

On the building blocks of mathematical logic

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These ideas were presented before the Göttingen Mathematical Society by Schönfinkel on 7 December 1920 but came to be written up for publication only in March 1924, by Heinrich Behmann. The last three paragraphs are given over to supplementary remarks of Behmann's own.

The initial aim of the paper is reduction of the number of primitive notions of logic. The economy that Sheffer's stroke function had wrought in the propositional calculus is here extended to the predicate calculus, in the form of a generalized copula " U " of mutual exclusiveness. Then presently the purpose deepens. It comes to be nothing less than the general elimination of variables.

Examples of how to eliminate variables had long been known in logic. The inclusion notation " $F \subseteq G$ " gets rid of the universally quantified " x " of " $(x)(Fx \supset Gx)$ ". The notation " $F | G$ " of relative product gets rid of an existentially quantified " x ", since " $(\exists x)(Fyx \cdot Gxz)$ " becomes " $(F | G)yz$ ". These devices and others figured already in Peirce's 1870 algebra of absolute and relative terms, thus even antedating any coherent logic of the variable itself, for such a logic, quantification theory, came only in Frege's work of 1879. The algebra of absolute and relative terms, however, or of classes and relations, is not rich enough to do the work of quantifiers and their variables altogether (see above, p. 229).

Schönfinkel's notions, which do suffice

to do the work of quantifiers and their variables altogether, go far beyond the algebra of classes and relations, for that algebra makes no special provision for classes of classes, relations of classes, classes of relations, or relations of relations. Schönfinkel's notions provide for these things, and for the whole sweep of abstract set theory. The crux of the matter is that Schönfinkel lets functions stand as arguments.

For Schönfinkel, substantially as for Frege,^a classes are special sorts of functions. They are propositional functions, functions whose values are truth values. All functions, propositional and otherwise, are for Schönfinkel one-place functions, thanks to the following ingenious device (which was anticipated by Frege (1893, § 36)). Where F is what we would ordinarily call a two-place function, Schönfinkel reconstrues it by treating Fxy not as $F(x, y)$ but as $(Fx)y$. Thus F becomes a one-place function whose value is a one-place function. The same trick applies to functions of three or more places; thus " $Fxyz$ " is taken as " $((Fx)y)z$ ". In particular, a dyadic relation, as an ostensibly two-place propositional function, thus becomes a one-place function whose value is a class. An example is U , above, where " UFG " means that F and G are exclusive classes; UFG

^a See Frege 1879, § 9. What prompts my qualification "substantially" is that in later writings Frege draws a philosophical distinction between a function and its *Wertverlauf*.

becomes $(UF)G$, so that U becomes a function whose value, for a class F as argument, is the class UF of all classes G that share no members with F . Or better, using general variables " x ", " y ", and so forth hereafter for classes and other functions and all other things as well, we may say that U is the function whose value Ux is the class of all classes that share no members with x .

Schönfinkel assumes one operation, that of application of a function to an argument. It is expressed by juxtaposition as in " Ux " above, or " $(Ux)y$ ", or in general " zy ". Also he assumes three specific functions, as objects: U above, C , and S . C is the *constancy* function, such that $(Cx)y$ is always x , and S is the *fusion* function, such that $((Sx)y)z$ is always $(xz)(yz)$. Any sentence that can be built up of truth functions and quantification and the " ϵ " of class membership can be translated into a sentence built up purely by the application operation, purely from " C ", " S ", " U ", and whatever free variables the given sentence may have had. This is made evident in the course of the paper. The elimination of quantification and bound variables is thus complete. Since sentences with free variables are wanted finally only as ingredients of closed sentences, the notion of variables may indeed be said at this point to have been analyzed away altogether. All we have is C , S , U , and application.

Variables seem to survive in rules of transformation, as when $(Cx)y$ is equated to x , and $((Sx)y)z$ to $(xy)(xz)$. But here the variables may be seen as schematic letters for metalogical exposition. If one cared to formalize one's metalanguage in turn, one could subject it too to Schönfinkel's elimination of variables.

