Network Algorithms and Stochastic Optimization: Exercises in the Properties and Stability of One Dimensional Queueing Processes

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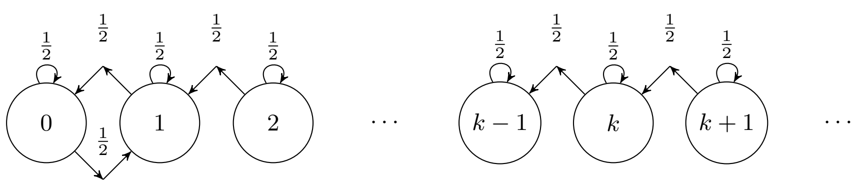
1 Recurrence, Communicating Classes, and Transition Probabilities

We consider a one-dimensional queue with arrivals $A(t)$ and service $M(t)$ filling a queue of length $X(t)$ (see Figure 1). $X(0)$ is a geometric random variable with $p > 0$ For $t \in \{0, 1, \dots\}$, X evolves according to:

$$X(t+1) = (X(t) - M(t+1))^+ + A(t+1).$$

1.1 Bernoulli Arrivals

For $t \in \{0, 1, \dots\}$, A(t) are i.i.d. Bernoulli with probability $\frac{1}{2}$. $M(t) = 1$. The state space is $\mathbb{Z}^+$ and the classes of communicating states are $\{0, 1\}$ (the only closed class with positive recurrence), $\{2\}, \{3\}, \dots$. For $k > 0$, the transition probability diagram is as follows:



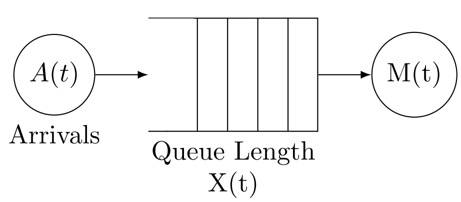


Figure 1: The model represented as a queuing system.

The transition probabilities can equivalently be represented in matrix form, where $\Pi_{i,j}$ is the transition probability from state j to i. That is, in general, we have:

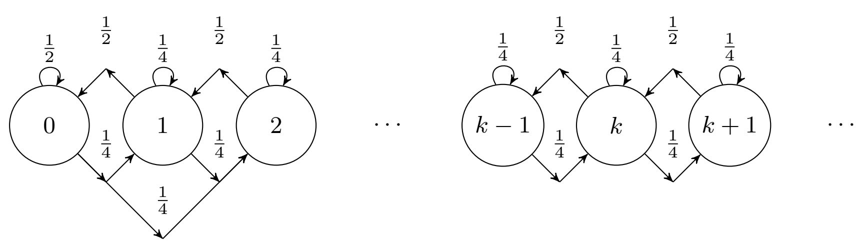
$$\Pi_{i,j} = \mathbb{P}[X(t+1) = i \mid X(t) = j].$$

In this case, we have:

$$\Pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & & \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & & & \\ \vdots & & & \ddots & & & \\ & & & & \frac{1}{2} & \frac{1}{2} & \\ & & & & & \ddots & \end{bmatrix}$$

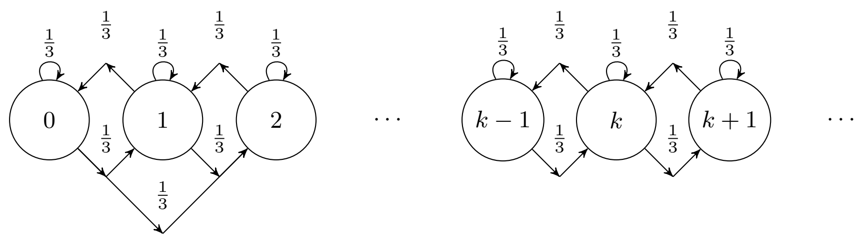
1.2 More Frequent Arrivals

Holding all else equal, we now consider the case where two arrivals occur at the same time. Specifically, $A(t)$ is independently distributed over support $\langle 0, 1, 2 \rangle$ with probabilities $\langle .5, .25, .25 \rangle$. The state space is still $\mathbb{Z}^+$, which now represents the only closed class of communicating states. All states are positive recurrent. Letting $k > 0$ be arbitrary, the transition probability diagram is specified as follows:



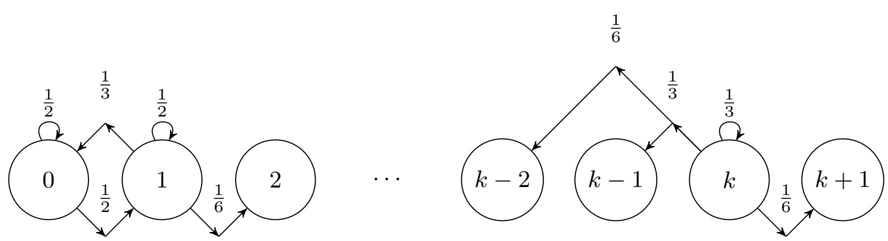
1.3 Arrival Rate Equal to Service Rate

There will not be a stable distribution for queues whose arrival rates are greater than or equal to the service rate. The following is such a process, featuring a symmetric random walk. Specifically, we have $A(t) \sim Unif(\{0, 1, 2\})$ iid. $M(t) = 1$, as before. $\mathbb{Z}^+$ is still the state space and the only (closed) class of communicating states, this time null recurrent. The transition probability can be specified for the case 0 and for all states $k > 0$, as follows:



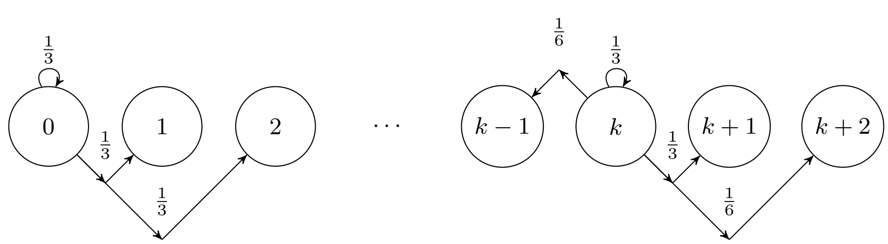
1.4 Uniform Distributed Stable Service

We'd also like to model systems where the service is random. Some of these systems are stable (generally if the expected service rate is larger than that of arrival). For example, let $A(t)$ be iid uniform in $\{0,1\}$, and $M(t)$ iid uniform in $\{0,1,2\}$. As before, $\mathbb{Z}^+$ is both the state space and the single closed positive recurrent communicating class. The transitions can be characterized for states 0 and 1 and in general for all states $k>1$ (some edges are not shown for clarity):



1.5 Too Many Arrivals

Sometimes, the service cannot keep up with the arrival load. Let $A(t)$ be iid uniform in $\{0,1,2\}$ and $M(t)$ uniform in $\{0,1\}$. As in the unstable symmetric random walk, $\mathbb{Z}^+$ are the state space and they are a null recurrent communicating class. The probability transition diagram edges fall into two cases, those from 0, and those from state $k>0$:



2 Queueing Process Stability

It is of interest to characterize the long term behavior of queueing processes. Given a state transition matrix Π, a stationary distribution $\bar{P}$ is one for which

$$\bar{P} = \Pi \bar{P}$$

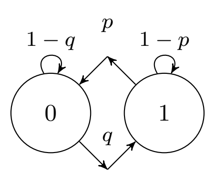
That is, the marginal probability mass is stable over the state space given another time step.

