Quantum Techniques for Jackson and Gordon Newell Networks

Dave B.H. Verweg

Bachelor thesis in Econometrics and Operational Research
Department of Economics and Business, University of Groningen, The Netherlands

Supervised by

John C. Baez

Department of Mathematics, University of California, Riverside CA
Centre for Quantum Technologies, National University of Singapore, Singapore

Nicky D. Van Foreest

Department of Economics and Business, University of Groningen, The Netherlands

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Abstract

Baez and Biamonte wrote the book "Quantum Techniques for Stochastic Mechanics" on the relation between quantum systems and stochastic processes, these processes can be understood as networks of $M/M/\infty$ queues. In this work, the Hamiltonian describing time-evolution for these stochastic processes is expressed in terms of creation and annihilation operators. The contribution of this paper is twofold. First, we develop a similar formulation as that of Baez and Biamonte for the transient behavior of networks of $M/M/c$ queues in terms of creation and annihilation operators, because dynamical properties of queueing networks can be quite involved. Second, a rigorous proof of the conserved quantity property of the number observable for a closed network of $M/M/c$ queues using Noether's theorem for Markov chains is provided.

Keywords: Stochastic mechanics, quantum techniques, queueing networks, Noether's theorem, transient behavior.

1. Introduction

The theory of dynamical systems generally considers massive classical objects, which is contrary to the field of quantum theory. However, both theories analyze many-particle systems in which transitions of states occur. In the 1970s a procedure for expressing relations in stochastic dynamical processes has been developed with the use of quantum techniques, see e.g. Doi [11, 12] or Suna [27]. Mattis and Glasse [24] state that the paradigm shift of "fitting quantum algebra to irreversible motion and chemical reactions of massive classical objects" can be thought to originate from Doi [11, 12] and Peliti [25]. There are several other important publications which cover stochastic systems in which objects undergo transitions using quantum algebra, see e.g. Cardy [9]. However, none of these contributions seem to raise significant attention to stochastic processes that yield restrictions on the maximum number of objects that can simultaneously be transitioned. Only a few decades later, Zeitak [30] seems to be the first who introduces the Doi-Peliti formalism for Markov processes in which such restrictions on the maximum number of simultaneous transitions are accounted for.

Baez and Biamonte [3] provide a refreshing overview of some important results of the paradigm shift of using quantum algebra for systems of classical objects. Refreshing in the sense that notation used by these authors is more insightful and straightforward than the formalism first used by Doi [11, 12] and Peliti [25]. The systems that Baez and Biamonte [3] study are stochastic processes where the objects, which we call *jobs*, undergo transitions. Before a job is transitioned, it remains stationed in some specific location, called a *station*, of the system. We can therefore interpret such a job waiting for its transition as being in *service* and if its service is completed the job is transitioned. These transitions occur randomly. More specifically, the service times, i.e. the duration of services, are assumed to be exponentially distributed. If a job arrives at a transition station, it can immediately start its service of undergoing the transition; the job never has to *wait* before starting its service. Since $M/M/\infty$ queues have an infinite amount of exponential servers with jobs arriving in accordance with a Poisson process,

Email addresses: dave@verweg.net (Dave B.H. Verweg), baez@math.ucr.edu (John C. Baez), n.d.van.foreest@rug.nl (Nicky D. Van Foreest)

we see that the models studied by Baez and Biamonte [3] are networks of $M/M/\infty$ queues.

Baez and Biamonte [3] do not consider (networks of) $M/M/c$ queues. Gordon and Newell [15] and Jackson [16] provide equations describing such networks and derive corresponding stationary distributions. In this paper, we take the stochastic description of Gordon and Newell [15] and Jackson [16] as given and try to develop a description of the transient behavior of networks of $M/M/c$ queues by using quantum techniques similar to Baez and Biamonte [3]. We furthermore verify that the number operator is a conserved quantity for a closed circuit of $M/M/c$ queues by invoking Noether's theorem for Markov chains.

The formalism of an operator-theoretic description of queueing networks is considered to be interesting, since the dynamical properties of queueing systems are often complex and advanced, see e.g. Zeitak [30]. Furthermore, our motivation for contributing to this formalism for $M/M/c$ queues is that the systems studied by Baez and Biamonte [3] are not representative for most operational systems. Because, jobs are independent from each other in an $M/M/\infty$ network as they do not interact with each other at all; each job that arrives at a queueing station gets an ample server. This is contrary to most $M/M/c$ networks where jobs do not always see an ample server, but are often required to wait; in general the jobs are interaction-dependent. One can imagine that the techniques discussed by Baez and Biamonte [3] are being simplified significantly by only considering networks of $M/M/\infty$ queues.

The remainder of this paper is organized as follows. In Section 2 we give an overview of prior work related to the use of quantum techniques for classical systems and its application to queueing networks. Next, in Section 3 we describe the mathematical model which will be studied. Throughout Section 4 we analyze the time evolution of the creation and annihilation processes of jobs. These derivations make it possible to derive expressions of the transient behavior of two kinds of networks: an open tandem and a closed series circuit of $M/M/c$ queues, which are discussed in Section 5. Then, Section 6 provides a verification that the number observables is a conserved quantity through the use of Noether's theorem for Markov chains. The paper is concluded by Section 7 in which we summarize the results and provide an discussion on the relevance of this work. In the last section, Section 8, we discuss interesting directions for further research.

2. Previous work

Stochastic mechanics finds its roots in the application of quantum field theory to quantum statistical mechanics, which was first done around the 1960s. Bloch, see e.g. [5], covers a systematic treatment of such formal methods. Furthermore, "Dr. Bloch always shows carefully how the classical theory is contained in the quantum theoretical formalism" (see De Boer et al. [8]), which is essential for the ideas formed a decade later by Doi [11, 12]. Among other things, De Boer [7] discusses the second quantization method for many-particle *quantum* systems. Naturally, it is for one to ask how this second quantization method and classical theory are interrelated. It turns out that Doi [11] took care of that question as Doi notices that creation and annihilation operators "can be introduced independently of quantum conditions." Doi [11] continues the paper by introducing the formalism of applying the second quantization method to classical systems with a large number of particles. To be specific, Doi [11, 12] together with Peliti [25] introduce a rather full description of this new formalism by studying reaction-diffusion systems, which are types of chemical reactions.

