Personal notes of a work in progress

An Introduction to Quantum Techniques for Queueing Systems

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Disclaimer: Please note that these are my personal notes and until now rather quick sketches of derivations, results and interpretation of the related subject. It may still contain misformulations.

Goal: We hope to write a text in the present format, which is meant for undergraduate students that have been introduced to basic queueing theory. We opt to write an enthusiastic text portraying readers the application of mathematics is interesting and exciting.

Abstract

This book tries to explain a way of describing how queueing systems evolve over time by invoking techniques found in quantum theory. The application of quantum techniques to systems which contain classical massive objects instead of quantum very small objects has been introduced by Doi and Peliti. The Doi-Peliti formulation stems from methods of quantum field theory, which makes the notations discussed everything but easy and insightful for people not familiar with quantum theory. Baez and Biamonte wrote a book which practically states the same results as Doi and Peliti but take a much more intuitive perspective. Therefore, we take the Baez-formalism as given and try to describe queueing systems with it. The description of the time evolution of such systems follows almost immediately. Hopefully, we can show this description of queueing systems can be understood by everyone who has seen some queueing theory before.

Preface

Contents

1 Introduction

1.1 Complex systems

A system is a collection of items and events where the items possibly undergo the considered events, known as processes. Studying systems is beneficial as it enables people to effectively gain insight into complex sets of data and underlying relations. Systems that probably occur the most in our every day life are complex ones in which things change randomly over time. Therefore, for us to describe these bundles of processes and items, we must look at models incorporating probability theory. Throughout the years different disciplines have tried to describe a huge variety of systems, which all relate to probabilities in some way or another. Two of which are the theory of queues and the theory of stochastic mechanics. The latter examines how things randomly change via interaction at a fundamental level. Queueing theorists regard an item undergoing a process as a job waiting in line for completion of its service. The processes can thus be thought of as the *queues*, where the underlying working force is said to be a server (e.g. a cashier in a supermarket). We assume that the arrival of jobs is independent of the state of the queue, formally that is we assume the arrival process to yield the *memoryless property*. Service times are regarded to be exponentially distributed. A complex system consisting of processes with several waiting lines is said to be a *queueing network*.

One can imagine that the number of items in a system can change over time, e.g. the number of candy bars in a factory. Systems where the number of items does not change are called *open*. Nevertheless, this need not be the case for any given system as the number of molecules undergoing a chemical reaction in a tube can be fixed, such systems are called *closed*. In this book, we start off by considering and describing a few basic open systems which helps us to analyse similar networks that are closed.

1.2 Problem definition

We are interested in expressing the *dynamical* properties of closed queueing networks, which will tell us exactly how systems behave over time. This is contrary to most of the properties discussed in introductions to queueing theory. The main target is to describe the dynamics of a considered system in equilibrium and providing expressions for expected values of observable properties, e.g. the expected waiting time for a job that arrives. This is due to the fact it analytically being difficult to model dynamic relationships, which hold for an arbitrary moment in time. Hence, we are aiming to identify a different method for describing queues to shed light upon these properties. Fortunately, a lot of hard work has already been done to develop such a theory describing dynamical properties of a certain kind of systems, namely stochastic mechanics. We will mainly follow the book by Baez and Biamonte [1] in order to see how we can manipulate derivations such that we can study the queues we are interested in. More specifically, we will try to modify and apply the theory of stochastic mechanics by exploiting techniques from quantum mechanics in order to study the (closed) systems that are frequently described in the theory of queues.

1.3 The aim of the project

Before we proceed, it is highly recommended that the reader is familiar with the theory of Markov processes, the theory of queues and that of quantum mechanics. Moreover, we suggest reading through Baez and Biamonte [1] on stochastic mechanics.

This book will hopefully be a contribution to strengthen the analogy between the theory of Markov processes and stochastic mechanics in order to shed light upon the dynamical properties of closed systems in queueing theory. The systems that are considered within stochastic mechanics are assumed to contain an infinite amount of servers. That means that for any job in the system it always undergoes some kind of service. In queueing theory, we usually examine finite server queues as well. The aim of the project is to describe the time evolution of the most important but basic queues and networks of queues that have a finite number of servers for each process.

The idea arose from the work of Baez and Biamonte [1], who only discuss processes which are interpretable as networks of M/M/∞ queues. We generalize several important notions discussed in this book for general networks of M/M/c queues. To describe the time evolution of the systems, we explicitly derive the Hamiltonian operators which is in contrast to most processes discussed by Baez and Biamonte [1]. Furthermore, we try to derive a general framework in order to study observable properties corresponding to such closed systems. We discuss the application of Noether's theorem on a closed network and show the number of jobs is indeed conserved. To summarize, we hope to answer the following **main research question**:

How can we generalize quantum techniques used for stochastic mechanics to open and closed networks of single and multi-server queues?

Since the main objective is to collide two disciplines that study systems, we hope that this project will portray, at least in a small way, that mathematics is a beautiful tool in order for us to describe intriguing phenomena of this earth.

1.4 Outline of the book

We first start in Section 2 with stating the most important notions of Markov processes, queueing theory and stochastic mechanics which will be used throughout this book.

In Section 3, we begin by analysing (networks of) queues with infinite amount of servers. One might recall that the steady-state of the systems which we consider are unique. This fact is used throughout the project in order to derive the equilibrium solution of several simple queues. With the theory of stochastic mechanics in hand, we know how to describe the change probabilities functions over time. We show that the time evolution can also be derived from a queueing perspective. We check this expression by verifying that the derived equilibrium distribution is indeed the state which does not change over time, i.e. the stationary state.

We follow a similar approach in Section 4 describing time evolution of finite server networks as was done in the previous section. To do so, we must develop a few new mathematical operators that help us describe the relevant processes that we are considering. We also give a general rule on how to portray the time evolution of a process with an arbitrary number of inputs and outputs where the number of servers is finite.

The previous two sections show how it is possible to describe the change of probabilities over time. It occurs that people are curious about a particular property of a dynamical system. In Section 5 we take a look at a specific class of these properties: conserved quantities. To study them, we invoke the theory of observables found in quantum mechanics. This makes that if one wants to know about an observable property of a closed system under the right assumptions, we are able to mathematically study the underlying property. This is done by applying the result of the Noether's theorem for Markov chains, which tells us what the necessary and sufficient conditions are for an observable to commute with the time evolution. We will see how Noether's theorem states that commutation of an observable with the Hamiltonian operator makes the observable a conserved quantity. Furthermore, we give an example of such observable, namely the number observable. Last but not least, we discuss how it might be possible to describe the entropy of a closed queueing network.

Throughout the report, we often derive notions by help of examples. However, we regard the results not to lose their generality as the examples have all been situated in general settings.

1.5 Notation

We discuss a few important notions regarding the mathematical notation used throughout this report. We frequently talk about vectors. A column vector $(v_1\ v_2\ \ldots\ v_k)^T$ will be denoted by the bold symbol $\boldsymbol{v}$ and sometimes explicitly written using comma's: $(v_1,\ v_2,\ \ldots,\ v_k)$. Unless stated otherwise, we assume that the k-dimensional natural numbers $\mathbb{N}^k_{\geq 0}$ is vector space in which $\boldsymbol{v}$ lives.

If we discuss probability functions of the form $\mathbb{P}(\boldsymbol{n}) = \psi(\boldsymbol{n})$, we write $\psi(\boldsymbol{n}) = \psi_n$ or sometimes more explicitly: $\psi(\boldsymbol{n}) = \psi_{n_1,\ldots,n_k}$. The context should provide the clarity needed to give interpretation of the state $\boldsymbol{n}$ in ψ_n. Furthermore, we denote the corresponding state space of state $\boldsymbol{n}$ by the calligraphic $\mathcal{S}$.

For the sake of the analogy between stochastic mechanics and queueing theory, we write the transpose of probability distribution columnvector $\boldsymbol{\psi}(t)$ as

$$\boldsymbol{\psi}^T(t) = \boldsymbol{\pi}(t).$$

Notice that $\boldsymbol{\pi}(t)$ is therefore a row vector.

We often use operators to denote certain changes in our systems. A specific operator that will show up is the differential operator

$$\frac{\partial}{\partial z},$$

which we denote by ∂_z.

Since changes in a system are related to multiple states, we use summations of which we assume it to be clear from the context what the lower and upper bounds are. For example, for a closed network with two appropiate queues and a total of N jobs where number of jobs at station i is $0 \leq n_i \leq N$, we write

$$\sum_{n_1,n_2} = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N}.$$

2 Introductory Theory

This section is here to refresh upon the most important theory which we will be using throughout this book. Furthermore, we define the model that will be of main interest. The preparatory theory consists of the theory of queues and that of Markov processes. We continue by treating the basics of stochastic mechanics as is introduced by Beaz and Biamonte [1].

2.1 Queueing theory

A queue consists of a waiting area and a service node, i.e. a station. Jobs, which can almost be any kind of objects like items or tasks, arrive as input at the waiting line. We assume the jobs to arrive following a Poisson process with rate β. There jobs line up until they can enter service at a specific server within the station. Jobs that arrive first will be served first, i.e. the First In First Out (FIFO) principle is assumed to hold. A job undergoes service of which the service time is assumed to be exponentially distributed with rate γ. When a job completes its service, it is annihilated from the queue and is send away as output. A queue can be represented by the diagram shown in Figure 2.1.

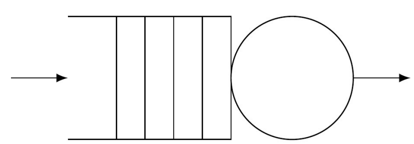

Figure 2.1: Representation of a queue in a diagram.

To describe the most important properties of a given queue, we use Kendall's notation. It classifies a queue by denoting four terms $A/S/c/K$, which A is the distribution of arrival times, S is the distribution of service times, c the number of servers and K the queue's capacity. Throughout this report, we will only look at cases where A and S are Markovian, which is denoted by M. If the fourth term is not mentioned it is that the specified queue has

A queueing network is a system of queues that are related to each other: every queue is connected to at least some other queue in the system. The simplest form of such networks are *open queueing networks*, these can be modelled as single-class systems, where each queue or (work)station can have multiple servers that can process jobs.

As has been introduced, we are interested in closed systems, which are systems that yield the property of not allowing any transfer in or out the system. In queueing theory, we call such systems *closed queueing networks* (CQNs) or just *closed networks*. These queueing networks have a fixed amount of jobs, so no jobs can leave nor enter. Gordon and Newell [2] found the unique equilibrium distribution for a certain kind of CQNs. This class of closed networks is defined as follows:

Definition 1. A **GN-network** is a single-class closed queueing network with $N \geq 1$ jobs in the system. Let the system have $M \geq 1$ stations, where the i-th station has $r_i \geq 1$ parallel servers each with a service time that is exponentially distributed with mean $1/\mu_i$. The jobs in the system have a Markovian routing, described by an irreducible routing matrix P.

2.2 Markov processes

Queueing networks are until now intriguing conceptual objects. For us to move forward with discussing applicable properties of these structures, we try to express these systems in the language of mathematics using Markov processes. Throughout the previous section it might have become clear that a queueing network could be expressed as a family of random variables $\{\boldsymbol{X}(t) \mid t \in T\}$, where $\boldsymbol{X}(t) = (X_1(t), \dots, X_k(t))$ is the random vector indexed by *continuous time parameter* $t \in T \subseteq \mathbb{R}_{\geq 0}$. Such a random vector we denote as $\boldsymbol{X}(t)$ or $\boldsymbol{X}_t$. Here element $X_i(t)$ is the random variable representing the number of jobs at server i. In other words, our queueing network is a *vector stochastic process*. Additionally, we are only to count the jobs in a system in integer numbers, thus we are dealing with *discrete-state processes*, also known as *chains*.

Since we narrow our perspective to GN-networks, see Definition 1, we consider processes with exponentially distributed service times and for this reason the counting processes $\{\boldsymbol{X}(t) \mid t \geq 0\}$ is a Poisson process. Notice that an exponentially distributed random variable is memoryless, and so the future state $\boldsymbol{X}(t_{n+1})$ does not conditionally dependent on the past given the present state $\boldsymbol{X}(t_n)$. This indicates that GN-networks yield the *Markov property* stated below.

Definition 2. Let state space $\mathcal{S}$ be a measurable space. A vector-stochastic process $\{\boldsymbol{Y}(t) \mid t \in T\}$ is said to be a **Markov process** if the conditional probability distribution of $\boldsymbol{Y}_{t_{n+1}}$ is independent of all its previous values besides $\boldsymbol{Y}_{t_n}$. More precisely, if it satisfies

$$\mathbb{P}(\boldsymbol{Y}_{t_{n+1}} \leq \boldsymbol{y}_{t_{n+1}} \mid \boldsymbol{Y}_{t_n} = \boldsymbol{y}_n, \ \dots, \ \boldsymbol{Y}_{t_0} = \boldsymbol{y}_0) = \mathbb{P}(\boldsymbol{Y}_{t_{n+1}} \leq \boldsymbol{y}_{t_{n+1}} \mid \boldsymbol{Y}_{t_n} = \boldsymbol{y}_n)$$

for all $t_0 < t_1 < \dots < t_n < t_{n+1}$ and all states $\boldsymbol{y}_i \in S$, where the events upon which we are condition must have a non-zero probability.

Consequently, we are dealing with *time-continuous Markov chains* (CTMCs). Note that we assume the Markov processes to be *time-homogenous*[1]. Recall the fact that Definition 1 states that the underlying Markov processes is irreducible. An important theorem related to this fact is given below.

Theorem 1. *If an irreducible Markov chain has a stationary distribution, then the stationary distribution is unique.*

A proof for this theorem can found in Levin et al. [3]. The notions on steady-states can be somewhat confusing. We therefore define the following:

Definition 3. A probability distribution row vector $\boldsymbol{\pi}$ is called **stationary** or a **steady-state vector** for the corresponding Markov chain if

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij},$$

for all $j \in \mathcal{S}$ and where $P = [p_{ij}]$ is the transition matrix.

This definition should make some sense as this implies that vector $\boldsymbol{\pi}$ *invariant* since $\boldsymbol{\pi} = \boldsymbol{\pi} P$. For proofs regarding the fact that an invariant vector is stationary and an equilibrium distribution, please refer to Norris [4, Ch. 1].

