Ch 3 - Relativistic gauge theory

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Abstract. -

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1 Relativistic gauge theory

1.1 Mathematical gauge theory for relativity

We introduce mathematical gauge theory in detail such that we can make the analogy to relativity precise.

1.1.1 Principal bundles

We introduce fibre bundles, which "can be thought of as twisted, non-trivial products between a base manifold and a fibre manifold" (**?**). In turn, principal bundles are fibre bunles whose fibres are Lie groups. This way the fibres have additional structure compared to other fibre bundles. For the definitions, we follow **?**.

Definition 1.1. Let E, F, M be smooth manifolds and $\pi : E \to M$ a surjective differentiable map. We call tuple (E, π, M, F) a *fibre bundle* if for every $p \in M$ there exists an open neighborhood $U \subset M$ around p and a diffeomorphism

$$\phi_U : \pi^{-1}(U) \to U \times F \qquad \text{such that} \qquad \mathrm{pr}_1 \circ \phi_U = \pi,$$

where $\mathrm{pr}_1 : U \times F \to U$ is the projection on the first factor.

Often we refer, by abuse of notation, $F \to E \xrightarrow{\pi} M$ to be the fibre bundle. Manifold E is called the *total space*, M the *base manifold* and F the *general fibre*. The map π is called the *projection*.

Definition 1.2. Let $G \times M \to M$ and $G \times N \to N$ be smooth left actions of a Lie group G on sets M and N. A *G-equivariant* map $f : M \to N$ is a continuous map satisfying

$$f(g \cdot p) = g \cdot f(p) \quad \forall\, p \in M,\ g \in G.$$

Definition 1.3. Let $G \to P \xrightarrow{\pi} M$ be a fibre bundle, where the general fibre G is a Lie group, and $\rho : P \times G \to P$ a smooth right action of G on P. We call P a *principal G-bundle* if the following two conditions are satisfied.

i. The action ρ restricts to $P_x \times G \to P_x$ and the orbit map $G \to P_x$ mapping $g \mapsto p \cdot g$ is a bijection for all $x \in sM$, $p \in P_x$.

ii. There exists a bundle atlas of G-equivariant bundle charts $\phi_i : P_{U_i} \to U_i \times G$ such that

$$\phi_i(p \cdot g) = \phi_i(p) \cdot g \quad \forall\, p \in P_{U_i},\ g \in G,$$

where in the latter expression G acts on $(x, a) \in U_i \times G$ via $(x, a) \cdot g = (x, ag)$.

We call such an atlas a *principal bundle atlas* for P.

Definition 1.4. Let $\pi : P \to M$ be a principal bundle. A *local gauge* for the principal bundle is a local section $s : U \to P$ for an open subset $U \subset M$. We call a local gauge a *gauge* if $U = M$.

So, a gauge is simply a global section $s \in \Gamma(P)$ of the total space P. Consequently, a *(local) gauge transformation* is to be understood as a transformation between (local) gauges, so that is a map $\gamma : \Gamma(P) \to \Gamma(P)$.

1.1.2 Frame bundle

Take frame bundle and derive the group $\mathcal{G}$ of gauge transformations. Then show $\mathcal{G} \cong \mathrm{Diff}(M)$ and that equations of motion for GR (EFE, Lagrangian, Hamiltonian) are invariant under $\mathcal{G}$, i.e. it is a left-action on $\mathcal{K}$ and satisfies (1.15).

For the reason that $\mathrm{Diff}(M) \cong \mathcal{G}$, diffeomorphism invariance can be identified with gauge invariance.

Prove: coordinate transformations are gauge transformations, etc.....

Proposition 1.5. The frame bundle is a principal bundle.

Proof. See **?**, Lemma 10.3. □

Proposition 1.6. Any diffeomorphism $\phi \in \mathrm{Diff}(M)$ induces a gauge transformation $\gamma \in \mathcal{G} \subset \mathrm{Aut}(LM)$.

