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GENERAL MODELS, DESCRIPTIONS, AND CHOICE IN TYPE THEORY

PETER B. ANDREWS[1]

§1. Introduction. In [4] Alonzo Church introduced an elegant and expressive formulation of type theory with λ-conversion. In [8] Henkin introduced the concept of a general model for this system, such that a sentence **A** is a theorem if and only if it is true in all general models. The crucial clause in Henkin's definition of a general model $\mathscr{M}$ is that for each assignment φ of values in $\mathscr{M}$ to variables and for each wff **A**, there must be an appropriate value $\mathscr{V}_\varphi\mathbf{A}$ of **A** in $\mathscr{M}$. Hintikka points out in [10, p. 3] that this constitutes a rather strong requirement concerning the structure of a general model. Henkin draws attention to the problem of constructing nonstandard models for the theory of types in [9, p. 324].

We shall use a simple idea of combinatory logic to find a characterization of general models which does not directly refer to wffs, and which is easier to work with in certain contexts. This characterization can be applied, with appropriate minor and obvious modifications, to a variety of formulations of type theory with λ-conversion. We shall be concerned with a language $\mathscr{L}$ with extensionality in which there is no description or selection operator, and in which (for convenience) the sole primitive logical constants are the equality symbols $Q_{o\alpha\alpha}$ for each type α.

We shall give two applications of this characterization. First, we show that the Axiom of Descriptions (D) is independent of $\mathscr{L}$. This axiom is very natural since a general model for $\mathscr{L}$ with a finite domain of individuals is standard if and only if D is true in it. Secondly, we show how the Fraenkel-Mostowski method [7], [11], [12] can be adapted to $\mathscr{L}$. We state our fundamental lemma concerning this method in fairly general form to facilitate possible future applications (analogous to those for axiomatic set theory mentioned in [11]), but confine ourselves here to simply showing that the Axiom of Choice is not derivable in $\mathscr{L}$, even if the Axiom of Descriptions is assumed.

When a description operator $\iota_{\iota(o\iota)}$ is included among the primitive symbols,[2] the Axiom of Descriptions may be taken in the form

$$\forall p_{o\iota}\centerdot\exists_1 x_\iota p_{o\iota}x_\iota \supset p_{o\iota}[\iota_{\iota(o\iota)}p_{o\iota}],$$

so that $\iota_{\iota(o\iota)}[\lambda\mathbf{x}_\iota\mathbf{A}_o]$ (which is abbreviated $(\imath\mathbf{x}_\iota\mathbf{A}_o)$) denotes the unique $\mathbf{x}_\iota$ such that $\mathbf{A}_o$, when there is such an $\mathbf{x}_\iota$. Church showed in [4] that description operators for higher types can be introduced by definition, using the operators for lower types. Specifically, $\iota_{\alpha\beta(o(\alpha\beta))}$ may be defined as

$$[\lambda h_{o(\alpha\beta)}\lambda x_\beta\iota_{\alpha(o\alpha)}\lambda y_\alpha\exists f_{\alpha\beta}\centerdot h_{o(\alpha\beta)}f_{\alpha\beta} \wedge y_\alpha = f_{\alpha\beta}x_\beta].$$

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[2] We follow the well established tradition of Church [4] in which the Greek letter ι is used as a type symbol for the type of individuals, and also, when given an appropriate type symbol subscript, as a primitive constant denoting a description operator. In practice this causes no confusion, since ι usually appears as a subscript in the former usage, but rarely in the latter.

R. O. Gandy has pointed out (in a private communication) that $\iota_{o\beta(o(o\beta))}$ can be defined as

$$[\lambda h_{o(o\beta)}\lambda x_\beta \exists f_{o\beta}\centerdot h_{o(o\beta)} f_{o\beta} \wedge f_{o\beta} x_\beta],$$

so description operators for certain higher types can be defined without using those for any other type. Also, Henkin noted in [9] that $\iota_{o(oo)}$ can be defined as

$$[\lambda h_{oo}\centerdot h_{oo} = [\lambda x_o x_o]].$$

(A number of other definitions of $\iota_{o(oo)}$ are also possible, of which the shortest is perhaps the closely related $Q_{o(oo)(oo)}[\lambda x_o x_o]$.) Thus it is seen that description operators for all types can be introduced once one has $\iota_{\iota(o\iota)}$. The argument in [2, pp. 22–24] shows that the description operator $\iota_{\iota(o\iota)}$ cannot be introduced by definition for the simple reason that there are no closed wffs of this type, and that the Axiom of Descriptions mentioned above is independent, since it is the sole axiom which describes the special characteristics of $\iota_{\iota(o\iota)}$.

If no description operator is included in the list of primitive symbols, the Axiom of Descriptions may be taken in the form

$$\exists i_{\iota(o\iota)} \forall p_{o\iota}\centerdot \exists_1 x_\iota p_{o\iota} x_\iota \supset p_{o\iota}[i_{\iota(o\iota)} p_{o\iota}],$$

or equivalently

(D) $$\exists i_{\iota(o\iota)} \forall x_\iota \centerdot i_{\iota(o\iota)}[Q_{o\iota\iota} x_\iota] = x_\iota.$$

(The equivalence results from the theorem $\exists_1 x_\iota p_{o\iota} x_\iota = \exists x_\iota \centerdot p_{o\iota} = \centerdot Q_{o\iota\iota} x_\iota$.) Since in many logical systems descriptions can be eliminated, it is very natural to ask whether the wff D, which asserts the existence of a description operator, is in fact derivable. It will be seen that our independence proof below is conceptually very simple, and is compatible with any axioms concerning the cardinality of the domain of individuals which permit it to have at least two members.

Church mentions in [5] an unpublished proof by Lagerström of a closely related independence result using a complete nonatomic Boolean algebra for the domain of truth values. It seems unlikely that Lagerström's proof applies to $\mathscr{L}$, since in $\mathscr{L}$, unlike the system treated by Lagerström, there is a strong axiom of extensionality for type o (Axiom 1 below) which permits one to derive $[p_o \equiv q_o] \supset \centerdot p_o = q_o$.

§2. **The language $\mathscr{L}$.** The language $\mathscr{L}$ is essentially the result of dropping the description operator from the language Q_o of [2], and is closely related to the system discussed in [9]. For the convenience of the reader we here provide a description of $\mathscr{L}$.

We use α, β, γ, etc., as syntactical variables ranging over *type symbols*, which are defined inductively as follows:

(a) o is a type symbol (denoting the type of truth values).

(b) ι is a type symbol (denoting the type of individuals).

(c) $(\alpha\beta)$ is a type symbol (denoting the type of functions from elements of type β to elements of type α).

The *primitive symbols* of $\mathscr{L}$ are the following:

(a) Improper symbols: [ ] λ.

(b) For each α, a denumerable list of *variables* of type α:

$$f_\alpha g_\alpha h_\alpha \cdots x_\alpha y_\alpha z_\alpha f^1_\alpha g^1_\alpha \cdots z^1_\alpha f^2_\alpha \cdots.$$

We shall use $\mathbf{f}_\alpha$, $\mathbf{g}_\alpha, \cdots, \mathbf{x}_\alpha$, $\mathbf{y}_\alpha$, $\mathbf{z}_\alpha$, etc., as syntactical variables for variables of type α.

(c) For each α, $Q_{((o\alpha)\alpha)}$ is a *constant* of type $((o\alpha)\alpha)$.

We write *wff*$_\alpha$ as an abbreviation for *wff of type* α, and use $\mathbf{A}_\alpha$, $\mathbf{B}_\alpha$, $\mathbf{C}_\alpha$, etc., as syntactical variables ranging over wffs$_\alpha$, which are defined inductively as follows:

(a) A primitive variable or constant of type α is a wff$_\alpha$.

(b) $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ is a wff$_\alpha$.

(c) $[\lambda\mathbf{x}_\beta\mathbf{A}_\alpha]$ is a wff$_{(\alpha\beta)}$.

