MATH/STAT 230S, SCRIPT 2: Foundations of Probability

2 Foundations of Probabilty

Definition 2.1. The following 4 definitions will be foundational to our study of probability.

- An *experiment* is a well-defined procedure.
- The *outcome space* or *sample space* is a set of all possible outcomes of the experiment. We will typically use the notation Ω to represent this set of events
- An *event* is a subset of possible outcomes.
- The *probability* of an event A is denoted $\mathbb{P}(A)$. When all events in an outcome space are equally likely, $\mathbb{P}(A) = \dfrac{\#(A)}{\#(\Omega)}$

Example 2.2. Apply the definitions in the following example.

1. Experiment: You flip a fair coin two times.
2. Outcome space: All of the following are outcome spaces.
   - $\Omega = \{0\, heads, 1\, head, 2\, heads\}$
   - $\Omega = \{TT, TH, HT, HH\}$
   - $\Omega = \{0\, tails, 1\, tail, 2\, tails\}$
3. Non-example: The following are NOT outcome spaces.
   - $\{1\, head, 2\, heads\}$ (does not cover all possible outcomes)
   - $\{0\, heads, 1\, head, 2\, heads, 0\, tails, 1\, tail, 2\, tails\}$ (contains repetitive outcomes)
4. Event: You flip at least one head.
5. Probability: What is the likelihood if you flip a fair coin twice, that the coin will land on heads at least once? Let A be the event of flipping at least one head. What is $\mathbb{P}(A)$?
   In the outcome space with equally likely outcomes ($\Omega = \{TT, TH, HT, HH\}$) there are 3 elements in event A. So,
   $$\mathbb{P}(A) = \frac{\#(A)}{\#(\Omega)} = \frac{3}{4}.$$

Example 2.3. Apply the definitions in the following example.

1. Experiment: Rolling a fair 6-sided die twice.
2. Outcome space: $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
3. Events: You roll a sum equal to 12.
4. Probability: What is the likelihood if you roll a fair 6-sided die twice, that you roll a sum equal to 12? Let A be the event of rolling a 12. What is $\mathbb{P}(A)$?

$$\mathbb{P}(A) = \frac{\#\text{ of ways to roll a 12}}{\#\text{ of ways to roll a 6-sided die twice}} = \frac{1}{36}$$

   As there is only one way to roll a 12 by rolling two 6s in a row and there are 36 possible ways to roll a fair 6-sided die twice (6 possible number of ways to roll a dice on the second roll for each of the 6 possible outcomes of the first roll or $6 \cdot 6$).

Definition 2.4. The *relative frequency* of event A out of n observations is denoted $\mathbb{P}_n(A)$. The relative frequency is computed as $\mathbb{P}_n(A) = \dfrac{\#\text{ of observations of event } A}{n}$.

Example 2.5. Let A be flipping heads on a single coin flip. You flip a coin 15 times. What is $\mathbb{P}\left(\mathbb{P}_{15}(A) = \dfrac{1}{2}\right)$?

Solution. As flipping a coin is a binary outcome, a relative frequency of event A out of 15 observations will be a distinct multiple of $\frac{1}{15}$. As $\frac{1}{2}$ is not a multiple of $\frac{1}{15}$, the probability of the relative frequency being $\frac{1}{2}$ will be 0. $\square$

Remark 2.6. In the "frequency" interpretation of probability, the probability of an event A is $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}_n(A)$. This applies well to things like flipping coins and drawing cards - experiments that can be easily repeated with the same conditions.
An alternate interpretation of probability is the "degree of belief" interpretation. In this sense, probability measures confidence in an opinion. This can apply to things like the probability that a particular candidate wins a particular election (that's an experiment that is difficult to repeat with the same conditions).

Example 2.7. Let A be rolling a 6 on a single 6-sided die. You roll this this dice 6 times. If you rolled a 6 two times, what is the relative frequency of A?

Solution. $\mathbb{P}_6(A) = \dfrac{2}{6} = \dfrac{1}{3}$ $\square$

What is the absolute probability of A?

Solution.

$$\lim_{n\to\infty} \mathbb{P}_n(A) = \mathbb{P}(A) = \frac{1}{6}$$

$\square$

Notation 2.8. We will use the following notation to make our writing more concise.