Proof. We follow **?**. Consider any $\phi \in \mathrm{Diff}(M)$. We know that through the pullback, any frame field $f : M \to LM$ is mapped

$$f_p \mapsto \phi^* f_{\phi(p)},$$

whereby $\phi^* f_{\phi(p)}$ is a new frame at $\phi(p) \in M$ [show this explicitly]. Define $\gamma \in \mathrm{Aut}(LM)$ by setting

$$G(f_{\phi(p)}) \coloneqq \phi^* f_{\phi(p)},$$

which is indeed a map $G : LM \to LM$ defined on any point $q \in M$ as there exists some $p \in M$ such that $\phi(p) = q$. □

1.1.3 Examples of black holes as gauges

Some intro.

Schwarzschild black hole.
We consider the spacetime manifold[1] $M = \mathbb{R} \times (E^3 - O) \cong \mathbb{R} \times (0, \infty) \times S^2$, where O is the the

[1] We take the Minkowski metric of flat spacetime to be $\eta = \mathrm{diag}(-1, +1, +1, +1)$.

center of a spherical object of mass M and radius r with no electric charge nor angular momentum. Schwarzschild **?** derives an exact solution of the Einstein field equations with a vanishing cosmological constant. The Schwarzschild metric in Schwarzschild coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{1.1}$$

describing the curvature of spacetime outside the spherical mass.

Lemma 1.7. The Schwarzschild metric is obtainable by a suitable choice of gauge. (Make this mathematically precise.)

Proof. Take any coordinate system (x^i), which induces a basis $\{\frac{\partial}{\partial x^i}|_p\}$ for each point p in chart neighborhood U. Then, any orthonormal frame field $e_{(a)} \in \Gamma(LM)$ can locally be expressed at an arbitrary point $p \in U$ by

$$e_{(a)}(p) = e^\mu{}_a(p) \frac{\partial}{\partial x^i}\big|_p, \tag{1.2}$$

for some map $e^\mu{}_a : M \to \mathbb{R}^{4\times 4}$ - called the *vierbein field* - assigning an invertible matrix to every point $p \in M$.

Recall that $e_{(a)}(p)$ is an orthonormal basis spanning the tangent space T_pM, i.e.

$$\big(e_{(a)}(p), e_{(b)}(p)\big) = g_{\mu\nu}(p) e^\mu{}_a(p) e^\nu{}_b(p) = \eta_{ab} \qquad \forall p \in M, \tag{1.3}$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric of flat spacetime. We are given the Schwarzschild metric (1.1), which characterized by $g_{\mu\nu} = 0$ for all $\mu, \nu = 0, 1, 2, 3$ such that $\mu \neq \nu$ and for the diagonal components,

$$g_{00} = -\left(1 - \frac{2M}{r}\right), \qquad g_{11} = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad g_{22} = r^2, \qquad g_{33} = r^2 \sin^2(\theta).$$

Conveniently writing $g_{\mu\nu}(p) = g_{\mu\nu}$ and $e^\mu{}_a(p) = e^\mu{}_a$, the diagonal inner-product constraints (1.3) become

$$\begin{aligned}
-\left(1 - \frac{2M}{r}\right)(e^0{}_0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_0)^2 + r^2(e^2{}_0)^2 + r^2\sin^2(\theta)(e^3{}_0)^2 &= -1, \\
-\left(1 - \frac{2M}{r}\right)(e^0{}_i)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(e^1{}_i)^2 + r^2(e^2{}_i)^2 + r^2\sin^2(\theta)(e^3{}_i)^2 &= 1, \qquad \text{for } i = 1, 2, 3, \\
-\left(1 - \frac{2M}{r}\right) e^0{}_a e^0{}_b + \left(1 - \frac{2M}{r}\right)^{-1} e^1{}_a e^1{}_b + r^2 e^2{}_a e^2{}_b + r^2 \sin^2(\theta) e^3{}_a e^3{}_b &= 0, \qquad \text{for } a \neq b,
\end{aligned} \tag{1.4}$$

which are 16 equations with 16 unknowns $e^\mu{}_a$. One can readily verify that

$$e^0{}_0 = \frac{1}{\sqrt{1 - 2M/r}}, \quad e^1{}_1 = \sqrt{1 - 2M/r}, \quad e^2{}_2 = \frac{1}{r}, \quad e^3{}_3 = \frac{1}{r\sin(\theta)}, \quad e^\mu{}_a = 0, \tag{1.5}$$