An occurrence of $\mathbf{x}_\alpha$ is *bound* (*free*) in $\mathbf{B}_\beta$ iff it is (is not) in a wf part of $\mathbf{B}_\beta$ of the form $[\lambda\mathbf{x}_\alpha\mathbf{C}_\delta]$. $\mathbf{A}_\alpha$ is *free for* $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$ iff no free occurrence of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$ is in a wf part of $\mathbf{B}_\beta$ of the form $[\lambda\mathbf{y}_\gamma\mathbf{C}_\delta]$ such that $\mathbf{y}_\gamma$ is a free variable of $\mathbf{A}_\alpha$.

Brackets, parentheses in type symbols, and type symbols may be omitted when no ambiguity is thereby introduced. A dot stands for a left bracket whose mate is as far to the right as is consistent with the pairing of brackets already present and with the formula being well formed. Otherwise brackets and parentheses are to be restored using the convention of association to the left.

We introduce the following definitions and abbreviations:

$[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$	stands for	$[Q_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha]$.
T_o	stands for	$[Q_{ooo} = Q_{ooo}]$.
F_o	stands for	$[\lambda p_o p_o] = [\lambda p_o T_o]$.
$[\forall\mathbf{x}_\alpha\mathbf{A}_o]$	stands for	$[\lambda\mathbf{x}_\alpha\mathbf{A}_o] = [\lambda\mathbf{x}_\alpha T_o]$.
$\wedge_{ooo}$	stands for	$[\lambda p_o\lambda q_o{\bullet}[\lambda g_{ooo}{\bullet}g_{ooo}p_oq_o] = [\lambda g_{ooo}{\bullet}g_{ooo}T_oT_o]]$.
$[\mathbf{A}_o \wedge \mathbf{B}_o]$	stands for	$[\wedge_{ooo}\mathbf{A}_o\mathbf{B}_o]$.
$\supset_{ooo}$	stands for	$[\lambda p_o\lambda q_o{\bullet}p_o \wedge q_o = p_o]$.
$[\mathbf{A}_o \supset \mathbf{B}_o]$	stands for	$[\supset_{ooo}\mathbf{A}_o\mathbf{B}_o]$.

Other connectives and quantifiers are introduced in familiar ways.

$K^{\alpha\beta}$ and $K_{\alpha\beta\alpha}$	stand for	$[\lambda x_\alpha\lambda y_\beta x_\alpha]$.
$S^{\alpha\beta\gamma}$ and $S_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$	stand for	$[\lambda x_{\alpha\beta\gamma}\lambda y_{\beta\gamma}\lambda z_\gamma{\bullet}x_{\alpha\beta\gamma}z_\gamma{\bullet}y_{\beta\gamma}z_\gamma]$.
$B^{\alpha\beta\gamma}$ and $B_{\alpha\gamma(\beta\gamma)(\alpha\beta)}$	stand for	$[\lambda f_{\alpha\beta}\lambda g_{\beta\gamma}\lambda x_\gamma{\bullet}f_{\alpha\beta}{\bullet}g_{\beta\gamma}x_\gamma]$.
$C^{\alpha\beta\gamma}$ and $C_{\alpha\gamma\beta(\alpha\beta\gamma)}$	stand for	$[\lambda f_{\alpha\beta\gamma}\lambda x_\beta\lambda y_\gamma{\bullet}f_{\alpha\beta\gamma}y_\gamma x_\beta]$.
$W^{\alpha\beta}$ and $W_{\alpha\beta(\alpha\beta\beta)}$	stand for	$[\lambda f_{\alpha\beta\beta}\lambda x_\beta{\bullet}f_{\alpha\beta\beta}x_\beta x_\beta]$.
$\mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta$	stands for	the result of substituting $\mathbf{A}_\alpha$ for $\mathbf{x}_\alpha$ at all free occurrences of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

$\mathscr{L}$ has a single rule of inference, which is the following:

Rule R. From $\mathbf{C}_o$ and $[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ to infer the result of replacing one occurrence of $\mathbf{A}_\alpha$ (which is not an occurrence of a variable immediately preceded by λ) in $\mathbf{C}_o$ by an occurrence of $\mathbf{B}_\alpha$.

The axioms and axiom schemata for $\mathscr{L}$ are the following:

(1) $[g_{oo}T_o \wedge g_{oo}F_o] = \forall x_o{\bullet}g_{oo}x_o$.

(2) $x_\alpha = y_\alpha \supset {\bullet}h_{o\alpha}x_\alpha = h_{o\alpha}y_\alpha$.

(3) $f_{\alpha\beta} = g_{\alpha\beta} = \forall x_\beta{\bullet}f_{\alpha\beta}x_\beta = g_{\alpha\beta}x_\beta$.

(4) $[\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]\mathbf{A}_\alpha = \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta$, where $\mathbf{A}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

Let us denote by $\mathscr{T}\mathscr{E}$ the system obtained when the axioms of extensionality (6.1.1 of [3]) are added to the list of axioms of the system $\mathscr{T}$ of [3]. This is essentially the system of [8] or [4] using axioms 1–6, 10^o, $10^{\alpha\beta}$, and with the selection operators deleted. $\mathscr{T}\mathscr{E}$ differs from $\mathscr{L}$ in having primitive constants $\sim_{oo}$, $\vee_{ooo}$, and $\Pi_{o(o\alpha)}$ instead of $Q_{o\alpha\alpha}$. There are natural translations Δ from $\mathscr{L}$ into $\mathscr{T}\mathscr{E}$ and ∇ from $\mathscr{T}\mathscr{E}$ into $\mathscr{L}$ which involve replacing the primitive constants of one language by appropriate closed wffs of the other language. For example, if $\mathbf{A}_\alpha$ is a wff of $\mathscr{L}$, $\Delta\mathbf{A}_\alpha$ is the result of replacing each occurrence of $Q_{o\beta\beta}$ in $\mathbf{A}_\alpha$ by the wff

$$[\lambda x_\beta \lambda y_\beta \forall f_{o\beta} \centerdot f_{o\beta} x_\beta \supset f_{o\beta} y_\beta]$$

of $\mathscr{T}\mathscr{E}$. It is easy to establish that $\mathscr{L}$ and $\mathscr{T}\mathscr{E}$ are equivalent in the sense that for each wff $\mathbf{A}_o$ of $\mathscr{L}$ and $\mathbf{B}_o$ of $\mathscr{T}\mathscr{E}$, $\vdash_{\mathscr{L}} \mathbf{A}_o$ iff $\vdash_{\mathscr{T}\mathscr{E}} \Delta\mathbf{A}_o$, and $\vdash_{\mathscr{T}\mathscr{E}} \mathbf{B}_o$ iff $\vdash_{\mathscr{L}} \nabla\mathbf{B}_o$; moreover, $\vdash_{\mathscr{L}} \mathbf{A}_o = \nabla\Delta\mathbf{A}_o$ and $\vdash_{\mathscr{T}\mathscr{E}} \mathbf{B}_o = \Delta\nabla\mathbf{B}_o$. Hence our independence proofs below apply also to $\mathscr{T}\mathscr{E}$.

DEFINITION. $\mathbf{C}_\gamma$ is *contractible* to $\mathbf{D}_\gamma$ ($\mathbf{C}_\gamma$ *contr* $\mathbf{D}_\gamma$) iff $\mathbf{D}_\gamma$ can be obtained from $\mathbf{C}_\gamma$ by a sequence of zero or more applications of the following two rules of λ-conversion:

I. (Alphabetic change of bound variables.) To replace any wf part $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]$ of a wff by $[\lambda \mathbf{y}_\alpha S^{\mathbf{x}_\alpha}_{\mathbf{y}_\alpha} \mathbf{B}_\beta]$, provided that $\mathbf{y}_\alpha$ is not free in $\mathbf{B}_\beta$ and $\mathbf{y}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

II. (λ-contraction.) To replace any wf part $[[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]\mathbf{A}_\alpha]$ of a wff by $\underset{\centerdot}{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta$, provided that $\mathbf{A}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

DEFINITION. $\mathbf{E}_\delta$ is a *KS-combinatorial wff* iff every occurrence of λ in $\mathbf{E}_\delta$ is in a wf part of $\mathbf{E}_\delta$ of the form $K^{\alpha\beta}$ or $S^{\alpha\beta\gamma}$.

$\mathbf{E}_\delta$ is a *KBCW-combinatorial wff* iff every occurrence of λ in $\mathbf{E}_\delta$ is in a wf part of $\mathbf{E}_\delta$ of the form $K^{\alpha\beta}$, $B^{\alpha\beta\gamma}$, $C^{\alpha\beta\gamma}$, or $W^{\alpha\beta}$.