C , S , and U , economical as they are, are further reducible. It is now known that we can get U from C , S , and the mere *identity relation*^b—which, by the doctrine of relations noted above, is the same as the unit-class function ι .

Schönfinkel himself contrives a more

drastic but very curious reduction of C , S , and U . He adopts a new function J , interpreted as having U as value for C as argument, C as value for S as argument, and S as value for other arguments. Then he defines " S " as " JJ ", " C " as " JS ", and " U " as " JC ". This trick reduces every closed sentence (of logic and set theory) to a string of " J " and parentheses.

In the second of his three added paragraphs, Behmann proposes an alternative reduction, which, at the cost of resting with C , S , U , and yet a fourth basic function instead of just J , would get rid of parentheses by maneuvering them all into a left-converging position, where they could be tacit. However, as Behmann recognized later in a letter to H. B. Curry,^c there is a fallacy here; the routine can generate new parentheses and not terminate.

But, if we want to get rid of parentheses, we can easily do so by adapting an idea that was used elsewhere in logic by Łukasiewicz (1929, footnote 1, pp. 610–612; 1929a; 1958; 1963). Instead of using mere juxtaposition to express the application of functions, we can use a preponent binary operator " o ". Thus " xy ", " $x(yz)$ ", and " $(xy)z$ " give way to " oxy ", " $oxoyz$ ", and " $ooxyz$ ". All Schönfinkel's sentences built of " J " and parentheses go over unambiguously into strings of " J " and " o ".

Schönfinkel's reduction to J , with parentheses or whatever, is interesting for its effortlessness and its broad applicability. One would like to rule it out as spurious reduction, but where can a line be drawn? The only contrast I think of between serious reduction and this reduction to J is that in serious reduction the axioms tend to diminish along with

^b This reduction figured in my seminar, from about 1952 on. The essential reasoning is recoverable from the last two pages of *Quine 1956*.

^c See Curry and Feys 1958, p. 184.

the primitive ideas: what had been axiomatic connections reduce in part to definitions. Schönfinkel does not get to axioms, but any axioms he might have adopted regarding C , S , and U would obviously have been diminished none by his reduction of these functions to J .

Axioms for C , S , and U pose, as it happens, a major problem. Since these three functions cover set theory, the question of axioms for them is as broad as axiomatic set theory itself. Moreover, this present angle of analysis is so radically different from the usual that we cannot easily adapt the hitherto known ways of getting around the set-theoretic paradoxes. The quest for an optimum axiomatization of the Schönfinkel apparatus has accounted for much work and many writings from 1929 onward, mainly by H. B. Curry, under the head of *combinatory logic*.^d

It was by letting functions admit functions generally as arguments that Schönfinkel was able to transcend the bounds of the algebra of classes and relations and so to account completely for quantifiers and their variables, as could not be done within that algebra. The same expedient carried him, we see, far beyond the bounds of quantification

theory in turn; all set theory was his province. His C , S , U , and application are a marvel of compact power. But a consequence is that the analysis of the variable, so important a result of Schönfinkel's construction, remains all bound up with the perplexities of set theory.

It proves possible, thanks to the perspective afforded by Schönfinkel's pioneer work, to separate these considerations. Ordinary quantification theory, the first-order predicate calculus, can actually be reworked in such a way as to get rid of the variables without thus increasing the power of the notation. The method turns on adopting a few operators attachable to predicates. They are reminiscent of Schönfinkel's functions, but with the difference that they do not apply to themselves or one another. An analysis of the variable is obtained that is akin to Schönfinkel's but untouched by the problems of set theory.^e

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^d See Curry and Feys 1958.

^e See Bernays 1957 and Quine 1960.

§ 1

It is in the spirit of the axiomatic method as it has now received recognition, chiefly through the work of Hilbert, that we not only strive to keep the axioms as few and their content as limited as possible but also attempt to make the number of fundamental undefined notions as small as we can; we do this by seeking out those notions from which we shall best be able to construct all other notions of the branch of science in question. Understandably, in approaching this task we shall have to be appropriately modest in our demands concerning the simplicity of the initial notions.

