Data Visualization From a Category Theory Perspective

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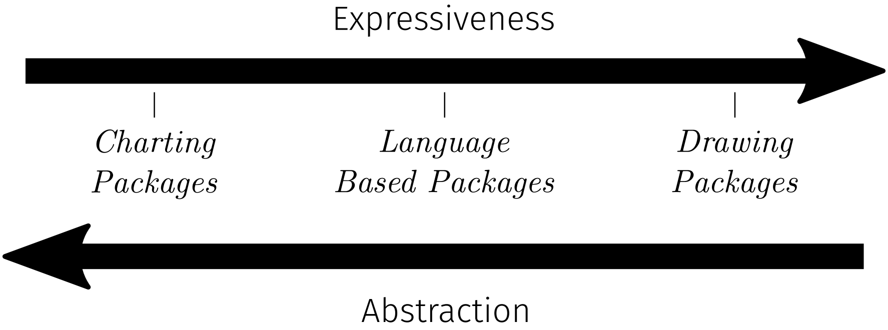
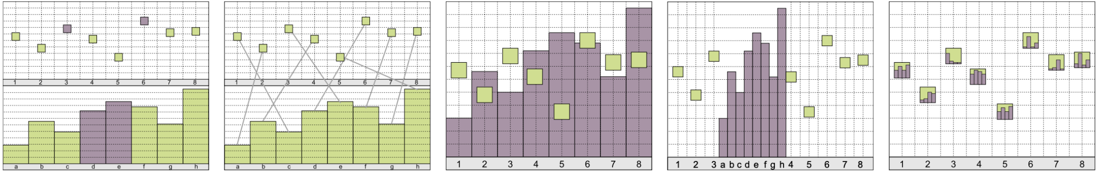
Introduction

Balance expressiveness and abstraction in data visualization frameworks.



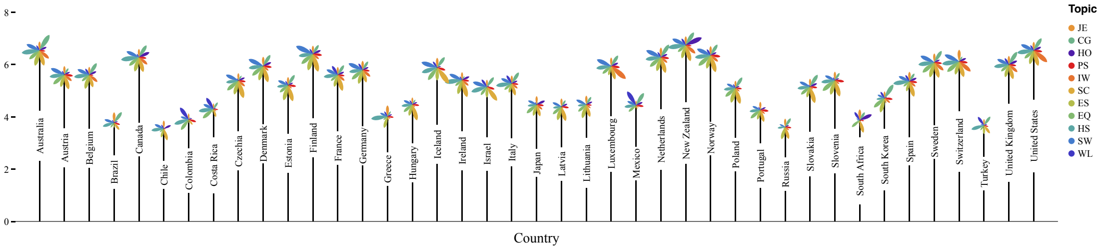
Introduction

Visualization grammars often come short in terms of expressiveness, e.g., composite visualizations and visualizations with custom marks.



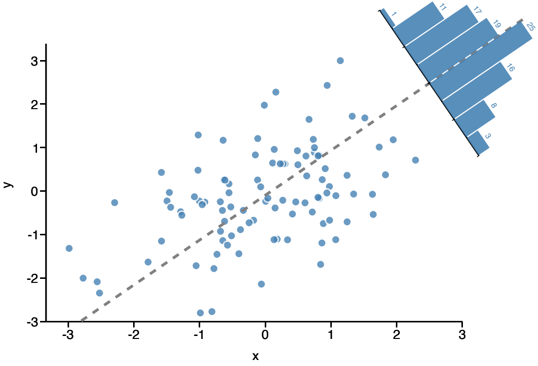
(a) Juxtaposed views. (b) Integrated views. (c) Superimposed views. (d) Overloaded views. (e) Nested views.

Introduction



Introduction

```
d = scatter_plot +
    T(x,y)R(θ)*histogram +
    S(:strokeDasharray=>2)*Line([p₁,p₂])
```