Arazi et al. [2] suggest adopting results from queueing theory for the study of chemical reaction networks. Subsequently, Gadgil et al. [13] addresses the relation between reaction-diffusion systems and queueing networks. Zeitak [30] takes this idea as starting point and applies the Doi-Peliti formalism to queues by working out the "connection between queueing networks and various physical models known as reaction diffusion models". Zeitak [30] describes any open Jackson networks of $M/M/1$ queues. In this paper, we expand upon the work of Zeitak [30] in the sense that we provide a description of an arbitrary open and closed network of $M/M/c$ queues. The notation is however directly based upon that of Baez and Biamonte [3] instead of the Doi-Peliti formalism, which is more in line with that of quantum field theory.

After writing a previous version of this work, awareness was raised to the fact that the main contribution of this paper, i.e. expressing the transient behavior of networks of $M/M/c$ queues, has also been introduced by Chernyak et al. [10] but in the Doi-Peliti formalism. Accordingly, throughout the upcoming sections we discuss similarities between results that are derived and those that are found in Chernyak et al. [10]. The present work is concluded by providing a discussion on the differences between the two some-what different formalism of Baez and Biamonte [3] seen in this paper and the Doi-Peliti formalism that is used in Chernyak et al. [10].

3. Model

Consider a network consisting of k queues, where jobs are assumed to arrive according to a Poisson process with rate β. This is a rather natural assumption, see e.g. Ross [26]. An arrival to one of the queue stations is transitioned by a single server of which the service times are exponentially distributed with rate γ. The number of servers per station $i = 1, \ldots, k$ is denoted by $c_i \in \mathbb{Z}_{>0} \cup \{\infty\}$. Bolch et al. [6, Ch. 1] discuss that the exponential distribution yields the memoryless property implying that the considered processes are irreducible Markov chains. They are irreducible in the sense that we assume that the system can go from any state to any other state with a non-zero probability. A state is a specific allocation of jobs in the system. Formally, a state is denoted by the k-dimensional column vector $\boldsymbol{n} = (n_1 \; \ldots \; n_k)^T$, where n_i denotes the number of jobs at queue i, which is also known as the number of jobs of species i. If not stated explicitly otherwise, we assume that state $\boldsymbol{n}$ is in the state space $\mathcal{S} = \mathbb{Z}^k_{\geq 0}$. When a closed network is considered, the number of jobs in the system is denoted by $N = \sum_{i=1}^{k} n_i$.

Let $\boldsymbol{N}(t) = \big(N_1(t) \ \ldots \ N_k(t)\big)^T$ be the vector random variable indicating the number of jobs at queues $1, \ldots, k$ at time $t \in \mathbb{R}_{\geq 0}$. The probability function of the system being in state $\boldsymbol{n} \in \mathcal{S}$ at time t is denoted by

$$\begin{aligned}\psi_n(t) &= \mathbb{P}\big(N_1(t) = n_1, \ldots, N_k(t) = n_k\big) \\ &= \mathbb{P}\big(\boldsymbol{N}(t) = \boldsymbol{n}\big),\end{aligned} \tag{3.1}$$

where we make use of multi-index $n = (n_1, \ldots, n_k)$. In analogy with the formulation of Peliti [25], any state, that a queueing system at time t can be in, is identified by the series

$$\big\{\psi_n(t)\big\}_{n\in\mathcal{S}}, \tag{3.2}$$

but can be identified more conveniently by the generating function $\Psi(t)$, which we define by

$$\Psi(t) = \sum_{n\in\mathcal{S}} \psi_n(t) z^n, \tag{3.3}$$

with complex formal variables $z_1, \ldots, z_k$ such that $|z_i| < 1$ for all $i = 1, \ldots, k$. Power series $\Psi(t)$ is convenient as it "stores" the probability function $\psi_n(t)$ for all states $\boldsymbol{n} = (n_1, \ldots, n_k)^T \in \mathcal{S}$ by means of forming a combination of formal variables $z^{n_1} \cdot \cdots \cdot z^{n_k}$ abbreviated by z^n.

From the theory of Markov processes, by e.g. Bolch et al. [6, Ch. 2] and Jackson [16], we know that the time derivative of $\psi_n(t)$ for any state $\boldsymbol{n} \in \mathcal{S}$ can be expressed as a system of differential equations of the transition rates of the process. This formalism is equivalent to the *master equation*, which is a probability balance equation expressing the transition probabilities of a Markov process, see Van Kampen [18, Ch. 5] and Korolkova et al. [20]. By Doi [11, 12] and Peliti [25], we know that the probability functions $\psi_n(t)$ for a Markov process, summarized in $\Psi(t)$, evolve according to the master equation of the form

$$\frac{d}{dt}\Psi(t) = H\Psi(t), \tag{3.4}$$

for some time-independent Hamiltonian operator H. Notice that (3.4) is simply the classical formulation of the Schrödinger equation. Since the time dynamics of the considered systems is of main interest, the master equation (3.4) and its corresponding Hamiltonian will be the pivot of the work throughout the remainder of this paper.

4. Creation and annihilation

In general open queueing systems see jobs arrive and depart, therefore we start off by deriving the transient behavior of an arrival and departure process. The formulation of the transient behavior of these two processes will be the building blocks to describe the time-evolution of more involved queueing systems. We derive the Hamiltonian of these two elementary processes by considering stochastic differential formulations similar to that of Jackson [16].

4.1. Creation process

As described above it is thought to be helpful to describe the transient behavior of a queueing system which only sees arrivals. More precisely, we identify an operator which implements the arrival of a job. Since an arrival is "new" to the system, we refer to this operator as the creation operator. The creation operator allows us to construct a process in which jobs arrive to some queueing system as a Poisson process. To find this operator, we consider a single station empty $M/M/c$ queue for which we assume there to be no departures. Assuming this provides the possibility to study arrivals in isolation as jobs are never served. Consequently, the operator that is obtained corresponds only to arrivals. We begin with considering differential equations that govern the transient behavior of the probability functions $\psi_n(t)$. After rewriting this stochastic formulation, the form of the creation operator will be evident.