2.1 Warm-Up: Selection Process

To warm up, we consider a Markov selection process $S(t)$, $t \in \mathbb{Z}^+$ over the state space $\{0, 1\}$. For $p, q \in [0, 1]$ (with at least one non-zero) $S(t)$ has transition probabilities

$$\Pi = \begin{bmatrix} 1-q & p \\ q & 1-p \end{bmatrix}$$

This can be visualized with the transition diagram:



Theorem 2.1. $\begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}$ *is a unique stationary distribution for the selection process.*

Proof. Because the process is finite and has a single closed class of positive recurrent communicating states (whichever index of (p, q) is nonzero, or both), a unique stationary distribution exists. The reader can verify that

$$\begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix} = \begin{bmatrix} 1-q & p \\ q & 1-p \end{bmatrix} \begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}.$$

□

2.2 Selected Arrivals and Burst Service

Suppose we have a queueing system where arrivals are chosen out of two streams via a selection process, and service is provide in bursts. Specifically, arrivals $A(t) = A_1(t)(1 - S(t)) + A_2(t)S(t)$, where $S(t)$ is as before with iid $A_i(t) \sim Bern(p_i) : i \in \{1, 2\}$). Service $M(t)$ are iid uniform in $\{0, 1, \dots, K\}$. As before, the queue length follows intuition about service and arrivals over discretized time, as:

$$X(t+1) = (X(t) - M(t+1))^+ + A(t+1), t \in \{0, 1, \dots\}$$

$X(t)$ is supported over $\mathbb{Z}^+$, and its behavior is of interest.

As defined, $X(t)$ is not a Markov process. This is because the arrivals $A(t)$ are not iid; they are selected among arrival streams by the $S(t)$ Markov process. But if we include $S(t)$ into a description of states, a Markov process with state space

$$\{\langle S(t), X(t)\rangle : S(t) \in \{0,1\}, X(t) \in \mathbb{Z}^+\}$$

can fully characterize the behavior of the queue.

Theorem 2.2. $\frac{K}{2} > \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}$ *is the necessary and sufficient condition for the stability of the queue.*

Lemma 2.3. *The stated condition is necessary for the stability of the queue.*

Proof. For positive recurrence of the states, it is necessary that the expected number of arrivals at each time step is less than the expected number of services. Since $M(t)$ is uniform iid in $\{0, 1, \dots, K\}$, the expected number of services is $\frac{K}{2}$. For arrivals, we have

$$\begin{aligned} \mathbb{E}[A(t)] &= \mathbb{E}[A_1(t)(1 - S(t)) + A_2(t)S(t)] \\ &= p_1\, \mathbb{E}[1 - S(t)] + p_2\, \mathbb{E}[S(t)] \end{aligned}$$

Which uses the assumption that the arrival streams are independent of the selection process. Continuing,

$$\begin{aligned} &= p_1\, \mathbb{E}[\mathbb{1}\{S(t) = 0\}] + p_2\, \mathbb{E}[\mathbb{1}\{S(t) = 1\}] \\ &= p_1 \frac{p}{p+q} + p_2 \frac{q}{p+q}, \end{aligned}$$

Which follows from the fact established in Theorem 2.1 that $\begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}$ is a stationary distribution for the selection process. □

To establish sufficiency, we seek a Lyapunov function with negative drift on the state space. Suppose we take V to be the queue length of a given state. Unfortunately, because of the hidden Markov process, requiring negative drift from a general state produces overly strict conditions on the size of K: one finds that K must be sufficient to overwhelm the worst of the A_1 and A_2 arrival streams, the drift not given information about how the selection process tempers any stronger stream by often selecting from the other one. If only our chain could treat the selection proportion directly for the sake of satisfying the negative drift criterion!

Towards that end, we introduce another Markov chain, $Y(t), t \in \{0, 1, \dots\}$, whose values are a sample over time of $X(t)$. Specifically, suppose without loss of generality that the chain begins with $S(0) = 0$. We can imagine times $n_0, n_1, n_2, \dots \in \mathbb{Z}^+$ such that, in an alternating manner, each n_t is the time of a transition from $S(n_t - 1) = 1$ to $S(n_t) = 0$ or vice-versa. Relatedly, let $L_{0,t}$ be iid First-Success[1] distributed with parameter q representing the length of the t^{th} run of S in state 0 given that $S(n_t - 1) = 1$ just before the run. Symmetrically for $L_{1,t}$. Visually, a run of the $S(t)$ process could be indexed as follows:

$$\overbrace{\underset{n_0}{0}\,,0\ldots}^{L_{0,0}}\overbrace{\underset{n_1}{1}\,,1\ldots}^{L_{1,1}}(\ldots)\overbrace{\underset{n_t}{0}\,,0\ldots}^{L_{0,t}}\overbrace{\underset{n_{t+1}}{1}\,,1\ldots}^{L_{1,t+1}}$$

Then $Y(t) = X(n_t)$ is a Markov chain supported over $\mathbb{Z}^+$.