In Bolch et al. [5, Ch. 2] it is stated that

$$\dot{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}(t) Q \tag{1}$$

with help of the *infinitesimal generator matrix* $Q = [q_{ij}]$ where $i, j \in \mathcal{S}$. The *transition rate* q_{ij} is the rate of the processes entering state j upon leaving state i. It is furthermore shown that $\dot{\boldsymbol{\pi}}(t)$ can be expressed by the system of differential equations

$$\dot{\pi}_j(t) = \sum_{i \in \mathcal{S}} q_{ij} \pi(t),$$

for all $j \in \mathcal{S}$.

[1]This means that for $s, t \in T$ with $s \leq t$ such that $\mathbb{P}(\boldsymbol{Y}(t) = \boldsymbol{y} \mid \boldsymbol{Y}(s) = \boldsymbol{x}) = \mathbb{P}(\boldsymbol{Y}(t - s) = \boldsymbol{y} \mid \boldsymbol{Y}(0) = \boldsymbol{x})$.

2.3 Stochastic mechanics

Stochastic mechanics is the study of the dynamics of complex systems. We describe such systems in stochastic Petri nets, which

Definition 4. A **stochastic Petri net** consists of a set S of *species* and a set T of *transitions* with the input function $i : S \times T \to \mathbb{N}_0$, the output function $o : S \times T \to \mathbb{N}_0$ and the rate function $r : T \to (0, \infty)$ giving the rate constant for each transition in T. We call a unit of a certain species an *organism*.

We always assume, unless explicitly stated otherwise, that for any stochastic Petri net a species i (which has just been transformed into a new species j) will immediately start the process of transforming it to some other species k . In queueing language we would say: we assume there to be an infinite amount of parallel servers at every station. Note that this is generally not the case for a given queueing networks.

The probability mass function is the probability that the system is in state $\boldsymbol{n} = (n_1 \ \ldots \ n_k) \in \mathcal{S}$ at time t, denoted by $\psi_n(t)$. This distribution lives in vector space

$$L^1(X) = \left\{ \psi : X \to \mathbb{R} \;\middle|\; \int_X \psi(x)dx < \infty \right\}.$$

Here X is some appropriate measure space. This definition is defined analogously to that of the amplitude of a quantum system. See Baez and Biamonte [1, Sec. 4] for explanation on this analogy.

Note that the vector random variable for this probability function is $\boldsymbol{x} = (x_1, \ \ldots, \ x_k) \in \mathbb{N}^k$, where x_i denotes the number of species i. The following transformation will come in handy with the upcoming derivations. The z-transformation[2] of $\boldsymbol{x}$ is given by

$$\Psi(t) = \sum_{n \in \mathcal{S}} \psi_n(t) z^n. \tag{2}$$

This is nothing more than hanging up our probability functions on a clothesline, which we call a *probability generating function*. Here formal variable z^n is defined by $z^n = z_1^{n_1} \cdot \cdots \cdot z_k^{n_k}$ where n_i is the i-th element of the state vector $\boldsymbol{n} \in \mathcal{S} \subseteq \mathbb{N}_{\geq 0}^k$ and z_i is a complex number such that $|z_i| \leq 1$.

We study these complex systems because we want to know something about their dynamics. This is why the master equation will be in the spotlight throughout this book. The *master equation* is "a differential equation describing precisely how the probability of the Petri net having a given labelling changes with time!"[3] We know that this master equation is of the form

$$\frac{d}{dt}\Psi(t) = H\Psi(t).$$

The attentive reader might recognize this equation from quantum mechanics, where the Hamiltonian is some operator consisting of the different operators like the annihilator, creation or number operator operating on Fock space. In stochastic mechanics, we apply a very similar way of thinking on defining our Hamiltonian H, which is explained in Baez and Biamonte [1, Sec. 4]. The creation operator $a^\dagger$ is defined as the operator that creates an organism of a specific species. This operator can be expressed in terms of the formal variable z. To see this consider the following: an arrival at time t for a system that is in state N means that the formal power series $\Psi(t)$ is of the form

$$\Psi(t) = \sum_{n=0}^{\infty} \psi_n(t) z^n = z^N.$$

Therefore, the following should hold

$$a^\dagger \Psi(t) = a^\dagger z^N = z^{N+1},$$

[2] To refresh upon the theory of z-transformations, read Appendix I of Kleinrock [6].

[3] See Section 3.1 of Baez and Biamonte [1].

since after the arrival the system contains $N+1$ organisms. We conclude that we can define $a^\dagger = z$. Similarly, we define the annihilator operator a as the operator that annihilates an organism. Assuming that there are N organisms in the system, we want

$$a\Psi(t) = az^N = Nz^{N-1}$$

as there are N organisms that can be annihilated. We can thus define $a = \frac{\partial}{\partial z}$.

Consider a system where organisms of a species are created one by one. The system can be depicted by the following stochastic Petri net:

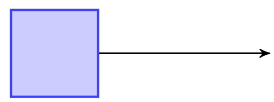

Figure 2.2: Stochastic Petri net depicting the creation of an job.

We desire to describe the time evolution of this system. As was introduced, we can do so by trying to describe the Hamiltonian operator corresponding to the stochastic Petri net. A guess for the Hamiltonian could simply be the creation operator. To see what happens suppose we have N organisms in the system at time t, that is $\psi_N(t) = 1$ and hence $\Psi(t) = z^N$. Letting the Hamiltonian operate on the power series gives

$$H\Psi(t) = z^{N+1}.$$

The master equation $H\Psi(t) = \frac{d}{dt}\Psi(t)$ tells us that the probability of having $N+1$ organisms changes to $\psi_{N+1} = 1$. Notice that the probability of having N organisms apparently stays the same. This contradicts with the law of total probability $\sum_{n\in\mathcal{S}} \psi_n(t) = 1$ for any moment in time. We can also verify the contradiction conceptually: we know that after the creation of an organism the probability of having being in state N is zero. More precisely, we should have that $\psi_N(t+\varepsilon) = 0$ for some sufficiently small[4] $\varepsilon > 0$ as there are $N+1$ organisms in the system right after the arrival. In other words, it must be that $\frac{d}{dt}\psi_N(t) = -1$. Hence, the probabilities change like

$$\frac{d}{dt}\psi_N(t) = -1 \quad \text{and} \quad \frac{d}{dt}\psi_{N+1}(t) = 1.$$

Turning back to the Hamiltonian H, we can conclude that $H = a^\dagger$ does not appropriately describe the time evolution of the system depicted in Figure 2.2. One can readily verify that the corresponding Hamiltonian is

$$H = a^\dagger - 1.$$

Consider the process of annihilation, i.e. the process where an organism is annihilated with a transition time that is exponentially distributed.

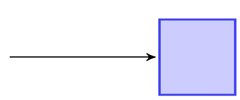

Figure 2.3: Stochastic Petri net depicting the annihilation of an organism.

Baez and Biamonte [1, Sec. 2] tell us that the corresponding Hamiltonian is

$$H = a - a^\dagger a,$$

[4]The arbitrary small $\varepsilon > 0$ makes time $t + \epsilon$ indicate the moment just after the arrival.

where $a^\dagger a =: \widehat{N}$ is the number operator which we recognize from quantum mechanics. With a similar reasoning which was used for the creation process, one can check that the Hamiltonian operator is of the correct form.

We have treated two transitions: one with one input, the other one with a single output. Baez and Biamonte [1, Sec. 5] explain how to treat transitions with multiple species as input and output. The *general rule* states that the Hamiltonian of a transition with k species as input and m species as output is of the form

$$H = (a^\dagger)^m a^k - \widehat{N} \cdot (\widehat{N} - 1) \cdot \dots \cdot (\widehat{N} - k + 1). \tag{3}$$

Now we know how to describe any system with one transition. However, most systems contain multiple transitions. Baez and Biamonte [1, Sec. 6] show that one can define the Hamiltonian of a multiple-transition system by breaking down the corresponding stochastic Petri net and find the Hamiltonian H_i for each transition i. After which we simply sum up all the Hamiltonian operators H_i weighted by the corresponding transition rates. This Hamiltonian H comes down to being the mechanical queueing version of the second quantized Schrödinger operator

$$\widehat{H} = \oplus_{i=0}^{\infty} \widehat{H}_i,$$

which operates in Fock space $\mathcal{F} := \oplus_{i=0}^{\infty} \mathcal{F}_i$.[5]

Analogously, we see that for an arbitrary mechanical queueing system with k transitions the Hamiltonian is

$$H = \oplus_{i=0}^{\infty} H_i,$$

operating on the vector space $L^1 := \oplus_{i=0}^{\infty} L^1_i$.

Recall from the theory of Markov processes that we are often interested in infinitesimal generator matrix of a system yielding information on how a system changes from state to state. In the theory of stochastic mechanics we have a notion that is very comparable:

Definition 5. An operator Q is called an **infinitesimal stochastic** if its non-diagonal entries are nonnegative and its columns sum to zero.

Baez and Biamonte [1, Sec. 4] show that the Kronecker delta function δ_{ij} can be used to prove that e^{tH} is a stochastic operator. This is of interest since the master equation tells us

$$\frac{d}{dt}\Psi(t) = H\Psi(t) \implies \Psi(t) = e^{tH}\Psi(0).$$

Along this line of thought, we can show that the Hamiltonian is an operator belonging to the class of stochastic operators, which is defined below.

Definition 6. A **stochastic operator** is a linear operator $U : L^1(X) \to L^1(X)$ such that for any distribution $\psi \in L^1(X)$

$$\int U\psi = \int \psi \qquad \text{and} \qquad \psi \geq 0 \implies U\psi \geq 0.$$

2.4 Stochastic mechanical queueing model

In this section we will formulate the general model that will be studied throughout this book. To do so, we must look into what it is that we actually want to study. As discussed before we are interested in queueing systems, but we would like to peek inside this world with the perspective of stochastic Petri nets. In Section 2.2 it was stated that Markov chains turn out to be a mathematical

[5] See Gustafson and Sigal [7] for precise explanations regarding these mathematical concepts of quantum mechanics.

tool in order fo us to describe queues. So, we consider a time-continuous Markov chain. Since the main target is to analyse closed systems, we restrict our process for some positive integer N by

$$n_1 + \cdots + n_k = N,$$

where n_i is the number of organisms we have of species i. In queueing terminology n_i would be the number of jobs at station i. Any vector

$$\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathcal{S} \subseteq \mathbb{N}^k_{\geq 0}$$

satisfying this restriction is called a *state*. Here the state space is the set

$$\mathcal{S} = \left\{ \boldsymbol{n} \in \mathbb{N}^k_{\geq 0} \;\middle|\; n_1 + \cdots + n_k = N \right\}.$$

We consider processes that allow going from any state $\boldsymbol{n} \in \mathcal{S}$ to any other state $m \in \mathcal{S}$ in the system. In other words, the Markov processes is *irreducible*.

Moreover, we assume that there are $k \geq 1$ stations, which will also be referred to as transitions. Notice that this may be confusing as a station refers to a waiting line and the service node including servers that process or *transition* the jobs that are in queue. Nevertheless, we make use of both terms interchangeably as we expect the reader to understand what is meant due to the context. The reason is that we have introduced the term 'transition' from the theory of stochastic mechanics, in which it is assumed that any job can immediately be served. Therefore, in a stochastic mechanical system there is no actual waiting line. The time it takes a server of station i to serve a job is assumed to be is exponentially distributed with a mean value of $1/\mu_i$ in queue i, where service rate $\mu_i > 0$.

As mentioned before, by construction stochastic Petri nets implicitly assume that any organism of some species can immediately start its service. Only considering networks of M/M/∞ queues will not satisfy queueing theorists, as interest can go out to single server queues as well. Hence, we will also consider systems where each station has only one server. The model we will study is defined below.

Definition 7. A **stochastic mechanical queueing system** is a single-class closed queueing network with $N \geq 1$ jobs, i.e. organisms in the system. Let the system have $M \geq 1$ stations, i.e. transitions, where the i-th station has only 1 server or an infinite number of servers each with a service time that is exponentially distributed with mean $1/\mu_i$ for server i. The jobs in the system have a Markovian routing, described by an irreducible routing matrix P.

Notice that Definition 1 implies that a stochastic mechanical queueing system, or mechanical queueing system for short, with single server stations is actually just a GN-network!

3 Time Evolution of Infinite Server Queueing Systems

We desire to study mechanical queueing systems which were defined in Definition 7. We will not try to reinvent the wheel and thus we take a look at what can help us analyse these processes. As commented on in the previous section, we know that the theory of stochastic mechanics considers processes where arrivals can immediately be served, in other words there are an infinite number of servers for each station.

On of the most interesting properties of any queuing system is the equilibrium solution. We know that this can be found by solving of the master equation equated to zero,

$$\frac{d}{dt}\Psi(t) = H\Psi(t) = 0.$$

The solution may be referred to as the *equilibrium probability distribution.* By Theorem 1, we know that such solution is unique since the Markov process is irreducible. This equilibrium solution is the probability distribution $\Psi(t) = \Psi$ such that and is therefore independent of time. For the case where there are infinite number of servers, we know how to derive the Hamiltonian H as was explained in Section 2.3. Therefore one might wonder how difficult solving this master equation can be, since we exactly now how to describe the dynamics of a system. Under certain circumstances it is indeed easy to derive the steady-state vector by only using the Hamiltonian. An example of that sort is shown below.

Example 1. *Finding the equilibrium distribution for a simple closed system with two jobs.*
In this example we will try to find the equilibrium with use the matrix representation of the Hamiltonian for the stochastic Petri net given in Figure 3.1. This is a stochastic Petri net consisting of two different species both undergoing a transition transforming them into the other species.[6] We can depict this as a closed system of two M/M/∞ queues, where the transition and preceding species node form a station. Thus, the node of species 1 and transition with transition rate β form one M/M/∞ queue. Let us assume that we always have two customers in our system, that is $N = 2$.

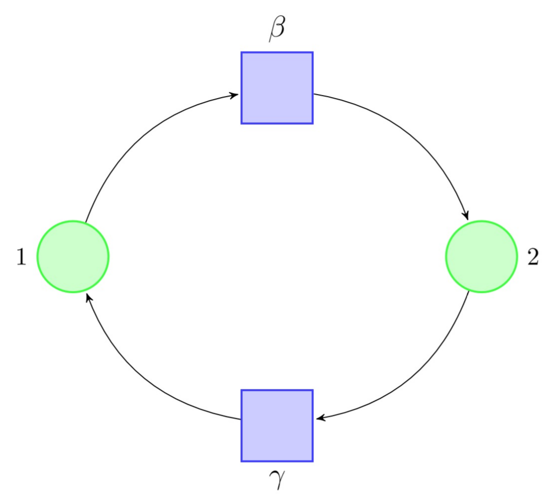

Figure 3.1: Stochastic Petri net of closed network of two M/M/∞ queues.

