Ch 4 - Cosmological backreaction

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Abstract. -

Contents

1   Cosmological backreaction: gauges and gravitational waves

1.1   Cosmological backreaction

Intro to the cosmological backreaction - give small history and concrete derivation + explanation. Follow buchert and ellis + Li.

Remark 1.1 (Comment on the claims in Buchert and Räsänen (2012)). The authors do not take diffeomorphism-invariance, i.e. gauge-invariance, into account as they only consider gauge invariance for cosmological perturbation theory and not $\mathrm{Diff}(M)$.

1.2   Gauge dependende of the averaging procedure

Buchert (2000); Buchert and Räsänen (2012) define the following averaging procedure.

Definition 1.2. Choose a hypersurface Σ which is orthogonal to the four-velocity u^α of observers comoving with the dust. Let $\psi : M \to \mathbb{R}$ be a function of Lagrangian coordinates x^i and time t. The *spatial averaging* of scalar field ψ on an arbitrary compact domain $\mathcal{D}$ on the hypersurface Σ is defined as the volume integral

$$\langle \psi \rangle_{\mathcal{D}}(t) := \frac{1}{V_{\mathcal{D}}} \int_D \psi(t, x^i) \sqrt{\det(h_{ij})} d^3 x,$$

with the *volume* of $\mathcal{D}$ defined as

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{\det(h_{ij})} d^3 x,$$

where h_{ij} is the metric on Σ.

Recall that a compact domain is a non-empty connected open subset such that any open covering of the set contains a finite subcovering. We prove that the averaging procedure defined above is gauge dependent. This can be done by using by a property of the pull-back of a smooth map between manifolds on a differential form.

Proposition 1.3. Let $F : M \to N$ be a smooth map between n-dimensional smooth manifolds with (x^i) and (y^i) smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and let $u : V \to \mathbb{R}$ be continuous. Then, on $U \cap F^{-1}(V)$,

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det DF) dx^1 \wedge \cdots \wedge dx^n,$$

where DF is the Jacobian matrix of F in these coordinates.

Proof. By the properties of the pull-back on differential forms, see e.g. Lemma 14.16 of Lee (2013), we have

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n,$$

where F^j is the j-th component of F with respect to the local coordinates (y^i), i.e. $F^j = y^j \circ F$. We see that $F^*(u dy^1 \wedge \cdots \wedge dy^n) \in \Omega^n(M)$, so it is a n-form on M.

Take any $v_1, \ldots, v_n \in T_p M$ for an arbitrary $p \in U \cap F^{-1}(V)$. We can decompose these tangent vectors into a linear combination of its basis components, i.e. $v_k = v_k^i \frac{\partial}{\partial x^i}\big|_p$ for any $k = 1, \ldots, n$ and some $v_k^i \in \mathbb{R}$. Hence,

$$
\begin{aligned}
F^*(u dy^1 \wedge \cdots \wedge dy^n)_p(v_1, \ldots, v_n) &= (u \circ F)_p dF_p^1 \wedge \cdots \wedge dF_p^n \left(v_1^i \frac{\partial}{\partial x^i}\big|_p, \ldots, v_n^i \frac{\partial}{\partial x^i}\big|_p \right) \\
&= (u \circ F)_p \det \left(dF_p^j \left(v_k^i \frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det \left(v_k^i dF_p^j \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det \left(d(y^j \circ F)_p \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= (u \circ F)_p \det(v_k^i) \det(DF),
\end{aligned}
$$

where the last step follows from writing out the Jacobian of F in local coordinates (x^i) and (y^i). Notice that (v_k^i) is simply the matrix of all the components of the vectors $v_1, \ldots, v_n$ with respect to local coordinates (x^i). As $dx^j \left(\frac{\partial}{\partial x^i} \right) = \delta_i^j$, we find

$$
\begin{aligned}
dx_p^1 \wedge \cdots \wedge dx^n(v_1, \ldots, v_n) &= dx^1 \wedge \cdots \wedge dx^n \left(v_1^i \frac{\partial}{\partial x^i}\big|_p, \ldots, v_n^i \frac{\partial}{\partial x^i}\big|_p \right) \\
&= \det \left(dx^j \left(v_k^i \frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= \det \left(v_k^i dx^j \left(\frac{\partial}{\partial x^i}\big|_p \right) \right) \\
&= \det \left(v_k^i \delta_i^j \right) \\
&= \det \left(v_k^j \right).
\end{aligned}
$$

