3 Conditional Probability and Bayes' Rule

Example 3.1. There are about 2800 people in the world taller than 7 feet. There are 450 players in the NBA and 23 of those players are list as 7 feet tall or taller. The global population is about 7.4 billion people.

1. If you select a person at random from the global population, what is the likelihood that the chosen person is greater than or equal to 7 feet tall?

   Solution. $\dfrac{2800}{7.4 \cdot 10^9} = 3.78 \cdot 10^{-7}$ □

2. Suppose you select a person at random from the global population. You are told, that the person you selected happens to be a player in the NBA. Now, what is the chance that the chosen person is greater than or equal to 7 feet tall?

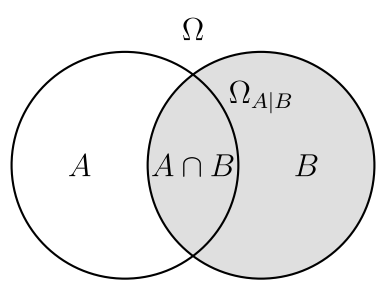
   Solution. $\dfrac{23}{450} = .05\overline{1}$ □

3. Given that a person is 7 feet tall, what are their chances of playing in the NBA?

   Solution. $\dfrac{23}{2800} = .0082$ □

Definition 3.2. A **conditional probability** of an event, is the probability of the event given the occurrence of another event (or given additional information). The probability of A given B is denoted $\mathbb{P}(A|B)$. The formula for $\mathbb{P}(A|B)$ is $\dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Remark 3.3. By stating B is a "given," we can treat B as becoming the new Ω or outcome space of $A|B$ where $\mathbb{P}(A|B)$, the probability of A given B occurring, is dictated by $\mathbb{P}(A \cap B)$ or the probability of A occurring within the "given" or restricted outcome space, B.



The figure above demonstrates how the outcome space changes given the occurrence of event B, the colored circle, where the only relevant probability of A for $A|B$ is $\mathbb{P}(A \cap B)$.

Example 3.4. A hat contains three cards. One card is red on both sides, one card is white on both sides, and one card is red on one side, white on the other. A single card is drawn and placed on a table. The visible side of the card is red. What is the chance that the other side is white?

Solution. Let A be the event the visible side of the card is red and B be the event the other side or the face down color of the card is white.

Given a card state is represented as $\dfrac{\text{Visible color}}{\text{Face down color}}$

$$\Omega = \{\frac{R}{R}, \frac{R}{R}, \frac{R}{W}, \frac{W}{R}, \frac{W}{W}, \frac{W}{W}\}$$

Using the outcome space above, we can determine

$$\begin{aligned}
\mathbb{P}(A) &= \frac{3}{6} \\
\mathbb{P}(A \cap B) &= \frac{1}{6} \\
\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{6} \cdot \frac{6}{3} = \frac{1}{3}
\end{aligned}$$

□

Theorem 3.5. *(Multiplication rule)* $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$.

Proof. Using Definition of Conditional prob and by multiplying both sides by $\mathbb{P}(A)$, we get

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \implies \mathbb{P}(A)\mathbb{P}(B|A) = \mathbb{P}(A \cap B)$$

□

Definition 3.6. A *Partition of* Ω is a list of events in Ω that cover all of Ω and do not have any overlap.

Example 3.7. Given the outcome space of a 6 sided die, the events that a number greater than 1 is rolled and that a 1 is rolled is an example of a partition, while the events that a 3 is rolled and that a number less than 6 is rolled is not an example of a partition.

3.1 Rule of average conditional probabilities

How can you compute $\mathbb{P}(A)$ using conditional probabilities, multiplication rule and partitions? We can use a partition to break up an outcome space into different cases and then compute a "weighted average," weighting by the likelihood of each case.

Example 3.8. About 36% of people in the US own a cat. A person with a cat has about a 20% chance of getting toxoplasmosis. A person without a cat has about a 5% chance of getting toxoplasmosis. What is the overall likelihood that a person selected at random from the US has toxoplasmosis?

Solution. Let C be the event a person in the US owns a cat and T be the event a person in the US has toxoplasmosis.

$$\begin{aligned}
\mathbb{P}(C) &= .36 \\
\mathbb{P}(C^C) &= 1 - \mathbb{P}(C) \\
&= 1 - .36 = .64 \\
\mathbb{P}(T|C) &= .2 \\
\mathbb{P}(T|C^C) &= .05
\end{aligned}$$

Using the multiplication rule, we can determine

$$\begin{aligned}
\mathbb{P}(T) &= \mathbb{P}(C \cap T) + \mathbb{P}(C^C \cap T) \\
&= \mathbb{P}(C)\mathbb{P}(T|C) + \mathbb{P}(C^C)\mathbb{P}(T|C^C) \\
&= (.36)(.2) + (.64)(.05) = .104 = 10.4\%
\end{aligned}$$