The problem formulation allows us to invoke the theory on Markov chains. Consequently, we take off and derive the steady-state probability vector $\boldsymbol{\pi} = (\pi_{(2,0)}\ \pi_{(1,1)}\ \pi_{(0,2)})$. Here we denote the states by state vector $(n_1,\ n_2)$. As previously stated, we know that this vector is unique.

[6]This is the same example given by Baez and Biamonte [1, Sec. 9].

Since we are after the vector describing a steady-state of the queueing process, we can derive it by solving the system of equations

$$\boldsymbol{\pi} Q = \frac{d}{dt}\boldsymbol{\pi} = \mathbf{0}. \tag{4}$$

This yields that the flux into a state must be equal to the flux out of a state. This is of course an obvious requirement of our steady-state vector $\boldsymbol{\pi}$! Recall that in (4) the matrix Q denotes the infinitesimal generator matrix.

The attentive reader would maybe have noticed the similarity with our problem of trying to find the equilibrium distribution regarding stochastic mechanics. To strengthen the analogy, we let the equilibrium distribution to be defined as the transpose of the steady-state vector $\boldsymbol{\pi}$ and define our Hamiltonian H as the transpose of the generator matrix Q. The system of equations given by (4) is equivalent to

$$H\boldsymbol{\psi} = \mathbf{0}.$$

Accordingly, we will try to solve this equation in a similar method as is normally done with Markov chains. First of all we set up a state transition diagram, found in Figure 3.2.

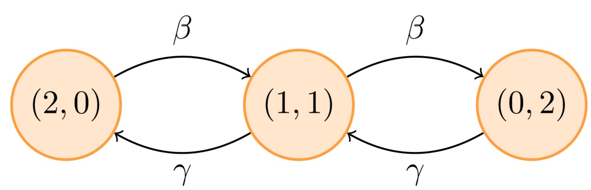

Figure 3.2: State-transition diagram for Figure 3.1 with $N = 2$.

Translating this information into our Hamiltonian H results in

$$H = \begin{pmatrix} -\beta & \gamma & 0 \\ \beta & -(\beta+\gamma) & \gamma \\ 0 & \beta & -\gamma \end{pmatrix}.$$

Promptly we substitute our Hamiltonian H into $H\boldsymbol{\psi} = 0$ from which the solution arises to be proportional in the following sense

$$\psi_{(2,0)} \propto \gamma^2, \quad \psi_{(1,1)} \propto \beta\gamma \quad \text{and} \quad \psi_{(0,2)} \propto \beta^2.$$

We readily verify that $\sum_{i=1}^{3} \psi_i = 1$ is indeed satisfied. And so, we have found our unique equilibrium distribution!

In this section our goal is to analyse the time evolution of some basic systems of queues. First of all, we like to find the steady-state vector for the process that is considered. More precisely, we try to discuss networks that help us analyse the process depicted in Figure 3.1, but for any number of jobs N. Similar to the example above, one can see that this can be done writing out the matrix form of the Hamiltonian and by recursively solving the equation $H\boldsymbol{\psi} = 0$. We, however, desire to find an alternative way. We thought that it would be a good idea to start analysing the normalization term and finding an explicit expression for it, like Gordon and Newell [2] did for the GN-networks. This attempt is shown in Appendix B. Unfortunately, it did not get us anywhere.

As mentioned before, we are interested specifically in networks of M/M/∞ queues in this section. Since we want to build up to analyse our mechanical queueing process introduced in Section 2.4, it would be a straightforward idea to consider a single open M/M/∞ queue at first. After that we will derive what happens if you combine two of such queues producing a tandem. At last, we will enclose the tandem resulting in a closed system of two queues. The latter one is the process showed in Figure 3.1, but now for any arbitrary number of jobs in contrast with Example 1.

3.1 Time evolution of the M/M/∞ queue

In this section we consider a single open M/M/∞ queue. To get an idea of to what state the system converges, we derive the equilibrium at fist. After that we specify the time evolution of the queue by deriving the Hamiltonian. We do the derivations by using a specific example. On the other hand, the results will not lose its generality.

Butterflies are intriguing animals: not only stunning, but also very diverse among all its species. A butterflies' life can be categorized into the typical four-stage life cycle of insects. This life cycle starts when the insect is born as a caterpillar. Eventually, such caterpillar pupates into a chrysalis undergoing the metamorphism phase. Notice that this process can be described as an M/M/∞ queue. We denote the birth rate (the arrival rate) by β and the rate of metamorphism (the service rate) by γ.

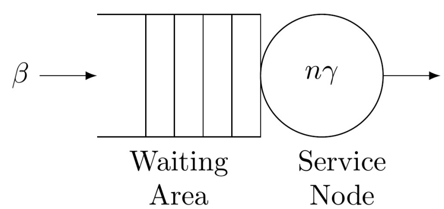

Figure 3.3: A queue with infinite number of servers.

In the figure above we formally portray the waiting area. But obviously there will never be any caterpillars waiting to go into metamorphism until some other caterpillar went through its metamorphism. Therefore we have no waiting lines. This makes that the service rate is $n\gamma$ in state n, where n are the number of caterpillars in the habitat.

Equilibrium solution

We define, as convention, that the load for a M/M/∞ process is the ratio of the arrival rate and service rate, that is

$$\rho = \frac{\beta}{\gamma}.$$

If at a moment in time the system contains n caterpillars the process has a performance of serving at a rate of $n\gamma$. This can be illustrated in the following diagram:

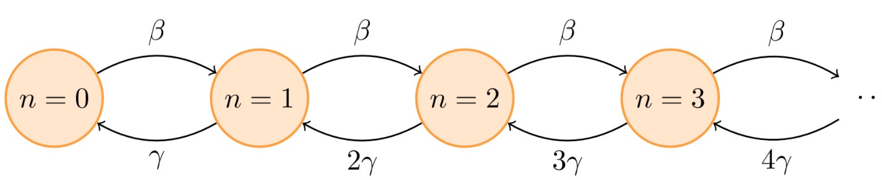

Figure 3.4: State-transition diagram for the M/M/∞ queue.

To derive the steady state we begin by stating the *global balance equations*, which identify that a network in equilibrium must be such that the total probability flux towards state i must be equal to the total flow from out state i to any other state. This relation can be expressed by

$$\psi_i = \sum_{i \in \mathbb{N}^k} h_{ij} \psi_j$$

for all i, where $\boldsymbol{\psi}$ denotes the steady-state vector. Here the probability flux from state j to state i is denoted by $h_{ij}\psi_j$. This conservation of flow can alternatively be represented by $H\boldsymbol{\psi} = \mathbf{0}$.

Chandy [8] showed that these global balance equations can be decomposed into a system of equations called the *local balance equations*, which are much easier to solve. Since deriving equilibrium state $\boldsymbol{\psi}$, this result is of great help. This state-transition diagram helps us visualize the process and in such a way deriving the local balance equations become much more intuitive. They are given by

$$\beta\psi_{n-1} = n\gamma\psi_n.$$

Rewriting these equations and by repeatedly substituting, we find

$$\psi_n = \frac{1}{n}\rho\psi_{n-1} = \frac{1}{n!}\rho^n\psi_0.$$

By the law of total probability, we know that

$$\sum_{n=0}^{\infty}\psi_n = \sum_{n=0}^{\infty}\frac{1}{n!}\rho^n\psi_0 = e^{\rho}\psi_0 = 1.$$

must hold if we assume that the arrival rate cannot exceed the service rate, more precisely $|\rho| < 1$. We are still left with two unknowns: ψ_0 and ψ_n. By the property stated above, we find that the initial probability distribution is given by

$$\psi_0 = e^{-\rho}.$$

Henceforth, the equilibrium distribution is

$$\psi_n = \frac{\rho^n}{n!}e^{-\rho}$$

for any integer $n \geq 0$. This distribution thus describes the steady-state of the birth and metamorphism process of caterpillars. One might have seen this distribution before: this is the Poisson distribution! This makes sense, since we have assumed the arrival process of the queue to be Poisson.

Master equation

The stochastic Petri net of this process is given below.

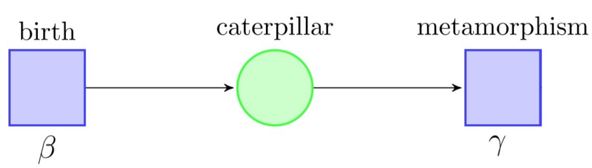

Figure 3.5: Stochastic Petri net of an M/M/∞ of caterpillars.

We want to examine the dynamics of the system, we do so by formulating the master equation of this queue and we must therefore come up with an corresponding Hamiltonian operator H. Defining an appropriate stochastic Petri net is important since deriving the Hamiltonian operator will be much easier with such depiction of the queue. We proceed by first showing an explicit derivation of the Hamiltonian operator, after which we confirm the result by looking at how Baez and Biamonte [1] do it.

We apply a clever trick which was introduced in Section 2.3, we may divide the stochastic Petri net of the queue into separate Petri nets of which we can more easily derive the corresponding Hamiltonian. The process shown in Figure 3.5 can be divided into two separate transitions:

Figure 3.6: The two separated transitions of the caterpillar queue.

Let us first examine the arrival process, which for this example is the birth transition of caterpillars. To do so, we set the service rate $\gamma = 0$. This makes that the considered system is solely determined by the arrival process. In Section 2.2, it was discussed that the time derivative of $\boldsymbol{\psi}(t)$ can be expressed in terms of the transition rates. More specifically, by the state-transition diagram drawn in Figure 3.4 we see

$$\begin{aligned}\dot{\psi}_n(t) &= \beta\big(\psi_{n-1}(t) - \psi_n(t)\big) \qquad \text{for } n \geq 1,\\ \dot{\psi}_0(t) &= -\beta\psi_0(t).\end{aligned}$$

For convenience, define $\psi_{-1}(t) = 0$ for all $t \in T$. Using the generating function Ψ, we can write

$$\dot{\Psi}(t) = \sum_{n=0}^{\infty} \dot{\psi}_n(t) z^n = \beta \sum_{n=0}^{\infty} \psi_{n-1}(t) z^n - \beta \sum_{n=0}^{\infty} \psi_n(t) z^n = \beta(z-1)\Psi(t). \tag{5}$$

We conclude that the Hamiltonian operator for the arrival process is given by $\beta\big(a^\dagger - 1\big)$, which tells us how the probabilities ψ_n change when a caterpillar is born. This should make sense as we see how operator $a^\dagger$ creates a caterpillar at rate β and the correction term -1 is there to make sure the law of total probability holds after an arrival.

Now we consider the departure process by setting the arrival rate $\beta = 0$. We can express the time derivatives of $\boldsymbol{\psi}(t)$ as

$$\begin{aligned}\dot{\psi}_n(t) &= \gamma\big((n+1)\psi_{n+1}(t) - n\psi_n(t)\big) \qquad \text{for } n \geq 1,\\ \dot{\psi}_0(t) &= \gamma\psi_1(t).\end{aligned}$$

Substituting these expressions in the derivative of power series $\Psi(t)$ gives

$$\dot{\Psi}(t) = \gamma \sum_{n=1}^{\infty} (n+1)\psi_{n+1}(t) z^n - \gamma \sum_{n=1}^{\infty} n\psi_n z^n.$$

Writing this in terms of the creation and annihilation operators makes

$$\dot{\Psi}(t) = \gamma \sum_{n=0}^{\infty} \partial_z \psi_{n+1}(t) z^{n+1} - \gamma \sum_{n=0}^{\infty} z\partial_z \psi_n z^n = \gamma\big(\partial_z - z\partial_z\big)\Psi(t),$$

since $a \sum_{n=0}^{\infty} \psi_{n+1}(t) z^{n+1} = a\Psi(t)$. Hence, we can describe the annihilation process by the Hamiltonian $a - a^\dagger a$. The annihilation operator a indeed makes sure a caterpillar departs from the queue, and the number operator $a^\dagger a$ is there such that the chance of being in the same state, as before the annihilation, goes down at an appropriate rate, namely $n\gamma$ if the system is in state $n \geq 0$. This Hamiltonian describes how the probabilities will change after a caterpillar leaves the queue by going into metamorphism.

We have seen in Section 2.3 that these operators are indeed the Hamiltonians Baez and Biamonte [1, Sec. 5] prescribe for the arrival and annihilation process, respectively. Having constructed the Hamiltonians of the arrival and annihilation process, we are just one step away from describing the time evolution of the whole system of caterpillars being born and going into metamorphism. In Section 2.2 we discussed that Baez and Biamonte [1, Sec. 6] tell us that we can define the Hamiltonian of a mutliple-transition system by simply summing the Hamiltonians corresponding to each individual transition. Since the caterpillar queue, which is an M/M/∞ queue, consists of the two processes we analysed above, we can conclude that the Hamiltonian is

$$H = \beta\big(a^\dagger - 1\big) + \gamma\big(a - a^\dagger a\big).$$

It follows that the master equation is given by

$$\frac{d}{dt}\Psi(t) = H\Psi(t) = \big(\beta(a^\dagger - 1) + \gamma(a - a^\dagger a)\big)\sum_{n=0}^{\infty}\psi_n(t)z^n. \tag{6}$$

This equation expresses how the probabilities of having n caterpillars changes over time.

We can use the master equation (6) in order to find a distribution ψ_n such that $\frac{d}{dt}\Psi(t) = 0$. By definition, this will yield an equilibrium solution as $\frac{d}{dt}\Psi(t) = \frac{d}{dt}\Psi = 0$ indicates ψ_n does not change over time. Trivially, by construction of our process, all states can be reached in the state space $\mathcal{S} = \mathbb{N}_{\geq 0}$, henceforth the the underlying Markov chain of this process is irreducible. By Theorem 1 we find that such distribution ψ_n is the unique equilibrium. Some algebra gives us that we can verify that the equilibrium solution derived in Section 3.1 given by

$$\psi_n = \frac{\rho^n}{n!}e^{-\rho},$$

for any integer $n \geq 0$, indeed gives $H\Psi = 0$. The corresponding derivation is shown in Appendix C. To derive this result we suppose that $|\rho| < 1$, which means that we assume the birth rate β of caterpillars to be smaller than the rate γ at which the caterpillars go into metamorphism.

