Problem Set 1

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1. (my linear algebra knowledge is literally non-existent so if the math stuff is wrong I'm sorry lol)

   The length $L_{jik} = a_{jik}f_{j1} + a_{jik}f_{j2}... + a_{jik}f_{jc}$, where j represents the jth experiment, i represents the ith cell type, k represents the kth cell, and c represents the cth force.

   In linear system form: $A\vec{f} = \vec{L}$, where A contains the variables for each cell, $\vec{f}$ contains the forces, and $\vec{L}$ contains the experimental forces.

   (a) This is an unconstrained optimization problem, as we have an over-determined system.
   Our objective is to find the variables a_i , such that we minimize:

   $$\sum_{jik}(L_{jik} - (a_i f_{j1} + a_i f_{j2}... + a_i f_{jc}))^2$$

   (the least-squares error between the experimental length and the modeled length)

   Notice that our variables a_i do not have j or k in them, as the variables are only dependent on the cell type, and are independent of the experiment and the individual cell number.

   From the lecture notes, we know that we can convert finding something that minimizes the least-squares into a full-rank system, and then solve that system via the GMRES method.

   (b) This is a linear system problem, as we have a full-rank system. Our objective is to find the variables a_{ik}, such that $A\vec{f} = \vec{L}$, or to minimize the least-squares like in part a. if equality is not achievable.

   Notice that this time, a_{ik} does contain k this time, as the model is now per cell, rather than per cell type like in part a. j is still not part of the variable, as the variables should still be independent of the experiment.

   This can be solved via the GMRES method, as it is a full-rank system.

   (c) This is a linear program problem, as we have a full-rank system, but are given some constraints on the possible solutions. Our objective is to find the variables a_{ik}, such that we minimize:

   $$\sum_{jik}(L_{jik} - (a_{ik} f_{j1} + a_{ik} f_{j2}... + a_{ik} f_{jc}))^2$$

   within the given constraints, which will take the form:

   $$x_{ik} \leq a_{ik} \leq y_{ik}$$

   for minima x and maxima y for each variable a_{ik}.

   As this is a full-rank linear program, we can solve this via the simplex method/algorithm.

(d) This is a non-linear program, as we are attempting to learn a quadratic model instead of a linear model. Our objective is to find the variables a_{jkl}, such that we minimize:

$$\sum_{jik}(L_{jik} - (a_{ik1}f_{j1} + a_{ik2}f_{j1}^2 + a_{ik1}f_{j2} + a_{ik2}f_{j2}^2 ... + a_{ik1}f_{jc} + a_{ik2}f_{jc}^2))^2$$

Again, the i represents the cell type, and the k represents the individual cell within the cell type. Note the addition of a new subscript, l, which denotes whether the constant is for the 1st power or the 2nd power. j is still not part of the variable as it should be independent of the experiment.

This can still be solved via interior point methods, such as the simplex method/algorithm.

2. Formulas:

$$E(x) = \frac{1}{2}x^2 - 2(1-x)^{1.5}$$

$$\frac{dE}{dt} = f(x) = x - 3\sqrt{1-x}$$

(a) We start with $x_{min} = 0$ and $x_{max} = 1$.

step	1	2	3
x_{min}	0	0.5	0.75
x_{max}	1	1	1
$f(x_{min})$	-3	-1.621	-0.75
$f(x_{max})$	1	1	1
x_{mid}	0.5	0.75	0.875
$f(x_{mid})$	-1.621	-0.75	-0.186

Our final estimate is $x = 0.875$.
The forward error is $f(x_{mid}) = -0.186$.
The maximum backward error is $x_{max} - x_{min} = 0.125$

(b) We start with $x_{min} = 0$ and $x_{max} = 1$.

step	1	2	3
x_{min}	0	0.75	0.857
x_{max}	1	1	1
$f(x_{min})$	-3	-0.75	-0.277
$f(x_{max})$	1	1	1
secant line	$y = 4x - 3$	$y = 7x - 6$	$y = 8.93x - 7.93$
x_{mid}	0.75	0.857	0.888
$f(x_{mid})$	-0.75	-0.277	-0.116

Our final estimate is $x = 0.888$.
The forward error is -0.116.
The maximum backward error is $x_{max} - x_{min} = 0.112$

(c) We start with an initial guess of $x = 0.75$. First, we will derive $f'(x)$.

$$f'(x) = 1 + \frac{3}{2\sqrt{1-x}}$$

step	1	2	3
x	0.75	0.9375	0.911
$f(x)$	-0.75	0.1875	0.0143
$f'(x)$	4	7	6.020
Next x	0.9375	0.911	0.908

Our final estimate is $x = 0.908$.
The forward error is $f(0.908) = 0.0000782$.
To estimate the backwards error, we need $f''(x)$, which is:

$$f''(x) = \frac{3}{4(1-x)^{\frac{3}{2}}}$$

We will estimate the error using $\xi = 0.91$. The backwards error using this ξ is:

$$-\frac{f''(0.91)}{2f'(0.911)}(0.908 - 0.911)^2 \approx 0.0000208$$

3. (a) Objective: maximize $D_1 + 3D_2$ subject to:

$$2D_1 + 8D_2 \leq 10$$
$$2D_1 + 4D_2 \leq 10$$
$$5D_1 + 2D_2 \leq 10$$
$$D_1 \geq 0, D_2 \geq 0$$

