Homework 0

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1 Conditional probability

Given $\Omega = \{(r \succ b \succ g), (r \succ g \succ b), (b \succ r \succ g), (b \succ g \succ r), (g \succ b \succ r), (g \succ r \succ b)\}$ where B is $r \succ g$ and A is $b \succ r$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$
$$\mathbb{P}(r \succ g|b \succ r) = \frac{\mathbb{P}(b \succ r \cap r \succ g)}{\mathbb{P}(b \succ r)}$$

As $\mathbb{P}(b \succ r \cap r \succ g)$ is the probability of b being preferred over r and r being preferred over g or the probability of b being preferred over r being preferred over g, it is the possibility of event $(b \succ r \succ g)$ or 1/6.

Likewise $\mathbb{P}(b \succ r)$ is the probability of b being preferred over r, meaning it is the probability of events $\{(b \succ r \succ g), (b \succ g \succ r), (g \succ b \succ r)\}$ or 3/6.

Thus the probability that in this second round the child will choose to play with the toy that we already offered in the first round is:

$$\frac{\mathbb{P}(b \succ r \cap r \succ g)}{\mathbb{P}(b \succ r)} = \frac{1/6}{3/6} = \frac{1}{3}$$

2 Probability density functions

2.1 Calculate expectation of bid

$$\begin{aligned} E[v_J] &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{1}^{3} x \cdot \frac{1}{2} dx \\ &= \frac{1}{2} \left[\frac{x^2}{2} \right]_1^3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} E[b_J] &= \int_{1}^{2} x \cdot \frac{1}{2} dx \\ &= \frac{1}{2} \left[\frac{x^2}{2} \right]_1^2 \\ &= \frac{3}{4} \end{aligned}$$

2.2 Calculate probability

Solution to first subproblem

$$\begin{aligned} E[b] &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{0}^{3} x \cdot \frac{1}{3} dx \\ &= \frac{1}{3} \left[\frac{x^2}{2} \right]_0^3 \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} F(x) &= \int_{-\infty}^{3} f(x) dx \\ &= \int_{-\infty}^{3} \frac{1}{3} dx \end{aligned}$$

Solution to second subproblem

$$\begin{aligned}
\mathbb{P}(b \leq 2) &= \int_0^2 x \cdot \frac{1}{3} dx \\
&= \frac{1}{3} \left[\frac{x^2}{2} \right]_0^2 \\
&= \frac{2}{3}
\end{aligned}$$

Solution to third subproblem

$$\begin{aligned}
\mathbb{P}\left(b \leq \frac{3}{4}\right) &= \int_0^{3/4} x \cdot \frac{1}{3} dx \\
&= \frac{1}{3} \left[\frac{x^2}{2} \right]_0^{3/4} \\
&= \frac{3}{32}
\end{aligned}$$

3 Combinatorics

Solution to first subproblem

$$\mathbb{P}(\text{No one}) = \frac{1}{10!}$$

Solution to second subproblem

Since there is only 9 possible students the first student can switch with, then 8 possible students the second student can switch with, and so forth to 1.

$$\mathbb{P}(\text{Exactly one student}) = \frac{45}{10!}$$

Solution to third subproblem

Since two swaps will be needed only if there are 3 out of place students in a chain or 4 out of place students in two separate chains, we can determine the probability of two swaps being needed by determining the probability of these outcomes.

$$\mathbb{P}(\text{Exactly two students}) = \frac{870}{10!}$$