3.2 Time evolution of a tandem of two M/M/∞ queues

Now we are familiar with the equilibrium of a single $M/M/\infty$ queue. Since we are interested in a closed network of such queues, it is a wise idea to try to understand what would happen if we put such queues in a network, also known as a *tandem*. We turn out to be lucky. The theory regarding finding an equilibrium of such a tandem has been well-developed. After discussing this, we explain how the time evolution of such system can be mathematically described.

Coming back to the process previously considered in Section 3.1 of caterpillars being born and going into metamorphism, we know that such caterpillars that enter metamorphism pupate into a chrysalis. Eventually, when the metamorphism is completed, a butterfly comes out. This process can be the described by the following stochastic Petri net:

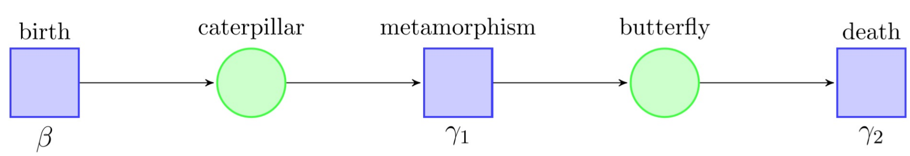

Figure 3.7: The life-cycle of a butterfly depicted in a stochastic Petri net.

One might recognize this as an extension of the stochastic Petri net in Figure 3.5. Notice that process can be thought of as a tandem of two M/M/∞ queues. In this section we denote the caterpillars and butterflies as species 1 and 2, respectively.

Equilibrium solution

Jackson [9] showed that for "an arbitrarily structured open queueing network with exponential processing times"[7] the stationary distribution is a product-form consisting of equilibrium distributions corresponding to each individual queue. We can mathematically express this by the

[7]See Chapter 2 on job shop manufacturing systems by Zijm [10].

following: assume that there are k stations, each with c_i servers. Then the stationary distribution is given by

$$\psi_{n_1,\dots,n_k} = \prod_{i=1}^{k} \phi_i(n_i),$$

where the marginal distributions are

$$\phi_i(n_i) = \frac{1}{G(i)} \frac{\rho_i^{n_i}}{n_i!} \quad \text{if } n_i < c_i \qquad \text{and} \qquad \phi_i(n_i) = \frac{1}{G(i)} \frac{c_i^{c_i}(\rho_i/c_i)^{n_i}}{c_i!} \quad \text{if } n_i < c_i.$$

Recall that we use the convention $\rho_i = \gamma_i/\beta_i$ since we are after M/M/∞ queues. Here $G(i)$ denotes the normalization constant which can be computed by invoking the law of total probability. Zijm [10] tells us that

$$G(i) = \sum_{n=0}^{c_i-1} \frac{\rho_i^n}{n!} + \frac{\rho_i^{c_i}}{c_i!}\big(1 - \rho_i/c_i\big)^{-1}.$$

We are interested in the specific case where we let c_i approach infinity for all $1 \leq i \leq k$. We for sure know that the number of jobs at any station is less than c_i, therefore by Jackson's derivations [9] we conclude that the stationary distribution for a tandem of k different M/M/∞ queues is

$$\psi_{n_1,\dots,n_k} = \prod_{i=1}^{k} e^{-\rho_i} \frac{\rho_i^{n_i}}{n_i!},$$

where we assume $|\rho_i| < 1$ for any $1 \leq i \leq k$.

Applying this to the network of the caterpillar and butterfly queue, we have that the steady-state is

$$\psi_{n_1,n_2} = e^{-\rho_1} \frac{\rho_i^{n_1}}{n_1!} \cdot e^{-\rho_2} \frac{\rho_2^{n_2}}{n_2!}, \tag{7}$$

where n_1, n_2 denotes the number of caterpillars and butterflies, respectively.

Master equation

We desire to describe the dynamics of system portrayed in Figure 3.7 and we therefore construct the master equation of this tandem. We derive the Hamiltonian operator by the same procedure executed in the previous section: we look at all transitions separately and sum their corresponding Hamiltonian operators. Notice that the Hamiltonian of the arrival process of this network is simply the operator describing the time evolution of the birth process of caterpillars, derived in Section 3.1. Since we denote caterpillars as species 1, the corresponding Hamiltonian is thus $= \beta(a_1^\dagger - 1)$. Similarly, we find that we can describe the death process of butterflies as the annihilation process we have analysed in the previous section. The Hamiltonian is given by $\gamma_2(a_2 - a_2^\dagger a_2)$. These two operators yield a similar interpretation as was discussed in the previous section.

There is however still one transition left to describe: the process of caterpillars going into metamorphism and come out as butterflies. The transition has one input and one output. By looking at the state-transition diagram of the network, one can readily verify that we can describe the time derivatives of the probability functions in terms of the transition rates by

$$\dot{\psi}_{n_1,n_2}(t) = \gamma_1\big((n_1+1)\psi_{n_1+1,n_2-1} - n_1\psi_{n_1,n_2}\big) \qquad \text{for } n_1, n_2 \geq 0, \tag{8}$$

where for convention we set $\psi_{n_1,-1} = 0$ for all $n_1 \geq 0$. Notice that (8) holds for the boundary case $n_2 = 0$, namely $\dot{\psi}_{n_1,0}(t) = -\gamma_1 n_1 \psi_{n_1,n_2}$ for any $n_1 \geq 0$. And the change in probability of there being no jobs in service is solely depend on the service process of the system being in state $(1, n_2 - 1)$ for $n_2 \geq 1$.

We want to derive the Hamiltonian of this transition. We do so by multiplying (8) by $z^n = z_1^{n_1} z_2^{n_2}$ and summing over all $n_1, n_2 \geq 0$. This gives

$$\dot{\Psi}(t) = \gamma_1 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(n_1+1)\psi_{n_1+1,n_2-1} z_1^{n_1} z_2^{n_2} - \gamma_1 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2}.$$

Rewriting this in terms of the creation and annihilation operators makes

$$\dot{\Psi}(t) = \gamma_1 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_2 \partial_{z_1} \psi_{n_1+1,n_2-1} z_1^{n_1+1} z_2^{n_2-1} - \gamma_1 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1 \partial_{z_1} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2}.$$

In Appendix D, Property 1, we show that

$$\partial_{z_1} \Psi(t) = \frac{\partial}{\partial z_1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \psi_{n_1+1,n_2-1} z_1^{n_1+1} z_2^{n_2-1}.$$

Invoking this identity results in

$$\dot{\Psi}(t) = \gamma_1 \big(z_2 \partial_{z_1} - z_1 \partial_{z_1} \big) \Psi(t).$$

We conclude that the Hamiltonian of the metamorphism-transition is $\gamma_1 \big(a_2^\dagger a_1 - a_1^\dagger a_1 \big)$. It makes sure that the coefficients of $\Psi(t)$ change according to the process of metamorphism where one caterpillar (species 1) is transformed into a butterfly (species 2) with a rate of γ_1. We can see this by reading off the operators appropriately: we see that a_1 annihilates a job of species 1 and $a_2^\dagger$ creates a job of species 2, this is exactly what happens at the metamorphism transition. The number operator $a_1^\dagger a_1$ for species 1 makes sure that the probability on being in the same stated as we started in after a transition goes down at an appropriate rate.

We have successfully derived the Hamiltonians of all individual transitions. We can conclude that the Hamiltonian of the stochastic Petri net depicted in Figure 3.7 of two M/M/∞ queues in tandem is given by

$$H = \beta \big(a_1^\dagger - 1 \big) + \gamma_1 \big(a_2^\dagger a_1 - a_1^\dagger a_1 \big) + \gamma_2 \big(a_2 - a_2^\dagger a_2 \big).$$

The master equation $\dot{\Psi}(t) = H\Psi(t)$ states how the probability function $\psi_{n_1,n_2}(t)$ of having n_1 caterpillars and n_2 butterflies changes over time. Some algebra makes it possible to verify that $H\Psi = 0$ holds for distribution (7), verifying that it indeed describes the steady-state.

3.3 Time evolution of a closed network of two M/M/∞ queues

Now we finally arrive at a mechanical queueing network. We analyse a simple system with help of an example regarding a process found in chemistry. We consider the chemical reaction where acetic acid is dissolves in water. That is the reaction where the hydroxy group of the acetic acid reacts with a water molecule turning into an acetate ion and a hydronium ion. The reaction is given by

$$CH_3COOH + H_2O \rightleftharpoons CH_3COO^- + H_3O^+. \tag{9}$$

Notice that the reverse reaction also happens but is not favored under normal circumstances. By taking a look at (9), the underlying queueing network might already seems very clear. We have a queue where acetic acid reacts, creating acetate ions. These acetate ions arrive in another queue where the ions react into acetic acid again. So we can interpret the chemical reaction 9 as a closed network of two M/M/∞ queues. We illustrated the process of the chemical reaction in Figure 3.8. We denote the acetic acid molecules as species 1 and the acetate ions as species 2. We denote the reaction rate of (9) from left to right and from right to left by β and γ, respectively.

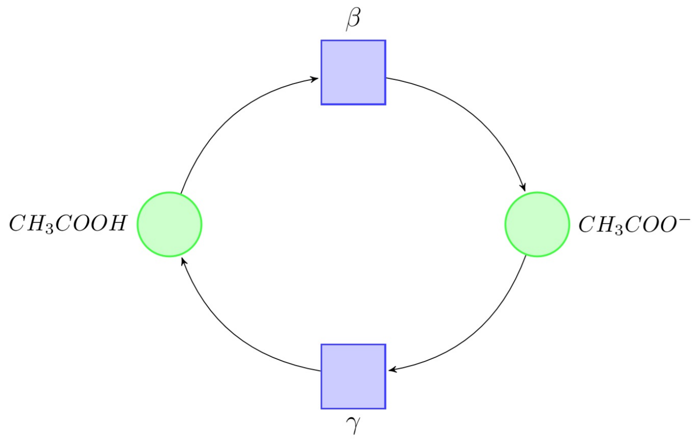

Figure 3.8: The closed stochastic Petri net of the chemical reaction 9.

Equilibrium solution

To derive the equilibrium solution of this closed network, we will be using the local balance equations again. First, notice that the state-space S is

$$S = \big\{(0, N),\ (1, N-1),\ \ldots,\ (N-1, 1)\ (N, 0)\big\}.$$

As convention we define the equilibrium vector $\boldsymbol{\psi}$ to be indexed as

$$\psi_n := \psi_{(n, N-n)},$$

for any integer $0 \leq n \leq N$. Let β and γ be the service rate of any server at station 1 and station 2, respectively. Denote the utilization of the whole system by $\rho = \beta/\gamma$. The state-transition diagram for a closed network of two M/M/∞ queues is given in Figure 3.9.

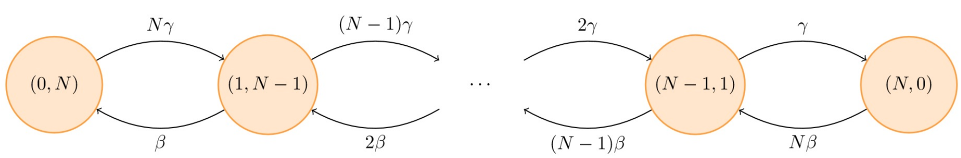

Figure 3.9: State-transition diagram for a closed network of two M/M/∞ queues.

The state-transition diagram helps us to write out the local balance equations

$$\big(N - (n-1)\big)\gamma\psi_{n-1} = n\beta\psi_n,$$

for all $0 < n \leq N$. By recursively substituting the derived result, it follows that

$$\psi_n = \big(N - (n-1)\big)\frac{\psi_{n-1}}{n\rho} = \frac{1}{n! \cdot \rho^n}\prod_{i=1}^{n}\big(N - (i-1)\big)\psi_0 = \binom{N}{n}\rho^{-n}\psi_0. \tag{10}$$

By the law of total probability, we find that the equilibrium for the initial state $(0, N)$ is given by

$$\psi_0 = \left(\sum_{n=0}^{N}\binom{N}{n}\rho^{-n}\right)^{-1} = \left(\frac{\rho}{1+\rho}\right)^N.$$

This last step follows from the binomial identity

$$(x+y)^N = \sum_{m=0}^{N} \binom{N}{n} x^{N-n} y^n,$$

by choosing $x = 1$ and $y = 1/\rho$. To conclude, we find that the equilibrium distribution is

$$\psi_n = \binom{N}{n} \frac{\rho^{N-n}}{(1+\rho)^N}.$$

To bring this into perspective, recall that we defined the convention $\psi_n = \psi_{(n,N-n)}$. With a more intuitive use of notation, we describe the equilibrium solution by

$$\psi_{n_1,n_2} = \binom{N}{n_1} \frac{\rho^{n_2}}{(1+\rho)^N}. \tag{11}$$

Thus, we derived the equilibrium solution for the process shown in Figure 3.1 representing an GN-network of two M/M/∞ queues. In contrast with what we derived in Example 1, solution (10) holds for an arbitrary number of total jobs in the system.