Introduction - Objective and Hypotheses

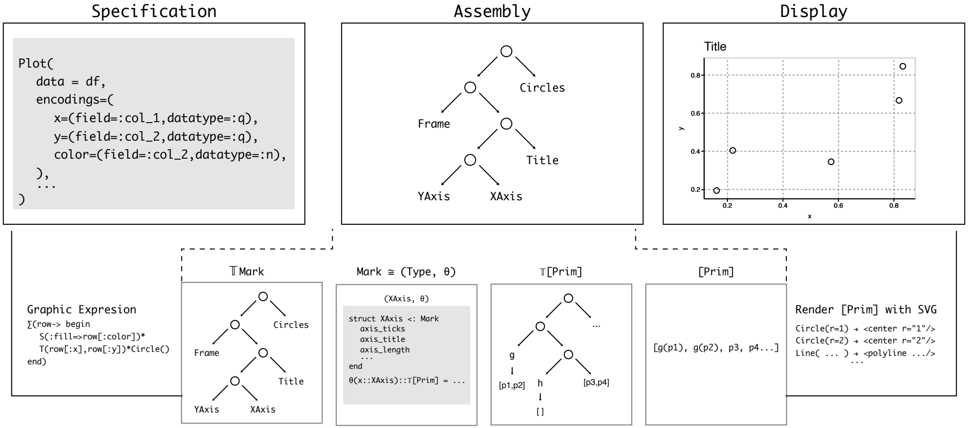
Main Objective: Develop a data visualization framework that extends the expressiveness of visualization grammars while preserving a high level of abstraction.

Hypothesis 1 (Cause): Limitations in expressiveness are due to a lack of integration between graphic specification and assembly.

Hypothesis 2 (Solution): Integration of diagramming and graphic specification via a unified constructive framework.

Formalization Tool: Category Theory.

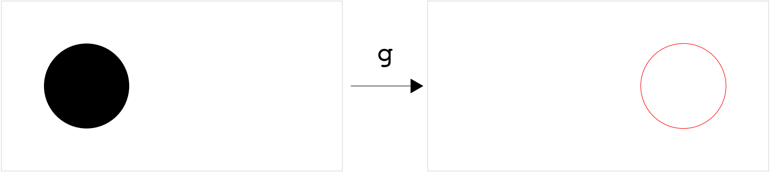
Introduction - Overview



Data Visualization + Category Theory: Diagrams as Monoids

The basic building block for any diagram is the primitive. A primitive is any geometry that can be drawn and manipulated via geometric or stylistic transformations.

```
g = (T(1,0), Dict(:fillOpacity =>0,:stroke =>:red))
```



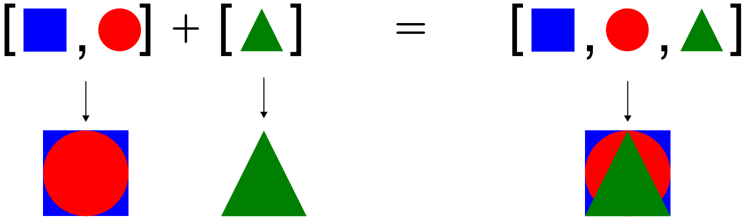
Data Visualization + Category Theory: Diagrams as Monoids

A diagram can be defined as an ordered list of primitives. $([\text{Prim}], +, [\,])$ can be interpreted as a monoid.

```julia
abstract type Prim end
struct Circle <: Prim
  radius::Real
  center::Tuple{Real, Real}
end
+(d1::Vector{Prim}, d2::Vector{Prim}) = vcat(d1,d2)
```

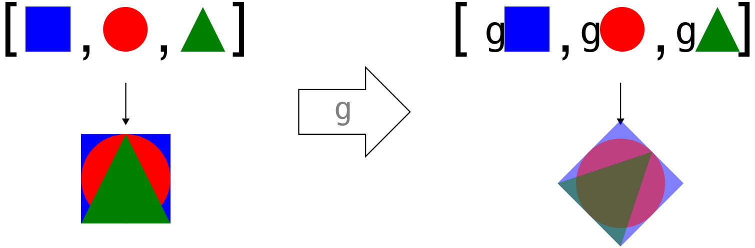
Data Visualization + Category Theory: Diagrams as Monoids

Diagrams are drawn by rendering the list of primitives on top of each other.

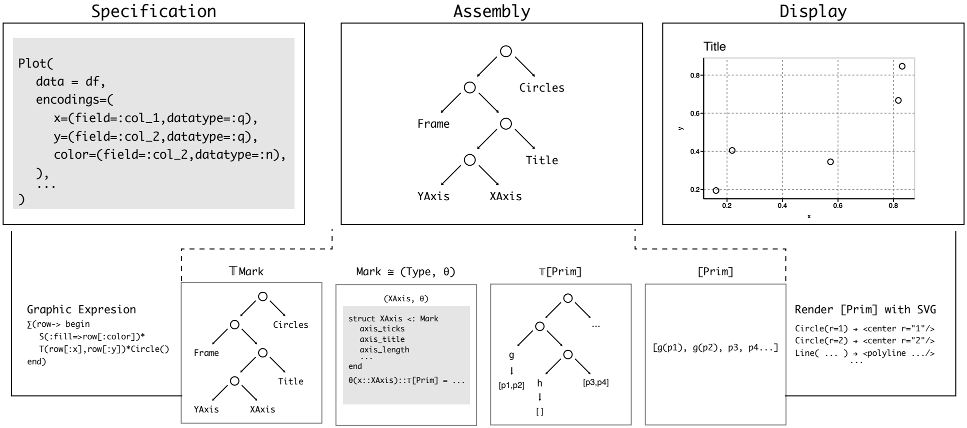


Data Visualization + Category Theory: Diagrams as Monoids

We can apply transformations to diagrams by applying the transformations to each primitive in the list.