for $\mu \neq a$, solves system (1.4). Notice that it is *not* unique as the solution (1.5) holds as well if one takes $e^2{}_2 = -\frac{1}{r}$ instead. In vector notation, (1.5) is described by

$$e_{(0)} = \frac{1}{\sqrt{1 - 2M/r}} \partial_t, \quad e_{(1)} = \sqrt{1 - 2M/r}\,\partial_r, \quad e_{(2)} = \frac{1}{r}\partial_\theta, \quad e_{(3)} = \frac{1}{r\sin(\theta)}\partial_\phi, \tag{1.6}$$

as is in accordance with the metric tensor (1.1). Observe that, by the inner-product signature constraint (1.3), $\{e_{(a)}\}$ described in (1.5) is an orthonormal basis by construction. □

Lemma 1.7, in short, tells us that by choosing an appropriate gauge $s : M \to LM$, e.g. setting $s(t, r, \theta, \phi) = e_{(a)}$ with $\{e_{(a)}\}$ defined as in (1.3), one constructs a Schwarzschild black hole in M and with it its corresponding metric.

If a gauge describing the Schwarzschild black hole is fixed, it turns out that no degrees of freedom is lost compared to an arbitrary gauge transformation on the spacetime manifold.

Proposition 1.8. There are four degrees of freedom in fixing gauges which describe the Schwarzschild black hole.

Proof. Note that the first column of matrix $(e^{\mu}{}_{a})$ cannot be 'altered' with respect to solution described in (1.3) and at most one variable can be changed per row. This makes that we can set four $e^{\mu}{}_{a} = 0$ to non-zero values. Hence, there is at least four degrees of freedom for fixing an appropriate gauge. By ...., we find that there is at most four degrees of freedom, which concludes the proof. $\square$

Kerr black hole.

Kerr **?** was the first to derive the metric tensor describing the curvature of spacetime outside of a rotating black hole of mass[2] M with angular momentum[3] J. In Boyer-Lindquist coordinates, the metric is

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\phi^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2} \sin^2\theta d\phi^2 - \frac{4Mar\sin^2\theta}{\rho^2} dtd\phi \tag{1.7}$$

Give intuitive description of the lemma below. Check the Hairy Ball Theorem on why we cannot describe a global frame field.

Lemma 1.9. There exists a gauge, i.e. a local section $s : U \subset M \to LM$, from which one can derive an explicit expression of the Kerr metric tensor on chart neighborhood U of spacetime manifold M.

Proof. Take any coordinate system (x^i), which induces a basis $\{\frac{\partial}{\partial x^i}|_p\}$ for each point p in chart neighborhood U. Then, any orthonormal frame field $e_{(a)} \in \Gamma(LM)$ can locally be expressed at an arbitrary point $p \in U$ by

$$e_{(a)}(p) = e^{\mu}{}_{a}(p) \frac{\partial}{\partial x^i}\Big|_p, \tag{1.8}$$

for some vierbein field $e^{\mu}{}_{a} : M \to \mathbb{R}^{4\times4}$.

Recall that $e_{(a)}(p)$ is an orthonormal basis spanning the tangent space T_pM, i.e.

$$g_{\mu\nu}(p) = e_{\mu}{}^{a}(p) e_{\nu}{}^{b}(p) \eta_{ab} \qquad \forall p \in M, \tag{1.9}$$

where we inverted the inner-product condition (1.3) using orthonormality $e_{\mu}{}^{a}(p)e_{\nu}{}^{a}(p) = \delta^{\mu}_{\nu}$. If we write $g_{\mu\nu}(p) = g_{\mu\nu}$ and $e_{\mu}{}^{a}(p) = e_{\mu}{}^{a}$ for convenience, system (1.9) becomes

$$\begin{aligned} g_{\mu\mu} &= -(e_{\mu}{}^{0})^2 + (e_{\mu}{}^{1})^2 + (e_{\mu}{}^{2})^2 + (e_{\mu}{}^{3})^2, && \text{for } \mu = 0,1,2,3, \\ g_{03} &= -e_0{}^0 e_3{}^0 + e_0{}^1 e_3{}^1 + e_0{}^2 e_3{}^2 + e_0{}^3 e_3{}^3, \\ g_{\mu\nu} &= -e_{\mu}{}^{0} e_{\nu}{}^{0} + e_{\mu}{}^{1} e_{\nu}{}^{1} + e_{\mu}{}^{2} e_{\nu}{}^{2} + e_{\mu}{}^{3} e_{\nu}{}^{3}, && \text{for } \mu \neq \nu \text{ and } (\mu,\nu) \neq (0,3)\,. \end{aligned} \tag{1.10}$$