Master equation

By taking a look at the state-transition diagram depicted in Figure 3.9, we can describe the time-derivative of the probability function $\psi_{n_1,n_2}(t)$ at any instant in time by using the transition rates as follows:

$$\dot{\psi}_{n_1,n_2}(t) = (n_2+1)\gamma\psi_{n_1-1,n_2+1}(t) + (n_1+1)\beta\psi_{n_1+1,n_2-1}(t) - \big(n_1\beta + n_2\gamma\big)\psi_{n_1,n_2}(t), \tag{12}$$

for any $0 \le n_1, n_2 \le N$, where we set $\psi_{-1,N+1}(t) = \psi_{N+1,-1}(t) = 0$. This makes sure that the boundary cases

$$\dot{\psi}_{0,N}(t) = \beta\psi_{1,N-1}(t) - N\gamma\psi_{0,N}(t) \quad \text{and} \quad \dot{\psi}_{N,0}(t) = \gamma\psi_{N-1,1} - N\beta\psi_{N,0}$$

are accurately described by (12). To derive the Hamiltonian we apply the same tricks we have used before: we first multiply (12) by $z^n = z_1^{n_1} z_2^{n_2}$ and sum over all $0 \le n_1, n_2 \le N$. Making use of the creation and annihilation operators, similar to what we did in the previous section, gives

$$\begin{aligned}\dot{\Psi}(t) = \beta z_2 \frac{\partial}{\partial z_1} \sum_{n_1,n_2} \psi_{n_1+1,n_2-1}(t) z_1^{n_1+1} z_2^{n_2-1} - \beta z_1 \frac{\partial}{\partial z_1} \sum_{n_1,n_2} \psi_{n_1,n_2}(t) z_1^{n_1} z_2^{n_2} \\ + \gamma z_1 \frac{\partial}{\partial z_2} \sum_{n_1,n_2} \psi_{n_1-1,n_2+1}(t) z_1^{n_1-1} z_2^{n_2+1} - \gamma z_2 \frac{\partial}{\partial z_2} \sum_{n_1,n_2} \psi_{n_1,n_2}(t) z_1^{n_1} z_2^{n_2}.\end{aligned}$$

Notice that we can rewrite

$$\frac{\partial}{\partial z_1} \sum_{n_1,n_2} \psi_{n_1+1,n_2-1}(t) z_1^{n_1+1} z_2^{n_2-1} = \partial_{z_1} \Psi(t)$$

and

$$\frac{\partial}{\partial z_2} \sum_{n_1,n_2} \psi_{n_1-1,n_2+1}(t) z_1^{n_1-1} z_2^{n_2+1} = \partial_{z_2} \Psi(t).$$

We can do so by a similar derivation as was shown in the proof of Property 1 in Appendix D, but we simply have to adjust the upper bound of the summations to the total number of organisms N in the system. Using these identities, we can write the time-derivatives of the probability functions $\psi_{n_1,n_2}(t)$ compactly as

$$\dot{\Psi}(t) = \big(\beta(a_2^\dagger a_1 - a_1^\dagger a_2) + \gamma(a_1^\dagger a_2 - a_2^\dagger a_2)\big)\Psi(t). \tag{13}$$

By noticing the form $\dot{\Psi}(t) = H\Psi(t)$, we conclude that the Hamiltonian describing the time-evolution of the closed network of two M/M/∞ queues is given by

$$H = \beta(a_2^\dagger - a_1^\dagger)a_1 + \gamma(a_1^\dagger - a_2^\dagger)a_2. \tag{14}$$

Intuitively, this operator makes sense since both terms create an organism and annihilate one. In other words, for every acetic acid molecule created an acetate ion disappears and the other way around. And so, the master equation (13) accurately describes the time-evolution of acetic acid dissolving in water. Since the result is derived in a general setting, the Hamiltonian (14) describes any closed network of two M/M/∞ queues.

Verification of the equilibrium solution

We know that the equilibrium solution for acetic acid dissolving in water must be given by (11). Since the total number of organisms in the system is given by $n_1 + n_2 = N$, we can write the steady state as

$$\psi_{n_1,n_2} = \binom{N}{n_1}\frac{\rho^{N-n_1}}{(1+\rho)^N}.$$

We would like to verify that the master equation $H\Psi(t) = \frac{d}{dt}\Psi(t)$ is zero for this distribution, confirming that it indeed is the equilibrium. First of all note that by the binomial identity, we have

$$\Psi = \sum_{n_1=0}^{N} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2} = \frac{(z_1 + \rho z_2)^N}{(1+\rho)^N}.$$

Note that for Ψ we have to sum over all states. Since $n_1 + n_2 = N$, it suffices to sum over all $0 \leq n_1 \leq N$. Now for the master equation we simply have to take the partial derivatives, which results in

$$H\Psi = \left\{\beta(z_2 - z_1) + \gamma\rho(z_1 - z_2)\right\}\frac{N(z_1 + \rho z_2)^{N-1}}{(1+\rho)^N} = 0,$$

since the load is defined by $\rho = \beta/\gamma$. This verifies that distribution (11) is indeed the steady-state vector expressing the equilibrium of acetic acid dissolving in water described by (9).

4 Time Evolution of Finite Server Queues

We have analyzed theory regarding networks of queues with infinite number of servers, i.e. the systems that we would normally consider in stochastic mechanics.

We would like to derive similar results as was done in Section 3 but for a more general setting. Namely for systems in which the number of servers need not be infinite. We begin with considering an queue with an arbitrary number of servers c: the M/M/c queue. We discuss the tandem of two M/M/c queues briefly and continue the discussion with taking a look at a closed network of two of such queues. We analyse the steady-state distributions and derive the master equation of the systems previously mentioned. The equilibria are derived in a more general setting than was done before.

The steady-state are derived in a more general setting in the sense of using the global balance equations instead of the local ones, which were used throughout Section 3. The global balance equations identify that a network in equilibrium must be such that the total probability flux towards state i must be equal to the total flow from out state i to any other state, that is

$$\psi_i = \sum_{j \in \mathcal{S}} h_{ij} \psi_j$$

for all i in state space $\mathcal{S}$, where $\boldsymbol{\psi}$ denotes the steady-state vector. Chandy [8] showed that these global balance equations can be decomposed into local balance equations, which are much easier to solve. Nevertheless, we will not consider these local balance equations for the upcoming derivations.

We use the global balance equations because of the following reason: Chandy [8] showed that decomposing the global balance equations into local balance equations is possible under *certain conditions*. These conditions will not always hold for all processes we might be interested in. More precisely, in Chapter 9 Kelly [11] proves that decomposing global balance equations into *partial balance equations*[8] can only be done if we consider so-called *spatial processes*. These processes must meet certain properties, one of which states a stronger version of irreducibility. The stochastic mechanical queueing model does not require this and therefore the processes that we consider are more general.

Since we know that we can always express the dynamics of our system in global balance equations, we regard it as useful to show how one can derive the desired equilibrium using solely global balance equations. Hopefully this will make the mathematics intuitively easier for more general cases, which may be considered in further studies.

Throughout this section we denote the arrival rate and departure rate by β and γ, respectively.

4.1 Time evolution of the M/M/1 queue

In this section we take a look at the most simple finite server queue there is: the M/M/1 queue. The state space of the system covers all the non-negative integers. The dynamics of the system can be portrayed in a simplified way by the state-transition diagram shown in Figure 4.1.

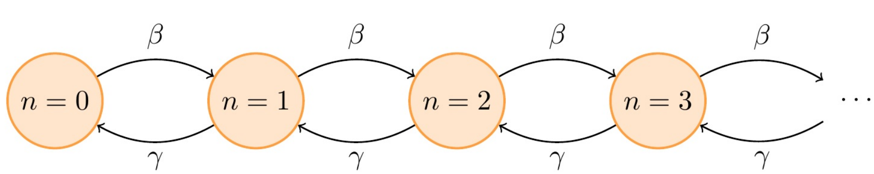

Figure 4.1: State-transition diagram for the M/M/1 queue.

[8]Partial balance equations is a synonym for local balance equations which Kelly [11] uses.

Equilibrium distribution

By taking a look at the state-transition diagram in Figure 4.1, we see that the equilibrium vector $\boldsymbol{\psi}$ must satisfy the global balance equations

$$\beta\psi_{n-1} + \gamma\psi_{n+1} = (\beta+\gamma)\psi_n,$$

for $n > 0$. Multiplying this expression on both sides by the formal variable z^{n-1} and summing over all positive integers n gives

$$\beta\sum_{n=1}^{\infty}\psi_{n-1}z^{n-1} + \gamma\sum_{n=1}^{\infty}\psi_{n+1}z^{n+1} = (\beta+\gamma)\sum_{n=1}^{\infty}\psi_n z^n,$$

where z is some formal variable. We recognize the formal power series $\Psi(z) = \sum_{n=0}^{\infty}\psi_n z^n$. Rewriting the equation with help of our generating function gives

$$\beta\Psi(z) + \frac{\gamma}{z^2}\big(\Psi(z) - \psi_0 - \psi_1 z\big) = \frac{\beta+\gamma}{z}\big(\Psi(z) - \psi_0\big).$$

Factoring out the steady-state vector results in

$$\Psi(z) = \frac{\gamma\psi_0 + \gamma\psi_1 z - (\beta+\gamma)\psi_0 z}{\beta z^2 - (\beta+\gamma)z + \gamma}. \tag{15}$$

But this expression does not always makes sense as the denominator can be zero, i.e. if $z = 1$. We seem to get into trouble here. Let us see if we can fix this problem with taking a good look at this expression. Since $\Psi(z)$ is a complex-valued function, we turn towards the theory of complex functions: complex analysis. We notice that the numerator of (15) is converging to zero as z is approached either from the right or left side on the real number line of $z = 1$. Assuming that our complex-valued function Ψ with respect to z is holomorphic[9], we find that (15) has a *removable singularity* at $z = 1$. And so, we can simply redefine our expression (15) such that $\Psi(1) = 0$. But then obviously

$$\gamma\psi_0 + \gamma\psi_1 - (\beta+\gamma)\psi_0 = 0 \implies \frac{\psi_1}{\psi_0} = \frac{\beta}{\gamma}$$

should hold. With the new and well-defined expression for Ψ in our hands, some straightforward algebra gives rise to

$$\Psi(z) = \gamma\psi_0\frac{1-z}{(\beta z-\gamma)(z-1)} = \frac{\gamma\psi_0}{\gamma - \beta z} = \psi_0\sum_{n=0}^{\infty}(\rho z)^n, \tag{16}$$

where the utilization is denoted by $\rho = \beta/\gamma$.

To simplify this last expression of $\Psi(z)$ even a bit more, we point out the *law of total probability*

$$\Psi(1) = \sum_{n=0}^{\infty}\psi_n = 1.$$

Evaluating (16) at $z = 1$ and equating it to one leads to $\psi_0 = 1 - \rho$, and therefrom

$$\Psi(z) = (1-\rho)\sum_{n=0}^{\infty}(\rho z)^n.$$

Observing that this expression yields a generating function quite similar to the definition of the formal power series $\Psi(z)$, we may deduce that the desired steady-state distribution is

$$\psi_n = (1-\rho)\rho^n$$

[9]In some neighborhood around any point of its domain the complex-valued function must be complex-differentiable.

for $n \in \mathbb{N}_{\geq 0}$. And so, we derived the equilibrium vector $\boldsymbol{\psi}$!

By a simple check we see that $\psi_n = (1-\rho)\rho^n$ indeed holds for the local balance equations $\beta\psi_n = \gamma\psi_{n+1}$ for any integer $n \geq 0$ as

$$\gamma\psi_{n+1} = \gamma(1-\rho)\rho^{n+1} = \gamma(1-\rho)\rho^n\frac{\beta}{\gamma} = \beta(1-\rho)\rho^n = \beta\psi_n.$$

Remark 1. *The equilibrium solution of a tandem of M/M/1 queues.*
Jackson [9] showed us that for a tandem of k M/M/1 queues the equilibrium distribution is simply the product of the equilibrium distribution of each individual queue. That would give the steady state vector

$$\psi_{n_1,n_2,\ldots,n_k} = \prod_{i=1}^{k}(1-\rho_i)\rho_i^{n_i},$$

where station i satisfies $\rho_i < 1$ and

$$\rho_i = \frac{\beta}{\gamma_i}$$

for any $i \in \mathbb{N}_{>0}$.

Master equation

Having derived the stationary state, we desire to analyse the time evolution of the queue. The arrival process of the queue has been investigated in Section 3.1. We showed that the Hamiltonian expressing the creation of jobs is given by $\beta(a^\dagger - 1)$. Due to there being only one server, the annihilation process is different than any other of the systems we previously discussed. To derive the operator of the departure process of the M/M/1 queue, we set the arrival rate $\beta = 0$. This makes that are constructed queue without arrivals can be interpreted as describing the departure process of any M/M/1 queue. The time derivatives of the probability functions $\psi_n(t)$ are

$$\begin{aligned}\dot{\psi}_n(t) &= \gamma\big(\psi_{n+1}(t) - \psi_n(t)\big) \quad \text{for } n \geq 0,\\ \dot{\psi}_0(t) &= \gamma\psi_1(t).\end{aligned}$$

Substituting these expression into the time-derivative of the power series $\Psi(t)$ gives

$$\dot{\Psi}(t) = \gamma\sum_{n=0}^{\infty}\psi_{n+1}(t)z^n - \gamma\sum_{n=1}^{\infty}\psi_n(t)z^n = \gamma\left(\frac{1}{z} - 1\right)\sum_{n=1}^{\infty}\psi_n(t)z^n. \tag{17}$$

Now we introduce some notation that will help us set (17) in the Schrödinger equation form which we are after.

Definition 8. The **projection operator**[10] p_n of generating function $\sum\psi_n(t)z^n$ is the operator which returns the probability function $\psi_n(t)$, i.e.

$$p_n\Psi(t) = \sum_{n\in\mathcal{S}}\psi_n(t)z^n = \psi_n(t).$$

In algebra terms, this makes p_n the operator projecting on the basis vector parallel to vector $z^n = z_1^{n_1}\cdot\ldots\cdot z_N^N$.

With this definition in hand, we can express the sum seen previously as

$$\sum_{n=1}^{\infty}\psi_n(t)z^n = \sum_{n=0}^{\infty}\psi_n(t)z^n - \psi_0(t)z^0 = (1-p_0)\Psi(t).$$

[10] These projection operators are based on the ideas of Wilf [12], who treats these kind of operators and their uses in much more detail.

This enables to rewrite (17) to the master equation

$$\dot{\Psi}(t) = \gamma\left(\frac{1}{z}(1 - p_0) - (1 - p_0)\right)\Psi(t) \tag{18}$$

from which we get the idea to define the annihilation operator for the M/M/1 queue by

$$a = \frac{1}{z}(1 - p_0).$$

Notice (18) is simply the Schrödinger equation for the M/M/1 queue where the Hamiltonian is of the form $\gamma\big(a - a^\dagger a\big)$. Using the property of being able to sum up Hamiltonians each corresponding to the individual transitions of the system, we can conclude that the master equation describing the time evolution of the M/M/1 queue is given by $\dot{\Psi}(t) = H\Psi(t)$ where the Hamiltonian is defined by

$$H = \beta(a^\dagger - 1) + \gamma(a - a^\dagger a).$$

The reader might recall that is the exact same Hamiltonian form we have derived for the M/M/∞ queue in Section 3.1!