Clearly $K^{\alpha\beta}$, $S^{\alpha\beta\gamma}$, and all primitive constants and variables are *KS*-combinatorial wffs. Also, $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ is such a wff iff $\mathbf{A}_{\alpha\beta}$ and $\mathbf{B}_\beta$ are.

We next show that every wff of $\mathscr{L}$ is convertible to a *KS*-combinatorial wff, and to a *KBCW*-combinatorial wff. This requires only a simple translation into the present context of familiar facts about combinatory logic (see [6], [13], for example).

LEMMA 1. *For any KS-combinatorial wff* $\mathbf{B}_\beta$ *and variable* $\mathbf{x}_\gamma$ *there is a KS-combinatorial wff* $\mathbf{P}_{\beta\gamma}$ *such that* $\mathbf{P}_{\beta\gamma}$ contr $[\lambda \mathbf{x}_\gamma \mathbf{B}_\beta]$.

PROOF. By induction on the number of occurrences of [in $\mathbf{B}_\beta$.

Case a. $\mathbf{B}_\beta$ is $\mathbf{x}_\gamma$. Let $\mathbf{P}_{\gamma\gamma}$ be $S^{\gamma(\gamma\gamma)\gamma} K^{\gamma(\gamma\gamma)} K^{\gamma\gamma}$. Thus $\mathbf{P}_{\gamma\gamma}$ contr $[\lambda z_\gamma \centerdot K^{\gamma(\gamma\gamma)} z_\gamma \centerdot K^{\gamma\gamma} z_\gamma]$ contr $[\lambda z_\gamma z_\gamma]$ contr $[\lambda \mathbf{x}_\gamma \mathbf{B}_\gamma]$.

Case b. $\mathbf{B}_\beta$ does not contain $\mathbf{x}_\gamma$ free. Let $\mathbf{P}_{\beta\gamma}$ be $K^{\beta\gamma}\mathbf{B}_\beta$. Then $\mathbf{P}_{\beta\gamma}$ contr $[\lambda \mathbf{x}_\gamma \mathbf{B}_\beta]$.

Case c. $\mathbf{B}_\beta$ has the form $[\mathbf{D}_{\beta\delta}\mathbf{E}_\delta]$. By inductive hypothesis there are *KS*-combinatorial wffs $\mathbf{G}_{\beta\delta\gamma}$ and $\mathbf{H}_{\delta\gamma}$ such that $\mathbf{G}_{\beta\delta\gamma}$ contr $[\lambda \mathbf{x}_\gamma \mathbf{D}_{\beta\delta}]$ and $\mathbf{H}_{\delta\gamma}$ contr $[\lambda \mathbf{x}_\gamma \mathbf{E}_\delta]$. Let $\mathbf{P}_{\beta\gamma}$ be $[S^{\beta\delta\gamma}\mathbf{G}_{\beta\delta\gamma}\mathbf{H}_{\delta\gamma}]$. Thus

$$\mathbf{P}_{\beta\gamma} \text{ contr } [S^{\beta\delta\gamma}[\lambda \mathbf{x}_\gamma \mathbf{D}_{\beta\delta}][\lambda \mathbf{x}_\gamma \mathbf{E}_\delta]] \text{ contr } [\lambda \mathbf{x}_\gamma \centerdot [\lambda \mathbf{x}_\gamma \mathbf{D}_{\beta\delta}]\mathbf{x}_\gamma \centerdot [\lambda \mathbf{x}_\gamma \mathbf{E}_\delta]\mathbf{x}_\delta] \text{ contr } [\lambda \mathbf{x}_\gamma \mathbf{B}_\beta].$$

Since every *KS*-combinatorial wff $\mathbf{B}_\beta$ falls under at least one of these three cases, this completes the proof of the lemma. □

PROPOSITION 1. *For every wff* $\mathbf{A}_\delta$ *of* $\mathscr{L}$ *there is a KS-combinatorial wff* $\mathbf{P}_\delta$ *such that* $\mathbf{P}_\delta$ contr $\mathbf{A}_\delta$.

PROOF. By induction on the number of occurrences of [in $\mathbf{A}_\delta$.

Case a. $\mathbf{A}_\delta$ is a primitive constant or variable. Let $\mathbf{P}_\delta$ be $\mathbf{A}_\delta$.

Case b. $\mathbf{A}_\delta$ has the form $[\mathbf{D}_{\delta\beta}\mathbf{E}_\beta]$. By inductive hypothesis there are *KS*-combinatorial wffs $\mathbf{D}'_{\delta\beta}$ and $\mathbf{E}'_\beta$ such that $\mathbf{D}'_{\delta\beta}$ contr $\mathbf{D}_{\delta\beta}$ and $\mathbf{E}'_\beta$ contr $\mathbf{E}_\beta$. Let $\mathbf{P}_\delta$ be $[\mathbf{D}'_{\delta\beta}\mathbf{E}'_\beta]$.

Case c. $\mathbf{A}_\delta$ has the form $[\lambda \mathbf{x}_\gamma \mathbf{B}_\beta]$.
By inductive hypothesis there is a *KS*-combinatorial wff $\mathbf{B}'_\beta$ such that $\mathbf{B}'_\beta$ contr $\mathbf{B}_\beta$. Then by Lemma 1 there is a *KS*-combinatorial wff $\mathbf{P}_{\beta\gamma}$ such that $\mathbf{P}_{\beta\gamma}$ contr $[\lambda \mathbf{x}_\gamma \mathbf{B}'_\beta]$. Thus $\mathbf{P}_{\beta\gamma}$ contr $\mathbf{A}_\delta$. □

PROPOSITION 2. *For every wff* $\mathbf{A}_\delta$ *of* $\mathcal{L}$ *there is a KBCW-combinatorial wff* $\mathbf{D}_\delta$ *such that* $\mathbf{D}_\delta$ contr $\mathbf{A}_\delta$.

PROOF. It can be verified that

$$B^{(\alpha\gamma(\beta\gamma))(\alpha\gamma(\beta\gamma)\gamma)(\alpha\beta\gamma)}[B^{(\alpha\gamma(\beta\gamma))(\alpha\gamma\gamma(\beta\gamma))(\alpha\gamma(\beta\gamma)\gamma)}\ [B^{(\alpha\gamma)(\alpha\gamma\gamma)(\beta\gamma)}\,W^{\alpha\gamma}]C^{(\alpha\gamma)(\beta\gamma)\gamma}][B^{(\alpha\gamma(\beta\gamma))(\alpha\beta)\gamma}\,B^{\alpha\beta\gamma}]$$

contr $S^{\alpha\beta\gamma}$. If one replaces $S^{\alpha\beta\gamma}$ by this wff everywhere in the wff $\mathbf{P}_\delta$ of Proposition 1, one obtains the desired wff $\mathbf{D}_\delta$. □

§3. General models for $\mathcal{L}$. We next define the general models for $\mathcal{L}$ by modifying appropriately the definition in [8].

DEFINITION. A *frame* is a collection $\{\mathcal{D}_\alpha\}_\alpha$ of nonempty domains (sets), one for each type symbol α, such that $\mathcal{D}_o = \{\mathfrak{t}, \mathfrak{f}\}$ and $\mathcal{D}_{\alpha\beta}$ is a collection of functions mapping $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$. The members of $\mathcal{D}_o$ are called *truth values* and the members of $\mathcal{D}_\iota$ are called *individuals.*

DEFINITION. Given a frame $\{\mathcal{D}_\alpha\}_\alpha$, an *assignment* (of values in the frame to variables) is a function φ defined on the set of variables of $\mathcal{L}$ such that for each variable $\mathbf{x}_\alpha$, $\varphi \mathbf{x}_\alpha \in \mathcal{D}_\alpha$. Given an assignment φ, a variable $\mathbf{x}_\alpha$, and an element $\mathfrak{z} \in \mathcal{D}_\alpha$, let $(\varphi\colon \mathbf{x}_\alpha/\mathfrak{z})$ be that assignment ψ such that $\psi \mathbf{x}_\alpha = \mathfrak{z}$ and $\psi \mathbf{y}_\beta = \varphi \mathbf{y}_\beta$ if $\mathbf{y}_\beta \neq \mathbf{x}_\alpha$.

