# 4 Voronoi Diagrams

A city builds a set of post offices, and now needs to determine which houses will be served by which office. It would be wasteful for a postman to go out of their way to make a delivery if another post office is closer. What is the right way to partition the post office ?

This is called the *post-office problem*, and is one of many examples of a problem whose solution requires us to partition space in some way. The space itself could be Euclidean space, or something more abstract. The resulting partition of space is called a *Voronoi* diagram[1].

**Definition 4.1.** *Given $n$* sites *$s_1, s_2, \ldots, s_n$ in a distance space $(X, d)$, partition $X$ into regions $S_1, S_2, \ldots, S_n$ such that*

$$S_i = \{x \in X | d(x, s_i) < d(x, s_j), i \neq j\}$$

In what follows, we will focus on Voronoi diagrams in Euclidean space. Later, we will generalize to other distance spaces.

## 4.1 Structure of a Voronoi diagram

In two dimensions, the Voronoi diagram partitions the plane into cells, edges and vertices. Understanding the structure of these pieces allows us to bound the complexity of the diagram and design algorithms to compute it.

**Voronoi edges.** An edge of a Voronoi diagram is where the closest neighbor of a point changes. By continuity of the distance function, this means that every point on a Voronoi edge is equidistance from at least two sites. In other words, the points satisfy the equation

$$\|p - s\| = \|p - s'\|$$

Expanding and squaring, we get

$$\|\mathbf{p}\|^2 + 2\langle \mathbf{p}, \mathbf{s} \rangle + \|\mathbf{s}\|^2 = \|\mathbf{p}\|^2 + 2\langle \mathbf{p}, \mathbf{s}' \rangle + \|\mathbf{s}'\|^2$$
$$2\langle \mathbf{p}, \mathbf{s} - \mathbf{s}' \rangle = \|\mathbf{s}'\|^2 - \|\mathbf{s}\|^2$$

This tells us that a Voronoi edge is a straight line, and that it's perpendicular to the line joining the two sites ($\mathbf{s} - \mathbf{s}'$). Also, since all points on the line are equidistant from the two sites, it's a bisector.

[1]It's named after the Russian mathematician Georgy Voronoi, and has also been called a Dirichlet tesselation, and dates back to Descartes.

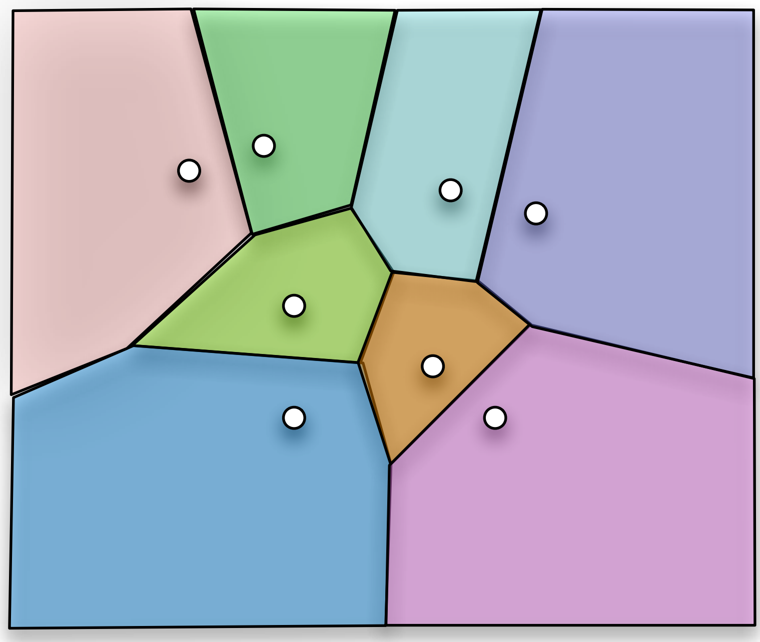

Figure 4.1: A Voronoi diagram in the plane

**Voronoi cells.** A Voronoi cell consists of all points closer to a fixed site than any other site. It can be expressed as the intersection of all halfspaces generated by the Voronoi edges for the site w.r.t all other sites. Since the bounding line is convex, the halfspaces are convex.

**Voronoi vertex** Finally, a vertex of a Voronoi diagram is the intersection of at least three edges, and so a Voronoi vertex is equidistant from at least three sites.

## 4.2 Voronoi diagrams as envelopes

There are two distinct ways in which the Voronoi diagram can be viewed as a level of an arrangement, and both of these are quite useful.