As is well known, the fundamental propositional connectives of mathematical logic, which I reproduce here in the notation used by Hilbert in his lectures,

$$\bar{a}, a \vee b, a \& b, a \rightarrow b, a \sim b$$

(read: "not a ", " a or b ", " a and b ", "if a , then b ", " a is equivalent to b "), cannot be defined at all in terms of any single one of them; they can be defined in terms of two of them only if negation and any one of the three succeeding connectives are

taken as undefined elements constituting the base. (Of these three kinds of reduction Whitehead and Russell employed the first and Frege the third.)

That the reduction to a single fundamental connective is nevertheless entirely possible provided we remove the restriction that the connective be taken only from the sequence above was discovered not long ago by Sheffer (1913). For, if we take as the base, say, the connective “not a or not b ”, that is, “of the propositions a and b at least one is false”, which can be written with the signs above in the two equivalent forms

$$\bar{a} \vee \bar{b} \quad \text{and} \quad \overline{a \& b},$$

and if we adopt

$$a | b$$

as the new sign for it, then obviously

$$\bar{a} = a | a \quad \text{and} \quad a \vee b = (a | a) | (b | b);$$

thus, because

$$a \& b = \overline{\bar{a} \vee \bar{b}}, \quad (a \rightarrow b) = \bar{a} \vee b, \quad \text{and} \quad (a \sim b) = (a \rightarrow b) \& (b \rightarrow a),$$

the reduction has in principle been accomplished.

It is remarkable, now, that it is possible to go beyond this and, by suitably modifying the fundamental connective, to encompass even the two higher propositions

$$(x)f(x) \quad \text{and} \quad (Ex)f(x),$$

that is, “all individuals have the property f ” and “there exists an individual having the property f ”, in other words, the two operations (x) and (Ex) , which, as is well known, together with those above constitute a system of fundamental connectives for mathematical logic that is complete from the point of view of the axiomatization.

For, if from now on we use

$$(x)[f(x) \vee \overline{g(x)}], \quad \text{or} \quad (x)\overline{f(x) \& g(x)},$$

as a fundamental connective and if we write

$$f(x) | x g(x)$$

for it, then evidently (since we can treat constants formally as functions of an argument)

$$\bar{a} = a | x a, \quad a \vee b = (x)(a \vee b) = \bar{a} | x \bar{b} = (a | y a) | x (b | y b),$$

and

$$(x)f(x) = (x)(\overline{f(x)} \vee \overline{f(x)}) = \overline{f(x)} | x \overline{f(x)} = (f(x) | y f(x)) | x (f(x) | y f(x));$$

thus, because

$$(Ex)f(x) = \overline{(x)\overline{f(x)}}$$

the last assertion is proved also.

The successes that we have encountered thus far on the road taken encourage us to attempt further progress. We are led to the idea, which at first glance certainly appears extremely bold, of attempting to eliminate by suitable reduction the remaining fundamental notions, those of proposition, propositional function, and variable,

from those contexts in which we are dealing with completely arbitrary, logically general propositions (for others the attempt would obviously be pointless). To examine this possibility more closely and to pursue it would be valuable not only from the methodological point of view that enjoins us to strive for the greatest possible conceptual uniformity but also from a certain philosophic, or, if you wish, aesthetic point of view. For a variable in a proposition of logic is, after all, nothing but a token *[[Abzeichen]]* that characterizes certain argument places and operators as belonging together; thus it has the status of a mere auxiliary notion that is really inappropriate to the constant, "eternal" essence of the propositions of logic.

It seems to me remarkable in the extreme that the goal we have just set can be realized also; as it happens, it can be done by a reduction to three fundamental signs.

§ 2

To arrive at this final, deepest reduction, however, we must first present a number of expedients and explain a number of circumstances.