If $\mathfrak{h}$ is a function of which $\mathfrak{x}$ is an argument, we write the value of $\mathfrak{h}$ at $\mathfrak{x}$ as $\mathfrak{h}\mathfrak{x}$ or $(\mathfrak{h}\mathfrak{x})$. If $\mathfrak{h}\mathfrak{x}$ is itself a function of which $\mathfrak{y}$ is an argument, we may write $(\mathfrak{h}\mathfrak{x})\mathfrak{y}$ simply as $\mathfrak{h}\mathfrak{x}\mathfrak{y}$, using the convention of association to the left in our meta-language. We shall use dots to denote parentheses in our meta-language in the manner of our convention for brackets in $\mathcal{L}$. We shall also use λ-notation informally in our meta-language. Thus when $\mathfrak{A}$ is an expression of our meta-language involving a variable $\mathfrak{x}$ of our meta-language, then $(\lambda \mathfrak{x}\mathfrak{A})$ shall serve as a name for the function whose domain is the range of the variable $\mathfrak{x}$ and whose value at each argument $\mathfrak{x}$ is $\mathfrak{A}$. In contexts where a frame has been specified, if α is a type symbol it will be understood that $\mathfrak{x}_\alpha$, $\mathfrak{y}_\alpha$, $\mathfrak{z}_\alpha$, etc., range over the domain $\mathcal{D}_\alpha$ of the frame. However, we reserve $\mathfrak{q}_{o\alpha\alpha}$ as a name for the identity relation over $\mathcal{D}_\alpha$; i.e., $\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}_\alpha\mathfrak{y}_\alpha = \mathfrak{t}$ if $\mathfrak{x}_\alpha = \mathfrak{y}_\alpha$, and $\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}_\alpha\mathfrak{y}_\alpha = \mathfrak{f}$ if $\mathfrak{x}_\alpha \neq \mathfrak{y}_\alpha$. We note for future reference that if $\mathfrak{x}_\alpha \in \mathcal{D}_\alpha$, then $\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}_\alpha$ is $\{\mathfrak{x}_\alpha\}$, the unit set whose only member is $\mathfrak{x}_\alpha$.

DEFINITION. A frame $\{\mathcal{D}_\alpha\}_\alpha$ is a *general model* for $\mathcal{L}$ iff there is a binary function $\mathcal{V}$ such that for each assignment φ and wff $\mathbf{A}_\alpha$, $\mathcal{V}_\varphi \mathbf{A}_\alpha \in \mathcal{D}_\alpha$ and the following conditions are satisfied for all assignments φ and all wffs:

(a) $\mathcal{V}_\varphi \mathbf{x}_\alpha = \varphi \mathbf{x}_\alpha$;
(b) $\mathcal{V}_\varphi Q_{o\alpha\alpha} = \mathfrak{q}_{o\alpha\alpha}$;
(c) $\mathcal{V}_\varphi[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta] = (\mathcal{V}_\varphi \mathbf{A}_{\alpha\beta})(\mathcal{V}_\varphi \mathbf{B}_\beta)$;
(d) $\mathcal{V}_\varphi[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] = (\lambda \mathfrak{y}_\alpha \mathcal{V}_{(\varphi:\mathbf{x}_\alpha/\mathfrak{y}_\alpha)}\mathbf{B}_\beta)$.

Remark. Clearly the crucial requirement above is that $\mathcal{V}_\varphi[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \in \mathcal{D}_{\beta\alpha}$. Note that in a general model the function $\mathcal{V}$ is uniquely determined.

DEFINITION. A frame $\{\mathscr{D}_\alpha\}_\alpha$ is a *standard model* for $\mathscr{L}$ iff for all α and β, $\mathscr{D}_{\alpha\beta}$ is the set of all functions from $\mathscr{D}_\beta$ into $\mathscr{D}_\alpha$.

Clearly a standard model is a general model, and is uniquely determined by $\mathscr{D}_\iota$.

A wff $\mathbf{A}_o$ is *valid* in a general model iff $\mathscr{V}_\varphi \mathbf{A}_o = t$ for all assignments φ. It can be shown by an easy modification of the argument in [8] that a wff $\mathbf{A}_o$ is a theorem of $\mathscr{L}$ iff it is valid in every general model. Also, the rule of inference of $\mathscr{L}$ preserves validity in a general model.

DEFINITION. A wff $\mathbf{A}_\alpha$ is *significant in* a frame $\{\mathscr{D}_\alpha\}_\alpha$ iff there is a function $\mathscr{V}$ such that for every assignment φ and for every wf part $\mathbf{B}_\beta$ of $\mathbf{A}_\alpha$ (including $\mathbf{A}_\alpha$ itself), $\mathscr{V}_\varphi \mathbf{B}_\beta \in \mathscr{D}_\beta$, and $\mathscr{V}$ satisfies conditions (a)–(d) (in the definition of general model).

Thus a frame is a general model iff every wff is significant in it.

Before proving the next proposition we state the following lemmas, which can be proved by straightforward induction on the construction of $\mathbf{B}_\beta$.

LEMMA 2. *If* $\mathbf{B}_\beta$ *is significant in a frame and* φ *and* ψ *are assignments which agree on the free variables of* $\mathbf{B}_\beta$, *then* $\mathscr{V}_\varphi \mathbf{B}_\beta = \mathscr{V}_\psi \mathbf{B}_\beta$.

LEMMA 3. *If* $\mathbf{A}_\alpha$ *and* $\mathbf{B}_\beta$ *are significant in a frame and* $\mathbf{A}_\alpha$ *is free for* $\mathbf{x}_\alpha$ *in* $\mathbf{B}_\beta$, *then* $\underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta$ *is significant and for any assignment* φ, $\mathscr{V}_\varphi \underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta = \mathscr{V}_{(\varphi:\mathbf{x}_\alpha/\mathscr{V}_\varphi \mathbf{A}_\alpha)} \mathbf{B}_\beta$.

PROPOSITION 3. *If* $\mathbf{C}_\gamma$ *is significant in a frame and* $\mathbf{C}_\gamma$ contr $\mathbf{D}_\gamma$, *then* $\mathbf{D}_\gamma$ *is significant, and for any assignment* φ, $\mathscr{V}_\varphi \mathbf{C}_\gamma = \mathscr{V}_\varphi \mathbf{D}_\gamma$.

PROOF. Clearly it suffices to prove this proposition for the case where $\mathbf{D}_\gamma$ is obtained from $\mathbf{C}_\gamma$ by a single application of Rule I or II of λ-conversion. In either case the proposition follows easily by induction on the construction of $\mathbf{C}_\gamma$ once one establishes it for the wf part of $\mathbf{C}_\gamma$ to which the rule is actually applied.

Thus in the case of Rule I one may suppose $\mathbf{C}_\gamma$ is $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]$ and $\mathbf{D}_\gamma$ is $[\lambda \mathbf{y}_\alpha \underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{y}_\alpha} \mathbf{B}_\beta]$, where $\mathbf{y}_\alpha$ is not free in $\mathbf{B}_\beta$ and $\mathbf{y}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$. We may assume that $\mathbf{y}_\alpha \neq \mathbf{x}_\alpha$. $\mathbf{B}_\beta$ is significant since $\mathbf{C}_\gamma$ is, so by Lemma 3, $\underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{y}_\alpha} \mathbf{B}_\beta$ is significant.

