Arrangements And Duality

3

3.1 Introduction

Point configurations are tbe most basic structure we study in computational geometry. But what about configurations of more complicated shapes ? For example, a configuration of lines, or curves, or surfaces ? Such a configuration is generally called an *arrangement* and is the object of study here.

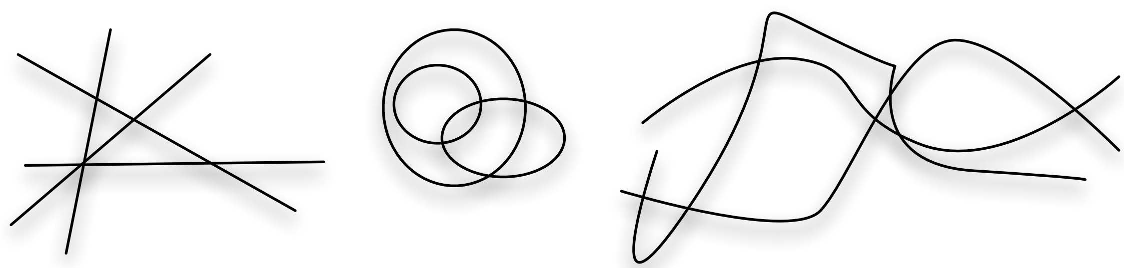

Figure 3.1: Examples of arrangements

An arrangement of $d-1$ dimensional objects in d dimensions generates objects of all dimensions that are at most d. Vertices are 0-dimensional, edges are 1 dimensional, and so on. The description of an arrangement consists of all the different dimensional structures present in the arrangement, as well as how they connect to each other. For example, consider the arrangement formed by four intersecting lines, show in Figure 3.2. The face f is bounded by edges e_1, e_2, e_3, e_4, and by vertices v_1, v_2, v_3, v_4. A complete description of this face requires all of this information, as well as their adjacency structure.

Examples of arrangements One example of an arrangement comes from robotics. Suppose you're trying to plan a path for a moving robot from a start to a finish in a region with obstacles. We can define a *configuration* space where each point is a placement of the robot (we assume for now that the robot can only translate). Now we can mark all points in the space where the robot can be placed without collision. This will be a collection of regions with boundaries that describe *critical* locations of the robot (where it touches an obstacle). Finding a path from start to finish then reduces to the problem of determining whether the cell containing the start posiition and the cell containing the finish are connected.

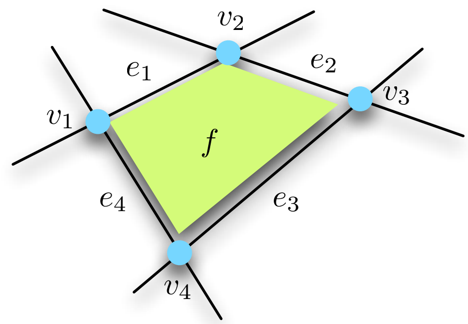

Figure 3.2: Labelled cell of an arrangement

In general, the robot may have more degrees of freedom than just its position. While this changes the configuration space and increases its dimension, it does not change the principle. Even in this higher-dimensional space, the *arrangement* can be used to solve the planning problem.

A second example of an arrangement arises from minimization problems. Suppose I am given a collection of functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ on the plane. The goal is to find a point $x \in reals^2$ such that $\max_i f_i(x)$ is minimized. For instance, $f_i(x)$ could be the distance from x to a fixed point p_i, in which case $\max_i f_i(x)$ is the radius of the ball centered at x that contains all the p_i.

A third example comes from collections of inequalities. Suppose we have a set of linear equations. As we have seen, we can think of these linear equations as hyperplanes. Now consider a single region in space carved out by these linear equations. All points in this region represent the *same set of inequalities* associated with the linear equations. In fact, a polyhedron is one cell of this arrangement; if we flip an inequality, we get a different adjacent cell in the arrangement, and so on.

In all of these cases, studying the arrangement of shapes allows us to break down a continuous domain into a *discrete* set of pieces, such that in each piece the problem either has a fixed solution or is easy to solve.

3.2 Duality

Perhaps the most important reason to study arrangements of lines is that they're the same as point configurations via a *duality mapping*.

What is duality ? It's a way to map between points and hyperplane (also called projective duality). Consider a point in the plane $p = (x, y)$. It has two parameters that describe it. If we consider a *line* in the plane (say $y = mx + c$), it also has two coordinates that describe it. This suggests that we can construct a mapping between points and lines.

Standard Dual

The "standard" dual mapping from points to lines takes the point $p = (a, b)$ to the line $p^* = \ell : y = ax - b$. Equivalently, the line $\ell : y = ax + b$ is mapped to the dual point $\ell^* = (a, -b)$.