Making

$$(u \circ F)_p (\det DF) dx_p^1 \wedge \cdots \wedge dx^n(v_1, \ldots, v_n) = (u \circ F)_p (\det DF) \det \left(v_k^j \right).$$

Since $p \in U \cap F^{-1}(V)$ was chosen arbitrarily, this concludes the proof.

An alternative proof can be formulated without explicitly evaluating on some collection of vectors $v_1, \ldots, v_n$, but rewriting $F^*(u dy^1 \wedge \ldots dy^n)$ and using Leibniz's formula for the determinant. $\square$

Proposition 1.3 allows us to prove that Buchert's averaging operation depends on the chosen gauge by invoking the theorem on global change of variables.

Proposition 1.4 (Global change of variables). Let M and N be smooth oriented manifolds of dimension n and $F : M \to N$ a diffeomorphism. If the differential form $\omega \in \Omega^n(N)$ has compact support, then the pull-back $F^*\omega$ has compact support and

$$\int_N \omega = \pm \int_M F^*\omega,$$

where we take the plus sign if F is orientation-preserving and the minus sign if F is orientation-reversing.

Proof. See Prop. 16.6 of Lee (2013). □

Theorem 1.5. The spatial averaging of a scalar field defined in Buchert (2000) is gauge dependent.

Proof. Let $T : M \to M$ be any gauge transformation and $\mathcal{D}$ an arbitrary compact domain on a hypersurface, which is orthogonal to the four-velocity vector of the observers comoving with the dust. Since T is a diffeomorphism, its inverse $T^{-1} : M \to M$ exists and is a diffeomorphism such that the commutative diagram (1.1) commutes.

$$
\begin{array}{ccc}
M & \xrightarrow{\;\;T^{-1}\;\;} & M \\[4pt]
\Big\uparrow{\scriptstyle\iota_{T(\mathcal{D})}} & & \Big\uparrow{\scriptstyle\iota_{\mathcal{D}}} \\[4pt]
T(\mathcal{D}) & \xrightarrow[\;T^{-1}|_{T(\mathcal{D})}\;]{} & \mathcal{D}
\end{array}
\tag{1.1}
$$

The result follows by invoking the theorem of global change of variables for the inverse $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ restricted to $T(\mathcal{D})$ on its domain M. For this we need to prove that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth oriented manifolds of the same dimension and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation preserving.

First, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are smooth manifolds of the same dimension. Since the image of a compact set under a continuous map between topological spaces is compact, $T(\mathcal{D})$ is a compact subspace. Since any subspace of a Hausdorff space is Hausdorff itself, cf. Sutherland (2009, p. 110), $\mathcal{D}$ and $T(\mathcal{D})$ are Hausdorff. By definition of compactness of a subspace, any open covering of $\mathcal{D}$ in M has a finite subcover, i.e. $\mathcal{D}$ is compact with the induced topology. Second countability follows by considering some open finite covering $\{B_i\}_{i \in I}$ of $\mathcal{D}$ and noting that $\{B_i \cap \mathcal{D}\}_{i \in I}$ is a countable basis for the topology of $\mathcal{D}$. Since M is a manifold, $\mathcal{D} \subset M$ is also locally Euclidean. We conclude that $\mathcal{D}$ is a smooth manifold, where smoothness follows from inducing a differentiable structure on $\mathcal{D}$ from one on M. Similarly, we can conclude that $T(\mathcal{D})$ is a smooth manifold. Noting that T is a diffeomorphism, we see $\dim \mathcal{D} = \dim T(\mathcal{D})$.