It will therefore be necessary to leave our problem at the point reached above and to develop first a kind of *function calculus* *[[Funktionskalkül]]*; we are using this term here in a sense more general than is otherwise customary.

As is well known, by function we mean in the simplest case a correspondence between the elements of some domain of quantities, the argument domain, and those of a domain of function values (which, to be sure, is in most cases regarded as coinciding with the former domain) such that to each argument value there corresponds at most one function value. We now extend this notion, permitting functions themselves to appear as argument values and also as function values. We denote the value of a function f for the argument value x by simple juxtaposition of the signs for the function and the argument, that is, by

$$fx.$$

Functions of several arguments can, on the basis of our extended definition of function, be reduced to those of a single argument in the following way.

We shall regard

$$F(x, y),$$

for example, as a function of the single argument y , say, but not as a fixed given function; instead, we now consider it to be a variable function that depends on x for its form. (Of course we are here concerned with a dependence of the *function*, that is, of the correspondence itself; we are not referring to the obvious dependence of the function *value* upon the argument.) In mathematics we would say in such a case that the function depends upon a parameter, too, and we would write, say,

$$G_x(y).$$

We can regard this function G itself—its form, so to speak—as the value (function value) of a new function f , so that $G = fx$.

We therefore write

$$(fx)y$$

in our symbolism, or, by agreeing, as is customary in the theory of infinite series for example, that parentheses around the left end of such a symbolic form may be omitted, more simply,

$$fxy,$$

where the new function, f , must be clearly distinguished from the former one, F .

I should like to make the transformation just described more intelligible by applying it to the specific number-theoretic function $x - y$. If we regard the expression as a function of y alone, this function has the "form" $x -$ and therefore means "the difference between x and any given quantity"; in that case the expression is to be taken as $(x -)y$. The essential point here is that we must not think of a simultaneous substitution of values for x and y , but, to begin with, of the substitution of a value— a , for instance—for x alone, so that we first obtain the function $a - y$ (in short: the function $a -$) as an intermediate step; only then does the replacement of y —by the fixed value b , say—become admissible.

In the foregoing case, therefore, fx is the value of a function that, upon substitution of a value for x , does not yet yield an object of the fundamental domain (if indeed such an object was intended as the value of $F(x, y)$) but yields another function, whose argument now is y ; that is, f is a function whose argument need not be subject to any restriction but whose function value is again a function. On functions of more than one variable we shall henceforth always carry out the transformation described above (or think of it as having been carried out), so that these will appear throughout in the form

$$fxyz \dots,$$

which, as we have already stated, is to be taken as an abbreviation of

$$(((fx)y)z) \dots$$

§ 3

Now a sequence of particular functions of a very general nature will be introduced. I call them the identity function [Identitätsfunktion] I , the constancy function [Konstanzfunktion] C , the interchange function [Vertauschungsfunktion] T , the composition function [Zusammensetzungsfunktion] Z , and the fusion function [Verschmelzungsfunktion] S .

1. By the *identity function* I we mean that completely determined function whose argument value is not subject to any restriction and whose function value always coincides with the argument value, that function, in other words, by which each object, as well as each function, is associated with itself. It is therefore defined by the equation

$$Ix = x,$$

where the equal sign is not to be taken to represent logical equivalence as it is ordinarily defined in the propositional calculus of logic but signifies that the expressions on the left and on the right mean the same thing, that is, that the function value Ix is always the same as the argument value x , whatever we may substitute for x . (Thus, for instance, II would be equal to I .)

2. Now let us assume that the argument value is again arbitrary without restriction, while, regardless of what this value is, the function value will always be the fixed value a . This function is itself dependent upon a ; thus it is of the form Ca . That its function value is always a is written

$$(Ca)y = a.$$

And by now letting a , too, be variable we obtain

$$(Cx)y = x, \quad \text{or} \quad Cxy = x,$$

as the defining equation of the **constancy function** C . This function C is obviously of the kind considered on page 360; for only when we substitute a fixed value for x does it yield a function with the argument y . In practical applications it serves to permit the introduction of a quantity x as a "blind" variable.