Consider the creation process of jobs governed by Poisson process with arrival rate β. Karlin and Taylor [19, Ch. 4] state that the time-derivatives of the probability functions $\psi_n(t)$ can be expressed in terms of the transition rates as follows:

$$\begin{aligned}&\dot{\psi}_0(t) = -\beta\psi_0(t), \\ &\dot{\psi}_n(t) = \beta\big(\psi_{n-1}(t) - \psi_n(t)\big) \qquad \text{if } n \geq 1.\end{aligned} \tag{4.1}$$

For convenience, we define ψ_{-1} to be the zero function such that (4.1) can be fully described by $\dot{\psi}_n(t) = \beta\big(\psi_{n-1}(t) - \psi_n(t)\big)$ for all $n \geq 0$. Substituting these expressions into the derivative of the generating function $\Psi(t)$ gives

$$\dot{\Psi}(t) = \beta\sum_{n=0}^{\infty}\psi_{n-1}(t)z^n - \beta\sum_{n=0}^{\infty}\psi_n(t)z^n. \tag{4.2}$$

By making use of the assumption $\psi_{-1} \equiv 0$, we find $\dot{\Psi}(t) = \beta(z-1)\Psi(t)$. We conclude that the Hamiltonian for the arrival process is given by

$$\beta\big(a^\dagger - 1\big), \tag{4.3}$$

where 1 is the identity operator and where the *creation* operator is defined to be $a^\dagger = z$. The motivation for this definition is as follows. Consider a system with n jobs, then $\psi_n(t) = 1$. We posit that the change to the system of a job arriving can be expressed by $a^\dagger\Psi = z\Psi$. Since $\Psi = z^n$, we have $a^\dagger\Psi = z^{n+1}$ implying that after the creation of a job we have $\psi_{n+1}(t)$. This is indeed correct since after an arrival, in a system with n jobs, we are sure there are $n+1$ jobs.

Note that (4.3) tells us how the probabilities ψ_n change when a job arrives within the system. This should make sense since operator $a^\dagger$ creates an arrival at rate β. Notice the correction term -1 to be there for normalization purposes such that the probabilities still sum up to one after a job has arrived.

Remark 4.1. (*Verification*). The Hamiltonian (4.3) describing the creation process is the same result Baez and Biamonte [3, Sec. 5] posit and is furthermore in accordance with the results of Peliti [25, Sec. 4] on the transient behavior of the birth-death processes on a lattice.

4.2. Annihilation process

Above we found an operator, i.e. the creation operator, by which it is possible to construct a process where jobs arrive to some queueing system in accordance with a Poisson process. Next we identify a similar operator; an operator that creates the possibility to construct a process where jobs depart from a queueing system. More specifically, the operator realizes the completion of a job's service and can therefore be regarded as an annihilation operator. Instructively, the annihilation operator implements the service process of jobs. Baez and Biamonte [3, Sec. 4] have already provided the annihilation operator for the $M/M/\infty$ queue. Here this operator will be generalized to describe the annhilation process of any $M/M/c$ queue. We generalize the operator with a similar approach as was seen in Section 4.1: we consider an $M/M/c$ queue for which there are no arrivals, so we set the arrival rate $\beta = 0$. Along these lines, the departure process of the queue is examined in isolation. We start by presenting the stochastic equations describing the queue similar to the formulations of Jackson [16]. After rewriting these equations, the form of the annihilation operator will become noticeable.

Let an $M/M/c$ queue be given, which never sees any arrivals and transitions the jobs that it contains with exponentially distributed service times with rate γ. Notice that the system is able to serve at most $c \geq 1$ jobs at the same time. The time-derivatives of the probability functions of the queue are

$$\begin{aligned} \dot{\psi}_0(t) &= \gamma\psi_1(t), \\ \dot{\psi}_n(t) &= \gamma\big((n+1)\psi_{n+1}(t) - n\psi_n(t)\big) && \text{if } 0 < n < c, \\ \dot{\psi}_n(t) &= c\gamma\big(\psi_{n+1}(t) - \psi_n(t)\big) && \text{if } n \geq c, \end{aligned} \tag{4.4}$$

stated in an equivalent form by e.g. Van Kampen [18, Ch. 6]. Write the probability function $\psi_n(t)$ as ψ_n for which we suppose time-dependency should be obvious from the context. Multiplying the time derivative $\dot{\psi}_n(t)$ by z^n and summing over all $n \geq 1$ results in

$$\begin{aligned} \sum_{n=1}^{\infty} \dot{\psi}_n z^n = \gamma \sum_{n=1}^{c-1} &\big((n+1)\psi_{n+1} - n\psi_n\big) z^n \\ &+ c\gamma \sum_{n=c}^{\infty} \big(\psi_{n+1} - \psi_n\big) z^n, \end{aligned} \tag{4.5}$$

according to (4.4). Gathering all the coefficients of ψ_n for every $n \geq 1$ and adding $\dot{\psi}_0 z^0$ to both sides of (4.5) makes

$$\dot{\Psi}(t) = \gamma \sum_{n=0}^{c} n\psi_n\big(z^{n-1} - z^n\big) + c\gamma \sum_{n=c+1}^{\infty} \psi_n\big(z^{n-1} - z^n\big). \tag{4.6}$$

In terms of the poly-differential operators (z, ∂_z), that is

$$\dot{\Psi}(t) = \gamma \sum_{n=0}^{c} \big(\partial_z - z\partial_z\big)\psi_n z^n + c\gamma \sum_{n=c+1}^{\infty} \left(\frac{1}{z} - 1\right)\psi_n z^n. \tag{4.7}$$

We define the *annihilation* operator to be

$$a = \begin{cases} \partial_z & \text{if } n \leq c, \\ c \cdot \frac{1}{z} & \text{if } n > c. \end{cases} \tag{4.8}$$

Our motivation for this definition can be illustrated by considering an $M/M/c$ queue, which is identified by state Ψ and has a total of $n \geq 0$ jobs. If the number of jobs is larger than the number of servers, we know that all servers are occupied and that there are exactly c jobs that can be annihilated. In other words, if $n > c$, we want $a\Psi = az^n = cz^{n-1}$ as there are $n-1$ jobs left in the system. If the number of jobs is at most c, we know that all jobs are in service and therefore there are n departure possibilities. That is to say, we want $a\Psi = nz^{n-1}$ if $n \leq c$. One can readily verify that $a\Psi$ gives the description of the effect of an annihiliation on the system for any $n \geq 0$. Definition (4.8) follows.

Recognizing the form of the Schrödinger equation in (4.7), we conclude that the Hamiltonian describing the time evolution of the annihilation process with c servers is

$$\gamma\big(a - a^\dagger a\big). \tag{4.9}$$

Remark 4.2. (*Verification*). Note that γ is the Poisson rate for the removal of a job from the queue. We see that the evolution operator (4.9) is equivalent to the Hamiltonian (35) derived by Chernyak et al. [10, Sec. II.C] by setting the injection rate and all other transfer rates equal to zero.

