|_p \right) \\
&= (u \circ F)_p \det \left(dF_p^j \left(v_k^i \frac{\partial}{\partial x^i}\Big|_p \right) \right) \\
&= (u \circ F)_p \det \left(v_k^i dF_p^j \left(\frac{\partial}{\partial x^i}\Big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det \left(d(y^j \circ F)_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det(DF),
\end{aligned}$$

where the last step follows from writing out the Jacobian of F in local coordinates (x^i) and (y^i). Notice that (v_k^i) is simply the matrix of all the components of the vectors $v_1, \ldots, v_n$ with respect to local coordinates (x^i). As $dx^j \left(\frac{\partial}{\partial x^i}\right) = \delta_i^j$, we find

$$\begin{aligned}
dx_p^1 \wedge \cdots \wedge dx^n(v_1, \ldots, v_n) &= dx^1 \wedge \cdots \wedge dx^n \left(v_1^i \frac{\partial}{\partial x^i}\Big|_p, \ldots, v_n^i \frac{\partial}{\partial x^i}\Big|_p \right) \\
&= \det \left(dx^j \left(v_k^i \frac{\partial}{\partial x^i}\Big|_p \right) \right) \\
&= \det \left(v_k^i dx^j \left(\frac{\partial}{\partial x^i}\Big|_p \right) \right) \\
&= \det \left(v_k^i \delta_i^j \right) \\
&= \det \left(v_k^j \right).
\end{aligned}$$

Making

$$(u \circ F)_p(\det DF)dx_p^1 \wedge \cdots \wedge dx^n(v_1, \ldots, v_n) = (u \circ F)_p(\det DF) \det \left(v_k^j \right).$$

Since $p \in U \cap F^{-1}(V)$ was chosen arbitrarily, this concludes the proof.

An alternative proof can be formulated without explicitly evaluating on some collection of vectors $v_1, \ldots, v_n$, but rewriting $F^*(udy^1 \wedge \ldots dy^n)$ and using Leibniz's formula for the determinant. $\square$

Proposition 1.3 allows us to prove that Buchert's averaging operation depends on the chosen gauge by invoking the theorem on global change of variables.

Proposition 1.4 (Global change of variables)**.** Let M and N be smooth oriented manifolds of dimension n and $F : M \to N$ a diffeomorphism. If the differential form $\omega \in \Omega^n(N)$ has compact support, then the pull-back $F^*\omega$ has compact support and

$$\int_N \omega = \pm \int_M F^*\omega,$$

where we take the plus sign if F is orientation-preserving and the minus sign if F is orientation-reversing.

Proof. See Prop. 16.6 of Lee (2013). □

Theorem 1.5. The spatial averaging of a scalar field defined in Buchert (2000) is gauge dependent.

Proof. Let $T : M \to M$ be any gauge transformation and $\mathcal{D}$ an arbitrary compact domain on a hypersurface, which is orthogonal to the four-velocity vector of the observers comoving with the dust. Since T is a diffeomorphism, its inverse $T^{-1} : M \to M$ exists and is a diffeomorphism such that the commutative diagram (1.1) commutes.

$$\begin{tikzcd} M \arrow[r, "T^{-1}"] & M \\ T(\mathcal{D}) \arrow[u, "\iota_{T(\mathcal{D})}"] \arrow[r, "T^{-1}|_{T(\mathcal{D})}"'] & \mathcal{D} \arrow[u, "\iota_{\mathcal{D}}"'] \end{tikzcd} \tag{1.1}$$

The result follows by invoking the theorem of global change of variables for the inverse $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ restricted to $T(\mathcal{D})$ on its domain M. For this we need to prove that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth oriented manifolds of the same dimension and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation preserving.

First, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth manifolds of the same dimension. Since the image of a compact set under a continuous map between topological spaces is compact, $T(\mathcal{D})$ is a compact subspace. Since any subspace of a Hausdorff space is Hausdorff itself, cf. Sutherland (2009, p. 110), $\mathcal{D}$ and $T(\mathcal{D})$ are Hausdorff. By definition of compactness of a subspace, any open covering of $\mathcal{D}$ in M has a finite subcover, i.e. $\mathcal{D}$ is compact with the induced topology. Second countability follows by considering some open finite covering $\{B_i\}_{i\in I}$ of $\mathcal{D}$ and noting that $\{B_i \cap \mathcal{D}\}_{i\in I}$ is a countable basis for the topology of $\mathcal{D}$. Since M is a manifold, $\mathcal{D} \subset M$ is also locally Euclidean. We conclude that $\mathcal{D}$ is a smooth manifold, where smoothness follows from inducing a differentiable structure on $\mathcal{D}$ from one on M. Similarly, we can conclude that $T(\mathcal{D})$ is a smooth manifold. Noting that T is a diffeomorphism, we see $\dim \mathcal{D} = \dim T(\mathcal{D})$.

Second, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are orientable and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving. Visser (1995, p. 286) shows that non-orientable spacetime manifolds are incompatible with current theories. Therefore, we take M to be orientable. Inducing an atlas on submanifolds $\mathcal{D}$ and $T(\mathcal{D})$ from the oriented atlas of M, makes that both submanifolds are orientable. It follows from Prop. 15.5 of Lee (2013) that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving.[1]

[1]For invoking the result Lee (2013, Prop. 15.5), note that every diffeomorphism is a local diffeomorphism.

Invoking Thm. 1.4 for $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ to rewrite $\langle\psi\rangle_{\mathcal{D}}(t)$,

$$\begin{aligned}
\langle\psi\rangle_{\mathcal{D}}(t) &= \frac{\int_{\mathcal{D}} \psi(t,x^i)\sqrt{\det(h_{ij})}d^3x}{\int_{\mathcal{D}} \sqrt{\det(h_{ij})}d^3x} \\
&= \frac{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \psi(t,x^i)\sqrt{\det(h_{ij})}dx^1 \wedge dx^2 \wedge dx^3}{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \sqrt{\det(h_{ij})}dx^1 \wedge dx^2 \wedge dx^3} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\sqrt{\det(h_{ij}\circ T^{-1})}d(x^1\circ T^{-1}) \wedge d(x^2\circ T^{-1}) \wedge d(x^3\circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij}\circ T^{-1})}d(x^1\circ T^{-1}) \wedge d(x^2\circ T^{-1}) \wedge d(x^3\circ T^{-1})} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\det T^{-1}\sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})}{\int_{T(\mathcal{D})} \det T^{-1}\sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})},
\end{aligned}$$

where the third equality follows form Prop. 1.3 and noting that $\sqrt{\det h_{ij}} : \mathcal{D} \to \mathbb{R}$, mapping $(t,x^i) \mapsto \sqrt{\det h_{ij}(t,x^i)}$, and ψ are both scalar fields. Seeing that

$$\langle\psi\circ T^{-1}\rangle_{T(\mathcal{D})}(t) = \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t,x^i)\sqrt{\det(h_{ij})\circ T^{-1}}d^3(x\circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij}\circ T^{-1})}d^3(x\circ T^{-1})},$$

we conclude $\langle\psi\rangle_{\mathcal{D}}(t) \neq \langle\psi\circ T^{-1}\rangle_{T(\mathcal{D})}(t)$ in general as $\det T^{-1} : M \to \mathbb{R}$ may possibly vary of value over M. □

1.2.2 Gauge invariant averaging procedure

The gauge invariant and covariant averaging of a scalar field proposed by Gasperini et al. (2009) amounts to the following. Consider

> "a cylinder-like domain Ω of M with temporal boundaries determined by the two space-like hypersurfaces on which a suitable scalar field $A(x)$ assumes the constant values A_1 and A_2; the region is bounded in space by the coordinate condition $B(x) < r_0$, where $B(x)$ is a suitable (positive) function of the coordinates with space-like gradient $\partial_\mu B$, and B_0 a positive constant."
> - Gasperini et al. (2009)

To average over a hypersurface on which $A(x) = A_0$ is constant, Gasperini et al. (2009) define the window function

$$W_\Omega(x) = \delta\big(A(x) - A_0\big)\mathbb{1}\big(B_0 - B(x)\big),$$

where δ and $\mathbb{1}$ are the delta and unit step function, respectively. The average of a scalar field $\psi : M \to \mathbb{R}$ over Σ is defined to be

$$\langle\psi\rangle_\Omega \coloneqq \frac{\int_M \psi\sqrt{\det(g_{ij})}W_\Omega d^4x}{\int_M \sqrt{\det(g_{ij})}W_\Omega d^4x}, \tag{1.2}$$

which is covariant and gauge invariant if $B(x) \mapsto B\big(\gamma^{-1}(x)\big)$ under any gauge transformation $\gamma \in \mathrm{Diff}(M)$ (Gasperini et al., 2009).

