MATH/STAT 230S, SCRIPT 4: INDEPENDENT EVENTS

4 Independent Events

Conditional probabilities define independent events. Heuristically, events are independent if knowledge of one event does not impact the probability of the other.

Definition 4.1. Two events A and B are *independent* if $\mathbb{P}(A) = \mathbb{P}(A|B)$, or $\mathbb{P}(B) = \mathbb{P}(B|A)$, or $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Example 4.2. The events of getting heads on a coin toss and a 6 in a fair 6-sided die are independent. However, the events of drawing an Ace from a standard deck of cards and then the drawing another Ace from the same deck, without replacements, are dependent.

Definition 4.3. Three events, A, B, and C are *independent* or *mutually independent.* If both of following hold

1. A and B are independent:
$$\mathbb{P}(B|A) = \mathbb{P}(B|A^c) = \mathbb{P}(B)$$
2. C does not depend on the occurrence of A or B:
$$\mathbb{P}(C) = \mathbb{P}(C|A \cap B) = \mathbb{P}(C|A^c \cap B) = \mathbb{P}(C|A \cap B^c)$$

Alternately, you can check that three events are independent by checking both of the following

1. All three events are pairwise independent (any two you pick are independent).
2. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$

If three events are independent, then the following multiplication rule applies:

Theorem 4.4. *(Multiplication Rule for Three Independent Events)*
$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

Example 4.5. Given we are rolling a fair 6-sided twice, let A be the event that you roll a 1 on the first die, B be the event that you roll a 2 on the first die, and C be the event that you roll a sum of 0 where
$$\begin{aligned}
\mathbb{P}(A) &= \frac{1}{6} \\
\mathbb{P}(B) &= \frac{1}{6} \\
\mathbb{P}(C) &= \frac{0}{36} = 0 \\
\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\
&= \frac{1}{6} \cdot \frac{1}{6} \cdot 0 = 0
\end{aligned}$$

This demonstrates that the multiplication rule does not imply independence as $\mathbb{P}(A \cap B \cap C)$ is 0 as rolling a sum of 0 is .

Example 4.6. Let Ω be an outcome space associated to rolling one die twice. Let A be the event that the first roll is even. Let B be the event that the second roll is even. Let C be the event that the sum of the rolls is 7.

Are these three events (mutually) independent? Are they *pairwise* independent? (pairwise independence means A and B are independent, A and C and independent, and B and C are independent)

Solution. Given

$$\Omega = \left\{ \begin{array}{cccccc} 11, & 12, & 13, & 14, & 15, & 16, \\ 21, & 22, & 23, & 24, & 25, & 26, \\ 31, & 32, & 33, & 34, & 35, & 36, \\ 41, & 42, & 43, & 44, & 45, & 46, \\ 51, & 52, & 53, & 54, & 55, & 56, \\ 61, & 62, & 63, & 64, & 65, & 66 \end{array} \right\}$$

$$\begin{aligned} \mathbb{P}(A) &= \frac{18}{36} = \frac{1}{2} \\ \mathbb{P}(B) &= \frac{18}{36} = \frac{1}{2} \\ \mathbb{P}(C) &= \frac{6}{36} = \frac{1}{6} \\ \mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24} \neq \frac{0}{36} \end{aligned}$$

As such, events A, B, and C are not mutually independent.

$$\begin{aligned} \mathbb{P}(A|B) &= \frac{9}{18} = \frac{1}{2} = \mathbb{P}(A) \\ \mathbb{P}(C|A) &= \frac{3}{18} = \frac{1}{6} = \mathbb{P}(C) \\ \mathbb{P}(C|B) &= \frac{3}{18} = \frac{1}{6} = \mathbb{P}(C) \end{aligned}$$

Thus, A, B, and C are all pairwise independent. $\square$