

## 2. Log-likelihood gradient

The *energy* of a Markov network is defined as

$$E(y; \alpha, \beta) = - \sum_i \alpha_i y_i - \sum_{i \neq j} \beta_{ij} y_i y_j.$$

The energy function can be used to define the log-likelihood of a given  $y$  vector as

$$\log \mathcal{L}(y; \alpha, \beta) = -E(y; \alpha, \beta) - \log(Z(\alpha, \beta)).$$

Here,  $Z(\alpha, \beta)$  is the *partition function*, a scaling factor defined as

$$Z(\alpha, \beta) = \sum_{y \in Y} e^{-E(y; \alpha, \beta)},$$

where  $Y$  is the set of all possible  $y$  vectors.

The partial derivative of the log-likelihood with respect to  $\alpha_i$  is

$$\frac{\partial}{\partial \alpha_i} \log \mathcal{L}(y; \alpha, \beta) = y_i - p(y_i; \alpha, \beta).$$

The first term,  $y_i$ , is zero if species  $i$  is absent in the observed assemblage and one if it's present. The latter term,  $p(y_i; \alpha, \beta)$ , describes the expected probability of observing species  $i$  under the current values of  $\alpha$  and  $\beta$ . It comes from the derivative of the partition function, as derived in (learning Boltzmann machines, Murphy 2012, etc.). Following the gradient of  $\alpha_i$  adjusts the expected probability of observing species  $i$  until it matches the observed value and the two terms in the gradient cancel one another out.

The partial derivative of the log-likelihood with respect to  $\beta_{ij}$  can be derived similarly as

$$\frac{\partial}{\partial \beta_{ij}} \log \mathcal{L}(y; \alpha, \beta) = y_i y_j - p(y_i y_j; \alpha, \beta).$$

Following this gradient adjusts the expected probability of co-occurrence between species  $i$  and species  $j$  until this value matches the observed co-occurrence frequency and the two terms in the gradient cancel one another out.

## References