Remark 4.3. ($M/M/\infty$ *queue*). Let the number of servers converge to infinity to observe the annihilation process of an $M/M/\infty$ queue. By looking at how the annihilation operator was defined in (4.8), we see that $a = \partial_z$ as $c \to \infty$. We conclude that the Hamiltonian (4.9) for the departure process of an $M/M/\infty$ queue is the same result found in Baez and Biamonte [3, Sec. 5].

4.3. $M/M/c$ queue process

Above we identified an operator that creates jobs in and another that departs jobs from a queueing system. Next we assemble the evolution operator for the $M/M/c$ queue, since we are after describing the time-evolution of $M/M/c$ networks. The transient behavior of this queue will be derived by combining the creation and annihilation operators discussed before.

Since the $M/M/c$ queue is completely determined by the combination of the arrival and departure process analyzed before, the probability function $\psi_n(t)$ evolves according to adding the right-hand side of the expressions (4.1) and (4.4) corresponding to state $\boldsymbol{n} \in \mathcal{S}$ for the creation and annihilation process, respectively. Noticing that the generating function $\dot{\Psi}(t)$ is linear in its coefficients $\dot{\psi}_n(t)$, we can add Hamiltonian operators (4.3) and (4.9) of the creation and annihilation process, respectively. The master equation $\dot{\Psi}(t) = H\Psi(t)$ therefore expresses the time dynamics of the $M/M/c$ queue, where the Hamiltonian is of the form

$$H = \beta(a^\dagger - 1) + \gamma\big(a - a^\dagger a\big). \tag{4.10}$$

Remark 4.4. (*Verification*). Note that β and γ are the Poisson rate for the injection and removal of a job from the queue, respectively. We see that the Hamiltonian (4.10) for an $M/M/c$ queue is equivalent to operator (35) derived by Chernyak et al. [10, Sec. II.C] by setting all other transfer rates equal to zero.

Worth noting are two special cases of the $M/M/c$ queue: where $c = 1$ and $c \to \infty$, since they are most often covered in queueing theory. We address these two cases in the remarks below.

Remark 4.5. ($M/M/\infty$ *queue*). First we let $c \to \infty$, i.e. we consider an $M/M/\infty$ queue. By (4.10), we have that the Hamiltonian of this queue is

$$\beta(z-1) + \gamma\big(\partial_z - z\partial_z\big), \tag{4.11}$$

as from (4.8) we see that $a = \partial_z$ since $n \leq c$ for all $n \in \mathbb{Z}_{\geq 0}$. This result can be verified by applying the general rule on Hamiltonian operators describing any transition with an infinite number of servers seen in Baez and Biamonte [3, Sec. 5].

Remark 4.6. ($M/M/1$ *queue*). Second we set the number of servers equal to $c = 1$, i.e. we consider an $M/M/1$ queue. We take a look at the annihilation process of this queue. Recall (4.7), which for the $M/M/1$ queue we can write differently as

$$\dot{\Psi}(t) = \gamma \sum_{n=1}^{\infty} \left(\frac{1}{z} - 1\right)\psi_n z^n. \tag{4.12}$$

We now introduce some notation that allows us to rewrite (4.12) into a Schrödinger form. In accordance with the formalism of Wilf [29], we define *projection* operator z_0 of formal power series $\Psi(t)$ to return the coefficient in front of z^0, which is $\psi_0(t)$. The projection operator z_0 enables to rewrite (4.12) such that we obtain the Hamiltonian for the M/M/1 queue:

$$H = \beta\big(z-1\big) + \gamma\left(1 - z\right)\frac{1}{z}\big(1 - z_0\big), \tag{4.13}$$

which is more explicit than (4.10) for $c = 1$.

By looking closely at the evolution operators (4.10) and (4.13), we define the explicit annihilation operator for the $M/M/1$ queue to be

$$a_1 = \frac{1}{z}\big(1 - z_0\big). \tag{4.14}$$

An $M/M/1$ queue with $n \geq 0$ jobs on time t is identified by generating function $\Psi(t) = z^n$. Suppose there is at least one job in the system, then there is only one possibility of seeing a departure within an arbitrary small time frame because there is only one job in service. In the case of $n > 0$, we indeed observe that a_1 removes a job from the queue as $a_1\Psi(t) = \frac{1}{z}(1 - z_0)z^n = z^{n-1}$ since $z_0 z^n = 0$ and z^{n-1} corresponds to the state where there are $n-1$ jobs in the system. Now suppose that the queue is empty. By definition of the projection operator z_0, we have $a_1\Psi(t) = \frac{1}{z}(1 - z_0)z^0 = 0$, just stating the fact that no jobs can be annihilated when the queue is empty. We conclude that the annihilation operator (4.14) indeed implements the service offered to jobs in an $M/M/1$ queue. Notice that this operator is equivalent to that of (4.8) for $c = 1$, but is conveniently state-independent.

Using the explicit annihilation operator a_1, we can write the Hamiltonian of an $M/M/1$ queue as

$$\beta\big(a^\dagger - 1\big) + \gamma\big(a_1 - a^\dagger a_1\big). \tag{4.15}$$

After rewriting this evolution operator to $\gamma\big(1 - a^\dagger\big)\big(a - \frac{\beta}{\gamma}\big)$, we see (4.15) is the exact same result provided by Zeitak [30, Sec. 3.2].

5. Networks of $M/M/c$ queues

Above we derived the evolution operator of an $M/M/c$ queue. Evidently, this description provides as a building block for describing $M/M/c$ networks. We derive explicit expressions for time evolution of two simple queueing networks: an open tandem and a closed system of two queues with an arbitrary amount of servers. To find the evolution operators, we follow the method seen before: we start with a stochastic formulation of the transient behavior from which the Hamiltonian will become apparent after some algebra. Furthermore, the results for the open and closed networks of two queues are generalized inductively to any number of queues in series. At last, we discuss the general rule of describing the Hamiltonian of any given $M/M/c$ queue with any number of inputs and outputs by analogy to the general rule for $M/M/\infty$ queues provided by Baez and Biamonte [3, Sec. 5]. The general rule makes it possible to find an expression for the transient behavior of any Jackson and Gordon Newell network.