The first approach is as a lower envelope of cones. As usual, we have $n$ sites in $\mathbb{R}^d$. At each site, draw a *cone* defined as the function $f_{\mathbf{p}}(\mathbf{x}) = (\mathbf{x} - \mathbf{p}, \|\mathbf{x} - \mathbf{p}\|)$. The important property of this function is that

$$\|f_{\mathbf{p}}(\mathbf{x}) - \mathbf{x}\| = \|\mathbf{p} - \mathbf{x}\|$$

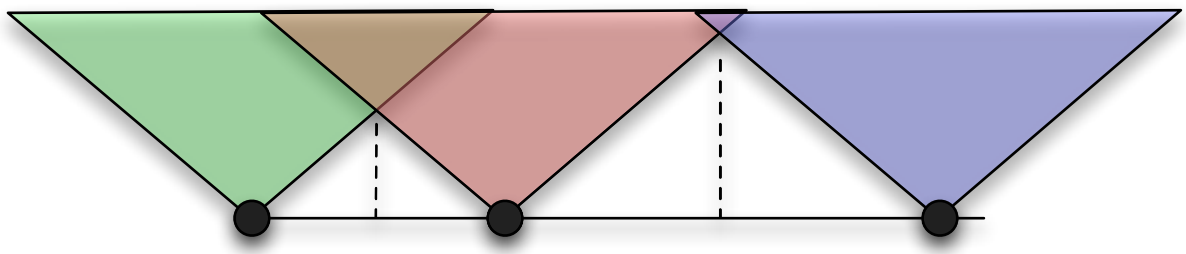

Figure 4.2: Cones

Now compute the lower envelope of the collection of functions $f_{\mathbf{p}}(\cdot)$. This is the function $g(\mathbf{x}) = \min_{\mathbf{p}} f_{\mathbf{p}})(\mathbf{x})$. Imagine that the pieces of the lower envelope are "colored" with the index of the site they come from: in other words, let $h(\mathbf{x}) = \arg\min_{\mathbf{p}} f_{\mathbf{p}})(\mathbf{x})$. By the relation above, we can conclude that $h(\mathbf{x})$ is the label of $\mathbf{p}$ minimizing $\|\mathbf{x} - \mathbf{p}\|$, which is the Voronoi diagram.

The idea of a Voronoi diagram as a lower envelope of cones was used in the OpenGL Red Book[**?**] as a way to render Voronoi diagrams (the envelope calculation was performed using the depth test). Later, this idea was generalized to render Voronoi diagrams for more complex shapes, in three dimensions, and even for moving shapes[**?**].

The second way to interpret the Voronoi diagram in terms of arrangement is by the duality lifting map. Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}\|^2)$ describe the paraboloid in $d+1$ dimensions. Now associate with each site $\mathbf{p}$ the tangent plane to $f$ at the point $(\mathbf{p}, f(\mathbf{p}))$. Then *the upper envelope of this arrangement of planes, when projected down to $\mathbb{R}^d$, is the Voronoi diagram*.

Here is a sketch of why this is true. Consider any point $\mathbf{x}$ and a site $p \in \mathbb{R}^d$. Then the vertical distance from the point $(\mathbf{x}, f(\mathbf{x}))$ to the tangent plane at $f(\mathbf{p})$ (which we denote as $d(\mathbf{x}, \mathbf{p})$ is given by the Taylor expansion

$$d(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}) - (f(\mathbf{p}) + \langle \nabla f(\mathbf{p}), \mathbf{x} - \mathbf{p} \rangle)$$

Observing that $\nabla f(\mathbf{x}) = \mathbf{x}$, the above expression simplifies to $d(\mathbf{x}, \mathbf{p}) = \|\mathbf{x} - \mathbf{p}\|^2$. Notice that the point $(\mathbf{x}, f(\mathbf{x}))$ is above all the tangent planes. Therefore, the closest plane to it (the one that minimizes $d(\mathbf{x}, \cdot)$. is the *highest* such plane. This explains why we take the upper envelope of the planes.

Notice further that by duality, the upper envelope of the collection of planes is the lower half of the convex hull of the dual collection of points. This yields the following result.

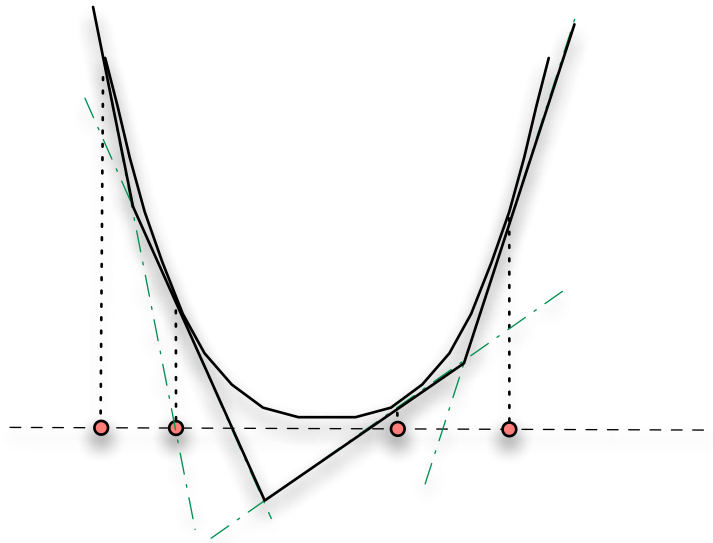

Figure 4.3: Voronoi diagram as upper envelope of tangents to parabola.