□

Theorem 3.9. ***Rule of Average Conditional Probabilities:***
For a partition $B_1, \ldots, B_n$ of Ω

$$\begin{aligned}
\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) &= \mathbb{P}(A)\mathbb{P}(B_1|A) + \cdots + \mathbb{P}(A)\mathbb{P}(B_n|A) \\
&= \mathbb{P}(B_1)\mathbb{P}(A|B_1) + \cdots + \mathbb{P}(B_n)\mathbb{P}(A|B_n)
\end{aligned}$$

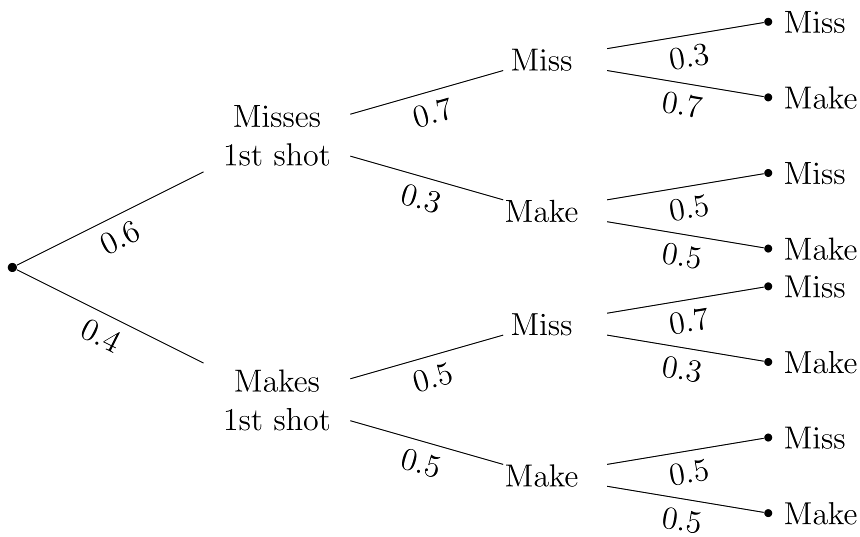
Multiplication Rule (general):
For n events:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \ldots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$$

3.2 Tree Diagrams

In a **tree diagram** possible outcomes are represented by branches. Probabilities/conditional probabilities are indicated along each branch.

Example 3.10. Aaron Nesmith is a basketball player for the Celtics. He's pretty good at making threes. If he makes a three, he has a 50% of making his next attempt. If he misses a three, he has a 30% of making his next attempt. Nesmith takes three shots. If he has a 40% chance of making his first shot, how likely is it that he makes all three?



Solution.
Let I, II , and III be the events that Nesmith made his first, second, and third shots respectively. Given

$$\begin{aligned}\mathbb{P}(I) &= .4\\ \mathbb{P}(II\,|I) &= .5 = \frac{\mathbb{P}(II\,\cap I)}{\mathbb{P}(I)}\\ \mathbb{P}(III\,|(II\,\cap I)) &= .5 = \frac{\mathbb{P}(III\,\cap (II\,\cap I))}{\mathbb{P}(II\,\cap I)}\end{aligned}$$

By solving for $\mathbb{P}(II\,\cap I)$ and $\mathbb{P}(III\,\cap (II\,\cap I))$, we can determine

$$\begin{aligned}\mathbb{P}(II\,\cap I) &= \mathbb{P}(II\,|I)\mathbb{P}(I)\\ &= .5 \cdot .4 = .2\\ \mathbb{P}(III\,\cap (II\,\cap I)) &= \mathbb{P}(III\,|(II\,\cap I))\mathbb{P}(II\,\cap I)\\ &= .5 \cdot .2 = .1 = \mathbb{P}(III\,\cap II\,\cap I)\end{aligned}$$

□

Given that Nesmith makes his first shot, what is the probability that he makes at least 2 shots?

Solution. Let $II\,\cup III\,|I$ be the event of making the at least two shots or making the second or/and third shot given Nesmith made his first shot where I, II , III are the events defined above. Using the subtree where Nesmith made his first shot, we can determine

$$\begin{aligned}\mathbb{P}(II\,\cup III\,|I)^C &= .5 \cdot .7 = .35\\ \mathbb{P}(II\,\cup III\,|I) &= 1 - \mathbb{P}(II\,\cap III\,|I)^C\\ &= 1 - .35 = .65\end{aligned}$$

□

Theorem 3.11. *(Multiplication Rule for Trees) After setting up a tree diagram whose paths represent disjoint outcomes, the multiplication rule can be used to define a distribution of probability over paths.* ***Multiply the probabilities along a path to find the probability of that outcome.***

Example 3.12. (The birthday problem) 30 people are in a room. What is the probability that at least two of them share the same birthday? (Assume there are 365 possible birthdays, and all birthdays are equally likely.)