Remark 1.6. Gasperini et al. (2009) state

> "that in the case of interest to this paper - i.e. for quasi-homogeneous cosmological backgrounds - a natural candidate for the scalar field $B(x)$ with space-like gradient is missing."

They conclude that one should take $B_0 \to \infty$ in general and thereby pursuing that one should average over all of the hypersurface $A(x) = A_0$. This might not be desirable for the analysis of the cosmological backreaction.

We provide a gauge invariant and covariant spatial averaging method for specific, but possibly finite, domains on a hypersurface of the spacetime foliation. We do so by adopting averaging procedure (1.2) and by specifying a scalar field $B(x)$ satisfying the conditions.

Consider a foliation $A(x) = \text{const.}$ of spacetime. Next we construct a compact and possibly finite domain $\mathcal{D}$. Consider the hypersurface $\Sigma \subset M$ specified by $A(x) = A_0$ over which one desire to average over. Note that the metric d of the metric space (M, d) corresponding to our Lorentzian spacetime manifold (M, g) exists. Such metric exists as M is second countable **?**p. 271]kobayashi1963foundations. Take a point $p \in \Sigma$ and define $B(x)$ to be

$$d_p : M \to \mathbb{R}, \qquad x \mapsto d_p(x) \coloneqq d(p, x).$$

The constants A_0, B_0 and the map $B(x)$ define the domain $\mathcal{D}$ as was done for domain Σ by Gasperini et al. (2009).

Throughout the rest op this paper, we assume such choice of domain $\mathcal{D}$ on a hypersurface has been constructed as was done above.

Proposition 1.7. Let $\mathcal{D}$ be any such constructed domain. In local coordinates, average (1.2) is equal to

$$\langle \psi \rangle_{\mathcal{D}}(t) = \frac{\int_\Sigma \psi \sqrt{\det(g_{ij})} W_\Omega(t) d^3x}{\int_\Sigma \sqrt{\det(g_{ij})} W_\Omega(t) d^3x}, \tag{1.3}$$

and is gauge invariant.[2]

Proof. Pick any such domain $\mathcal{D}$. Since $B : M \to \mathbb{R}$ is a scalar field, it transforms as $B(x) \mapsto B\big(\gamma^{-1}(x)\big)$, thus average (1.2) is gauge invariant.

Consider flow-orthogonal coordinate system $x^\mu \coloneqq (t, x^i)$ and let constant t_0 be such that $t = t_0$ specifies hypersurface Σ. Accordingly, we can write

$$W_\Omega(t, x^i) = \delta\big(t - t_0\big) \mathbb{1}\big(B_0 - B(x^i)\big).$$

Adopting this into (1.2), the average is

$$\begin{aligned}
\langle \psi \rangle_\Omega &= \frac{\int_M \psi(t_0, x^i) \sqrt{\det\big(g_{ij}(t_0, x^i)\big)} W_\Omega(t_0, x^i) dt \wedge d^3x}{\int_M \sqrt{\det\big(g_{ij}(t_0, x^i)\big)} W_\Omega(t_0, x^i) dt \wedge d^3x} \\
&= \frac{\int_\Sigma \psi(t_0, x^i) \sqrt{\det\big(g_{ij}(t_0, x^i)\big)} W_\Omega(t_0, x^i) d^3x}{\int_\Sigma \sqrt{\det\big(g_{ij}(t_0, x^i)\big)} W_\Omega(t_0, x^i) d^3x} \\
&= \frac{\int_\Sigma \psi \sqrt{\det(g_{ij})} W_\Omega(t_0) d^3x}{\int_\Sigma \sqrt{\det(g_{ij})} W_\Omega(t_0) d^3x},
\end{aligned}$$

as $t = t_0$ is constant and where at the least step we leave out the explicit dependence on the spatial coordinates x^i. By t for t_0 and by noting that $\langle \psi \rangle_{\mathcal{D}}(t) \coloneqq \langle \psi \rangle_\Omega(t) \equiv \langle \psi \rangle_\Omega$, we conclude that (1.3) holds. □