This mapping has many properties:

Self-inverse $(p^*)^* = p$

Incidence If p lies on ℓ, then ℓ^* lies on p^*

Ordering If p lies above (below) ℓ, then p^* lies below (above) ℓ^*.

But not all lines have a well defined dual. Consider any line parallel to the y-axis. It cannot be written in the form $y = mx + c$. It is helpful to think of what happens in the limit as the line slope increases. Consider the line $y = mx + c$ and let $m \to \infty$ (this corresponds to swinging the line around a pivot at $(0, c)$. In the dual, this corresponds to the point $\ell^* = (m, -c)$ which moves along the horizontal line $y = -c$ as m increases. In other words, the dual of the line $x = 0$ is the *point at infinity*. We will return to this topic later when we talk about the projective plane.

Polar Dual

There are other dual mappings that often come up in practice. The *polar dual* takes a point (a, b) and maps it to the line $p^* : ax + by + 1 = 0$. Geometrically, this dual can be visualized as the line we get when we draw a ray from p to the origin, extend it further by an amount $1/\|p\|$, and then draw a line perpendicular to it. You can verify that this mapping also satisfies the incidence property. It also satisfies an order mapping, in that ℓ lies between p and the origin, then p^* lies between ℓ^* and the origin.

Dual Constructions

To gain intuition about dual mappings, it helps to examine some simple cases. We know that a point maps to a line and vice versa. Suppose we have a configuration with two points. The corresponding dual configuration consists of two *lines*. What does the intersection of these two (dual) lines correspond to in the primal plane ? Since the dual point intersects both lines, its dual *line* must intersect both priaml *points*. Therefore, an intersection of two lines in the dual corresponds to the (unique) line passing through two points in the primal.

Figure 3.3 also illustrates the ordering principle. The red point is above the line in the primal, and therefore its dual line is *below* the dual point.

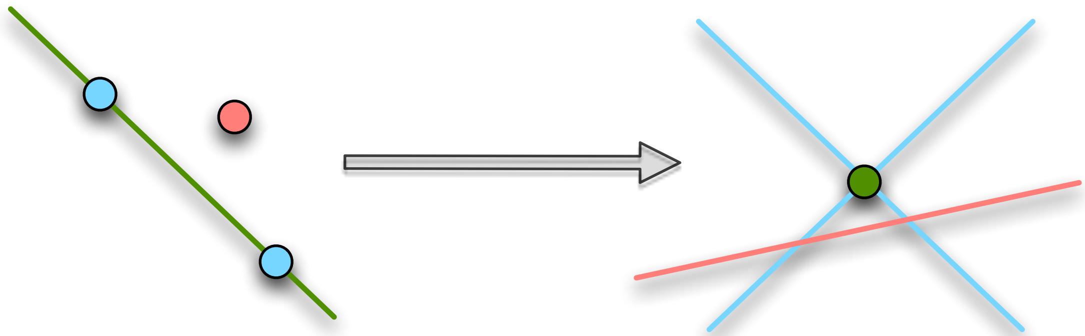

Figure 3.3: Dual Mapping example

Another interesting example is a *line segment*. We know that the dual of a point is a line. But what the dual of a collection of points (i.e a line segment). Imagine walking from one endpoint of the line segment to the other. Since at any stage, we are on the line connecting the two points, we know from the above observation that the dual *line* corresponding to our position must pass through the (dual) point corresponding to the primal line supporting the line segment.

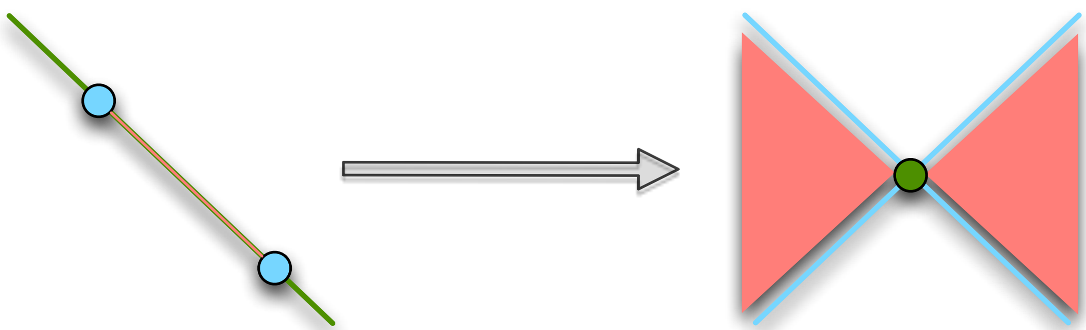

Figure 3.4: Double Wedge

This means that in the dual, we can imagine a line sweeping around a pivot point. As a result, the dual of a line segment is a *double wedge*.