A solution is described by symmetric $e_{\mu}{}^{a}$ with components

$$\begin{aligned} e_0{}^0 &= \frac{1}{\sqrt{1 - 2M/r}}, \quad e_1{}^1 = \sqrt{1 - 2M/r}, \quad e_2{}^2 = \frac{1}{r}, \\ e_3{}^3 &= \sqrt{\frac{\Sigma}{\rho^2}\sin^2\theta + \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1 - 2Mr/\rho^2}}}, \quad e_3{}^0 = \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1 - 2Mr/\rho^2}}. \end{aligned} \tag{1.11}$$

[2] To be specific, M is the Schwarzschild mass **?**.

[3] The spheroidal coordinates (r, θ, ϕ) are equivalent to Cartesian coordinates (x, y, z), where one of the transformations is $z = r\cos(\theta)$. The angular momentum is to be considered about this z-axis **?**.

In vector notation, that is

$$e_{(0)} = \frac{1}{\sqrt{1-2M/r}}\partial_t + \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1-2Mr/\rho^2}}\partial_\phi, \qquad e_{(1)} = \sqrt{1-2M/r}\partial_r,$$
$$e_{(2)} = \frac{1}{r}\partial_\theta, \qquad e_{(3)} = \frac{4Mar\sin^2\theta}{\rho^2\sqrt{1-2Mr/\rho^2}}\partial_t + \frac{1}{r\sin(\theta)}\partial_\phi. \quad (1.12)$$

□

Proposition 1.10. There are no degrees of freedom in fixing gauges which describe the Kerr black hole.

Proof. The degrees of freedom of a system are the number of parameters of which one is free to choice the parameter value without altering the state of the system. In the proof of Lemma 1.9 we showed that the frame field described by (1.12) is an explicit gauge describing the Kerr geometry of spacetime. Suppose there was at least one degree of freedom, then we could change at least one variable value. Do so for any of the components $e_\mu{}^a$ and one of the product terms in the inner-product constraint (1.10) changes, leading to a change in the system. Hence, one cannot alter any parameter value, i.e. there are no degrees of freedom in fixing gauges describing the Kerr geometry. □

Note: There is **one** degrees of freedom! Namely, we can choose ϕ to be anything we like.
Q: What if the solution holds up to sign? What happens to the degrees of freedom?

1.2 Invariance and covariance

To combine the mathematical approach to gauges and the concepts of gauge invariance in cosmology, we start at the beginning: invariance itself. It relates to objects etc. Follow gauge - diff - topics.

Covariant = form invariant. So one can only speak of a covariant formalism and not an invariant one, e.g. a gauge covariant formalism is one in which the equations are form invarint under gauge transformations.

1.3 General framework of invariance and covariance

To combine the mathematical approach to gauges and the concepts of gauge invariance in cosmology, we start at the beginning: invariance itself.[4] We follow **?** and **?**; providing us with a rigorous treatment of the topic. To be precise in our reasoning, we provide a generalization of a topological space.

Definition 1.11 (**?**)**.** A *mathematical structure* is a set S of mathematical objects such that the objects are described axiomatically and the set has at least one relation $R \subset S \times S$ defined on it.

Remark 1.12 (A topological space is a mathematical structure). Define the equivalence relation $\sim$ on any topological space $(S, \mathcal{T})$ by letting any two open sets $S_1, S_2 \subset S$ be equivalent if and only if $S_1, S_2 \in \mathcal{T}$. From this one sees that indeed a topological space is a mathematical structure making Def. 1.11 a generalized of a topological space.