3. Conversely, we can obviously always think of an expression like

$$fxy$$

as having been obtained from

$$F(x, y),$$

where F is uniquely determined by the given f . If, however, we now rewrite this expression as

$$gyx,$$

taking y as a parameter, then this new function, too, is uniquely given by F and therefore indirectly also by f .

Hence we may think of the function g as the value of a function T for the argument value f . This **interchange function** T has as its argument a function of the form φxy , and the function value

$$\psi = T\varphi$$

is that function ψxy whose value ψxy coincides with φyx for all argument values x and y for which φyx has meaning. We write this definition briefly as

$$(T\varphi)xy = \varphi yx,$$

where the parentheses may again be omitted.

The function T makes it possible to alter the order of the terms of an expression, and in this way it compensates to a certain extent for the lack of a commutative law.

4. If in the argument place of a function f there occurs the value (dependent upon x) of another function, g , then

$$f(gx)$$

obviously also depends upon x and can in consequence be regarded as the value of a third function, F , which is uniquely determined by f and g . In analysis, as is well known, we speak loosely in such cases of a "function of a function"—strictly, it should be a "function of a function value"—and we call F the function "compounded" from f and g . The function F is thus itself the value of a certain function Z' of f and g .

We could therefore define

$$[Z'(\varphi, \chi)]x = \varphi(\chi x).$$

But, following our earlier convention, we prefer to replace Z' by the corresponding function of one argument, and we consequently obtain

$$Z\varphi\chi x = \varphi(\chi x)$$

as the defining equation of the *composition function* Z .

By means of the function Z parentheses can be shifted (not really eliminated, since they must always be thought of as still being there) within a more comprehensive expression; its effect is therefore somewhat like that of the associative law, which is not satisfied here either.

5. If in

$$fxy$$

we substitute the value of a function g for y , and in particular the value taken for the same x as that which appears as argument of f , we come upon the expression

$$fx(gx),$$

or, as we shall write it for the moment to make it clearer,

$$(fx)(gx).$$

This, of course, is the value of a function of x alone; thus

$$(fx)(gx) = Fx,$$

where

$$F = S'(f, g)$$

again depends in a completely determined way upon the given functions f and g . Accordingly we have

$$[S'(\varphi, \chi)]x = (\varphi x)(\chi x),$$

or, if we carry out the same transformation as in the preceding case,

$$S\varphi\chi x = (\varphi x)(\chi x),$$

as the defining equation of the *fusion function* S .

It will be advisable to make this function more intelligible by means of a practical example. If we take for fxy , say, the value $\log_x y$ (that is, the logarithm of y to the base x) and for gz the function value $1 + z$, then $(fx)(gx)$ obviously becomes $\log_x (1 + x)$, that is, the value of a function of x that is univocally associated with the two given functions precisely by our general function S .

Clearly, the practical use of the function S will be to enable us to reduce the number of occurrences of a variable—and to some extent also of a particular function—from several to a single one.

§ 4

It will prove to be relevant to the solution of the problem that we have raised concerning the symbolism of logic that the five particular functions of the function

calculus that were defined above, I , C , T , Z , and S , are not mutually independent, that, rather, two of them, namely C and S , suffice to define the others. In fact, the following relations obtain here.

1. According to the definitions of the functions I and C ,

$$Ix = x = Cxy.$$

Since y is arbitrary, we can substitute any object or any function for it, hence, for example, Cx . This yields

$$Ix = (Cx)(Cx).$$

According to the definition of S , however, this means

$$SCCx,$$

so that we obtain

$$I = SCC.^1$$

The last C , incidentally, does not occur in the expression SCC in an essential way. For, if above we put for y not Cx but an arbitrary function φx , we obtain in a similar way