The fact that $Y(t)$ is indeed a Markov chain is interesting, and deserves some minor discussion. Remember that, on its own, $X(t)$ was not a Markov chain. This was because information about $S(t)$ was to be found nowhere in X, yet it was crucial to specifying the state transition probabilities of X. Neither is the information to be found in Y. However, the value of t for the Y process encodes the state of S. $S(n_t) = \mathbb{1}\{t \text{ is odd}\}$. The Markov condition holds:

$$\begin{aligned}&\mathbb{P}[Y(t+1) = i \mid Y(t) = j_t, Y(t-1) = j_{t-1}, \ldots Y(0) = j_0]\\=&\mathbb{P}[Y(t+1) = i \mid Y(t) = j_t].\end{aligned}$$

Lemma 2.4. $\frac{K}{2} > \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}$ *is sufficent to stabilize* Y.

Proof. We propose the Lyapunov function $V : \mathbb{Z} \to \mathbb{R}$ such that $V(y) = y$. This function passes the first part of the Lyapunov criterion for stability, that for all δ,

$$|\{y \mid V(y) \leq \delta\}| < \infty.$$

We now consider the drift of the function on process $Y(t)$:

$$\begin{aligned}\text{drift} =& \mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i]\\=& \frac{1}{2}\,\mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i, \text{t is odd}]\\+& \frac{1}{2}\,\mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i, \text{t is even}].\end{aligned}$$

We take the case that t is even, and $S(n_t) = 0$, before proceeding by symmetry.

$$\mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i, \text{t is even}] \tag{1}$$
$$= \mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i, S(n_t) = 0] \tag{2}$$
$$= \mathbb{E}[V(X(n_{t+1})) - V(X(n_t)) \mid X(n_t) = i, S(n_t) = 0] \tag{3}$$
$$= \mathbb{E}[L_{0,t}]\,\mathbb{E}[V(X(n_t+1)) - V(X(n_t)) \mid X(n_t) = i, S(n_t) = 0] \tag{4}$$

[1] The First-Success distribution describes $1+X$ where X a geometrically distributed random variable with the same parameter; this is to count the success itself.

The last move (4) is a little tricky. It follows from the Markov assumption on the $\langle X(t), S(t)\rangle$ process: in lay terms, l steps of the Markov chain would drift as much as l *times* the drift of just one step. Finally, we know that $L_{0,t}$ is first-success distributed with parameter $\mathbb{E}[L_{0,t}] = \frac{1}{q}$. Recall that when S is 0, arrivals occur via A_1 with parameter p_1. Continuing, we have that

$$\mathbb{E}[V(Y(t+1)) - V(Y(t)) \mid Y(t) = i, \text{t is even}] = \frac{1}{q}(p_1 - \frac{K}{2})$$

By symmetry with the above, one can derive that

$$\text{drift} = \frac{1}{2}[\frac{1}{q}(p_1 - \frac{K}{2}) + \frac{1}{p}(p_2 - \frac{K}{2})] \tag{5}$$

Straightforward algebra shows that the drift will be negative if and only if:

$$\frac{K}{2} > \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{p}{p+q} \\ \frac{q}{p+q} \end{bmatrix}.$$

□

Proof of Theorem 2.2. By Lemma 2.4, the states of Y are positive recurrent. Therefore, so are the states in the $\langle S, X\rangle$ chain. The stated condition is necessary and sufficient to stabilize the queue length $X(t)$. □