Solution. Let A be the probability that at least two of the people in the room share the same birthday.

$$\begin{aligned}
\mathbb{P}(A^C) &= \frac{365}{365} + \frac{364}{365} + \cdots + \frac{335}{365} \\
&= \frac{365!}{(365-30)!} \cdot \frac{1}{365^{30}} = .2936 \\
\mathbb{P}(A) &= 1 - \mathbb{P}(A^C) \\
&= 1 - .2936 = .7063 = 70.63\%
\end{aligned}$$

□

3.3 Bayes' Rule

Main idea: We can use the multiplication rule to translate a difficult-to-calculate conditional probability into an easy-to-calculate conditional probability.

Example 3.13. Suppose there are three similar boxes. Box i contains i white balls and one black ball for $i = 1, 2, 3$. I pick a box at random (I don't tell you which one). From that box, I remove a ball and show it to you. I offer you a prize if you can guess correctly which box the ball came from. Which box would you guess if the ball is white? What is your chance of guessing correctly?

Big idea: One conditional is easy to calculate - $\mathbb{P}(\text{White}|\text{Box}_3)$. The opposite condition is hard to calculate - $\mathbb{P}(\text{Box}_3|\text{White})$. But we can translate between the two!

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \underline{\frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(A|B) \cdot \frac{\mathbb{P}(B)}{\mathbb{P}(A)}}$$

Which box would you guess if the ball is white? What is your chance of guessing correctly? (Hint: Draw a tree diagram)

Solution. Given

$$
\begin{aligned}
\mathbb{P}(White|Box_3) &= \frac{3}{4}\\
\mathbb{P}(White|Box_2) &= \frac{2}{3}\\
\mathbb{P}(White|Box_1) &= \frac{1}{2}\\
\mathbb{P}(Box_3) = \mathbb{P}(Box_2) = \mathbb{P}(Box_1) &= \frac{1}{3}\\
\mathbb{P}(White) &= \frac{1}{3}(\mathbb{P}(White|Box_3) + \mathbb{P}(White|Box_2) + \mathbb{P}(White|Box_1))\\
&= \frac{1}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{2} = \frac{23}{36}
\end{aligned}
$$

I would choose Box_3 as it has the highest probability of having a white ball. Using the conditional probability formula, we can determine

$$
\begin{aligned}
\mathbb{P}(Box_3|White) &= \mathbb{P}(White|Box_3)\cdot\frac{\mathbb{P}(Box_3)}{\mathbb{P}(White)}\\
&= \frac{3}{4}\cdot\frac{1}{3}\cdot\frac{36}{23} = \frac{9}{23} = .3913 = 39.13\%
\end{aligned}
$$

□

Theorem 3.14. *(Bayes' Rule) For a partition $B_1, \ldots, B_n$ of an outcome space Ω*

$$P(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)}$$

Proof of Bayes' Rule. By using the Rule of Average Conditional Probabilities for a partition, we can substitute the denominator of Bayes' Rule and find the conditional probability theorem of event B_i given A.

$$
\begin{aligned}
P(B_i|A) &= \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)}\\
&= \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{j=1}^{n}\mathbb{P}(A\cap B_j)}\\
&= \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A)}
\end{aligned}
$$

□

Example 3.15. A blood test for a certain disease returns a value of either positive or negative. 95% of people with the disease test positive. 2% of people without the disease test positive (false positive). Only 1% of the population has the disease.
If a person is chosen at random from the population, tested, and the result comes back positive, what is the likelihood that the person has the disease?

Solution. Let D be the event that a person has the disease and T be the event that a person test positive for that disease.

	Test +	Test -	Total
Disease +	.95	.05	1
Disease -	.02	.98	1

Given $\mathbb{P}(D) = .01$ Using the conditional probability formula and values above, we can determine

$$\begin{aligned}
\mathbb{P}(D \cap T) &= \mathbb{P}(T|D)\mathbb{P}(D) \\
&= .95 \cdot .01 = .0095 \\
\mathbb{P}(D^C \cap T) &= \mathbb{P}(T|D^C)\mathbb{P}(D) \\
&= .02 \cdot .01 = .0002 \\
\mathbb{P}(T) &= \mathbb{P}(D \cap T) + \mathbb{P}(D^C \cap T) \\
&= .0095 + .0002 = .0097 \\
\mathbb{P}(D|T) &= \frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} \\
&= \frac{.0095}{.0097} = .9794 = 97.94\%
\end{aligned}$$

$\square$