Second, we show that $\mathcal{D}$ and $T(\mathcal{D})$ are orientable and that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving. Visser (1995, p. 286) shows that non-orientable spacetime manifolds are incompatible with current theories. Therefore, we take M to be orientable. Inducing an atlas on submanifolds $\mathcal{D}$ and $T(\mathcal{D})$ from the oriented atlas of M, makes that both submanifolds are orientable. It follows from Prop. 15.5 of Lee (2013) that $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ is orientation-preserving.[1]

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[1]For invoking the result Lee (2013, Prop. 15.5), note that every diffeomorphism is a local diffeomorphism.

Invoking Thm. 1.4 for $T^{-1}|_{T(\mathcal{D})} : T(\mathcal{D}) \to \mathcal{D}$ to rewrite $\langle \psi \rangle_{\mathcal{D}}(t)$,

$$
\begin{aligned}
\langle \psi \rangle_{\mathcal{D}}(t) &= \frac{\int_{\mathcal{D}} \psi(t, x^i) \sqrt{\det(h_{ij})} d^3 x}{\int_{\mathcal{D}} \sqrt{\det(h_{ij})} d^3 x} \\
&= \frac{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \psi(t, x^i) \sqrt{\det(h_{ij})} dx^1 \wedge dx^2 \wedge dx^3}{\int_{T(\mathcal{D})} \left(T^{-1}\right)^* \sqrt{\det(h_{ij})} dx^1 \wedge dx^2 \wedge dx^3} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t, x^i) \sqrt{\det(h_{ij} \circ T^{-1})} d(x^1 \circ T^{-1}) \wedge d(x^2 \circ T^{-1}) \wedge d(x^3 \circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij} \circ T^{-1})} d(x^1 \circ T^{-1}) \wedge d(x^2 \circ T^{-1}) \wedge d(x^3 \circ T^{-1})} \\
&= \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t, x^i) \det T^{-1} \sqrt{\det(h_{ij} \circ T^{-1})} d^3(x \circ T^{-1})}{\int_{T(\mathcal{D})} \det T^{-1} \sqrt{\det(h_{ij} \circ T^{-1})} d^3(x \circ T^{-1})},
\end{aligned}
$$

where the third equality follows form Prop. 1.3 and noting that $\sqrt{\det h_{ij}} : \mathcal{D} \to \mathbb{R}$, mapping $(t, x^i) \mapsto \sqrt{\det h_{ij}(t, x^i)}$, and ψ are both scalar fields. Seeing that

$$
\langle \psi \circ T \rangle_{T(\mathcal{D})}(t) = \frac{\int_{T(\mathcal{D})} \psi \circ T^{-1}(t, x^i) \sqrt{\det(h_{ij}) \circ T^{-1}} d^3(x \circ T^{-1})}{\int_{T(\mathcal{D})} \sqrt{\det(h_{ij} \circ T^{-1})} d^3(x \circ T^{-1})},
$$

we conclude $\langle \psi \rangle_{\mathcal{D}}(t) \neq \langle \psi \circ T \rangle_{T(\mathcal{D})}(t)$ in general as $\det T^{-1} : M \to \mathbb{R}$ may possibly vary of value over M. $\qquad \square$

1.3 Backreaction is not a Newtonian phenomenon

The backreaction is an effect on the curvature of spacetime determined by the deviation from homogeneity or isotropy in some finite domain of spacetime. This effect must, therefore, be containted in "the part of the spacetime curvature not determined locally by the matter at a point, but rather determined by conditions elsewhere."[2] That part of the spacetime curvature is exactly represented by the the Weyl tensor and, thus, we expect the backreaction effect to be fully captured within this tensor.

Adopting the definition from Buchert et al. (2015) of the *backreaction* to be

"deviation of spatial average properties of an inhomogeneous universe model from the values predicted by a homogeneous-isotropic universe model."

1.3.1 Newtonian and general relativistic evolution of expansion rate.

Comparing the modified Raychaudhuri equation for a Newtonian cosmology derived in Buchert and Ehlers (1995, Eq. 8) with its general relativistic representative showed in e.g. Buchert (2000, Eq. 11c), we observe that they are both of the exact same form:

$$
\partial_t \langle \theta \rangle = \Lambda - 4\pi G \langle \rho \rangle + \frac{2}{3} \left(\langle \theta^2 \rangle - \langle \theta \rangle^2 \right) - 2\langle \sigma^2 \rangle, \tag{1.2}
$$

for an irrotational model.