Convex Hulls and Envelopes

Let us take an even more complex example. What is the dual of a convex hull ? It's easier to use the properties of the dual mapping, rather than direct reasoning. Let's consider the lower hull of the set of points. Consider each of the support lines of the lower hull (i.e lines defined by pairs of adjacent points on the lower hull). Each support line by definition lies *below* all the points in the set. By the order property of the standard dual, this means that the dual of such a line must lie *above* all the lines in the dual. Moreover, by the incidence property, these (dual) points will be incident on the dual lines corresponding to the points of the lower hull.

Now let us move to the dual and consider the arrangement of lines formed from the duals of the primal points. Imagine a ray being shot from the top of the diagram vertically down. It will intersect the arrangement at some line ℓ. Let's call this point of intersection p. Now we know that p^* is some primal line passing through ℓ^*, and by the order property p^* is below all the points of the set. Thus, if we mark off all such points in the dual, we will trace out the dual of the lower hull.

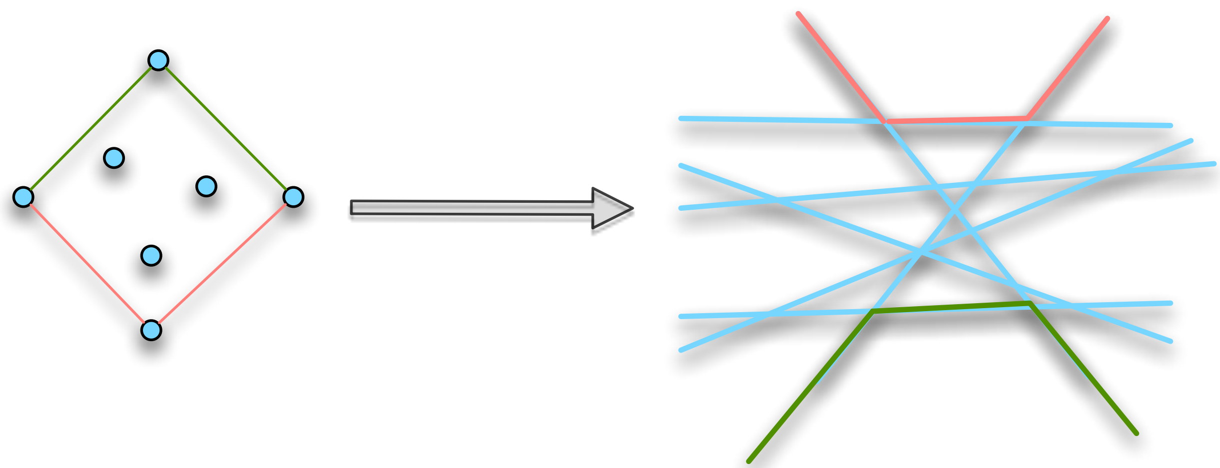

Figure 3.5: Convex hull dual envelopes

This set of points is called the *upper envelope* of the arrangements. Formally, it is the set of all points from which we can draw a vertical ray that does not intersect any line of the arrangement. Similarly, we can define the *lower envelope* as the set of points from which rays drawn *downward* do not cross any line of the arrangement. By a similar argument, we can show that the lower envelope of the arrangement is the dual of the *upper* hull of the primal set of points.

3.3 Constructing Arrangements

An arrangement is described by the collection of all faces of all dimensions, and how they connect to each other. What is the total complexity of the description of an arrangement ? Let us consider the case of an arrangement of lines. Firstly, assume that no two lines are parallel (because this can only decrease the complexity). Then there are $\binom{n}{2}$ vertices (every pair of lines can intersect). Since each line eventually extends down to $-\infty$, the n lines form $n+1$ faces that are unbounded below. Similarly, there are $n+1$ faces unbounded above. Each *bounded* face has a unique vertex at its base, and no vertex is the base for two faces. Therefore, the number of bounded faces is precisely the number of vertices. Finally, since the arrangement is planar, we can apply Euler's formula to bound the number of edges. In summary, an arrangement of n lines has $\binom{n}{2}$ vertices, $O(n^2)$ faces and $O(n^2)$ edges.

What about the complexity of a *single cell* ? Any cell in the arrangement is the intersection of at most n halfplanes and so has complexity at most n.

Another important structure in an arrangement is the *zone* of a line ℓ. This is the set of faces, edges and vertices intersected by the line.