Note that for any $\mathfrak{d}_\alpha \in \mathscr{D}_\alpha$ we have $\mathscr{V}_{(\varphi:\mathbf{y}_\alpha/\mathfrak{d}_\alpha)} \mathbf{y}_\alpha = \mathfrak{d}_\alpha$ so

$$\begin{aligned} \mathscr{V}_{(\varphi:\mathbf{y}_\alpha/\mathfrak{d}_\alpha)} \underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{y}_\alpha} \mathbf{B}_\beta &= \mathscr{V}_{((\varphi:\mathbf{y}_\alpha/\mathfrak{d}_\alpha):\mathbf{x}_\alpha/\mathfrak{d}_\alpha)} \mathbf{B}_\beta \quad \text{(by Lemma 3)} \\ &= \mathscr{V}_{(\varphi:\mathbf{x}_\alpha/\mathfrak{d}_\alpha)} \mathbf{B}_\beta \quad \text{(by Lemma 2).} \end{aligned}$$

Hence

$$\mathscr{V}_\varphi \mathbf{C}_\gamma = (\lambda \mathfrak{d}_\alpha \mathscr{V}_{(\varphi:\mathbf{x}_\alpha/\mathfrak{d}_\alpha)} \mathbf{B}_\beta) = (\lambda \mathfrak{d}_\alpha \mathscr{V}_{(\varphi:\mathbf{y}_\alpha/\mathfrak{d}_\alpha)} \underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{y}_\alpha} \mathbf{B}_\beta),$$

which is the desired value for $\mathscr{V}_\varphi \mathbf{D}_\gamma$, so $\mathbf{D}_\gamma$ is significant and $\mathscr{V}_\varphi \mathbf{C}_\gamma = \mathscr{V}_\varphi \mathbf{D}_\gamma$.

In the case of Rule II one may suppose that $\mathbf{C}_\gamma$ is $[[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]\mathbf{A}_\alpha]$ and $\mathbf{D}_\gamma$ is $\underset{\bullet}{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta$, where $\mathbf{A}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$. Since $\mathbf{C}_\gamma$ is significant, $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are, so by Lemma 3, $\mathbf{D}_\gamma$ is significant. Also $\mathscr{V}_\varphi \mathbf{C}_\gamma = (\mathscr{V}_\varphi [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]) \mathscr{V}_\varphi \mathbf{A}_\alpha = \mathscr{V}_{(\varphi:\mathbf{x}_\alpha/\mathscr{V}_\varphi \mathbf{A}_\alpha)} \mathbf{B}_\beta = \mathscr{V}_\varphi \mathbf{D}_\gamma$ by Lemma 3. □

Remark. It is not true that if $\mathbf{C}_\gamma$ is significant in a frame and $\mathbf{D}_\gamma$ contr $\mathbf{C}_\gamma$, then $\mathbf{D}_\gamma$ must be significant. For x_ι is always significant, but $[[\lambda x_\iota x_\iota] x_\iota]$ might not be.

PROPOSITION 4. *For any frame* $\mathscr{M}$, *the following conditions are equivalent*:

(a) $\mathscr{M}$ *is a general model for* $\mathscr{L}$.

(b) *every KS-combinatorial wff of* $\mathscr{L}$ *is significant in* $\mathscr{M}$.

(c) *Every KBCW-combinatorial wff of* $\mathscr{L}$ *is significant in* $\mathscr{M}$.

PROOF. By Propositions 1, 2, and 3. □

We now rephrase condition (b) to obtain a simple criterion for a frame to be a general model.

THEOREM 1. *A frame $\{\mathscr{D}_\alpha\}_\alpha$ is a general model for $\mathscr{L}$ iff it satisfies all of the following conditions (for all type symbols α, β, γ):*

(a) $\mathfrak{q}_{o\alpha\alpha} \in \mathscr{D}_{o\alpha\alpha}$.

(b) *For all* $\mathfrak{x}_\alpha \in \mathscr{D}_\alpha$, $(\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) \in \mathscr{D}_{\alpha\beta}$.

(c) $(\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) \in \mathscr{D}_{\alpha\beta\alpha}$.

(d) *For all* $\mathfrak{x}_{\alpha\beta\gamma} \in \mathscr{D}_{\alpha\beta\gamma}$ *and* $\mathfrak{y}_{\beta\gamma} \in \mathscr{D}_{\beta\gamma}$, $(\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}_{\alpha\beta\gamma}\mathfrak{z}_\gamma\centerdot\mathfrak{y}_{\beta\gamma}\mathfrak{z}_\gamma) \in \mathscr{D}_{\alpha\gamma}$.

(e) *For all* $\mathfrak{x}_{\alpha\beta\gamma} \in \mathscr{D}_{\alpha\beta\gamma}$, $(\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}_{\alpha\beta\gamma}\mathfrak{z}_\gamma\centerdot\mathfrak{y}_{\beta\gamma}\mathfrak{z}_\gamma) \in \mathscr{D}_{\alpha\gamma(\beta\gamma)}$.

(f) $(\lambda\mathfrak{x}_{\alpha\beta\gamma}\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}_{\alpha\beta\gamma}\mathfrak{z}_\gamma\centerdot\mathfrak{y}_{\beta\gamma}\mathfrak{z}_\gamma) \in \mathscr{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$.

PROOF. Clearly if the frame is a general model, the conditions (a)–(f) must be satisfied. To show they are sufficient, we show they imply condition (b) of Proposition 4. Since every variable is significant in every frame, and a wff $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ is significant in a frame iff $\mathbf{A}_{\alpha\beta}$ and $\mathbf{B}_\beta$ are, it suffices to show that the wffs $Q_{o\alpha\alpha}$, $K^{\alpha\beta}$, and $S^{\alpha\beta\gamma}$ are significant in the frame. This is assured by conditions (a)–(f). (We note that condition (a) implies that for all $\mathfrak{x}_\alpha \in \mathscr{D}_\alpha$, $(\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}_\alpha) \in \mathscr{D}_{o\alpha}$.) $\square$

Remark. We leave it to the reader to state the analogous theorem using $B^{\alpha\beta\gamma}$, $C^{\alpha\beta\gamma}$, and $W^{\alpha\beta}$ in place of $S^{\alpha\beta\gamma}$. Such a theorem may be useful since $B^{\alpha\beta\gamma}$, $C^{\alpha\beta\gamma}$, and $W^{\alpha\beta}$ are each conceptually simpler than $S^{\alpha\beta\gamma}$.

§4. The Axiom of Descriptions. We remind the reader that the Axiom of Descriptions is

$$\text{(D)} \qquad \exists i_{\iota(o\iota)}\forall x_\iota\centerdot i_{\iota(o\iota)}[Q_{o\iota\iota}x_\iota] = x_\iota.$$

THEOREM 2. D *is not a theorem of* $\mathscr{L}$.

PROOF. We partition the type symbols into two sets, $\mathscr{T}_o$ and $\mathscr{T}_\iota$ as follows: $o \in \mathscr{T}_o$ but $o \notin \mathscr{T}_\iota$; $\iota \in \mathscr{T}_\iota$ but $\iota \notin \mathscr{T}_o$; $(\alpha\beta)$ is in whichever set contains α. We then let $\mathscr{C} = \{(\alpha\beta) \mid \alpha \in \mathscr{T}_\iota \text{ and } \beta \in \mathscr{T}_o\}$.

We next define a frame $\mathscr{M} = \{\mathscr{D}_\alpha\}_\alpha$ by induction on α. $\mathscr{D}_o = \{\mathfrak{t}, \mathfrak{f}\}$. $\mathscr{D}_\iota = \{\mathfrak{m}, \mathfrak{n}\}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are distinct individuals. (Actually $\mathscr{D}_\iota$ may be taken to have any cardinality greater than one.) If $(\alpha\beta) \in \mathscr{C}$, $\mathscr{D}_{\alpha\beta}$ is the set of all constant functions (i.e., functions with the same value for all arguments) from $\mathscr{D}_\beta$ into $\mathscr{D}_\alpha$. If $(\alpha\beta) \notin \mathscr{C}$, $\mathscr{D}_{\alpha\beta}$ is the set of all functions from $\mathscr{D}_\beta$ into $\mathscr{D}_\alpha$.

We next use Theorem 1 to verify that $\mathscr{M}$ is a general model for $\mathscr{L}$.

(a) Since $(o\alpha) \notin \mathscr{C}$ and $(o\alpha\alpha) \notin \mathscr{C}$, $\mathfrak{q}_{o\alpha\alpha} \in \mathscr{D}_{o\alpha\alpha}$.

(b) $(\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha)$ is a constant function, and so is in $\mathscr{D}_{\alpha\beta}$.

(c) $(\alpha\beta\alpha) \notin \mathscr{C}$ whether $\alpha \in \mathscr{T}_\iota$ or $\alpha \in \mathscr{T}_o$. Hence $(\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) \in \mathscr{D}_{\alpha\beta\alpha}$.