For a given physical theory, its equations of motion take the general form

$$\mathfrak{E}[\mathcal{D}, \mathcal{B}] = 0, \quad (1.13)$$

[4]We will not advance the discussion by trying to define 'symmetries' as these have different meanings in physics than they have in mathematical literature. In the literature of physics, an invariant of a set of equations of motion need not be a symmetry, while mathematical contributions most often define a symmetry to be an invariant **?**.

where $\mathcal{D}$ is the collection of *dynamical structures* - i.e. mathematical structures that have no a priori values and must be solved to get assigned values to - and the set $\mathcal{B}$ of *background structures* contains all non-dynamical structures, excluding For example in GR, the metric tensor is a dynamical structure and a coordinate system is a background structure. Let $\mathcal{K}$ be the space of kinematically possible field configurations, also known as field histories, of the theory. Note that the solutions of (1.13) give rise to the subset $\mathcal{P} \subset \mathcal{K}$ of dynamically possible configurations.

Definition 1.13. Let G be a group acting on $\mathcal{K}$ from the left-hand side, i.e. $G \times \mathcal{K} \to \mathcal{K}$. The equations (1.13) of motion are called *covariant* under G if for any $\gamma \in G$,

$$\mathfrak{E}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{E}[\gamma \cdot \mathcal{D}, \gamma \cdot \mathcal{B}] = 0, \tag{1.14}$$

and *invariant* if

$$\mathfrak{E}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{E}[\gamma \cdot \mathcal{D}, \mathcal{B}] = 0. \tag{1.15}$$

—– Some clear intuitive argument on invariance and possible relate this to manifestly invariance. ——-

These concepts allow us to define diffeomorphism invariance rigorously.

Definition 1.14. Let G be a group with an operation $\cdot : G \to \mathcal{D} \to \mathcal{D}$. Equations of motion $\mathfrak{E}[\mathcal{D}, \mathcal{B}] = 0$ are called *G-invariant* if and only if it allows group G as invariance group, i.e. G defines a left-action on $\mathcal{K}$ and for any $\gamma \in G$ statement (1.15) is satisfied.

For convenience, one speaks of diffeomorphism invariance if $G = \mathrm{Diff}(M)$, gauge invariance if $G = \mathcal{G}$ and Lorentz invariance if $G = SO^+(3,1)$ in the above definition.

Remark 1.15 (Invariant equivalent to covariant in GR). Note that $\mathcal{B} = \emptyset$ as it does not include the manifold, atlas, etc. This makes that the Einstein field equations are invariant under G if and only if they are covariant under G.

Remark 1.16 (Manifestly invariant). For a theory with equations of motion invariant under some group G, it is desirable to formulate the theory such that any equation preserves the invariance as else the degrees of freedom could be reduced; such theory is called *manifestly G-invariant*.

Give some explicit argument for manifestly invariant, i.e. any equation

$$\mathfrak{E}[\mathcal{D}, \mathcal{B}] = 0 \iff \mathfrak{E}[\gamma \cdot \mathcal{D}, \mathcal{B}] = 0,$$

for any $\gamma \in G$.

I would now say: you need to categorize different concepts: invariant tensor fields and invariant equations. The former is the requirement

$$\tau = \gamma \cdot \tau,$$

for $\tau \in \mathcal{T}(M)$ and the latter is in line with the theory being manifestly invariant.

1.4 Invariants in relativity

The equations of motion for relativity are the Einstein field equations

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right) g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.16}$$

We show several groups under which (1.16) are invariant.

Proposition 1.17. The field equations are invariant under $\mathrm{Diff}(M)$.

Proof. Take any $\phi \in \mathrm{Diff}(M)$, then

$$\phi^* R_{\mu\nu} + \left(\Lambda - \frac{1}{2}\phi^* R\right)\phi^* g_{\mu\nu} = \kappa \phi^* T_{\mu\nu}$$

if and only if (1.16) holds as both are tensorial field equations defined on any point of M. To see this argumentation more clearly, notice that for any $q \in M$ there exists $p \in M$ such that $\phi(p) = q$ for any fixed $\phi \in \mathrm{Diff}(M)$. $\square$

Proposition 1.18. The field equations are invariant under the group of coordinate transformations on M.

Don't think this is a group.

1.5 Equations of motion for relativity

(First provide derivation: argumentation for the EH-action - principle of least action - Einstein field equations, i.e. the equations of motion of relativity.)