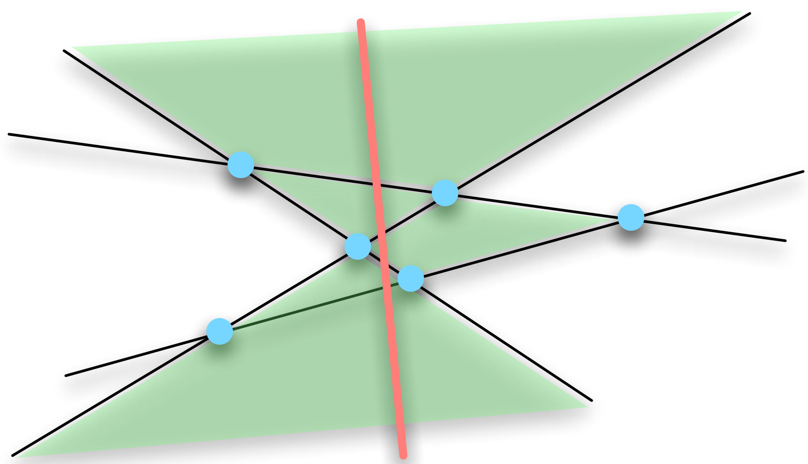

Figure 3.6: The zone of a line

Theorem 3.1 (Zone Theorem)**.** *In an arrangement of n lines, the zone of a line has complexity $O(n)$.*

The proof is by a careful induction. Note that this is not entirely obvious. A line clearly intersects at most n edges, but maybe it cuts many faces. The zone theorem says that this can't happen.

The zone theorem leads to a simple algorithm for computing an arrangement: add each line in one at a time, updating the arrangement each time. The zone theorem guarantees that only a linear number of cells are changed in each iteration, and these can also be found in linear time (see Exercise 6).

3.4 Levels of an arrangement

We've already encountered the lower and upper envelope of an arrangement. If we imagine a ray shooting upwards from $-\infty$ in the vertical dimension, then the lower envelope is the set of all points of the arrangement that are hit first, and the upper envelope is the set of all points of the arrangement hit last.

We can also define intermediate *levels* of the arrangement. The k-level of an arrangement is the set of all points on the shapes of the arrangement such that a ray shooting upwards from $-\infty$ hits $k-1$ curves of the arrangement before hitting the point. By this definition, the lower envelope of a set of shapes is the 1-level.

Levels of an arrangement are very important. Many problems can be formulated as an optimization over a set of levels of an arrangement.

Near Neighbors For example, if we wanted to compute the nearest neighbor of a point, this can be viewed as a query on the lower envelope of an arrangement of 45° cones with apex at each point.

MEBs If we want to find the *minimum enclosing ball (MEB* of a set of points, we can draw the cones as above. For any point x, the height of the *upper envelope* of this arrangement is the size of an MEB centered there. Therefore, the center of the optimal enclosing ball is found by finding the point corresponding to the minimum of the upper envelope.

Best Fit Circle Imagine you're given a set of points $\{p_1, \ldots, p_n\}$, and you wish to find a circle of radius r centered at c such that $\sum_i |\|p_i - c\| - r|$ is minimized. This is one version of the *best-fit circle* problem. As before, we draw cones centered at each of the input points. Now consider a candidate center (x, y) and a candidate radius r. The distance of any point from this circle is $|\|p_i - c\| - r|$. This is equivalent to the vertical distance of the point (x, y, r) from the cone corresponding to p_i. Therefore, if we fix c and draw the corresponding vertical line through $(c, 0)$, the optimal radius can be found by marking the intersections of the cones along the line, and picking a point that minimizes the sum of distances to these intersections.

The problem of finding r such that $\sum_i |r_i - r|$ is minimized (where here r_i is the height at which p_i's cone is intersected) is easy to solve. The solution is the *median* of the r_i. In other words, the optimal center and radius can be found by *finding the lowest point on the medial level* of the arrangement of cones[2].

We know that the lower and upper envelope of an arrangement can be bounded by the complexity of the convex hull. But what about intermediate levels ? It's not hard to see that the complexity of the k-level of an arrangement of lines in the plane is $O(nk)$, but it remains one of the biggest open problems in computational geometry to give a tight bound. The best known upper bound is $O(nk^{1/3})$[1].

3.5 Quantifier Elimination and Tarski's Theorem

Coming soon...

3.6 After Notes

Topics for later

- Duality via lifting map and tangents
- The topology of the projective plane

Exercises

6. Describe a linear time algorithm for inserting a line into an arrangement. You may assume the Zone theorem.

7. Let a *bitonic* function be one that increases monotonically upto a point, and then decreases monotonically. Show that the k-level of an arrangement of lines in the plane can be expressed as the upper envelope of k bitonic functions, and then use this fact to show that the complexity of the k-level of an arrangement of lines is $O(nk)$.

3.7 Bibliography

[1] T. Dey. Improved bounds for planar k-sets and related problems. *Discrete & Computational Geometry*, 19(3):373–382, 1998.

[2] S. Har-Peled. Approximation algorithm for the l1-fitting circle problem. In *EuroCG*, pages 103–106. Technische Universiteit Eindhoven, 2005.