5.1. *Tandem of $M/M/c$ queues*

We start by analyzing the most basic $M/M/c$ network: an open system of two $M/M/c$ queues working in series, i.e. a tandem. Since we are already familiar with the arrival and departure process of the system, we define an auxiliary queue in which we observe the transitioning of jobs from the first to the second queue in isolation. Once the transient behavior of this auxiliary queue has been described in terms of a Hamiltonian, the description of a tandem of $M/M/c$ queues follows almost immediately.

Consider two queues in tandem of which the arrivals follow a Poisson process with rate β, where queue i has c_i servers with exponentially distributed service times with rate γ_i for $i = 1, 2$. Since the arrival and departure process of this network has already been discussed in Section 4, we set the creation rate $\beta = 0$ and the annihilation rate $\gamma_2 = 0$. The time derivatives of the probability functions of this auxiliary queueing tandem can be expressed by

$$\begin{aligned} \dot{\psi}_{n_1,n_2} &= \gamma_1\big((n_1+1)\psi_{n_1+1,n_2-1} - n_1\psi_{n_1,n_2}\big) && \text{if } n_1 < c_1, \\ \dot{\psi}_{n_1,n_2} &= c_1\gamma_1\big(\psi_{n_1+1,n_2-1} - \psi_{n_1,n_2}\big) && \text{if } n_1 \geq c_1, \end{aligned} \tag{5.1}$$

with $\psi_{n_1,-1} \equiv 0$. Substitution of (5.1) into $\dot{\Psi}(t)$ makes

$$\begin{aligned}\dot{\Psi}(t) = \gamma_1 \sum_{n_1=0}^{c_1-1} \sum_{n_2=0}^{\infty} \big((n_1+1)\psi_{n_1+1,n_2-1} - n_1 \psi_{n_1,n_2}\big) z^n \\ + c_1\gamma_1 \sum_{n_1=c_1}^{\infty} \sum_{n_2=0}^{\infty} \big(\psi_{n_1+1,n_2-1} - \psi_{n_1,n_2}\big) z^n. \end{aligned} \quad (5.2)$$

In terms of operators in the z-space, we therefore have

$$\begin{aligned}\dot{\Psi}(t) = \gamma_1 \sum_{n_1=0}^{c_1} \sum_{n_2=0}^{\infty} \big(z_2\partial_{z_1} - z_1\partial_{z_1}\big)\psi_{n_1,n_2} z^n \\ + c_1\gamma_1 \sum_{n_1=c_1+1}^{\infty} \sum_{n_2=0}^{\infty} \left(z_2 \frac{1}{z_1} - 1\right)\psi_{n_1,n_2} z^n. \end{aligned} \quad (5.3)$$

Rewriting this with help of the creation and annihilation operators, we obtain

$$\dot{\Psi}(t) = \gamma_1 \big(a_2^\dagger - a_1^\dagger\big) a_1 \Psi(t). \quad (5.4)$$

We have seen before that the linearity of the coefficients $\dot{\psi}_n(t)$ in $\dot{\Psi}(t)$ makes it possible to sum the evolution operator of the auxiliary queue examined above with the Hamiltonians of the arrival process (4.3) and the departure process (4.9) to describe the transient behavior of the tandem completely. We conclude that the Hamiltonian describing the time evolution of an $M/M/c$ network of two queues in tandem is

$$H = \beta\big(a_1^\dagger - 1\big) + \gamma_1\big(a_2^\dagger - a_1^\dagger\big)a_1 + \gamma_2\big(1 - a_2^\dagger\big)a_2. \quad (5.5)$$

Remark 5.1. (*Generalization*). Generalizing result (5.5) to a tandem consisting of an arbitrary number of $M/M/c_i$ queues with annihilation rate γ_i is done straightforwardly by induction, considering the derivations above. The Hamiltonian of such tandem with $k \geq 2$ queues is

$$\begin{aligned} H = \beta\big(a_1^\dagger - 1\big) + \gamma_k\big(a_k - a_k^\dagger a_k\big) \\ + \sum_{i=2}^{k-1} \gamma_i \big(a_i^\dagger - a_{i-1}^\dagger\big) a_{i-1}. \end{aligned} \quad (5.6)$$

Remark 5.2. (*Verification*). For a tandem of k queues, we assumed that jobs are transferred from queueing station $1 \leq i < k$ to $1 < i+1 \leq k$ with Poisson rate $\gamma_i =: \gamma_{i,i+1}$ and β and γ_k are the Poisson rate for the injection and removal of a job from the queueing system, respectively. Analogously, we see that the Hamiltonian (5.6) for a tandem of $M/M/c_i$ queues is equivalent to operator (35) derived by Chernyak et al. [10, Sec. 2C] by setting all other transferring rates equal to zero.

Remark 5.3. ($M/M/\infty$ *queue*). Consider a tandem of k number of $M/M/c_i$ queues where the number of servers of both queues are assumed to be infinite, i.e. let $c_i \to \infty$ for $i = 1, 2$. Recall that the annihilation operator is now the differentiable operator $a_i = \partial_z$. Hamiltonian (5.6) can be verified by applying the general rule by Baez and Biamonte [3, Sec. 5].

5.2. Closed network of $M/M/c$ queues

Above we looked at a class of open $M/M/c$ networks. Next we consider a closed network of such queues. To be specific, we consider the queueing stations to be in a series circuit. This makes generalizing the description to an arbitrary Gordon Newell network more obvious.

Consider a closed network of $M/M/c_i$ queues for $i = 1, 2$, i.e. a simple reversible process consisting of two transitions. Denote the service rates by γ_i and the total number of jobs by N. The probability functions of the system change according to

$$\begin{aligned} \dot{\psi}_{n_1,n_2} = \gamma_1\big((n_1+1)\wedge c_1 \cdot \psi_{n_1+1,n_2-1} - n_1 \wedge c_1 \cdot \psi_{n_1,n_2}\big) \\ + \gamma_2\big((n_2+1)\wedge c_2 \cdot \psi_{n_1-1,n_2+1} - n_2 \wedge c_2 \cdot \psi_{n_1,n_2}\big), \end{aligned} \quad (5.7)$$

where ψ_{-1,n_2} and $\psi_{n_1,-1}$ are zero functions. Here map $\wedge : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined to be the minimum $(x,y) \mapsto x \wedge y = \min(x,y)$.