1.3 Backreaction is not a Newtonian phenomenon

The backreaction is an effect on the curvature of spacetime determined by the deviation from homogeneity or isotropy in some finite domain of spacetime. This effect must, therefore, be containted in "the part of the spacetime curvature not determined locally by the matter at a point, but rather determined by conditions elsewhere."[3] That part of the spacetime curvature is exactly represented by the the Weyl tensor and, thus, we expect the backreaction effect to be fully captured within this tensor.

Adopting the definition from Buchert et al. (2015) of the *backreaction* to be

[2] We take $d^3x \coloneqq dx^1 \wedge dx^2 \wedge dx^3$.

[3] The quotation is the intuitive definition of Ellis et al. (2012, Sec. 2.7.6) for the Weyl tensor.

"deviation of spatial average properties of an inhomogeneous universe model from the values predicted by a homogeneous-isotropic universe model."

1.3.1 Newtonian and general relativistic evolution of expansion rate.

Comparing the modified Raychaudhuri equation for a Newtonian cosmology derived in Buchert and Ehlers (1995, Eq. 8) with its general relativistic representative showed in e.g. Buchert (2000, Eq. 11c), we observe that they are both of the exact same form:

$$\partial_t \langle \theta \rangle = \Lambda - 4\pi G \langle \rho \rangle + \frac{2}{3}\left(\langle \theta^2 \rangle - \langle \theta \rangle^2 \right) - 2\langle \sigma^2 \rangle, \tag{1.4}$$

for an irrotational model.

Considering the commutation identity for any scalar tensor field ψ,

$$\partial_t \langle \psi \rangle - \langle \partial_t \psi \rangle = \langle \theta \psi \rangle - \langle \theta \rangle \langle \psi \rangle, \tag{1.5}$$

where "the fluctuation part on the right-hand side of this rule produces the *kinematical backreaction*" Ellis and Buchert (2005).

1.3.2 Weyl tensor

The problem reduces to showing that there is some part of the backreaction effect contained within the Weyl tensor, i.e. by showing that the commutator $\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle$ solely depends on the Weyl tensor C_{abcd}.

Notice that if the backreaction is not captured entirely within the Weyl tensor, also called the *conformal tensor*, then there must exist a spacetime with a vanishing Weyl tensor $C_{abcd} = 0$, but wherein the backreaction effect is still present. If we find it is not, we can conclude that the backreaction is not a Newtonian phenomenon.

Recall that $h^a{}_b = g^a{}_b + u^a u_b$ is a projection tensor projecting into the three-dimensional tangent space orthogonal to u^a (Ellis et al., 2012, Sec. 4.4). For any vector $X^a \in T_pM$ at some point $p \in M$, the projection $X^a_\perp$ orthogonal to u^a is

$$X^a_\perp = h^a{}_b X^b,$$

making that $X^a_\perp u_a = 0$, and

$$\perp E'_{ab} = h_a{}^d h_b{}^e E'_{de}.$$

Lemma 1.8. Let (M, g) be a spacetime with its matter fields $\{\tau_i : M \to \mathcal{T}(M) \mid i \in I\}$ such that the rotation tensor vanishes, i.e. $\omega^\mu{}_\nu = 0$. The electric component E_{ab} of the Weyl tensor C_{abcd} is depended on the commutator $\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle$ via the relation

$$\begin{aligned}\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle \\ - \left(\frac{2}{\rho+p}\right)^2 \Big\langle \Big(\perp E'^a{}_b + E^a{}_b\theta - E^{c(a}\sigma_{b)c} - \eta^a{}_{cde}\eta_{bpqr}u^c u^p \sigma^{dq} E^{er} + 2H^{d(a}\eta_{b)cde}u^c u'^e \Big) \\ \cdot \Big(\perp E'_a{}^b + E_a{}^b\theta - E_{c(a}\sigma^{b)}{}_c - \eta_{acde}\eta^b_{pqr}u^c u^p \sigma^{dq}E^{er} + 2H^d{}_{(a}\eta^{b)}{}_{cde}u^c u'^e \Big) \Big\rangle_D . \end{aligned} \tag{1.6}$$