4.2 Time evolution of the M/M/c queue

In this section we generalize the notions of time evolution of a single queue, which we have seen before. Consider the M/M/c queue, that is the queue with a Poisson point process and a server capacity of processing c number of jobs. The dynamics of the states corresponding to this queue can be portrayed as:

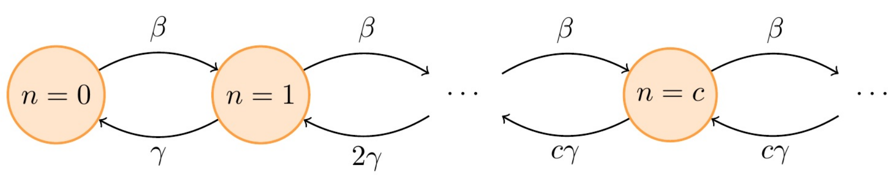

Figure 4.2: State-transition diagram for the M/M/c queue.

We want to derive the master equation which describes in detail how the probability functions $\psi_n(t)$ will change over time for any state $n \in \mathbb{N}_{\geq 0}$. Before we do so, we find the stationary distribution of the simplest queue within the class of M/M/c queues: the queue where there is only a single server.

Derivation of the master equation of the M/M/c queue

We now go back to analysing the M/M/c queue in general and derive the master equation of this system. Since the M/M/c queue has the same arrival process of which we derived the Hamiltonian in Section 3.1, we only have to look at the annihilation process in order to describe the time evolution of the queue. Therefore we assume that there are no arrivals by setting arrival rate $\beta = 0$. Then the derivatives of the coefficients of $\boldsymbol{\psi}(t)$ are

$$\begin{aligned}
\dot{\psi}_n(t) &= c\gamma\big(\psi_{n+1}(t) - \psi_n(t)\big) && \text{if } n \geq c,\\
\dot{\psi}_n(t) &= \gamma\big((n+1)\psi_{n+1}(t) - n\psi_n(t)\big) && \text{if } 0 < n < c,\\
\dot{\psi}_0(t) &= \gamma\psi_1(t).
\end{aligned}$$

For convention, we write $\psi_n(t) = \psi_n$. With the derivatives of the probability functions expressed with help of transition rates, we multiply the time derivative $\dot{\psi}_n$ by z^n and sum over all $n \leq 1$ making

$$\begin{aligned}\sum_{n=1}^{\infty} \dot{\psi}_n z^n &= \gamma\big(2\psi_2 - \psi_1\big)z^1 + \ \dots \ + \gamma(c\psi_c - (c-1)\psi)z^{c-1} + c\gamma(\psi_{c+1} - \psi_c)z^c + \dots \\ &= \gamma \sum_{n=1}^{c-1} \big(\partial_z - z\partial_z\big)\psi_n z^n + \gamma \sum_{n=c}^{\infty} \left(\frac{c}{z} - z\frac{c}{z}\right)\psi_n z^n.\end{aligned}$$

So, we conclude that the Hamiltonian for the annihilation process of an M/M/c queue is given by

$$H = \gamma\big(a - a^\dagger a\big), \tag{19}$$

where the corresponding annihilation operator is defined by

$$a = \begin{cases} \partial_z & \text{if } n < c, \\ c\frac{1}{z}(1 - p_0) & \text{if } n \geq c. \end{cases} \tag{20}$$

To see this, note that for all $n < c$ we have that

$$\gamma\big(a - a^\dagger a\big)\Psi = \gamma\big(\partial_z - z\partial_z\big)\Psi - \gamma\big(1 - z\big)\partial_z\psi_0 = \gamma\big(\partial_z - z\partial_z\big)\Psi.$$

Thus, the Hamiltonian operator of an M/M/c queue is

$$H = \beta(a^\dagger - 1) + \gamma\big(a - a^\dagger a\big).$$

Remark 2. *Verification of the Hamiltonians corresponding to the M/M/∞.*
Recall that for the M/M/1 and the M/M/∞ queue, we have the Hamiltonian operators

$$H_1 = \beta\big(a^\dagger - 1\big) + \gamma\left(\frac{1}{z} - 1\right)\big(1 - p_0\big) \quad \text{and} \quad H_\infty = \beta\big(a^\dagger - 1\big) + \gamma\big(a - a^\dagger a\big),$$

respectively.

We can indeed directly see that if $c = 1$, the constructed Hamiltonian coincides with that of the M/M/1 queue. Now if $c \to \infty$, then clearly $n < c$ for all n and by looking at the definition of annihilation operator a_c, we can readily verify that the Hamiltonian is that of the M/M/∞ queue.

4.3 Time evolution of a closed network of two M/M/1 queues

Consider a closed network of two M/M/1 queues, i.e. a closed system with just two transitions and two species. The service rate of species 1 being transformed into species 2 is β and the reverse rate is γ, that is the arrival rate of species 1 is γ and that of species 2 is β. This is a closed network of two M/M/1 queues. The utilization is $\rho = \frac{\beta}{\gamma}$.

Equilibrium distribution

As stated in the introduction of Section 4.2, we will derive the equilibrium solution by only using the global balance equations. To help us constructing these equations, we portray the corresponding state-transition diagram in Figure 4.3.

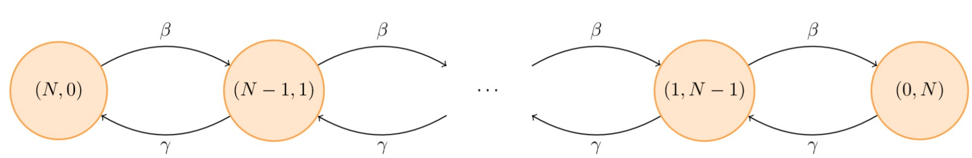

Figure 4.3: State-transition diagram for a closed network of two M/M/1 queues.

Formally, we know that the balance equations are of the form

$$\psi_i = \sum_{i \in \mathcal{S}} h_{ij} \psi_j \tag{21}$$

for any state $\boldsymbol{i} \in S$. Here S denotes the usual state-space

$$\mathcal{S} = \big\{ (N,0),\ (N,1),\ \ldots,\ (1,N-1)\ (0,N) \big\}. \tag{22}$$

We desire to derive an equivalent system of linear equations yielding the same information as (21). The key difference regarding this sum compared to the global balance equations of the M/M/1 queue is that we are now dealing with a finite series instead of an infinite series. This makes that the global balance equations (21) somewhat less simple, since we have two "end-point states" where these states only have one state to transform into. These states are the most right and the most left states drawn in the state-transition diagram in Figure 4.3.

As indicated, everything turns just a slightly bit more complex as in the M/M/1-case. Avoiding harsh expressions, we introduce almost the same notation as was used in Section 3.3. We denote the equilibrium distribution by $\psi_n := \psi(N-n,n) = \psi_{N-n,n}$ for any integer $0 \leq n \leq N$.

As visually portrayed in Figure 4.3, we see that the global balance equations are

$$\begin{aligned} \beta\psi_0 &= \gamma\psi_1, \\ \beta\psi_{n-1} + \gamma\psi_{n+1} &= (\beta+\gamma)\psi_n, \\ \beta\psi_{N-2} &= \gamma\psi_{N-1}, \end{aligned}$$

for $1 \leq n \leq N-2$. As we have seen, the method used in the previous section to determine the steady-state vector was successful. So let us just hope that along the same way of thought we will discover a comparable result for our closed system. And so we begin by multiplying $\beta\psi_{n-1} + \gamma\psi_{n+1} = (\beta+\gamma)\psi_n$ on both sides by formal variable z^{n-1} and by summing over $n=1$ up to $n = N-2$, which yields

$$\beta \sum_{n=1}^{N-2} \psi_{n-1} z^{n-1} + \frac{\gamma}{z^2} \sum_{n=1}^{N-2} \psi_{n+1} z^{n+1} = \frac{\beta+\gamma}{z} \sum_{n=1}^{N-2} \psi_n z^n.$$

By rewriting the sum expressions into the form of our complex-valued function $\Psi(z) = \sum_{n=0}^{N-1} \psi_n z^n$, the equation above must be equivalent to

$$\begin{aligned} \Psi(z)\left(\beta + \frac{\gamma}{z^2} - \frac{\beta+\gamma}{z}\right) &= \beta\big(\psi_{N-2} z^{N-2} + \psi_{N-1} z^{N-1}\big) \\ &\quad + \frac{\gamma}{z^2}\big(\psi_0 + \psi_1 z\big) - \frac{\beta+\gamma}{z}\big(\psi_0 + \psi_{N-1} z^{N-1}\big). \end{aligned}$$

Invoking the global balance equations and after some algebra we find

$$\Psi(z) = \frac{(\beta+\gamma)\psi_0(1-z) + \beta\psi_{N-1}\big(z^{N+1} - z^N\big)}{\beta z^2 - (\beta+\gamma)z + \gamma}. \tag{23}$$

We pause our derivation for a moment and try to investigate (23) a bit. We see that the denominator of the right hand-side yields roots $z_1 = 1$ and $z_2 = \frac{\gamma}{\beta}$. And so if we multiply this denominator, we can express it as

$$\beta z^2 - (\beta+\gamma)z + \gamma = (1-z)(1-\rho z).$$

We conclude that (23) can be written as

$$\Psi(z) = \frac{(\beta+\gamma)\psi_0(1-z) + \beta\psi_{N-1}\big(z^{N+1} - z^N\big)}{(1-z)(1-\rho z)} = \frac{(\beta+\gamma)\psi_0 - \beta\psi_{N-1} z^N}{1-\rho z}, \tag{24}$$

where at the last step we divide the term $(1-z)$ out of the numerator.

This leaves us with only one singularity: $z^* = \frac{1}{\rho}$. As we have seen before in the previous section, we can easily deal with this problem. Simply redefining our complex function $\Psi(z)$ to be equal to zero at $z^* = \frac{1}{\rho}$ makes us remove the removable singularity! But we then must have that the numerator of (24) is such that

$$(\beta+\gamma)\psi_0 - \beta\psi_{N-1}(z^*)^N = (\beta+\gamma)\psi_0 - \beta\psi_{N-1}\left(\frac{1}{\rho}\right)^N = 0.$$

Multiplying with ρ^{N+1} provides

$$\beta\psi_{N-1} = (\beta+\gamma)\psi_0\rho^N.$$

We seem to get closer to our desired explicit expression of the probability mass function ψ_n. However, we still have an unknown in our derivation, namely ψ_0. The generating function $\Psi(z)$ seems to be this abstract clothesline upon we hang up our probability functions. But we do know something about the combination of all probability functions, more specifically we have that $\psi_0 + \ldots \psi_{N-1} = 1$. Relating this to the function $\Psi(z) = \sum_{n=0}^{N-1} \psi_n z^n$, we must have

$$\Psi(1) = \sum_{n=0}^{N-1} \psi_n = \psi_0 + \ldots \psi_{N-1} = 1.$$

Returning to our ongoing derivation, we can combine this result with the expression given in (24). Evaluating at $z = 1$ yields

$$\Psi(1) = \frac{(\beta+\gamma)\psi_0 - \beta\psi_{N-1}}{1-\rho} = 1,$$

indicating that $(\beta+\gamma)\psi_0 - \beta\psi_{N-1} = 1-\rho$. Inasmuch as we found earlier that $\beta\psi_{N-1} = (\beta+\gamma)\psi_0\rho^N$, we get

$$(\beta+\gamma)\psi_0 = \frac{1-\rho}{1-\rho^N}.$$

Substituting these expressions into (24) gives

$$\Psi(z) = \frac{(\beta+\gamma)\psi_0\big(1-(\rho z)^N\big)}{1-\rho z} = \frac{1-\rho}{1-\rho^N}\sum_{n=0}^{N-1}(\rho z)^n.$$

As we defined $\Psi(z)$ as $\Psi(z) = \sum_{n=0}^{N+1} \psi_n z^n$, we know that the desired probability function ψ_n is given by

$$\psi_n = \psi_{N-n,n} = \rho^n \frac{1-\rho}{1-\rho^N} \quad \text{for } n = 0, 1, 2, \ldots, N.$$

To be sure of our result we check whether this expression holds for our local balance equations. Notice that these are given by

$$\beta\psi_{n-1} = \gamma\psi_n.$$

We can readily verify that these local balance equations for the equilibrium distribution indeed hold!

Master equation

We now derive the equation expressing the time evolution of the mechanical queueing system consisting of two queues with a capacity of a single job. Recall that we are able to describe the time derivatives $\dot{\psi}_n(t)$ by

$$\dot{\psi}_{n_1,n_2}(t) = \gamma\psi_{n_1-1,n_2+1}(t) + \beta\psi_{n_1+1,n_2-1}(t) - (\beta+\gamma)\psi_{n_1,n_2}(t), \tag{25}$$

for any state $(n_1, n_2) \in \mathcal{S}$, where we define $\psi_{-1,N+1}(t) = \psi_{N+1,-1}(t) = 0$ for convenience. Here the state space is given by (22). To derive the desired Schrödinger equation of this network, we apply the same trick we have seen before: we multiply (25) by the formal expression z^n and sum over all states $(n_1, n_2) \in \mathcal{S}$, i.e. all pairs n_1, n_2 such that $0 \le n_1, n_2 \le N$, resulting in

$$\dot{\Psi}(t) = \gamma \sum_{n_1,n_2} \psi_{n_1-1,n_2+1}(t) z_1^{n_1} z_2^{n_2} + \beta \sum_{n_1,n_2} \psi_{n_1+1,n_2-1}(t) z_1^{n_1} z_2^{n_2} - (\beta+\gamma) \sum_{n_1,n_2} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2}.$$

Rewriting this equation with help of the creation and annihilations operators such that the probability function $\psi_n(t)$ matches the formal vector z^n gives

$$\begin{aligned}\dot{\Psi}(t) = {} & \gamma \frac{z_1}{z_2} \sum_{n_1=1}^{N} \sum_{n_2=0}^{N} \psi_{n_1-1,n_2+1}(t) z_1^{n_1-1} z_2^{n_2+1} \\ & + \beta \frac{z_2}{z_1} \sum_{n_1=0}^{N} \sum_{n_2=1}^{N} \psi_{n_1+1,n_2-1}(t) z_1^{n_1+1} z_2^{n_2-1} \\ & - (\beta+\gamma) \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2}.\end{aligned}$$

Using the convention $\psi_{-1,N+1}(t) = \psi_{N+1,-1}(t) = 0$ and the projection operator p_n defined in Definition 8,

$$\dot{\Psi}(t) = \left(\gamma \frac{z_1}{z_2} (1 - p_{(N,0)}) + \beta \frac{z_2}{z_1} (1 - p_{(0,N)}) - (\beta+\gamma) \right) \Psi(t).$$

We conclude that the Hamiltonian describing the time evolution of the mechanical queueing network of two queues with single job capacity is given by

$$H = \beta(a_2^\dagger - a_1^\dagger) a_1 + \gamma(a_1^\dagger - a_2^\dagger) a_2.$$

5 Conserved Quantities and Noether's Theorem

In current section a new topic will be addressed. We provided results on the equilibrium solutions of basic mechanical queueing systems in Section 3 and 4, but such systems have many more interesting properties one might study. That is the reason why we hope to lay the mathematical foundations for examining measurable properties, also called *observables*, within the theory of mechanical queueing systems. The notion of observables one might recall from quantum mechanics. By taking a look on how Baez and Biamonte [1] tried to reinforce the analogy between the quantum and stochastic mechanical world, we make it possible to create operators for retrieving observable properties of complex systems that fit the mechanical queueing model.