Notice that the γ_1-term of (5.7) is equivalent to (5.1), which yielded the Hamiltonian

$$\gamma_1\big(a_2^\dagger a_1 - a_1^\dagger a_1\big). \quad (5.8)$$

Examining the γ_2-term in (5.7) with help of an auxiliary network by equating $\gamma_1 = 0$. Substitution of the expressions $\dot{\psi}_{n_1,n_2}$ into $\dot{\Psi}(t)$ and some algebra gives

$$\begin{aligned} \dot{\Psi}(t) = \gamma_2 \sum_{n_1=0}^{N} \sum_{n_2=0}^{c_2} \big(z_1\partial_{z_2} - z_2\partial_{z_2}\big)\psi_{n_1,n_2} z^n \\ + c_2\gamma_2 \sum_{n_1=0}^{N} \sum_{n_2=c_2+1}^{N} \left(z_2\frac{1}{z_1} - 1\right)\psi_{n_1,n_2} z^n. \end{aligned} \quad (5.9)$$

Recalling the definition (4.8) of the annihilation operator, we see that the corresponding Hamiltonian for the auxiliary queueing network with $\gamma_1 = 0$ is

$$\gamma_2\big(a_1^\dagger - a_2^\dagger\big)a_2. \quad (5.10)$$

Summing up (5.8) and (5.10), we conclude that the Hamiltonian of a closed network of the two considered queues is

$$H = \gamma_1\big(a_2^\dagger - a_1^\dagger\big)a_1 + \gamma_2\big(a_1^\dagger - a_2^\dagger\big)a_2. \quad (5.11)$$

Remark 5.4. (*Verification*). If we let $c_i \to \infty$ for both queues $i = 1, 2$, we see that this process is the exact same as the reversible reaction considered in Baez and Biamonte [3, Sec. 9]. Accordingly, we verify its corresponding Hamiltonian (5.11) to the result found in [3, Sec. 9].

Remark 5.5. (*Generalization*). The resulting Hamiltonian (5.11) can be generalized for a series circuit of $M/M/c_i$ queues. Suppose the i-th queue has c_i servers with service rate γ_i. By induction, we can conclude that the Hamiltonian of a closed network of $k \geq 2$ queues in series is

$$H = \sum_{i=1}^{k-1} \gamma_i\big(a_{i+1}^\dagger - a_i^\dagger\big)a_i + \gamma_k\big(a_{k-1}^\dagger - a_k^\dagger\big)a_k. \quad (5.12)$$

Remark 5.6. (*Verification*). Verifying Hamiltonian (5.12) is done almost identical to what was done for open $M/M/c$ networks in Section 5.1. Consider an open $M/M/c_i$ network of $k+1$ queues for which we set the arrival rate β and removal rate γ_{k+1} to zero. Observe that this simply describes a closed series circuit of k queues by letting station k transfer jobs to station 1 instead of station $k+1$. The verification follows from Chernyak et al. [10, Sec. 2C] by carefully looking at the indices of the operators and transferring rates.

5.3. The general rule

In this section we generalize the *general rule* provided by Baez and Biamonte [3, Sec. 5], which specifies the Hamiltonian operator of any transition within a process. Recall that such transition in stochastic mechanics is assumed to yield an infinite amount of servers. We generalize this general rule for processes where each transition has an arbitrary number of servers. We do so analogously to the work of Baez and Biamonte [3, Sec. 5]. Notice that with this general rule it is possible to describe the time evolution of any Jackson or Gordon Newell network.

Consider an $M/M/c$ queue with service rate γ. Denote the number of different inputs by k and outputs by j. Set $k=0$ if the queue conceives arrivals in the system. Likewise, set $j=0$ if the queue annihilates jobs from the network. To count the number of jobs in the system, define $\widehat{N}=a^\dagger a$ to be the *number* operator. The time evolution of the queue can then be described by Hamiltonian

$$H=\left(a^\dagger\right)^j a^k-\widehat{N}\cdot\ \ldots\ \cdot(\widehat{N}-k+1). \tag{5.13}$$

One can readily verify (5.13) by supposing there to be n jobs in the queue at time $t=t_0$. Notice that $\Psi(t_0)=z^n$ and hence

$$a^k\Psi(t_0)=\begin{cases}\partial_z^k z^n & \text{if} \quad n\le c,\\ \partial_z^{c+k-n}\left(\frac{c}{z}\right)^{n-c}z^n & \text{if } c<n\le c+k,\\ \left(\frac{c}{z}\right)^k z^n & \text{if} \quad n>c+k.\end{cases} \tag{5.14}$$

Working out the operators,

$$a^k\Psi(t_0)=\begin{cases}n\cdot\ \ldots\ \cdot(n-k+1)z^{n-k} & \text{if} \quad n\le c,\\ c^{n-c}\cdot\overline{n}^{c+k-n}z^{n-k} & \text{if } c<n\le c+k,\\ c^k z^{n-k} & \text{if} \quad n>c+k,\end{cases} \tag{5.15}$$

where $\overline{n}^{c+k-n}$ is the $(c+k-n)$-th falling power of n, i.e.

$$\overline{n}^{c+k-n}=n\cdot\big(n-1\big)\cdot\ \ldots\ \cdot\big(n-(c+k-n)\big). \tag{5.16}$$

Consider the case where the number of jobs in the queue is such that $c<n\le c+k$. The master equation with Hamiltonian (5.13) at time $t=t_0$ becomes

$$\dot{\Psi}(t_0)=c^{n-c}\cdot\overline{n}^{c+k-n}\big(z^{n+j-k}-z^n\big), \tag{5.17}$$

implying that the time derivatives of ψ_{n+j-k} and ψ_n are equal up to sign. Since this necessary condition for (5.13) to be an infinitesimal stochastic operator holds, we verified that the Hamiltonian is of the right form. A similar argument is given for the remaining cases, where $n\le c$ and $n>c+k$.

If we consider a queue where an arrival always finds an ample server, then observe that (5.13) is the exact same result seen in Baez and Biamonte [3, Sec. 5].

6. Observables and Noether's theorem

Above the time evolution of a closed system of $M/M/c$ queues was derived. Since the number of jobs in a closed network is always the same, we do a sanity check by verifying that the expected values of the number observable, that is the operator counting the number of jobs in a system, does not change over time. In other words, we prove that the counting opreator is a conserved quantity. The verification is done by invoking Noether's theorem for Markov processes and by rigorously writing out the operators.