$$I = SC\varphi,$$

where any function can then be substituted for φ .²

2. According to the definition of Z ,

$$Zfgx = f(gx).$$

Furthermore, by virtue of the transformations already employed,

$$f(gx) = (Cfx)(gx) = S(Cf)gx = (CSf)(Cf)gx.$$

Fusion over f yields

$$S(CS)Cf gx;$$

therefore

$$Z = S(CS)C.$$

3. In an entirely analogous way,

$$Tf yx = fxy$$

can be further transformed thus:

$$\begin{aligned} fx(Cyx) &= (fx)(Cy x) = Sf(Cy)x = (Sf)(Cy)x = Z(Sf)Cy x \\ &= ZZSfCy x = (ZZSf)Cy x = (ZZSf)(CCf)yx = S(ZZS)(CC)f yx. \end{aligned}$$

Therefore we have

$$T = S(ZZS)(CC).$$

If we here substitute for Z the expression found above, T too will have been reduced to C and S .

¹ This reduction was communicated to me by Boskowitz; some time before that, Bernays had called the somewhat less simple one $(SC)(CC)$ to my attention.

² Only such a function, of course, as has meaning for every x .

§ 5

Let us now apply our results to a special case, that of the calculus of logic in which the basic elements are individuals and the functions are propositional functions. First we require an additional particular function, which is peculiar to this calculus. The expression

$$fx |^x gx,$$

where f and g are propositional functions of one argument—in view of an earlier remark we may confine ourselves to these—is obviously a definite function of the two functions f and g ; thus it is of the form $U(f, g)$, or, by our principle of transformation, Ufg . Thus we have

$$Ufg = fx |^x gx,$$

where f and g , of course, now are propositional functions, as the defining equation of the *incompatibility function* U .

It is a remarkable fact, now, that every formula of logic can be expressed by means of our particular functions I , C , T , Z , S , and U alone, hence, in particular, by means solely of C , S , and U .

First of all, every formula of logic can be expressed by means of the generalized stroke symbol, with the bound variables (apparent variables) at the upper ends of the strokes. This holds without restriction; hence it holds for arbitrary orders of propositions *[[Aussagenordnungen]]* and also if relations occur. Furthermore, we can introduce the function U step by step in place of the stroke symbol by suitable use of the remaining constant functions.

We will not give the complete demonstration here but only explain the role of the different particular functions in this reduction.

By means of the function C we can see to it that the two expressions standing to the left and the right of the stroke become functions of the same argument.

Thus, for example, the expression

$$fx |^x gy,$$

which depends upon f , g , and y and in which x does not occur on the right, would have to be rewritten as

$$fx |^x C(gy)x.$$

If, however, x occurs on the right at some place other than the end, we can move it there by means of the function T ; in doing this we must use the function Z to extricate it from parentheses, if there are any, and the function S to fuse it, if it occurs several times. Thus, for example,

$$fx |^x gxy = fx |^x Tgyx = Uf(Tgy).$$

Or, to take a somewhat more involved example,

$$(fxy |^y gxy) |^x (hxx |^z kxz) = U(fx)(gx) |^x U(hx)(kx).$$

Here, for instance, the expression preceding the stroke must be dealt with further as follows:

$$U(fx)(gx) = ZUfx(gx) = S(ZUf)gx.$$

The entire expression thus becomes

$$S(ZUf)gx \mid^x S(ZUh)kx,$$

which is equal to

$$U[S(ZUf)g][S(ZUh)k].$$

If in the last example f and g were identical, we would obtain the expression

$$S(ZUf)f.$$

To be able to achieve the fusion over f here, we make use of the function I by calculating further:

$$S(ZUf)f = S(ZUf)(If) = [ZS(ZU)f](If) = S[ZS(ZU)]If.$$

To give a practical example of the claim of this section we shall deal with the following proposition: “For every predicate there exists a predicate incompatible with it”, that is, “For every predicate f there exists a predicate g such that the propositional function $fx \ \& \ gx$ is not true of any object x ”.