Data Visualization + Category Theory



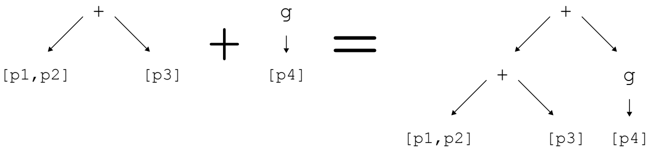
Data Visualization + Category Theory: Free Monads

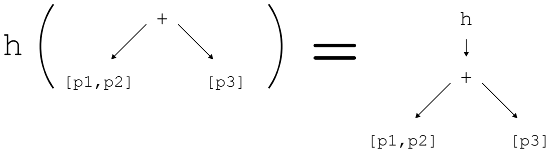
Let us improve our representation using expression trees. To do this, first we define a functor F that encodes diagram composition (i.e. $+$) and diagram transformations.

```
@data F{a} begin
Comp(::a, ::a)
Act(::H, ::a)
end
fmap(f::Function, x::Comp) = Comp(f(x._1),f(x._2))
fmap(f::Function, x::Act) = Act(x._1,f(x._2))
```

Data Visualization + Category Theory: Free Monads

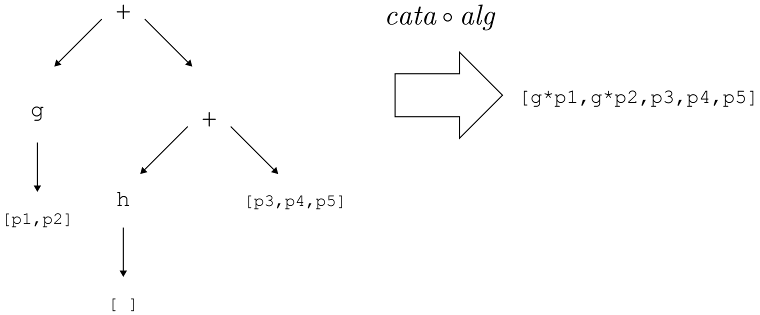
Using our functor F, we can define a free monad over such functor, i.e $\mathbb{T} := \mathrm{Free}F$. Thus, $(\mathbb{T}, \eta, \mu)$ is our monad, where $\mathbb{T}$ is a parametric type representing trees with F as possible operations.
To ease the process of expressing trees, we overload the $+$ and the $*$ operators to represent composition and transformation application respectively.



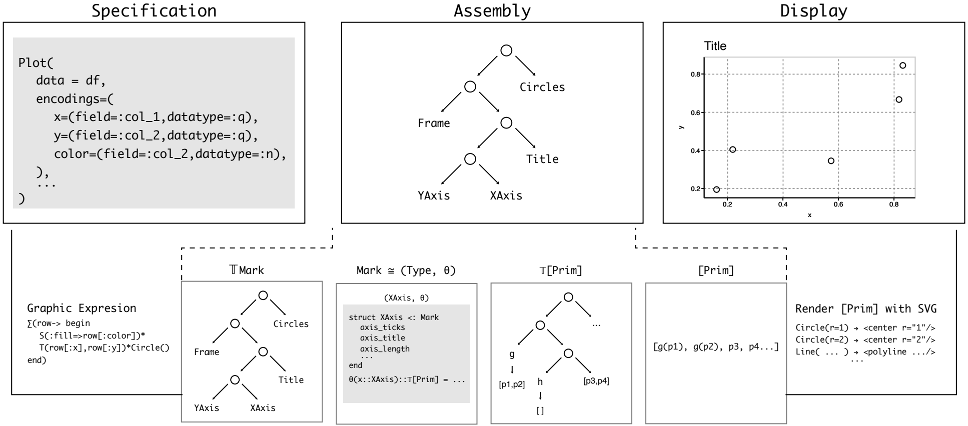


Data Visualization + Category Theory: Free Monads

A diagram tree is a value of type $\mathbb{T}$ [Prim]. For this representation to be useful, we must be able to flatten it, i.e. we must define a function $f : \mathbb{T}[\text{Prim}] \to [\text{Prim}]$. We can do this using F-algebras and catamorphism.