To the first order, that is,

$$\partial_t\langle\theta\rangle_D = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \left(\frac{2}{\rho+p}\right)^2 \big\langle \left(E'^a{}_b + E^a{}_b\theta\right)\left(E'^b{}_a + E^b{}_a\theta\right) \big\rangle_D . \tag{1.7}$$

Proof. Let ρ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that the evolution of E_{ab} can be described by

$$-\frac{1}{2}(\rho+p)\sigma_{ab} = \perp E'_{ab} + E_{ab}\theta + h_{(a}{}^{f}\eta_{b)cde}u^c H_f{}^{d;e} \\ - E^c{}_{(a}\sigma_{b)c} - E^c{}_{(a}\omega_{b)c} - \eta_{acde}\eta_{bpqr}u^c u^p \sigma^{dq}E^{er} + 2H^d{}_{(a}\eta_{b)cde}u^c u'^e, \tag{1.8}$$

hereby $\perp E'_{ab}$ is the projection of E'_{ab} under the projection tensor $h_a{}^b = g_a{}^b + u_a u^b$, i.e. $\perp E'_{ab} = h_a{}^d h_b{}^e E'_{de}$, and where the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\rho+p)\eta_{abcd}\omega^{cd}u^b, \tag{1.9}$$

of the magnetic component H_{ab} of the Weyl tensor. As we consider an irrotational dust, we see that $\omega^{cd} = g^{di}\omega^c{}_i = 0$, implying that $E^c{}_{(a)}\omega_{b)c} = 0$ and that covariant derivative (1.9) also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (1.8) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\rho+p}\left(\perp E'_{ab} + E_{ab}\theta - E^c{}_{(a}\sigma_{b)c} - \eta_{acde}\eta_{bpqr}u^c u^p \sigma^{dq}E^{er} + 2H^d{}_{(a}\eta_{b)cde}u^c u'^e\right), \tag{1.10}$$

if $\rho + p \neq 0$. Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\rho+p)^2}\left(\perp E'^a{}_b + E^a{}_b\theta - E^{c(a}\sigma_{b)c} - \eta^a{}_{cde}\eta_{bpqr}u^c u^p\sigma^{dq}E^{er} + 2H^{d(a}\eta_{b)cde}u^c u'^e\right) \\ \cdot\left(\perp E'_a{}^b + E_a{}^b\theta - E_{c(a}\sigma^{b)}{}_c - \eta_{acde}\eta^b_{pqr}u^c u^p\sigma^{dq}E^{er} + 2H^d{}_{(a}\eta^{b)}{}_{cde}u^c u'^e\right) \tag{1.11}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{1.12}$$

of Buchert (2000), one sees that (1.6) holds.

To the first order.
Let ρ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that to the first order, the evolution of E_{ab} can be described by

$$E'_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)}{}_{cde}u^c H_f{}^{d;e} = -\frac{1}{2}(\rho+p)\sigma_{ab}, \tag{1.13}$$

with the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\rho+p)\eta_{abcd}\omega^{cd}u^b, \tag{1.14}$$

of the magnetic component H_{ab} of the Weyl tensor. Since $\omega^{cd} = g^{di}\omega^c{}_i = 0$, its covariant derivative also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (1.13) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\rho+p}\left(E'_{ab} + E_{ab}\theta\right). \tag{1.15}$$

Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\rho+p)^2}\left(E'^a{}_b + E^a{}_b\theta\right)\left(E'^b{}_a + E^b{}_a\theta\right). \tag{1.16}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{1.17}$$

of Buchert (2000), one sees that (1.7) holds. ∎

Remark 1.9 (Covariance and gauge invariance). It should be noted that the result of Lem. 1.8 is not covariant as the partial differential operator $\frac{\partial}{\partial t}$ is depended on the coordinate system.