(d) We need consider only the case where $(\alpha\gamma) \in \mathscr{C}$.

We must show that if $\mathfrak{x} \in \mathscr{D}_{\alpha\beta\gamma}$ and $\mathfrak{y} \in \mathscr{D}_{\beta\gamma}$, then $(\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}\mathfrak{z}_\gamma\centerdot\mathfrak{y}\mathfrak{z}_\gamma)$ is a constant function. So we let $\mathfrak{z}^1, \mathfrak{z}^2 \in \mathscr{D}_\gamma$ and show that $(\mathfrak{x}\mathfrak{z}^1\centerdot\mathfrak{y}\mathfrak{z}^1) = (\mathfrak{x}\mathfrak{z}^2\centerdot\mathfrak{y}\mathfrak{z}^2)$. Since $\alpha \in \mathscr{T}_\iota$ and $\gamma \in \mathscr{T}_o$, $(\alpha\beta\gamma) \in \mathscr{C}$ so $\mathfrak{x}\mathfrak{z}^1 = \mathfrak{x}\mathfrak{z}^2$.

Case 1. $\beta \in \mathscr{T}_\iota$. Then $(\beta\gamma) \in \mathscr{C}$ so $\mathfrak{y}\mathfrak{z}^1 = \mathfrak{y}\mathfrak{z}^2$ so $(\mathfrak{x}\mathfrak{z}^1\centerdot\mathfrak{y}\mathfrak{z}^1) = (\mathfrak{x}\mathfrak{z}^2\centerdot\mathfrak{y}\mathfrak{z}^2)$.

Case 2. $\beta \in \mathscr{T}_o$. Then $(\alpha\beta) \in \mathscr{C}$. Since $\mathfrak{x}\mathfrak{z}^1 = \mathfrak{x}\mathfrak{z}^2 \in \mathscr{D}_{\alpha\beta}$, $\mathfrak{x}\mathfrak{z}^1(\mathfrak{y}\mathfrak{z}^1) = \mathfrak{x}\mathfrak{z}^2(\mathfrak{y}\mathfrak{z}^2)$.

(e) Suppose $(\alpha\gamma(\beta\gamma)) \in \mathscr{C}$ and $\mathfrak{x} \in \mathscr{D}_{\alpha\beta\gamma}$.

We must show that $(\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}\mathfrak{z}_\gamma\centerdot\mathfrak{y}_{\beta\gamma}\mathfrak{z}_\gamma) \in \mathscr{D}_{\alpha\gamma(\beta\gamma)}$. So suppose $\mathfrak{y}^1, \mathfrak{y}^2 \in \mathscr{D}_{\beta\gamma}$. We must show that $(\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}\mathfrak{z}_\gamma\centerdot\mathfrak{y}^1\mathfrak{z}_\gamma) = (\lambda\mathfrak{z}_\gamma\centerdot\mathfrak{x}\mathfrak{z}_\gamma\centerdot\mathfrak{y}^2\mathfrak{z}_\gamma)$. To do this we show that for an arbitrary $\mathfrak{z} \in \mathscr{D}_\gamma$, $(\mathfrak{x}\mathfrak{z}\centerdot\mathfrak{y}^1\mathfrak{z}) = (\mathfrak{x}\mathfrak{z}\centerdot\mathfrak{y}^2\mathfrak{z})$. But $\alpha \in \mathscr{T}_\iota$ and $\beta \in \mathscr{T}_o$ so $(\alpha\beta) \in \mathscr{C}$ and $\mathfrak{x}\mathfrak{z} \in \mathscr{D}_{\alpha\beta}$, which contains only constant functions. Hence $\mathfrak{x}\mathfrak{z}(\mathfrak{y}^1\mathfrak{z}) = \mathfrak{x}\mathfrak{z}(\mathfrak{y}^2\mathfrak{z})$.

(f) $(\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)) \notin \mathscr{C}$ whether $\alpha \in \mathscr{T}_\iota$ or $\alpha \in \mathscr{T}_o$, so

$$(\lambda \mathfrak{x}_{\alpha\beta\gamma}\lambda \mathfrak{y}_{\beta\gamma}\lambda \mathfrak{z}_\gamma . \mathfrak{x}_{\alpha\beta\gamma}\mathfrak{z}_\gamma . \mathfrak{y}_{\beta\gamma}\mathfrak{z}_\gamma) \in \mathscr{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}.$$

Now $\mathfrak{q}_{o\iota\iota}\mathfrak{m}$ and $\mathfrak{q}_{o\iota\iota}\mathfrak{n}$ are elements of $\mathscr{D}_{o\iota}$, so in order that D be valid in $\mathscr{M}$ there must be a function $\mathfrak{h} \in \mathscr{D}_{\iota(o\iota)}$ such that $\mathfrak{h}(\mathfrak{q}_{o\iota\iota}\mathfrak{m}) = \mathfrak{m}$ and $\mathfrak{h}(\mathfrak{q}_{o\iota\iota}\mathfrak{n}) = \mathfrak{n}$. However, $(\iota(o\iota)) \in \mathscr{C}$, so there is no such function in $\mathscr{D}_{\iota(o\iota)}$. Thus D is not valid in the general model $\mathscr{M}$, and so is not a theorem of $\mathscr{L}$. □

The idea behind the following theorem is contained in [9], but the proof is short, so we give it here.

THEOREM 3. *Let $\mathscr{M} = \{\mathscr{D}_\alpha\}_\alpha$ be a general model for $\mathscr{L}$ in which $\mathscr{D}_\iota$ is finite. Then $\mathscr{M}$ is a standard model iff* D *is valid in $\mathscr{M}$.*

PROOF. The domains $\mathscr{D}_\alpha$ must, of course, all be finite. If $\mathscr{M}$ is standard one can enumerate the elements in $\mathscr{D}_{\iota(o\iota)}$ to see that D is valid in $\mathscr{M}$.

Suppose D is valid in $\mathscr{M}$. We show that $\mathscr{D}_{\alpha\beta}$ must contain all functions from $\mathscr{D}_\beta$ to $\mathscr{D}_\alpha$. So let $\mathfrak{g}$ be any such function. Let $\mathscr{D}_\beta = \{\mathfrak{m}_\beta^1, \cdots, \mathfrak{m}_\beta^k\}$. By the methods mentioned in §1 one sees that there must be a description operator $\mathfrak{h}_{\alpha(o\alpha)} \in \mathscr{D}_{\alpha(o\alpha)}$ such that for each $\mathfrak{n}_\alpha \in \mathscr{D}_\alpha$, $\mathfrak{h}_{\alpha(o\alpha)}[\mathfrak{q}_{o\alpha\alpha}\mathfrak{n}_\alpha] = \mathfrak{n}_\alpha$. Let φ be an assignment with values on the variables $i_{\alpha(o\alpha)}, w_\beta^1, \cdots, w_\beta^k, \cdots, z_\alpha^1, \cdots, z_\alpha^k$ as follows: $\varphi i_{\alpha(o\alpha)} = \mathfrak{h}_{\alpha(o\alpha)}$, $\varphi w_\beta^1 = \mathfrak{m}_\beta^1, \cdots, \varphi w_\beta^k = \mathfrak{m}_\beta^k$, $\varphi z_\alpha^1 = \mathfrak{g}\mathfrak{m}_\beta^1, \cdots$, and $\varphi z_\alpha^k = \mathfrak{g}\mathfrak{m}_\beta^k$. Then

$$\mathfrak{g} = \mathscr{V}_\varphi[\lambda x_\beta . i_{\alpha(o\alpha)} . \lambda y_\alpha . [x_\beta = w_\beta^1 \wedge y_\alpha = z_\alpha^1] \vee \cdots \vee [x_\beta = w_\beta^k \wedge y_\alpha = z_\alpha^k]],$$

so $\mathfrak{g}$ must be in $\mathscr{D}_{\alpha\beta}$ since $\mathscr{M}$ is a general model. □

Remark. Theorem 3 provides a strong argument for always assuming the Axiom of Descriptions. If one does this by introducing a description operator $\iota_{\iota(o\iota)}$ and modifies the definition of general model in the natural way by introducing an appropriate requirement for $\mathscr{V}_\varphi\iota_{\iota(o\iota)}$ (thus getting closer to the definition in [8]), one can again prove that the theorems are precisely the wffs valid in all general models. Thus it appears that the language Q_o of [2] is more natural than $\mathscr{L}$.