In Hilbert’s symbolism this sentence is written

$$(f)(Eg)(x)fx \ \& \ gx.$$

This becomes, first,

$$(f)(Eg)(fx \mid^x gx)$$

and then, if we write the particular judgment as the negation of a universal one,

$$(f)(g)fx \mid^x gx,$$

or

$$(f)(g)fx \mid^x gx \ \& \ fx \mid^x gx.$$

This is

$$(f)(fx \mid^x gx) \mid^g (fx \mid^x gx).$$

If we proceed similarly for f as well, we obtain, further,

$$(f)(fx \mid^x gx) \mid^g (fx \mid^x gx) \ \& \ (fx \mid^x gx) \mid^g (fx \mid^x gx),$$

which is equal to

$$[(fx \mid^x gx) \mid^g (fx \mid^x gx)] \mid^f [(fx \mid^x gx) \mid^g (fx \mid^x gx)].$$

From this point on the stroke symbol occurs as the sole logical connective. If we now introduce the incompatibility function U , we obtain first

$$[(Ufg) \mid^g (Ufg)] \mid^f [(Ufg) \mid^g (Ufg)]$$

and then

$$[U(Uf)(Uf)] \mid^f [U(Uf)(Uf)].$$

But now

$$U(Uf)(Uf) = (ZUUf)(Uf) = S(ZUU)Uf;$$

hence the expression above becomes

$$[S(ZUU)Uf] \mid^f [S(ZUU)Uf];$$

but this is

$$U[S(ZUU)U][S(ZUU)U].$$

§ 6

So far as we can see, we cannot carry the reduction to anything beyond the symbols C , S , and U without doing violence to it.

Purely schematically, to be sure, we could replace even C , S , and U by a single function if we were to introduce the new function J through the definitions

$$JC = U, \quad JS = C, \quad \text{and} \quad Jx = S,$$

where x is any object distinct from C and S . We ascertain, first, that J is itself distinct from C and S , since J takes on only three function values whereas C and S take on infinitely many. Consequently we have

$$JJ = S, \quad J(JJ) = JS = C, \quad \text{and} \quad J[J(JJ)] = JC = U,$$

which in fact accomplishes the reduction. But on account of its obvious arbitrariness it is probably without any real significance.

However,³ we can in a certain sense free ourselves at least from the sign U in another, more natural way. Every formula of logic [when transformed as indicated] certainly contains the sign U and—quite in accordance with our earlier conclusion about an arbitrary symbol—can be written in terms of the particular functions of the general function calculus (hence, in particular, by means of C and S) in such a way that U occurs as the argument of the entire expression; the expression thus assumes the form FU , where F itself no longer contains U . If in writing the expression down we omit the U , regarding it as understood, we can in fact manage with C and S .

On the other hand we could, while relinquishing the most extreme reduction of the basic function symbols, demand that parentheses be entirely avoided. If now we take the form FU as a point of departure, then, by means of Z alone, F can be transformed in such a way that all parentheses disappear. By means of C , Z , and S , therefore, every formula of logic can be written without parentheses as a simple sequence of these signs and can therefore be characterized completely by a number written to the base 3.

So far as the uniqueness of the reduction considered is concerned, it is, purely from the point of view of the symbolism, quite out of the question, since every formula both of the old and of the new calculus can be transformed in various ways. Yet in a certain more limited sense we can ascertain a uniqueness of correspondence here. For if we use the term “equivalent”, on the one hand, for the formulas of the old calculus that can be reduced to one another purely on the basis of the definitions, that is, without use of the logical axioms (in which, of course, the generalized Sheffer connective would now have to figure as the fundamental connective), and, on the other hand, for those that differ from one another only in the typography of the variables occurring in them, then to a given formula of the new calculus and, likewise, to any one that can be obtained from it through symbolic calculation there in fact correspond precisely those formulas of the old calculus that are equivalent to one another in the sense just defined. The reduction here considered of the formulas of logic has the remarkable peculiarity, therefore, of being independent of the axioms of logic.

³ The considerations that follow are the [German] editor's.