Lemma 1.10. For an irrotational dust with congruence $\mathcal{O}$, we have

$$\perp E'_{ab} = \frac{1}{6}\rho\theta h_{ab} - E_{ab}\theta - \frac{1}{3}\theta E^c{}_{(a}h_{b)c} - \frac{1}{3}\theta\eta_{acde}\eta_{bpqr}u^c u^p h^{dq} E^{er} - 2H^d{}_{(a}\eta_{b)cde}u^c u'^e. \tag{1.18}$$

Proof. An ideal fluid is pressure free, i.e. $p = 0$. The result is immediate from Eq. (3) of Hawking (1966) by noting that

$$\sigma_{ab} = -\frac{1}{3}\theta h_{ab}$$

as $\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta h_{ab}$, where $B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$ with $\omega_{ab} = 0$ (Poisson, 2004, Sec. 2.3). □

For a given cosmology, one assumes that the field equations hold, which come with some metric tensor g_{ab}. The 3-metric h_{ab} is then derived by $h_{ab} = g_{ab} + u_a u_b$, implying that the propagation of gravitational waves is influenced by the mass density ρ and the expansion rate θ. However, the evolution of the expansion rate θ is fully determined by the (evolution of) electric part E_{ab} of the Weyl tensor and the mass density ρ.

The evolution of the expansion rate θ is solely described by the density ρ given an initial value of θ.

Lemma 1.11. For an irrotational dust with congruence $\mathcal{O}$, we have

$$\frac{d\theta}{d\tau} = -\frac{4}{9}\theta^2 - 8\pi\rho\left(\xi^0\xi^0 + \frac{1}{2}\right). \tag{1.19}$$

Proof. For an irrotational dust, the Raychaudhuri equation reduces to

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{cd}\xi^c\xi^d. \tag{1.20}$$

In the proof of Lem. 1.10, we found that $\sigma = -\frac{1}{3}\theta h_{ab}$. By the field equations, the last term in (1.20) is

$$R_{cd}\xi^c\xi^d = 8\pi\left(T_{ab} + \frac{1}{2}T\right) = 8\pi\rho\left(\xi^0\xi^0 + \frac{1}{2}\right), \tag{1.21}$$

where $T \coloneqq T^a{}_a = \rho$ is the trace of T_{ab}. In the last step we use $T_{ab} = \rho\delta^0{}_a\delta^0{}_b$. □

The result of Lem. 1.11 is a manifestation of the physical effect of gravity: a positive mass density $\rho(p)$ at some $p \in M$ makes the geodesics converge to another as according to (1.19) the derivative $\frac{d\theta}{d\tau}$ of expansion rate with respect to the proper time τ is negative (given an initial value $\theta_0 < 0$).

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.
2. $\phi \in \mathrm{Diff}(M)$ diffeomorphism.
3. $\gamma \in \mathcal{G}$ gauge transformation.
4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.
5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.

References

Buchert, T. (2000). On average properties of inhomogeneous fluids in general relativity: dust cosmologies. *General Relativity and Gravitation*, 32(1):105–125.

Buchert, T., Carfora, M., Ellis, G. F. R., Kolb, E. W., MacCallum, M. A. H., Ostrowski, J. J., Räsänen, S., Roukema, B. F., Andersson, L., Coley, A. A., and et al. (2015). Is there proof that backreaction of inhomogeneities is irrelevant in cosmology? *Classical and Quantum Gravity*, 32(21):215021.

Buchert, T. and Ehlers, J. (1995). Averaging inhomogeneous newtonian cosmologies. *arXiv preprint astro-ph/9510056*.

Buchert, T. and Räsänen, S. (2012). Backreaction in late-time cosmology. *Annual Review of Nuclear and Particle Science*, 62:57–79.

Ellis, G. F. and Buchert, T. (2005). The universe seen at different scales. *Physics Letters A*, 347(1-3):38–46.

Ellis, G. F., Maartens, R., and MacCallum, M. A. (2012). *Relativistic cosmology*. Cambridge University Press.

Gasperini, M., Marozzi, G., and Veneziano, G. (2009). Gauge invariant averages for the cosmological backreaction. *Journal of Cosmology and Astroparticle Physics*, 2009(03):011.

Hawking, S. (1966). Perturbations of an expanding universe. *The Astrophysical Journal*, 145:544.

Lee, J. M. (2013). Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer.

Poisson, E. (2004). *A relativist's toolkit: the mathematics of black-hole mechanics*. Cambridge university press.

Sutherland, W. A. (2009). *Introduction to metric and topological spaces*. Oxford University Press.

Visser, M. (1995). Lorentzian wormholes. from einstein to hawking. *Woodbury*.