§5. The Axiom of Choice. The Axiom of Choice (for individuals) is

(E) $$\exists i_{\iota(o\iota)}\forall p_{o\iota} . \exists x_\iota p_{o\iota}x_\iota \supset p_{o\iota} . i_{\iota(o\iota)}p_{o\iota}.$$

Clearly [E ⊃ D] is a theorem of $\mathscr{L}$. We use the Fraenkel-Mostowski method to show that its converse is not. Thus E is not provable in $\mathscr{L}$, even if D is added to the list of axioms.

We first establish the following lemma, which is fundamental for applications of the Fraenkel-Mostowski method to $\mathscr{L}$. The lemma is true but trivial if $\mathscr{D}_\iota$ is finite, since in this case the conditions on $\mathscr{F}$ assure that $\mathscr{M}$ will be the standard model over $\mathscr{D}_\iota$. We use $\circ$ to denote the composition of functions.

LEMMA 4. *Let $\mathscr{D}_\iota$ be an infinite set of individuals and P a set of permutations σ of $\mathscr{D}_\iota$ such that $\sigma \circ \sigma = (\lambda \mathfrak{x}_\iota \mathfrak{x}_\iota)$. Let $\mathscr{F}$ be a family of subsets of P such that*

(a) *for each $\mathfrak{m} \in \mathscr{D}_\iota$ there is a set $K \in \mathscr{F}$ such that $\sigma\mathfrak{m} = \mathfrak{m}$ for all $\sigma \in K$, and*

(b) *for all $H, K \in \mathscr{F}$ there is a set $J \in \mathscr{F}$ such that $J \subseteq H \cap K$.*

Let the frame $\mathscr{M} = \{\mathscr{D}_\alpha\}_\alpha$ be defined, and each permutation $\sigma \in P$ be extended to a permutation of $\mathscr{D}_\alpha$ (which we may denote by σ^α) such that $\sigma^\alpha \circ \sigma^\alpha = (\lambda \mathfrak{x}_\alpha \mathfrak{x}_\alpha)$ for each α, as follows by induction on α:

$$\mathscr{D}_o = \{\mathfrak{t}, \mathfrak{f}\}; \qquad \sigma^o = (\lambda \mathfrak{x}_o \mathfrak{x}_o) \quad \textit{for all } \sigma \in P.$$

Given $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$ and any function $\mathfrak{h}$ from $\mathscr{D}_\beta$ into $\mathscr{D}_\alpha$, let $\sigma\mathfrak{h} = \sigma^\alpha \circ \mathfrak{h} \circ \sigma^\beta$, and let $\mathscr{D}_{\alpha\beta}$ be the set of all functions $\mathfrak{h}$ from $\mathscr{D}_\beta$ into $\mathscr{D}_\alpha$ such that there is some $K \in \mathscr{F}$ such that $\sigma\mathfrak{h} = \mathfrak{h}$ for all $\sigma \in K$.

Then $\mathscr{M}$ is a general model for $\mathscr{L}$ in which D *is valid.*

PROOF. For notational convenience, if $\mathfrak{h} \in \mathscr{D}_\gamma$ we let $K_\mathfrak{h}$ denote some $K \in \mathscr{F}$ such that $\sigma\mathfrak{h} = \mathfrak{h}$ for all $\sigma \in K$. Clearly such a set $K_\mathfrak{h}$ always exists. Note that if $\mathfrak{h} \in \mathscr{D}_{\alpha\beta}$ and $\mathfrak{x} \in \mathscr{D}_\beta$, then $(\sigma^{\alpha\beta}\mathfrak{h})(\sigma^\beta\mathfrak{x}) = \sigma^\alpha(\mathfrak{h}\mathfrak{x})$.

We use Theorem 1 to verify that $\mathscr{M}$ is a general model.

(a) If $\mathfrak{x}, \mathfrak{y} \in \mathscr{D}_\alpha$ and $\sigma \in K_\mathfrak{x}$ then $(\sigma^{o\alpha}.\mathfrak{q}_{o\alpha\alpha}\mathfrak{x})\mathfrak{y} = \sigma^o(\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}.\sigma\mathfrak{y}) = \mathfrak{q}_{o\alpha\alpha}\mathfrak{x}(\sigma\mathfrak{y})$, which is t iff $\sigma\mathfrak{y} = \mathfrak{x} = \sigma\mathfrak{x}$ iff $\mathfrak{x} = \mathfrak{y}$, so $(\sigma^{o\alpha}.\mathfrak{q}_{o\alpha\alpha}\mathfrak{x})\mathfrak{y} = (\mathfrak{q}_{o\alpha\alpha}\mathfrak{x})\mathfrak{y}$ for all $\mathfrak{y} \in \mathscr{D}_\alpha$ so $\sigma(\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}) = \mathfrak{q}_{o\alpha\alpha}\mathfrak{x}$. Thus $(\mathfrak{q}_{o\alpha\alpha}\mathfrak{x}) \in \mathscr{D}_{o\alpha}$, and $\mathfrak{q}_{o\alpha\alpha}$ at least maps $\mathscr{D}_\alpha$ into $\mathscr{D}_{o\alpha}$. Also, for any $\sigma \in P$, $(\sigma\mathfrak{q}_{o\alpha\alpha})\mathfrak{x}\mathfrak{y} = (\sigma^{o\alpha}.\mathfrak{q}_{o\alpha\alpha}.\sigma^\alpha\mathfrak{x})\mathfrak{y} = \mathfrak{q}_{o\alpha\alpha}(\sigma^\alpha\mathfrak{x})(\sigma^\alpha\mathfrak{y})$, which is t iff $\sigma\mathfrak{x} = \sigma\mathfrak{y}$ iff $\mathfrak{x} = \mathfrak{y}$, so $\sigma\mathfrak{q}_{o\alpha\alpha} = \mathfrak{q}_{o\alpha\alpha}$ and $\mathfrak{q}_{o\alpha\alpha} \in \mathscr{D}_{o\alpha\alpha}$.

(b) For any $\mathfrak{x}_\alpha \in \mathscr{D}_\alpha$ and $\sigma \in K_\mathfrak{x}$, $\sigma(\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) = (\lambda\mathfrak{y}_\beta\sigma\mathfrak{x}_\alpha) = (\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha)$, so $(\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) \in \mathscr{D}_{\alpha\beta}$.

(c) For any $\sigma \in P$, $\sigma(\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) = (\lambda\mathfrak{x}_\alpha\sigma.\lambda\mathfrak{y}_\beta\sigma\mathfrak{x}_\alpha) = (\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\sigma\sigma\mathfrak{x}_\alpha) = (\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha)$, so $(\lambda\mathfrak{x}_\alpha\lambda\mathfrak{y}_\beta\mathfrak{x}_\alpha) \in \mathscr{D}_{\alpha\beta\alpha}$.

Before checking (d)–(f) we observe that if $\mathfrak{x} \in \mathscr{D}_{\alpha\beta\gamma}$, $\mathfrak{y} \in \mathscr{D}_{\beta\gamma}$, and $\mathfrak{z} \in \mathscr{D}_\gamma$, then $\sigma^\alpha.(\sigma\mathfrak{x})(\sigma\mathfrak{z}).(\sigma\mathfrak{y}).\sigma\mathfrak{z} = \sigma^\alpha.(\sigma^{\alpha\beta}.\mathfrak{x}\mathfrak{z}).\sigma^\beta.\mathfrak{y}\mathfrak{z} = \sigma^\alpha.\sigma^\alpha.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z} = \mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}$.

(d) Suppose $\mathfrak{x} \in \mathscr{D}_{\alpha\beta\gamma}$ and $\mathfrak{y} \in \mathscr{D}_{\beta\gamma}$. Let J be a member of $\mathscr{F}$ such that $J \subseteq K_\mathfrak{x} \cap K_\mathfrak{y}$. For any $\sigma \in J$, $\sigma\mathfrak{x} = \mathfrak{x}$ and $\sigma\mathfrak{y} = \mathfrak{y}$ so $\sigma(\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}) = \sigma(\lambda\mathfrak{z}_\gamma.(\sigma\mathfrak{x})\mathfrak{z}.(\sigma\mathfrak{y})\mathfrak{z}) = (\lambda\mathfrak{z}_\gamma.\sigma^\alpha.(\sigma\mathfrak{x})(\sigma\mathfrak{z}).(\sigma\mathfrak{y}).\sigma\mathfrak{z}) = (\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z})$, which must therefore be in $\mathscr{D}_{\alpha\gamma}$.