- $\cap$: For two events A and B, the overlap or intersection of A and B is written as $A \cap B$. This is also written as "A and B."
  Example - A fair 6-sided die is rolled. A is the event of rolling an even number. B is the event of rolling a number greater than 3. Then $A \cap B$ is $\underline{\text{ 4,6 }}$
- $\cup$: For two events A and B, the union of A and B is written as $A \cup B$. This is also written as "A or B." This includes all events in A and in B and in both.
  Example - A fair 6-sided die is rolled. A is the event of rolling an even number. B is the event of rolling a number greater than 3. Then $A \cup B$ is $\underline{\text{ 2,4,5,6 }}$
- $\emptyset$: The empty set, or the set of no outcomes is $\emptyset$. $\mathbb{P}(\emptyset) = 0$.
- A^c: The complement of A or the opposite of A is written as A^c. The complement includes all the outcomes that are not in A.

Theorem 2.9. *Useful Equations for Calculations*

1. $\mathbb{P}(\Omega) = \underline{1}$ *for an outcome space* Ω.
2. $\underline{0} \leq \mathbb{P}(B) \leq \underline{1}$ *for any event* B.
3. *Addition rule for disjoint sets: If* $B_1 \cap B_2 = \emptyset$, *then* $\mathbb{P}(B_1 \cup B_2) = \mathbb{P}(B_1) + \mathbb{P}(B_2)$.
4. *Inclusion-Exclusion:* $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Proof. Given A is $(A \cap B^C) \cup (A \cap B)$, using the addition rule for disjoint sets,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}((A \cap B^C) \cup (A \cap B)) \\ &= \mathbb{P}(A \cap B^C) + \mathbb{P}(A \cap B)\end{aligned}$$

Likewise, by switching events A and B, we can get that for B:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A^C) + \mathbb{P}(B \cap A)$$

By then solving for $\mathbb{P}(A \cap B^C)$ and $\mathbb{P}(B \cap A^C)$, we get

$$\begin{aligned}\mathbb{P}(A \cap B^C) &= \mathbb{P}(A) - \mathbb{P}(A \cap B) \\ \mathbb{P}(B \cap A^C) &= \mathbb{P}(B) - \mathbb{P}(A \cap B)\end{aligned}$$

Now, given the event $A \cup B$ is $(A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$ or the three disjoint parts of a Venn diagram of two events, by using the disjoint sets as before and then substituting the equations above, we get

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}((A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)) \\ &= \mathbb{P}(A \cap B^C) + \mathbb{P}(B \cap A^C) + \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\end{aligned}$$

$\square$

5. *Complement rule:* $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

   Proof. $\mathbb{P}(A) + \mathbb{P}(A^c) = 1 = \mathbb{P}(\Omega)$ □

6. *Multiplication rule for independent events:* $\mathbb{P}(B \cap A) = \mathbb{P}(B)\mathbb{P}(A)$

Example 2.10. Given that there are 10 cats and dogs in total where 3 are male cats and 3 are dogs with 2 of them being female, what is the probability one randomly chosen pet is a cat or a male?

Solution. Let M be the event of picking a male pet, C be the event of picking a cat, and D be that for dogs out of 10 pets.
Given

$$\begin{aligned} \mathbb{P}(C \cap M) &= \frac{3}{10} \\ \mathbb{P}(D) &= \frac{3}{10} \\ \mathbb{P}(D \cap M^C) &= \frac{2}{10} \end{aligned}$$

Using the complement rule, we can determine that

$$\begin{aligned} \mathbb{P}(C) &= 1 - \mathbb{P}(C^C) \\ &= 1 - \mathbb{P}(D) \\ &= 1 - \frac{3}{10} = \frac{7}{10} \end{aligned}$$

$$\begin{aligned} \mathbb{P}(D \cap M) &= 1 - \mathbb{P}(D \cap M)^C \\ &= 1 - (\mathbb{P}(D \cap M^C) + \mathbb{P}(C)) \\ &= 1 - \left(\frac{2}{10} + \frac{7}{10}\right) = \frac{1}{10} \end{aligned}$$

Using the addition rule for disjoint sets, given that there can't be a male pet that is both a cat and a dog, we can then determine that

$$\begin{aligned} \mathbb{P}(M) &= \mathbb{P}((C \cap M) \cup (D \cap M)) \\ &= \mathbb{P}(C \cap M) + \mathbb{P}(D \cap M) \\ &= \frac{3}{10} + \frac{1}{10} = \frac{4}{10} \end{aligned}$$

Using the Inclusion-Exclusion rule, we can determine the probability of a random pet being a cat or a male to be:

$$\begin{aligned} \mathbb{P}(C \cup M) &= \mathbb{P}(C) + \mathbb{P}(M) - \mathbb{P}(C \cap M) \\ &= \frac{7}{10} + \frac{4}{10} - \frac{3}{10} = \frac{8}{10} \end{aligned}$$

□