Considering the commutation identity for any scalar tensor field ψ,

$$
\partial_t \langle \psi \rangle - \langle \partial_t \psi \rangle = \langle \theta \psi \rangle - \langle \theta \rangle \langle \psi \rangle, \tag{1.3}
$$

where "the fluctuation part on the right-hand side of this rule produces the *kinematical backreaction*" Ellis and Buchert (2005).

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[2]The quotation is the intuitive definition of Ellis et al. (2012, Sec. 2.7.6) for the Weyl tensor.

1.3.2 Weyl tensor

The problem reduces to showing that there is some part of the backreaction effect contained within the Weyl tensor, i.e. by showing that the commutator $\partial_t \langle \theta \rangle - \langle \partial_t \theta \rangle$ solely depends on the Weyl tensor C_{abcd}.

Notice that if the backreaction is not captured entirely within the Weyl tensor, also called the *conformal tensor*, then there must exist a spacetime with a vanishing Weyl tensor $C_{abcd} = 0$, but wherein the backreaction effect is still present. If we find it is not, we can conclude that the backreaction is not a Newtonian phenomenon.

Recall that $h^a{}_b = g^a{}_b + u^a u_b$ is a projection tensor projecting into the three-dimensional tangent space orthogonal to u^a (Ellis et al., 2012, Sec. 4.4). For any vector $X^a \in T_p M$ at some point $p \in M$, the projection $X_\perp^a$ orthogonal to u^a is

$$X_\perp^a = h^a{}_b X^b,$$

making that $X_\perp^a u_a = 0$, and

$$\perp E'_{ab} = h_a{}^d h_b{}^e E'_{de}.$$

Lemma 1.6. Let (M, g) be a spacetime with its matter fields $\{\tau_i : M \to \mathcal{T}(M) \mid i \in I\}$ such that the rotation tensor vanishes, i.e. $\omega^\mu{}_\nu = 0$. The electric component E_{ab} of the Weyl tensor C_{abcd} is depended on the commutator $\partial_t \langle \theta \rangle - \langle \partial_t \theta \rangle$ via the relation

$$\partial_t \langle \theta \rangle - \langle \partial_t \theta \rangle = \Lambda - 4\pi G \langle \rho \rangle_D + \frac{2}{3} \langle \theta^2 \rangle_D - \langle \partial_t \theta \rangle$$

$$- \left(\frac{2}{\mu + p} \right)^2 \left\langle \left(\perp E'^a{}_b + E^a{}_b \theta - E^{c(a} \sigma_{b)c} - \eta^a{}_{cde} \eta_{bpqr} u^c u^p \sigma^{dq} E^{er} + 2H^{d(a} \eta_{b)cde} u^c u'^e \right) \right.$$

$$\left. \cdot \left(\perp E'^{\,b}_a + E_a{}^b \theta - E_{c(a} \sigma^{b)}{}_c - \eta_{acde} \eta^b_{pqr} u^c u^p \sigma^{dq} E^{er} + 2H^d{}_{(a} \eta^{b)}{}_{cde} u^c u'^e \right) \right\rangle_D. \quad (1.4)$$

To the first order, that is,

$$\partial_t \langle \theta \rangle - \langle \partial_t \theta \rangle = \Lambda - 4\pi G \langle \rho \rangle_D + \frac{2}{3} \langle \theta^2 \rangle_D - \langle \partial_t \theta \rangle - \left(\frac{2}{\mu + p} \right)^2 \left\langle \left(E'^a{}_b + E^a{}_b \theta \right) \left(E'^b{}_a + E^b{}_a \theta \right) \right\rangle_D. \tag{1.5}$$