In analogy to quantum theory, the number observable for a network of k queues is defined to be

$$\widehat{N}=\widehat{N}_1+\cdots+\widehat{N}_k=z_1\partial_{z_1}+\cdots+z_k\partial_{z_k}. \tag{6.1}$$

Our motivation for this definition can be illustrated by considering a queue with n jobs at time t. The system can be described by state $\Psi(t)=z^n$ and letting the number observable operate on $\Psi(t)$ returns

$$\widehat{N}\Psi(t)=n\Psi(t), \tag{6.2}$$

which is simply the multiplicative operator of the number of jobs in the system. We wish to proof that the expected value of the number observable does not change over time. The following result comes in handy. Baez and Fong [4] propose Noether's Theorem for Markov processes as the following:

Theorem 1. (*Noether's Theorem for Markov processes*). Let O be an observable and H an infinitesimal stochastic operator on the vector space of the probability functions $\psi: X\to\mathbb{R}$ for some appropriate state space. Then

$$[O,H]=0$$

if and only if for all probability functions $\psi(t)$ obeying the master equation

$$\frac{d}{dt}\psi(t)=H\psi(t),$$

the expected values of O and O^2 do not change in state $\psi(t)$.

Proof. Refer to Baez and Fong [4]. □

Since a Hamiltonian is an infinitesimal stochastic operator, Noether's Theorem creates the possibility to prove that the number observable and the Hamiltonian commute in order to show that the counting operator is indeed a conserved quantity.

6.1. Creation and annihilation operator identities

Rigorously proving the commutation relation of the number of jobs and the time evolution requires some operator identities, which will now be derived.

The commutation of the creation and the annihilation operators depend on the number of jobs within the system. For convenience consider a queueing system with n jobs at time t, i.e. $\Psi(t) := \Psi = z^n$, then

$$\big(aa^\dagger - a^\dagger a\big)\Psi = \begin{cases} \partial_z z\Psi - z\partial_z\Psi & \text{if } n < c, \\ \frac{c}{z} z\Psi - z\partial_z\Psi & \text{if } n = c, \\ \frac{c}{z} z\Psi - z\frac{c}{z}\Psi & \text{if } n > c. \end{cases} \tag{6.3}$$

By the product rule we obtain the following identity:

Identity 1. The commutation of the creation and annihilation operator for $M/M/c$ queue with n jobs is

$$\big[a, a^\dagger\big] = \begin{cases} 1 & \text{if } n < c, \\ 0 & \text{if } n \geq c. \end{cases} \tag{6.4}$$

The relation expressed in (6.4) can be interpreted as follows. If the number of jobs in the system is below the number of servers, all the jobs have been assigned a server and are in service. In this situation there is one more way to observe an arrival and then a departure, than to see a departure and then an arrival. But if there would be at least as many jobs in the system as there are servers, all the servers are busy. This implies that there are exactly as many possibilities of a job waiting in line entering service and then to see a departure as there are ways to first see a departure and then to observe a job entering service.

Obviously, the number of jobs after seeing a departure and then an arrival in some given queueing station does not change. This basic relation can be expressed indirectly by the following statement:

Identity 2. The annihilation operator and the number observable can be written as

$$aN = \partial_z a^\dagger a. \tag{6.5}$$

Proof. Suppose we are considering a system with n jobs, that is the system is the state $\Psi = z^n$, from which we see

$$aNz^n = naz^n = \begin{cases} n\partial_z z^n & \text{if } n \leq c, \\ cnz^{n-1} & \text{if } n > c. \end{cases} \tag{6.6}$$

In terms of the minimum function $(x, y) \mapsto x \wedge y = \min(x, y)$, we find

$$aNz^n = \partial_z (n \wedge c) z^n = \partial_z a^\dagger a z^n. \tag{6.7}$$

The result follows. □

Notice that Identity 2 implies $a^\dagger aN = Na^\dagger a$, which indeed expresses that the number of jobs stays the same if a departure and an arrival are observed directly after each other.

The following identity does not have an immediate interpretation, but will come in handy when proving that the number observable is a conserved quantity for a closed network of $M/M/c$ queues.

Identity 3.

$$Na^\dagger = a^\dagger + a^\dagger N. \tag{6.8}$$

Proof. Consider a queueing network which is described by state ψ. By definition of the number operator $N = z\partial_z$, the product rule makes

$$Na^\dagger\psi = z\big(1 + z\partial_z\big)\psi. \tag{6.9}$$

The result follows. □

6.2. Number observable in a closed $M/M/c$ network

In Section 5.2 the Schrödinger equation for a closed circuit of $M/M/c$ queues was derived for which we found Hamiltonian (5.12). We now prove that the number observable commutes with the evolution operator, from which we can conclude that the number observable is a conserved quantity. More specifically, consider a closed queueing network which is a circuit of k transition stations with the i-th queue being an $M/M/c_i$ queue. By definition, the total number of jobs $N = \sum_i^k n_i$ is never changing. Therefore, it is hypothesized that the number observable for a closed circuit of $M/M/c_i$ queues is a conserved quantity. This result will now be verified.

Property 1. For a closed circuit queueing network of $M/M/c_i$ queues where $1 \leq i \leq k$, the number observable $\widehat{N}$ is a conserved quantity.

Proof. Pick some $i \neq j \in \{1, \dots, k\}$. First, we prove that $a_i^\dagger a_j$ commutes with the counting operator $\widehat{N}$. Recall that Identity 2 states $a_i N_i = \partial_{z_i} a_i^\dagger a_i$ and so by the product rule,

$$a_j^\dagger a_i N_i = a_j^\dagger \partial_{z_i}\big(a_i^\dagger a_i\big) = a_j^\dagger \Bigg(1 + N_i\Bigg) a_i, \tag{6.10}$$

as $\partial_{z_1} z_1$ is simply the identity operator 1 and where we denoted $\widehat{N}_i$ to be the number operator counting all the number of jobs at queueing station i. Adding $a_j^\dagger N_j a_i$ to both sides of (6.10) and invoking Identity 3 gives

$$N_i a_j^\dagger a_i + N_j a_j^\dagger a_i = a_i N_i a_j^\dagger + a_j^\dagger N_j a_i. \tag{6.11}$$

Noticing that the number operator N_l for any $l \neq i, j$, i.e. $l \neq i$ nor $l \neq j$, commutes with $a_j^\dagger a_i$ makes

$$N_i a_j^\dagger a_i + N_j a_j^\dagger a_i + \sum_{l \neq i,j} N_l a_j^\dagger a_i = a_i N_i a_j^\dagger + a_j^\dagger N_j a_i + \sum_{l \neq i,j} a_j^\dagger a_i N_l. \tag{6.12}$$

As the number operator N_i commutes with the creation and annihilation operators $a_j^\dagger$ and a_j for any $i \neq j$, we see that

$$[N, a_j^\dagger a_i] = 0, \tag{6.13}$$

since $N = N_i + N_j + \sum_{l \neq i,j} N_l$. Hence, the number operator N and $a_j^\dagger a_i$ commute.