1. Principle of least action
2. So, we need the action for GR. Give an argumentation for the EH-action
3. Use variational approach by invoking Hamilton's principle
4. Derive the Euler-Lagrange - or simply Lagrange's - equations of motions for the system
5. Derive Hamilton's equations of motion of GR and prove that they are equivalent to the Lagrange equations.
6. Derive the Einstein field equations from the Lagrange equations

1.6 Invariants in relativity part II

Discuss here the invariants in relativity in terms of Section 1.

Remark 1.19 (Invariants in general relativity). Consider the equations of motion $\mathfrak{C}[\mathcal{D}, \mathcal{B}] = 0$ for a relativistic system. Observe that the diffeomorphism group $\mathrm{Diff}(M)$ is not the only invariant appearing in relativity; the Lorentz group is an invariant as well.

Prove: (proper, restricted) Lorentz / Poincare group is a

? generalizes the definition of invariance from **?** as follows.

Definition 1.20. The n-ary relation $R \subset A^n \coloneqq A \times \cdots \times A$ is *invariant* under group G of bijections $A \xrightarrow{\phi} A$ if for any bijection $\phi \in G$,

$$(a_1, \ldots, a_n) \in R \iff (\phi(a_1), \ldots, \phi(a_n)) \in R,$$

for any $a_1, \ldots, a_n \in A$.

The Einstein-Hilbert action

$$S(g) = \int_M R\sqrt{-\det g}\, d^4x$$

determines the field equations of relativity and therefore describe its equations of motion. We can define a relation on the space $\mathrm{Met}(M)$ of Lorentzian metric tensors on M by

$$g \sim \hat{g} \iff S(g) = S(\hat{g}).$$

So $(g, \hat{g}) \in R_\sim \subset \mathrm{Met}(M) \times \mathrm{Met}(M)$ if and only if $S(g) = S(\hat{g})$.

Proposition 1.21. The relation $R_\sim$ on the Einstein-Hilbert action is invariant under the group $\mathrm{Diff}(M)$ of all diffeomorphisms on M.

Proof. Let $(g, \hat{g}) \in R_\sim$ and consider any $\phi \in \mathrm{Diff}(M)$. Since

$$\begin{aligned}
\int_M \phi^* \left(R\sqrt{-\det(g)} d^4 x \right) &= \int_M (R \circ \phi) \sqrt{-\det(g \circ \phi)} d^4(x \circ \phi) \\
&= \int_{\phi(M)} (\phi^{-1})^* \left((R \circ \phi) \sqrt{-\det(g \circ \phi)} d^4(x \circ \phi) \right) \\
&= \int_M R\sqrt{-\det(g)} d^4 x
\end{aligned}$$

□

1.7 Gauge theory for cosmological perturbation theory

Consider a spacetime manifold M and its background M_b with a one-to-one correspondence (bijection) $c : M_b \to M$. Do not confuse: background gauges and frame gauges. However, the former can be understood as a special case of the latter.

Proposition 1.22. A transformation $c \mapsto \phi(c)$ of the background correspondence, where $\phi \circ c : M_b \to M$ is bijective, induces a gauge transformation.

Proof. Consider an arbitrary $p \in M$. There exists an $b \in M_b$ such that $p = c(b)$, and so $M \ni c(b) \mapsto \phi \circ c(b) \in M$, i.e. $p \mapsto \phi(p)$ bijectively. □

1.8 Gauge invariance for backreaction

—— Maybe for next chapter ———-
So what is gauge invariance? How can we be *check* whether the equations written down in GR are gauge invariant? (Look at the Hamiltonian?)

By Definition 1.14 we have that the equations of motion of GR are gauge invariant if $\mathcal{G}$ is a left action on $\mathcal{K}$ and for any gauge transformation $\gamma \in \mathcal{G} \cong \mathrm{Diff}(M)$ the

Notation

Summary of notation used corresponding to its context.

1. $\tau \in \mathcal{T}(M)$ tensor field.
2. $\phi \in \mathrm{Diff}(M)$ diffeomorphism.
3. $\gamma \in \mathcal{G}$ gauge transformation.
4. $(U, \varphi), (V, \psi) \in \mathcal{A}$ smooth charts in smooth atlas $\mathcal{A}$.
5. $F : \mathcal{T}(M) \to \mathbb{R}$ functional.