Proof. Let μ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that the evolution of E_{ab} can be described by

$$-\frac{1}{2}(\mu + p)\sigma_{ab} = \perp E'_{ab} + E_{ab} \theta + h_{(a}{}^f \eta_{b)cde} u^c H_f{}^{d;e} \tag{1.6}$$

$$- E^c{}_{(a} \sigma_{b)c} - E^c{}_{(a} \omega_{b)c} - \eta_{acde} \eta_{bpqr} u^c u^p \sigma^{dq} E^{er} + 2H^d{}_{(a} \eta_{b)cde} u^c u'^e, \tag{1.7}$$

hereby $\perp E'_{ab}$ is the projection of E'_{ab} under the projection tensor $h_a{}^b = g_a{}^b + u_a u^b$, i.e. $\perp E'_{ab} = h_a{}^d h_b{}^e E'_{de}$, and where the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\mu + p)\eta_{abcd} \omega^{cd} u^b, \tag{1.8}$$

of the magnetic component H_{ab} of the Weyl tensor. As we consider an irrotational dust [**define this precisely**], we see that $\omega^{cd} = g^{di} \omega^c{}_i = 0$, implying that $E^c{}_{(a} \omega_{b)c} = 0$ and that covariant derivative (1.8) also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (1.6) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\mu + p} \left(\perp E'_{ab} + E_{ab}\theta - E^c{}_{(a} \sigma_{b)c} - \eta_{acde} \eta_{bpqr} u^c u^p \sigma^{dq} E^{er} + 2H^d{}_{(a} \eta_{b)cde} u^c u'^e \right), \tag{1.9}$$

Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\mu+p)^2}\left(\perp E'^a{}_b + E^a{}_b\theta - E^{c(a}\sigma_{b)c} - \eta^a{}_{cde}\eta_{bpqr}u^c u^p \sigma^{dq}E^{er} + 2H^{d(a}\eta_{b)cde}u^c u'^e\right) \tag{1.10}$$

$$\cdot \left(\perp E'_a{}^b + E_a{}^b\theta - E_{c(a}\sigma^{b)}{}_c - \eta_{acde}\eta^b_{pqr}u^c u^p \sigma^{dq}E^{er} + 2H^d{}_{(a}\eta^{b)}{}_{cde}u^c u'^e\right) \tag{1.11}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{1.12}$$

of Buchert (2000), one sees that (1.4) holds.

To the first order.
Let μ and p, respectively, be the density and pressure of the fluid. Hawking (1966) shows that to the first order, the evolution of E_{ab} can be described by

$$E'_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e} = -\frac{1}{2}(\mu+p)\sigma_{ab}, \tag{1.13}$$

with the covariant derivative,

$$H_{ab}{}^{;b} = \frac{1}{2}(\mu+p)\eta_{abcd}\omega^{cd}u^b, \tag{1.14}$$

of the magnetic component H_{ab} of the Weyl tensor. Since $\omega^{cd} = g^{di}\omega^c{}_i = 0$, its covariant derivative also vanishes: $H_{ab}{}^{;b} = 0$. Recapitulating on (1.13) gives an expression for the shear tensor,

$$\sigma_{ab} = -\frac{2}{\mu+p}\left(E'_{ab} + E_{ab}\theta\right). \tag{1.15}$$

Raising the first index of σ_{ab} and substituting it provides the rate of shear $\sigma^2 : M \to \mathbb{R}$,

$$\sigma^2 = \frac{1}{2}\sigma^a{}_b\sigma^b{}_a = \frac{2}{(\mu+p)^2}\left(E'^a{}_b + E^a{}_b\theta\right)\left(E'^b{}_a + E^b{}_a\theta\right). \tag{1.16}$$

From the commutation identity

$$\partial_t\langle\theta\rangle - \langle\partial_t\theta\rangle = \Lambda - 4\pi G\langle\rho\rangle_D + \frac{2}{3}\langle\theta^2\rangle_D - \langle\partial_t\theta\rangle - 2\langle\sigma^2\rangle_D \tag{1.17}$$

of Buchert (2000), one sees that (1.5) holds. $\qquad\square$

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.

2. $\phi \in \text{Diff}(M)$ diffeomorphism.

3. $\gamma \in \mathcal{G}$ gauge transformation.

4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.

5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.

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