(b) In standard form, this is equivalent to:
minimize $-D_1 - 3D_2$ subject to:

$$2D_1 + 8D_2 + L = 10$$
$$2D_1 + 4D_2 + K = 10$$
$$5D_1 + 2D_2 + B = 10$$
$$D_1 \geq 0, D_2 \geq 0, L \geq 0, K \geq 0, B \geq 0$$

(c) We start with $D_1 = 0$ and $D_2 = 0$, such that we have 3 variables for 3 equality constraints. We will then rearrange the equations to place the non-zero variables on the left side.

$$2D_1 + 8D_2 + L = 10 \implies L = 10 - 2D_1 - 8D_2$$
$$2D_1 + 4D_2 + K = 10 \implies K = 10 - 2D_1 - 4D_2$$
$$5D_1 + 2D_2 + B = 10 \implies B = 10 - 5D_1 - 2D_2$$

Solving for L, K, and B, we get:

$$L = 10, K = 10, B = 10$$

So we are starting at $D_1 = 0$, $D_2 = 0$, $L = 10$, $K = 10$, $B = 10$.
We will now increase D_2 as much as possible, since it has the largest negative multiplier.
We then rewrite the equations from above to take into account that $D_1 = 0$ still.

$$L = 10 - 8D_2$$
$$K = 10 - 4D_2$$
$$B = 10 - 2D_2$$

When increasing D_2, we find that the first variable to hit zero is L, at $D_2 = 1.25$. We now set $D_2 = 1.25$ and $L = 0$, and rewrite the equations and objective in terms of the zeroed variables.

$$2D_1 + 8D_2 + L = 10 \implies 8D_2 = 10 - 2D_1 - L \implies D_2 = \frac{10 - 2D_1 - L}{8} = 1.25 - 0.25D_1 - 0.125L$$

$$K = 10 - 2D_1 - 4D_2 \implies K = 10 - 2D_1 - \frac{10 - 2D_1 - L}{2} = 5 - D_1 + 0.5L$$

$$B = 10 - 5D_1 - 2D_2 \implies B = 10 - 5D_1 - \frac{10 - 2D_1 - L}{4} = 7.5 - 4.5D_1 + 0.25L$$

$$\text{minimize } -D_1 - 3D_2 \implies \text{minimize } -D_1 - 3\tfrac{10-2D_1-L}{8} = -0.25D_1 + 0.375L - 3.75$$

Solving for our non-zero variables, we find that:

$$D_2 = 1.25 - 0 - 0 = 1.25$$

$$K = 5 - 0 + 0 = 5$$

$$B = 7.5 - 0 + 0 = 7.5$$

So our current values are: $D_1 = 0$, $D_2 = 1.25$, $L = 0$, $K = 5$, $B = 7.5$.

Recall the objective function: minimize $-0.25D_1 + 0.375L - 3.75$

We will now increase D_1 as much as possible, as it is the only variable in our objective function with a negative coefficient. The constraints on D_1 are:

$$D_2 = 1.25 - 0.25D_1 - 0.125L$$

$$K = 5 - D_1 + 0.5L$$

$$B = 7.5 - 4.5D_1 + 0.25L$$

As we increase D_1, we find that the largest we can increase it before turning one of the left variables to zero is $\frac{5}{3}$. At $D_1 = \frac{5}{3}$, $B = 0$. We set D_1 and B to those values, and rearrange equations again, moving non-zero variables (D_1, D_2, K) to the left and zeroed variables (L, B) to the right.

$$D_2 = 1.25 - 0.25D_1 - 0.125L \implies D_2 + 0.25D_1 = 1.25 - 0.125L$$

$$K = 5 - D_1 + 0.5L \implies D_1 + K = 5 + 0.5L$$

$$B = 7.5 - 4.5D_1 + 0.25L \implies 4.5D_1 = 7.5 + 0.25L - B$$

Solving for our non-zero variables, we get: $D_1 = \frac{5}{3}$, $D_2 = \frac{5}{6}$, and $K \approx 3.33$. $L = B = 0$.

Substituting the 3rd formula into our objective, we get:

$$\text{minimize } -0.25D_1 + 0.375L - 3.75 \longrightarrow \text{minimize } 0.319L + 0.0555B - 4.1675$$

As there are no more negative coefficients in our objective function, we stop here, as this is the optimum.

So the optimal values are: $D_1 = \frac{5}{3}$, $D_2 = \frac{5}{6}$

(d) The total effectiveness is: $D_1 + 3D_2 \approx 4.167$

The total toxicity to the liver is: $2D_1 + 8D_2 = 10$

The total toxicity to the kidney is: $2D_1 + 4D_2 \approx 6.67$

The total toxicity to the brain is: $5D_1 + 2D_2 = 10$

(e) A drug that increased liver tolerance would likely allow us to get a higher total effectiveness, by lowering D_1 and increasing D_2. D_2 is more effective, so the tradeoff of losing D_1 should be worth it. The liver is highly sensitive to D_2, so a higher liver tolerance would allow more D_2 to be used.

An increase in kidney tolerance wouldn't do much, since we are currently limited by the toxicity to the liver and brain, while the kidney still has some wiggle room.

An increase in brain tolerance also wouldn't do much, since although it would allow us to increase D_1 and lower D_2(the brain is sensitive to D_1, so a higher tolerance means we could probably increase the amount of D_1 used at the expense of losing some D_2). However, the loss of the D_2 would likely not be worth it, since D_2 is much more effective than D_1.

4. no idea what the hell im doing

   (a) shit's fucked