Second, we prove that $Na_i^\dagger a_i = a_i^\dagger a_i N$. Notice that N_j and $a_i^\dagger a_i$ commute. The result folows immediately from Identity 2 as

$$a_i^\dagger a_i(N_i + \sum_{l\neq i} N_l) = a_i^\dagger \partial_{z_i} a_i^\dagger a_i + \sum_{l\neq i} N_l a_i^\dagger a_i$$
$$= (N_i + \sum_{l\neq i} N_l) a_i^\dagger a_i. \quad (6.14)$$

Third, we rewrite the commutation $[N, H]$ as

$$\sum_{i=1}^{k-1} \gamma_i \Big(\big[N, a_{i+1}^\dagger a_i\big] - \big[N, a_i^\dagger a_i\big] \Big)$$
$$+\gamma_k \Big(\big[N, a_{k-1}^\dagger a_k\big] - \big[N, a_k^\dagger a_k\big] \Big). \quad (6.15)$$

We do so because substituting results (6.13) and (6.14) proves the fact that the number operator commutes with the time evolution of the system. By Noether's Theorem for Markov processes, the result follows. □

7. Conclusions and discussion

In this paper, we generalized the quantum techniques discussed by Baez and Biamonte [3] to describe the transient behavior of networks of $M/M/c$ queues. To do so, we first identify two operators - the creation and annihilation operators - which implement the arrival of and the service offered to a job in a queueing system with exponentially distributed service times. By isolating creations and annihilations to the system, an arrival and departure process is constructed by considering well-know stochastic formulation of the transition probabilities of Markov processes. Noticing that an $M/M/c$ queue is completely determined by its arrival and departure process, the evolution operator of this queue follows almost immediately. Following the analogous scheme seen before, the transient behavior of $M/M/c$ networks are described. We furthermore discuss a general rule on describing transient behavior of any $M/M/c$ queue with multiple inputs and outputs. The paper is concluded by verifying, in terms of operators, that the number observable is a conserved quantity for a closed $M/M/c$ network as it is proven to commute with the evolution operator of the system.

Our motivation of introducing an operator-theoretic description of the transient behavior of $M/M/c$ networks originates from the fact that dynamical properties of queueing systems can turn out to be complex. The description is derived by looking at the methods discussed by Baez and Biamonte [3] and the stochastic formulation of Jackson [16]. After having written down all of the most important results, the author has been made aware that a similar operator description has also been introduced by Chernyak et al. [10]. From this we learned probably one of the most important lessons for doing research: diving into the work upon the subject is essential. Nevertheless, reinventing things can provide a new perspective. Thus, making the goal of this paper providing a new perspective in queueing theory in the following sense: the Doi-Peliti formalism is based upon methods seen in quantum theory applied to *classical* many-body problems. The direct relation between quantum theory makes authors contribute to the classical problems in notation seen in quantum (field) theory. It is in this formalism Chernyak et al. [10] derive their results, leading to rather simple notions being treated in a very mathematical rigorous way; e.g. the generating function which stores all the probability distributions of a system is introduced via “a Hilbert space of $\mathcal{G}_0$-dimensional analytic functions". The rigorous mathematical formalism and the bra-ket notation cause the meaning of results to lose their transparency and clarity at first sight, especially for students who are not well-read in quantum theory. However, we regard the Baez-formalism to be more intuitive and equivalent to that of Doi and Peliti. By implementing the Baez-formualism, we therefore introduce an operator-theoretic description of the transient behavior of queueing systems that is more insightful and intuitive than seen until now.

The purpose of this paper will be reached if the present work contributes to simplifying the analyzes of the transient behavior of queueing systems. The approach taken in this paper is by starting to examine two elementary processes - the arrival and departure process - well-known by almost all natural science students in some form or another. By taking a close look at the stochastic formulation, identifying corresponding operators follows naturally without having to be familiar with operators operating on Fock space. These operators form the building blocks for expressing the evolution of more involved queueing systems over time, which is derived according to an almost identical method as of that for the arrival and departure processes. This systematic and basic approach makes analyzing the transient behavior accessible for a broader public.

8. Suggestions for further research

The present work is straightforwardly generalized by deriving the Hamiltonian of any open or closed $M/M/c$ queueing network, since we only covered the explicit expressions of such networks where queues are in a closed series circuit or in an open tandem. These generalization will be formulated in the ongoing project in progress of the author [28]. In his own notation, Massey [23] describes the transient behavior of a class-dependent network of single-server queues, with Poisson arrivals and exponentially distributed service times, in terms of operators. An interesting direction for further research could be to generalize these results for networks of queues with an arbitrary number of queues in Baez-formalism without needing to address algebraic technicalities like “a tensor product of Fock spaces". In another paper, Massey [22] shows that operator methods can be used to analyze the time evolution of the mean and variance for the queue length of an $M/M/1$ queue in an arbitrary Jackson network. Another interesting direction could be to derive similar descriptions for networks of $M/M/c$ queues.

Simplifying, to an appropriate extend, of formalism and interpretation regarding the transient behavior of queueing systems is enriching for the above mentioned reasons in Section

7. Since Chernyak et al. [10] provide several interesting applications of the operator description of $M/M/c$ networks, it might be possible to simplify the derivations and results regarding what we have done so far for the Hamiltonian of $M/M/c$ networks. More specifically, the authors of this paper analyze currents, which can informally be understood as the rate of job-flow. An interesting direction for further research is to simplify the operator-theoretic description of the time evolution of currents between queues in an $M/M/c$ network. To do so, we suggest using Baez-formalism and implementing the systematic approach discussed in Section 7. This could come down identifying operators for the amount of current between queueing stations seen in Eq. (43) and (44) of [10, Sec. 3A], by looking at a perhaps more intuitive, but potentially less general, stochastic description of the transient behavior of currents. Second, defining a more insightful generating function equivalent to (47) of [10] of which the coefficients involve expressions of the joint probability distribution function of the currents and queue size will especially contribute to increasing the overall transparency of the results. Particularly, because the newly defined operators in Eq. (43) and (44) and the generating function (47) in [10] are the building blocks for the analysis of currents done in [10, Sec. 3].

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