After that we will take a look at Noether's theorem. This theorem will be stated specifically for observables in closed systems. The theorem links such observables to *symmetries* of the Hamiltonian operator H. In the last section we give an example of such observable: the number observable.

5.1 Observables

Baez and Biamonte [1] try to make the analogy between quantum theory and stochastic mechanics "seem as strong as possible". In quantum mechanics we are particularly interested in some observable property, or *observable* for short, of the quantum system. In Chapter 13 Cresser [13] explains that we are able to "construct a Hermitian operator to represent a particular measurable property of a physical system". To confuse the reader, the associated Hermitian operator O is also referred to as an observable.

This intuitive definition of observables must sound pretty good for us queueing theorists wanting to know more about the properties of our queues. This namely allows us to describe measurable properties of a system mathematically. Baez and Biamonte [1] thought the exact same thing. They defined the following:

Definition 9. An **observable** is a real-valued function O on some measure space X. The **expected value** of observable O in some stochastic state $\psi \in L^1(X)$ is

$$\langle O\psi \rangle = \int O\psi.$$

Here we assume that observable O is a bounded function on X. Notice that this assumption for sure will hold if measure space X yields a finite set. In this definition we see a strange form of bra-ket notation we have not seen before. Think of this notation as the stochastic mechanical variant of the quantum version. This notation will make some derivations and concepts easier to grasp and it emphasizes the analogy between stochastic and quantum mechanics. Since we defined the stochastic mechanical queueing model in such away that the stochastic mechanics developed is still applicable, we simply take on Definition 9 ourselves.

We are mainly interested in closed systems throughout this report. Having defined observables in Definition 9, the reader might wonder what kind of properties might be fascinating to study. A closed system raises the idea of some things staying the same within the system. And if certain things stay the same within a system, this definitely tells us quite a lot about the system itself. We therefore will discuss measurable properties of which the expected value does *not* change over time. This gives rise to the following definition:

Definition 10. A **conserved quantity** of a stochastic mechanical queueing system is an observable O whose expected value does not change over time[11], that is

$$\frac{d}{dt}\langle O\psi(t) \rangle = \frac{d}{dt}\int O\psi(t) = 0.$$

[11] Notice that we still us the convention by integrating over some state space $S \subseteq X$ and not over the time variable t.

The quantum version of Noether's theorem tells us that an observable is a conserved quantity if and only if it commutes with any self-adjoint operator for a finite set X. The reader might suspect that this would be a similar result in stochastic mechanics. In Section 10.3 Baez and Biamonte [1] give a simple counterexample showing that

$$\frac{d}{dt}\int O\psi(t) = 0$$

is not sufficient for commutation $[O, S] = 0$, where S is some stochastic operator (see Definition 6). Therefore we need to slightly modify Noether's theorem to be made applicable for the Markov chains. This will be discussed in the upcoming section.

5.2 Noether's theorem for Markov chains

As was introduced in the preceding section, we are after a similar statement of Noether's theorem for Markov chains which satisfy the mechanical queueing model. Most of the hard work has already been done by Baez and Biamonte [1], they stated Noether's theorem for Markov chains as the following:

Theorem 2. *For any observable O and stochastic operator S, observable O and stochastic operator S commute if and only if the expected value of observable O and its square O^2 do not change over time, i.e.*

$$[O, S] = 0 \iff \frac{d}{dt}\int O\psi(t) = 0 \quad \text{and} \quad \frac{d}{dt}\int O^2\psi(t) = 0.$$

This tells us that if a conserved quantity yields the property that the expected value of its square does not change over time, the conserved quantity commutes with the Hamiltonian operator H. For a proof refer to Baez and Biamonte [1].

5.3 The number observable

This section covers one of the most important examples of conserved quantities that commute with time evolution, namely the number observable. This is the mechanical queueing equivalent of the *total particle number operator* found in quantum mechanics. This quantum observable operates on the tensor product $\mathcal{H}$ of the Hilbert factor spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k.$$

Audretsch [14] states that $\mathcal{H}$ itself is a Hilbert space. In a corresponding *composite quantum system*[12] the particle number operator counts the number of particles in the system by summing up all *occupation number observables* $\widehat{N}_i = a_i^\dagger a$ $(1 \leq i \leq k)$ such that for some quantum product state $|n_1, n_2, \ldots, n_k\rangle$ we indeed have[13]

$$\sum_{i\in S} \widehat{N}_i|n_1, n_2, \ldots, n_k\rangle = (n_1 + n_2 + \cdots + n_k)|n_1, n_2, \ldots, n_k\rangle, \tag{26}$$

where set $S := \{1, 2, \ldots, k\}$. Recall from quantum mechanics that for some composite quantum system composed of two subsystems $|m, n\rangle$, we have $a_m|m, n\rangle = \sqrt{m}|m-1, n\rangle$ and $a_m^\dagger|m, n\rangle = \sqrt{m+1}|m+1, n\rangle$.

Analogously, we define the number observable for some mechanical queueing system:

[12] A composite system is a quantum system in which the observer can categorize the system into two different subsystems. See Audretsch [14] for a more elaborate explanation.

[13] See Section 14.6 on 'Occupation numbers' by Schumacher and Westmoreland [15].

Definition 11. Let some stochastic queueing system be given with k different species. The **number observable** for the system is sum of the number operators of each species, that is

$$\widehat{N} := \widehat{N}_1 + \widehat{N}_2 + \cdots + \widehat{N}_k,$$

where the number operator for species i is defined as $\widehat{N}_i = a_i^\dagger a_i$.

Analogously to the quantum observable, this number observable operates on the tensor product L^1 of the vector spaces $L_1^1, L_2^1, \ldots, L_k^1$,

$$L^1 = L_1^1 \otimes L_2^1 \otimes \cdots \otimes L_k^1.$$

One can easily prove that L^1 is also a vector space.

Returning once again to the quantum version of the number observable: Schumacher and Westmoreland [15] explain that if "we have a given number N of particles, then the system's state is restricted to the eigenspace" of the total particle number operator with eigenvalue $N = n_1 + n_2 + \cdots + n_k$. And this is exactly what we have in (26)! This sounds abstract, but it comes down to saying that if we find a system with a specific amount of particles, then throughout time the system can only be found in states such that the total number of particles is N. This makes sense since we only consider closed systems.

We suspect that we are able to find a similar result for mechanical queueing systems, since these are also closed. Suppose that for some mechanical queueing system we know for sure that the number of organisms of k different species is $n_1 + n_2 + \cdots + n_k = N$. Therefore our power series is of the form

$$\Psi(t) = \sum_{n \in \mathbb{N}_0^k} \psi_n z^n = \sum_{n \in S} \psi_n z^n,$$

where state space S is defined to be

$$S = \left\{ \boldsymbol{n} \in \mathbb{N}_0^k \;\middle|\; n_1 + n_2 + \cdots + n_k = N \right\}.$$

Implying that if $\Psi(0) := \Psi_0$ is a solution to the master equation

$$\frac{d}{dt}\Psi(t) = H\Psi(t),$$

then at any instance in time the solution will remain to be an element of subspace

$$L_N = \left\{ \Psi(t) \;\middle|\; \Psi(t) = \sum_{n \in S} \psi_n z^n \right\}.$$

Since we are after the eigenspaces of the number observable, it is time we try to think what this subspace tries to tell us in terms of $\widehat{N}$. We are simply after the set of all generating functions $\Psi(t)$ such that

$$\widehat{N}\Psi(t) = N\Psi(t)$$

for any time $t \in T$. In other words, that is the set of all $\Psi(t)$ that are in the eigenspace of the number observable $\widehat{N}$ with eigenvalue N. This is the exact equivalent of the statement by Schumacher and Westmoreland [15]. Mathematically this allows us to rewrite subspace L_N using the number observable $\widehat{N}$ as

$$L_N = \left\{ \Psi(t) \;\middle|\; \widehat{N}\Psi(t) = N\Psi(t) \right\}. \tag{27}$$

Stating that if "we have a given number N of particles, then the system's state is restricted to the eigenspace" of the number observable with eigenvalue N for our mechanical queueing systems might seem obvious, but will be of huge importance when proving the fact that the number observable is a conserved quantity. The corresponding claim is stated and proved down below.

Lemma 1. For any stochastic mechanical queueing system, the number observable $\widehat{N}$ commutes with the Hamiltonian operator H and is therefore a conserved quantity.

Proof. Let a stochastic mechanical queueing system be given. Choose an arbitrary stochastic operator S. By definition, the system is closed and thus the total number of organisms

$$n_1 + n_2 + \dots n_k = N$$

is conserved. We therefore have that any initial generating function $\Psi(0) = \Psi_0$ will for sure be in eigenspace L_N given by (27).

We know that the time dynamics of the system can be expressed by the master equation

$$\frac{d}{dt}\Psi(t) = H\Psi(t).$$

In Section 2.3, we noted that the Hamiltonian H is an infinitesimal stochastic operator and hence e^{tH} is a stochastic operator. This means that we can rewrite the master equation to

$$\Psi(t) = e^{tH}\Psi_0.$$

Since we know that initial Ψ_0 is in eigenspace L_N, we know that we are restricted to this eigenspace, i.e. we stay in this eigenspace. This implies that $e^{tH}\Psi_0 \in L_N$. In other words,

$$\widehat{N}e^{tH}\Psi_0 = Ne^{tH}\Psi_0.$$

Since N is simply an integer, we see that

$$\widehat{N}e^{tH}\Psi_0 = e^{tH}N\Psi_0 = e^{tH}\widehat{N}\Psi_0. \tag{28}$$

We are after the commutation of the number observable and the time evolution, that is $\widehat{N}H = H\widehat{N}$. This form we can already notice in (28), but we are not quite there yet. We are still left with the exponential term and most importantly (28) does not directly imply that $\widehat{N}e^{tH} = e^{tH}\widehat{N}$. However, note that we can choose the initial generating function $\Psi(0) = \Psi_0$ such that it forms a basis for number observable $\widehat{N}$. In this specific case we for sure know that by (28):

$$\widehat{N}e^{tH} = e^{tH}\widehat{N}.$$

Differentiating both expressions with respect to time gives the desired result

$$\widehat{N}H = H\widehat{N}.$$

Since the considered process is a Markov chain, we may invoke Noether's theorem for Markov chains, that is Theorem 2. This tells us that

$$\frac{d}{dt}\int \widehat{N}\psi(t) = 0 \quad \text{and} \quad \frac{d}{dt}\int \widehat{N}^2\psi(t) = 0,$$

hence we can conclude that $\widehat{N}$ is a conserved quantity. This concludes the proof! □

So we know that for any closed networks of M/M/1 or M/M/∞ queues the number observable is a conserved quantity. A concrete example showing this property is by considering the process of acetic acid dissolving in water discussed in Section 3.3, which is a mechanical queueing network of two M/M/∞ queues. Here n_1 denotes the number of acetic acid molecules and n_2 the number of acetate ions. The number observable $\widehat{N}$ operates on the formal power series $\Psi(t)$ as follows

$$\widehat{N}\Psi(t) = \left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\sum_{n_2=0}^{N}\psi_{n_1,n_2}(t)z_1^{n_1}z_2^{n_2} = \sum_{n_2=0}^{N}(n_1+n_2)\psi_{n_1,n_2}(t)z_1^{n_1}z_2^{n_2} = N\Psi(t).$$

This tells us that system always contains $n_1 + n_2 = N$ particles, which is indeed the thing we assumed. The reader may verify that $[\widehat{N}, H]\Psi(t) = 0$.

We have seen a simple example for the case where the number operator exactly tells us what we already knew. Lemma 1 states that for appropriate closed systems the number observable commutes with the Hamiltonian operator and is therefore a conserved quantity. One might wonder if we can say anything about the number observable with regards to an open system. One can possibly do so, but must be careful since giving interpretations to the commutation $[\widehat{N}, H]$ is not trivial and is some cases the commutation is not even defined. An attempt is done in the following example.

Example 2. *The number operator for the M/M/∞ queue.*
Consider an M/M/∞ queue with arrival rate β and service rate γ. The commutator of the number observable with the Hamiltonian can be written as

$$[\widehat{N}, H] = \left[a^\dagger a,\ \beta\big(a^\dagger - 1\big) + \gamma\big(a - a^\dagger a\big)\right] = \beta a^\dagger \left[a, a^\dagger\right] - \gamma\left[a, a^\dagger\right]a = \beta z - \gamma \frac{\partial}{\partial z},$$

since Baez and Biamonte [1] show that the commutator of the annihilation operator with the creation operator is $[a, a^\dagger] = 1$. We can see that the number observable does not commute with the time evolution of the queue. This makes since the number of jobs in the system in not conserved.

Formally, the commutator depicts to what extend the number observable and the Hamiltonian do not commute. Let $\Psi(t)$ satisfy the master equation, then an oversimplified perspective on the commutator may be given by considering

$$[\widehat{N}, H]\Psi(t) = \widehat{N}H\Psi(t) - H\widehat{N}\Psi(t)$$

term by term. This might give some intuition of what is going on. The term $\widehat{N}H\Psi(t)$ shows that we first let H operate on the coefficient of $\Psi(t)$ after which $\widehat{N}$ operates on $H\Psi(t)$. Since $H\Psi(t) = \frac{d}{dt}\Psi(t)$, operating $\widehat{N}$ on $H\Psi(t)$ can be thought of as the 'gross growth rate'. We see that $\widehat{N}H\Psi(t) = \beta z\Psi(t)$ and so all states increase by a rate of β, hence the 'gross growth rate' is β. This makes sense, since the arrival rate is indeed given by β. The more tricky one is the latter term: $H\widehat{N}\Psi(t)$. Here we let the number observable operate on $\Psi(t)$ first, which one can think of as counting the current number of jobs in the system. Finally, one can think of operating H on $\widehat{N}\Psi(t)$ as the time evolution of the 'constant stream of jobs' that are being processed, which turns out to be γ. That confirms the assumption of γ being the service rate of the queue. This leaves us that the commutator $[\widehat{N}, H]$ roughly tells us that the discrepancy between the 'gross growth rate' and the 'rate of the constant stream of jobs' is $\beta - \gamma$.