**Theorem 4.1.** *The complexity of the Voronoi diagram of n points in d dimensions equals (asymptotically) the complexity of the convex hull of n points in* $d+1$ *dimensions, and is* $O(n^{\lceil d/2 \rceil})$.

There are two practical consequences of this result. Firstly, we can use a convex hull algorithm in $d+1$ dimensions to compute a Voronoi diagram in $d$ dimensions. Secondly, the Voronoi diagram of $n$ points in the plane has the same complexity as the convex hull of $n$ points in space, which is $O(n)$.

## 4.3 Constructing a Voronoi diagram

From the above remarks, we can construct a Voronoi diagram in the plane by constructing a convex hull in *three* dimensions. However, there's a more direct divide-and-conquer based method that also reveals interesting structure in the Voronoi diagram. We state two propositions without proof.

**Proposition 4.1.** *Every infinite edge in the Voronoi diagram of n points is a bisector of an edge of the convex hull.*

**Proposition 4.2.** *If a point set P is divided into two sets* $L, R$ *and we construct the Voronoi diagrams of L and R, then edges of* $Vor(P)$ *that are not present in* $Vor(L)$ *or* $Vor(R)$ *must be bisectors of a pair of points, one from L and one from R.*

Armed with these two propositions, we now describe the divide and conquer approach. Firstly, we divide the point set into two equal parts $L$ and $R$ that can be separated by a vertical line. This can be done in linear time by finding the median $x$-coordinate of the points. We recursively compute $\mathrm{Vor}(L)$ and $\mathrm{Vor}(R)$. In addition, we also recursively compute $CH(L)$ and $CH(R)$.

The idea is to walk down the set of Voronoi edges that merge the two diagrams. By Proposition 4.2, we know that each such edge is a part of a bisector of one point from $L$ and one from $R$. To find the topmost edge (that extends to infinity) we compute the two bridge edges connecting the two convex hulls. By Proposition 4.1, a part of the bisector of this edge must be one of the infinite edges of the Voronoi diagram. Call this bisector $\ell$.

Let the two endpoints of the bridge edge be $l, r$. We compute the intersection of the Voronoi cells of $l$ (in $L$) and $r$ (in $R$) with $\ell$. Note that these cells already exist by recursion. Suppose that an edge of $r$'s Voronoi cell intersects $\ell$ earlier (at a higher point) than $l$'s Voronoi cell. Let $r'$ be the site in $R$ defining (together with $r$) the edge that intersects $\ell$. We replace $r$ by $r'$, $\ell$ by the bisector of $l$ and $r'$, and repeat the process. A similar process occurs if an edge of $l$'s Voronoi cell intersected $\ell$ higher, in which case $l$ is replaced by $l'$ in a similar manner. This process terminates when we reach the lower bridge edge of the convex hull of $L \cup R$.

Each advance takes constant time to compute, and so the overall merge step takes linear time (because at each step we consume at least one point from one of the two convex hulls). Thus, the overall recurrence looks like

$$T(n) = 2T(n/2) + O(n)$$

or $T(n) = \Theta(n \log n)$.

## 4.4 Other Voronoi diagrams

The idea of a Voronoi diagram as an *envelope* of an appropriately constructed arrangement is very powerful. It allows us to define and construct many different kinds of diagrams, some of which we will now discuss.

### Higher order Voronoi diagrams

Let us return to the arrangement of cones in which the Voronoi diagram is the projection of the lower envelope. Suppose instead of the lower envelope (level 1) we were to take the second level of the arrangement and project it ? What we get is a *second-order* Voronoi diagram. In each cell of this diagram, the *second* nearest neighbor of a point in the cell is the same. Similarly, if we overlaid the first-and second-order Voronoi diagrams, we get an arrangement in which all points of a cell have the same first *and* second nearest neighbors (in order).

Higher order Voronoi diagrams are useful for many problems when you want to search over *sets* of points that might satisfy some properties. The Voronoi structure restricts the number of possible such sets, thus reducing the complexity of a search procedure. For example, the $k$-nearest neighbors of a point are often used as part of classification scheme. In general, there are $O(n^k)$ subsets of size $k$ from an $n$ point set. But using a $k^{\text{th}}$ order Voronoi diagram we know that the complexity of the $k$-level is polynomial in both $n$ and $k$, reducing the number of potential subsets to search over.