(e) If $\mathfrak{x} \in \mathscr{D}_{\alpha\beta\gamma}$ and $\sigma \in K_\mathfrak{x}$, then

$$\begin{aligned}\sigma(\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}) &= \sigma(\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.(\sigma\mathfrak{x})\mathfrak{z}.\mathfrak{y}\mathfrak{z})\\ &= (\lambda\mathfrak{y}_{\beta\gamma}.\sigma.\lambda\mathfrak{z}_\gamma.(\sigma\mathfrak{x})\mathfrak{z}.(\sigma\mathfrak{y})\mathfrak{z})\\ &= (\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\sigma^\alpha.(\sigma\mathfrak{x})(\sigma\mathfrak{z}).(\sigma\mathfrak{y}).\sigma\mathfrak{z})\\ &= (\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}),\end{aligned}$$

which must therefore be in $\mathscr{D}_{\alpha\gamma(\beta\gamma)}$.

(f) For any $\sigma \in P$,

$$\begin{aligned}\sigma(\lambda\mathfrak{x}_{\alpha\beta\gamma}\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}) &= (\lambda\mathfrak{x}_{\alpha\beta\gamma}\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\sigma^\alpha.(\sigma\mathfrak{x})(\sigma\mathfrak{z}).(\sigma\mathfrak{y}).\sigma\mathfrak{z})\\ &= (\lambda\mathfrak{x}_{\alpha\beta\gamma}\lambda\mathfrak{y}_{\beta\gamma}\lambda\mathfrak{z}_\gamma.\mathfrak{x}\mathfrak{z}.\mathfrak{y}\mathfrak{z}),\end{aligned}$$

which must therefore be in $\mathscr{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$. Thus $\mathscr{M}$ is a general model for $\mathscr{L}$.

We next verify that D is valid in $\mathscr{M}$. Let $\mathfrak{n} \in \mathscr{D}_\iota$. We shall construct a description operator $\mathfrak{h}$ mapping $\mathscr{D}_{o\iota}$ to $\mathscr{D}_\iota$ as follows. For each unit set $\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota$, we let $\mathfrak{h}(\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota) = \mathfrak{x}_\iota$. If $\mathfrak{g} \in \mathscr{D}_{o\iota}$ is not a unit set, let $\mathfrak{h}\mathfrak{g} = \mathfrak{n}$. Now we verify that $\mathfrak{h} \in \mathscr{D}_{\iota(o\iota)}$. Let $\sigma \in K_\mathfrak{n}$. For each unit set $\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota$, $\sigma(\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota) = (\sigma\mathfrak{q}_{o\iota\iota})(\sigma\mathfrak{x}_\iota) = \mathfrak{q}_{o\iota\iota}(\sigma\mathfrak{x}_\iota)$, so $(\sigma\mathfrak{h})(\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota) = \sigma(\mathfrak{h}.\sigma.\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota) = \sigma(\mathfrak{h}.\mathfrak{q}_{o\iota\iota}.\sigma\mathfrak{x}_\iota) = \sigma\sigma\mathfrak{x}_\iota = \mathfrak{x}_\iota = \mathfrak{h}(\mathfrak{q}_{o\iota\iota}\mathfrak{x}_\iota)$. If $\mathfrak{g}_{o\iota}$ is not a unit set, then $\sigma\mathfrak{g}_{o\iota}$ (i.e. $\mathfrak{g}_{o\iota} \circ \sigma^\iota$) is not either, so $(\sigma\mathfrak{h})\mathfrak{g}_{o\iota} = \sigma(\mathfrak{h}.\sigma\mathfrak{g}_{o\iota}) = \sigma\mathfrak{n} = \mathfrak{n} = \mathfrak{h}\mathfrak{g}_{o\iota}$. Thus $\sigma\mathfrak{h} = \mathfrak{h}$, and $\mathfrak{h} \in \mathscr{D}_{\iota(o\iota)}$. It is now easy to see that D is valid in $\mathscr{M}$. □

THEOREM 4. [D ⊃ E] *is not a theorem of $\mathscr{L}$.*

PROOF. Let $\mathscr{J}$ be an infinite index set and for all $j \in \mathscr{J}$ let $\mathfrak{m}^j$ and $\mathfrak{n}^j$ be distinct individuals, so chosen that $\mathfrak{m}^j \neq \mathfrak{m}^i$ and $\mathfrak{n}^j \neq \mathfrak{n}^i$ if $j \neq i$. Let $\mathscr{D}_\iota = \{\mathfrak{m}^j \mid j \in \mathscr{J}\} \cup \{\mathfrak{n}^j \mid j \in \mathscr{J}\}$. Let P be the set of all mappings σ from $\mathscr{D}_\iota$ to $\mathscr{D}_\iota$ such that for all $j \in \mathscr{J}$, $\sigma\mathfrak{m}^j = \mathfrak{m}^j$ and $\sigma\mathfrak{n}^j = \mathfrak{n}^j$, or $\sigma\mathfrak{m}^j = \mathfrak{n}^j$ and $\sigma\mathfrak{n}^j = \mathfrak{m}^j$. Thus for each $\sigma \in P$ we have $\sigma \circ \sigma = (\lambda\mathfrak{x}_\iota\mathfrak{x}_\iota)$. Let $\mathscr{F}$ be the family of all subsets K of P such that there is a finite subset $\mathscr{I}$ of $\mathscr{J}$ such that $K = \{\sigma \in P \mid \text{for all } j \in \mathscr{I}, \sigma\mathfrak{m}^j = \mathfrak{m}^j \text{ and } \sigma\mathfrak{n}^j = \mathfrak{n}^j\}$. It is easily checked that $\mathscr{F}$ satisfies the conditions of Lemma 4, so let $\mathscr{M}$ be the general model constructed as in Lemma 4.

We must see that E is false in $\mathscr{M}$. Suppose it were true. Then there would be a choice function $\mathfrak{h} \in \mathscr{D}_{\iota(o\iota)}$ such that for every nonempty set $\mathfrak{g} \in \mathscr{D}_{o\iota}$, $\mathfrak{h}\mathfrak{g}$ is in $\mathfrak{g}$, i.e., $\mathfrak{g}(\mathfrak{h}\mathfrak{g}) = \mathfrak{t}$. For each $j \in \mathscr{J}$, let $\mathfrak{g}^j = (\lambda \mathfrak{x}_\iota . \mathfrak{x}_\iota = \mathfrak{m}^j \text{ or } \mathfrak{x}_\iota = \mathfrak{n}^j)$, i.e., $\mathfrak{g}^j = \{\mathfrak{m}^j, \mathfrak{n}^j\}$. It is easy to see that $\sigma\mathfrak{g}^j = \mathfrak{g}^j$ for all $\sigma \in P$, so each $\mathfrak{g}^j \in \mathscr{D}_{o\iota}$. Now for any $K \in \mathscr{F}$ there is some $j \in \mathscr{J}$ which is not in the finite subset of $\mathscr{J}$ which determines K, and hence some $\sigma \in K$ such that $\sigma\mathfrak{m}^j = \mathfrak{n}^j$ and $\sigma\mathfrak{n}^j = \mathfrak{m}^j$. Then $(\sigma\mathfrak{h})\mathfrak{g}^j = \sigma(\mathfrak{h} . \sigma\mathfrak{g}^j) = \sigma(\mathfrak{h}\mathfrak{g}^j) \neq \mathfrak{h}\mathfrak{g}^j$, so $\sigma\mathfrak{h} \neq \mathfrak{h}$. Thus there can be no choice function $\mathfrak{h} \in \mathscr{D}_{\iota(o\iota)}$, so E is false in $\mathscr{M}$.

Thus $[\mathrm{D} \supset \mathrm{E}]$ is not valid in the general model $\mathscr{M}$ and so is not a theorem of $\mathscr{L}$. □

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CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213