As warned before, the proceeding example is solely an intuitive and oversimplified perspective on the commutator of an open system. To be clear, note that $\widehat{N}\Psi(t)$ does not represent the actual number of jobs in the system, since for an arbitrarily open queue the number observable does not have an eigenvalue and therefore not an eigenstate. Furthermore, for $H\widehat{N}\Psi(t)$ to be mathematically rigorous and make any sense at all, it must be that $\widehat{N}\Psi(t)$ satisfies the master equation. This is generally not the case.

6 Conclusion

Appendix

Appendix A. Property of the steady-state for a simple closed mechanical queueing system

Consider the mechanical queueing system consisting of two M/M/∞ queues. An example of such a system was treated in Section 3.3. We will derive a property worth noting that in equilibrium, we see that the probability mass function ψ_n will

Note that we are considering indeed a closed system, and thus we set $n_1 + n_2 = N$. The formal power series corresponding to this Petri net is given by

$$\Psi(t) = \sum_{n \in S} \psi_n(t) z_1^{n_1} z_2^{n_2}, \tag{29}$$

where $\psi_n(t)$ is the probability that the system is in state $n = (n_1\ n_2)^T$ at time t. Since we know that the number of each species is restricted in some way, there is only a limited number of possible states n. The set containing all possibilities is the state space, mathematically that is,

$$S = \left\{ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{N}_0^2 \,\middle|\, n_1 + n_2 = N \right\}. \tag{30}$$

Now we will actually use the rules discussed in Section 2.3 regarding the Hamiltonian H, we find that we can formulate the Hamiltonian as

$$H = \beta(a_2^\dagger - a_1^\dagger)a_1 + \gamma(a_1^\dagger - a_2^\dagger)a_2 = (z_2 - z_1)\left(\beta \frac{\partial}{\partial z_1} - \gamma \frac{\partial}{\partial z_2}\right).$$

We are looking for the equilibrium solution to the master equation, that is a solution that does not change over time: $H\Psi = 0$. By some algebra, we find that we can write

$$\begin{aligned} H\Psi &= (z_2 - z_1)\left(\beta \frac{\partial}{\partial z_1} - \gamma \frac{\partial}{\partial z_2}\right) \sum_{n \in S} \psi_n(t) z_1^{n_1} z_2^{n_2} \\ &= (z_2 - z_1) \sum_{n \in S} \left(\beta n_1 z_1^{n_1 - 1} z_2^{n_2} - \gamma n_2 z_1^{n_1} z_2^{n_2 - 1}\right) \psi_n(t) \\ &= \sum_{n \in S} \left(\beta n_1 z_1^{n_1 - 1} z_2^{n_2 + 1} - (\beta n_1 + \gamma n_2) z_1^{n_1} z_2^{n_2} + \gamma n_2 z_1^{n_1 + 1} z_2^{n_2 - 1}\right) \psi_n(t). \end{aligned}$$

This last expression might seem a monster, but is actually of great use determining the form of an equilibrium solution. Let us examine this monster a bit more. We see that it is still of the form of our formal power series: it has certain terms attached to each $z_1^{n_1} z_2^{n_2}$ for all combinations of n in state space S.

As we want to find Ψ such that $H\Psi = 0$, we want this monster to be equal to zero. By linear independence of the formal variables $z_1^N z_2^0, z_1^{N-1} z_2^1, \ldots, z_1^0 z_2^N$, we must have that each individual coefficient of any formal variable $z_1^x z_2^y$ must be zero. Translating this into equations: for $H\Psi = 0$, we need that coefficient of $z^x z^y$ is equal to zero, thus

$$\sum_{n \in S_{xy}^1} \beta n_1 - \sum_{n \in S_{xy}^2} (\beta n_1 + \gamma n_2) + \sum_{n \in S_{xy}^3} \gamma n_2 = 0. \tag{31}$$

Here we remind ourselves that formal variable $z^x z^y$ may appear for different $n \in S$. Thus, for any coefficient to be written in a somewhat clear way we let

$$\begin{aligned} S_{xy}^1 &:= \left\{ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \,\middle|\, x = n_1 - 1,\ y = n_2 + 1 \right\}, \\ S_{xy}^2 &:= \left\{ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \,\middle|\, x = n_1,\ y = n_2 \right\}, \\ S_{xy}^3 &:= \left\{ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \,\middle|\, x = n_1 + 1,\ y = n_2 - 1 \right\}. \end{aligned}$$

Obviously, we need this for any coefficient of $z_1^x z_2^y$. And thus, Equation (31) must hold for any x and y such that $x + y = N$.

Appendix B. Normalization of equilibrium distribution for closed systems

Since we formed a model that is exactly in line with the idea of closed queueing networks (CQNs), we turn to Gordon and Newell [2]. This paper contributes to "the problem of determining the equilibrium distribution of customers in closed queuing systems composed of M interconnected stages of service".[14] Gordon and Newell derive the unique equilibrium distribution for classical closed networks, which is given by

$$P(n_1, \ldots, n_m) = \frac{1}{G(N)} \prod_{i=1}^{m} \frac{y_i^{n_i}}{\beta_i(n_i)},$$

where $P(n_1, \ldots, n_m)$ is the equilibrium probability that the system is in state $\boldsymbol{n} = (n_1, \ldots, n_m)$ and $G(N)$ is the normalization constant given by

$$G(N) = \sum_{n \in \mathcal{S}} \prod_{i=1}^{m} \frac{y_i^{n_i}}{\beta_i(n_i)}. \tag{32}$$

Here y_i is a formal variable and the set $\mathcal{S}$ is the state space defining all possible vectors $\boldsymbol{n}$ such that the restriction $n_1 + \cdots + n_m = N$ is satisfied and therefore yields all possible states the system can be in. Mathematically, that is

$$S = \left\{ \boldsymbol{n} \in \mathbb{N}_{\geq 0}^m \,\middle|\, \sum_{i=1}^{m} n_i = N \right\}. \tag{33}$$

We will now try to repeat a similar derivation as done by Gordon and Newell [2], but we will consider a stochastic closed queueing network. The only thing that changes is the expression $\mathbb{P}(x_1 = n_1, \ldots, x_m = n_m) =: \psi(x_1 = n_1, \ldots, x_m = n_m)$. Furthermore, for the mechanical queueing systems, we are particularly interested in the probability $\psi(x_1(t) = n_1, \ldots, x_m(t) = n_m)$ at some time t. In words, Gordon and Newell do the derivation for the probability that there are n_i units of species i for all $t \in T$. We are interested in a somewhat more precise probability expression, namely: the probability that there are n_i units of species i at some specific time t.

This difference is easy to cope with, which makes that we can still use the results that are found in Gordon and Newell [2]. We simply have to modify the perspective we have upon the derivation itself. As the equilibrium distribution is derived for all t, we inter-change $P(x_1 = n_1, \ldots, x_m = n_m)$ for $P(x_1(t) = n_1, \ldots, x_m(t) = n_m)$. Meaning that, we now derive the equilibrium distribution initialized at time t. Therefore, we can just use the normalization constant $G(N)$ expressed by (32) at time t. This implies that the equilibrium solution $\psi(\boldsymbol{n}) = \psi_n$ for $\boldsymbol{n} \in S$ now satisfies

$$\int_S \psi_n = 1,$$

where ψ_n is the probability that the system is in state $\boldsymbol{n} \in S$. We would like to derive whether or not ψ stays normalized for any $t \in T$ if it was already so for some $t = t_0$. Let us figure this out. Assume that ψ is normalized at time $t = t_0$, i.e.

$$\int_S \psi_n(t_0) = 1.$$

Recalling that the master equation is

$$\frac{\partial}{dt} \psi_n(t) = H \psi_n(t),$$

[14]From the abstract of Gordon and Newell [2].

where H is an infinitesimal stochastic operator. This makes $\exp(tH)$ a stochastic operator, wherefore, by definition, we have that

$$\int_S \exp(tH)\psi_n = \int_S \psi_n.$$

Differentiating both sides of this equation with respect to t and evaluating at $t = 0$ yields

$$\int_S \frac{d\exp(tH)}{dt}\psi_n(t) + \exp(tH)\frac{\partial \psi_n(t)}{\partial t}\bigg|_{t=0} = \int_S \frac{\partial \psi_n(t)}{\partial t}\bigg|_{t=0}.$$

Applying the master equation gives

$$\int_S H\exp(tH)\psi_n(t) + \exp(tH)H\psi_n(t)\bigg|_{t=0} = \int_S H\psi_n(t)\bigg|_{t=0}$$

and therefore we must have that

$$2\int_S H\psi_n(0) = \int_S H\psi_n(0) \implies \int_S H\psi_n(0) = 0.$$

This gives us an trivial result: the integral of ψ on the state space S at time $t = 0$ does not change over time, as

$$\int_S H\psi_n(0) = \frac{d}{dt}\int_S \psi_n(0) = 0.$$

This makes sense, because state space $\mathcal{S}$ given by (33) is independent of time t and $\psi_b(0)$ is simply a constant.

Appendix C. Verification of the equilibrium of an M/M/∞ queue using the master equation

Recall that we derived the Hamiltonian of the M/M/∞ queue to be

$$H = \beta\big(a^\dagger - 1\big) + \gamma\big(a - a^\dagger a\big).$$

We know that we can write the annihilation operator as a differential operator of z and the creation operator as the multiplication by z, making that the master equation is of the form

$$H\Psi(t) = \left\{\beta\big(z-1\big) + \gamma\left(\frac{\partial}{\partial z} - z\frac{\partial}{\partial z}\right)\right\}\sum_{n=0}^{\infty}\psi_n(t)z^n.$$

Taking care of the operators gives

$$H\Psi(t) = \beta\sum_{n=0}^{\infty}\psi_n(t)z^{n+1} + \gamma\sum_{n=0}^{\infty}n\psi_n(t)z^{n-1} - \beta\sum_{n=0}^{\infty}\psi_n(t)z^n - \gamma\sum_{n=0}^{\infty}n\psi_n(t)z^n. \tag{34}$$

Let us now investigate the master equation (34) in order to verif the distribution ψ_n such that $\frac{d}{dt}\Psi(t) = 0$. We showed that the equilibrium is

$$\psi_n = \frac{\rho^n}{n!}e^{-\rho}. \tag{35}$$

Using this expression, we rewrite the sum of first two series in (34) as

$$\begin{aligned}
\beta\sum_{n=0}^{\infty}\psi_n z^{n+1} + \gamma\sum_{n=0}^{\infty}n\psi_n z^{n-1} &= \beta\sum_{n=0}^{\infty}\frac{\rho^n}{n!}e^{-\rho}z^{n+1} + \gamma\sum_{n=0}^{\infty}n\frac{\rho^n}{n!}e^{-\rho}z^{n-1} \\
&= \beta z\sum_{n=0}^{\infty}\frac{(\rho z)^n}{n!}e^{-\rho} + \frac{\gamma}{z}\sum_{n=0}^{\infty}n\frac{(\rho z)^n}{n!}e^{-\rho} \\
&= \beta(z+1)e^{\rho(z-1)}.
\end{aligned}$$

Here the last step follows from the identity

$$\frac{\gamma}{z}\sum_{n=0}^{\infty} n\frac{(\rho z)^n}{n!} = \frac{\gamma}{z}\sum_{n=1}^{\infty} n\frac{(\rho z)^n}{n!} = \frac{\gamma}{z}\rho z\sum_{n=1}^{\infty} \frac{(\rho z)^{n-1}}{(n-1)!} = \gamma\rho\sum_{m=0}^{\infty} \frac{(\rho z)^m}{m!} = \beta e^{\rho z},$$

where we substitute $m = n - 1$.

Similarly, for the sum of the latter two sums in (34), we have

$$\beta\sum_{n=0}^{\infty} \psi_n(t)z^n + \gamma\sum_{n=0}^{\infty} n\psi_n(t)z^n = \beta\sum_{n=0}^{\infty} \frac{(\rho z)^n}{n!}e^{-\rho} + z\frac{\gamma}{z}\sum_{n=1}^{\infty} n\frac{(\rho z)^n}{n!}e^{-\rho} = \beta(z+1)e^{\rho(z-1)}.$$

Hence, we find that for $\psi_n = \frac{1}{n!}\rho^n e^{-\rho}$ the master equation is such that $H\Psi = 0$. And thus by verification, we found that (35) is the unique equilibrium distribution!

Appendix D. Derivations of properties of the geometric series $\Psi(t)$

In this section we state and prove several properties regarding the power series $\Psi(t)$ for certain (networks of) queues.

Property 1. Consider a tandem of two M/M/∞ queues, where we denote the species by species 1 and 2. The following property holds:

$$\partial_{z_1}\Psi(t) = \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2} = \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \psi_{n_1+1,n_2-1} z_1^{n_1+1} z_2^{n_2-1}.$$

Proof. First, we let ∂_{z_1} operate on the series and we rewrite the index of the first sum:

$$\begin{aligned}
\partial_{z_1}\Psi(t) &= \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \psi_{n_1,n_2} z_1^{n_1} z_2^{n_2} \\
&= \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} n_1\psi_{n_1,n_2} z_1^{n_1-1} z_2^{n_2} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} (n_1+1)\psi_{n_1+1,n_2} z_1^{n_1} z_2^{n_2} \\
&= \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \psi_{n_1,n_2} z_1^{n_1+1} z_2^{n_2}.
\end{aligned}$$

Now playing with the index of the second sum,

$$\begin{aligned}
\partial_{z_1}\Psi(t) &= \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \psi_{n_1,n_2} z_1^{n_1+1} z_2^{n_2} \\
&= \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} \psi_{n_1,n_2-1} z_1^{n_1+1} z_2^{n_2-1} \\
&= \frac{\partial}{\partial z_1}\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} \psi_{n_1,n_2-1} z_1^{n_1+1} z_2^{n_2-1},
\end{aligned}$$

as we assumed $\psi_{n_1,-1} = 0$ for any $n_1 \geq 0$. This proves the property.

□ □

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