## Weighted Voronoi diagrams

One way to think of a Voronoi diagram is as marking out regions of influence. If you were to imagine invading armies of equal strength leaving from each site *at the same speed* in all directions, they'd meet each other at the Voronoi edges.

So how can we incorporate weights, or strength of influence, into the Voronoi diagram. We will now see three different ways of doing this. As usual, we will have $n$ sites $s_1, \ldots, s_n$, and with each site $s_i$ will be associated a weight $w_i$.

### Additively weighted diagrams

In an additively weighted Voronoi diagram, the "distance" from a point $p$ to a cell $s_i$ is

$$d(p, s_i) = \|p - s_i\| - w_i$$

Thus, the boundary between two sites has the general form

$$d(x, p_i) = d(x, p_j) + \omega$$

We can solve for the general form of the equation. However, observe that the bisector is the set of points whose distance from two fixed points differ by a constant. The curve satisfying this property is the hyperbola. Thus, the edges of the additively weighted Vorono diagram are sections of a hyperbola.

There is a natural arrangement that generates an additively weighted Voronoi diagram. Take a distance cone as before. Its apex is on the $xy$ plane. Now imagine the cone moving downwards by an amount $w$. If we take any point $p$, it is easy to see that the vertical distance from $p$ to the cone is precisely the modified distance described above. Thus, the additively weighted Voronoi diagram can be represented as the lower envelope of a collection of cones, where each cone is moved downwards by an amount that equals the weight of the associated point.

### Multiplicatively weighted Voronoi diagram

Suppose the influence function is not additive but multiplicative. Specifically, we will define the distance from a point to a site as

$$d(p, s_i) = \|p - s_i\| / w$$

. Intuitively, this means that a site with large weight is "close" to many things.

Now you can verify that the bisectors are *circular* arcs. To obtain an envelope formulation, observe that if we draw a cone whose angle to the plane is $\theta$, then the vertical distance to the cone (which should represent $d(p,s)$) is precisely $\|p-s\|\tan\theta$. Therefore, capturing a site with influence $w$ requires drawing a cone with angle $\frac{1}{\tan\theta}$. As before, the lower envelope of these cones yields the desired Voronoi diagram.

### Power Diagrams

Both of the above diagrams allow us to capture the idea of influence. But they yield diagrams that have nonlinear edges. Is it possible to construct an influence diagram without curves edges. It turns out that you can, and the resulting diagram is called the *power diagram*.The trick here is to use the *square* of the Euclidean distance in an additively weighted Voronoi diagram.

Fix two sites $s, s'$ with weights $w, w,'$. The equation of the bisector is now the set of points $x$ such that

$$\begin{aligned} \|x-s\|^2 - w &= \|x-s'\|^2 - w' \\ \|x\|^2 + \|s\|^2 - 2\langle x, s\rangle &= \|x^2\| + \|s'\|^2 - 2\langle x, s'\rangle + w - w' \\ 2\langle x, s'-s\rangle &= \|s'\|^2 - \|s\|^2 \end{aligned}$$

This equation is linear in $x$, and therefore defines a straight line.

The power diagram also has an envelope-based formulation via the paraboloid lifting map. As before, lift each site to the paraboloid one dimension higher. Draw tangent planes at each of these points, and then *raise* the plane for $s_i$ by $w_i$. You can verify that the vertical distance from a point to a plane (below it) is precisely the above distance. Thus, computing the *upper* envelope of these lifted planes yields the power diagram.

There is a geometric interpretation of the influence in a power diagram. Consider a site $s$ with weight $w$, and fix a point $p$. Draw the sphere around $s$ with radius $\sqrt{w}$. Then the distance $\|\mathbf{p}-\mathbf{s}\|^2 - w^2$ is the distance from $\mathbf{p}$ to the point $\mathbf{q}$ such that $\overline{\mathbf{pq}}$ is a tangent to the sphere.

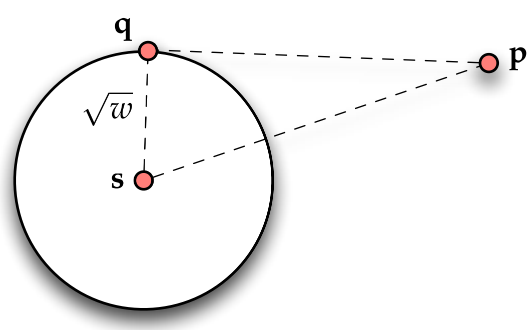


Figure 4.4: A geometric interpretation of the power distance