Data Visualization + Category Theory



Data Visualization + Category Theory

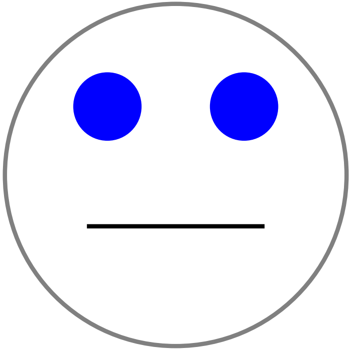
The current definitions of marks equate them with primitives. We propose a new definition:

Def: A **Graphical Mark** is a tuple (A, θ_A), where A is a type and θ_A is a function $\theta_A : A \to \mathbb{T}[\mathrm{Prim}]$.

```
struct Face <: Mark
  size::Real
  smile::Real
end

function θ(face::Face)
  eyes = Circle(...) + Circle(...)
  smile = Bezier(...)
  head = Circle(...)

  return head + eyes + smile
end
```



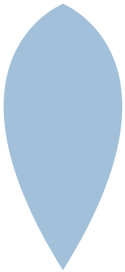
Data Visualization + Category Theory

We want to be able to define marks using previously defined marks. In other words, instead of specifying $\theta_A : A \to \mathbb{T}[Prim]$, we want to define a function $\zeta_A : A \to \mathbb{T}\mathrm{Mark}$. In other words, we want use ζ_A to infer θ_A.

Petal

```
struct Petal <: Mark
  height::Real
  width::Real
  color::String
end

function θ(p::Petal)
  return η([Prim(
    BezierPolygon(...),
    Dict(:fill=>p.color)
  )])
end
```



Flower

```
struct Flower <: Mark
  heights::Vector{Real}
  widths::Vector{Real}
  colors::Vector{String}
end

function ζ(fl::Flower)
  """
  Create petals and
  apply rotations.
  """
  ...
end
```

Data Visualization + Category Theory

Our ζ function can be formalized through the use of slice categories.

Def. The category of marks $\mathcal{M}$ is a subcategory of $\mathbf{Set}_{\mathbb{T}}/[\mathrm{Prim}]$.

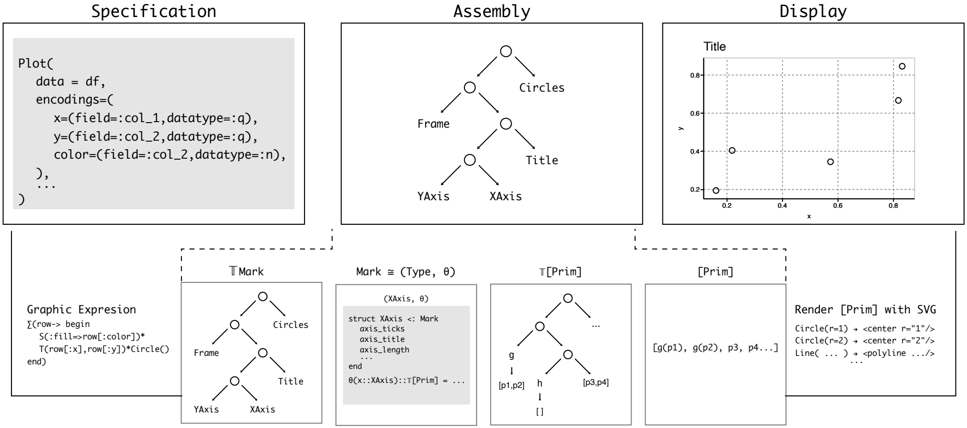
Assuming this category, a morphism $\zeta_A \in \mathrm{Hom}_{\mathcal{M}}(A, B)$ induces

$$\theta_A = \theta_{\mathbb{T}B} \circ \zeta_A = \mu \circ \mathbb{T}\theta_B \circ \zeta_A$$

This means that given an existing mark (B, θ_b), we can define a new mark (A, θ_A) by defining ζ_A and inferring of θ_A.

Programmatically, we can define an abstract type 'Mark', such that every subtype of 'Mark' has a *theta* function.

Data Visualization + Category Theory



Data Visualization + Category Theory

In summary:

- A diagram is a value of $[\text{Prim}]$;
- A diagram tree is a value of $\mathbb{T}[\text{Prim}]$;
- A mark is a value of type `A <: Mark` for which $\exists \theta : A \to \mathbb{T}[\text{Prim}]$;

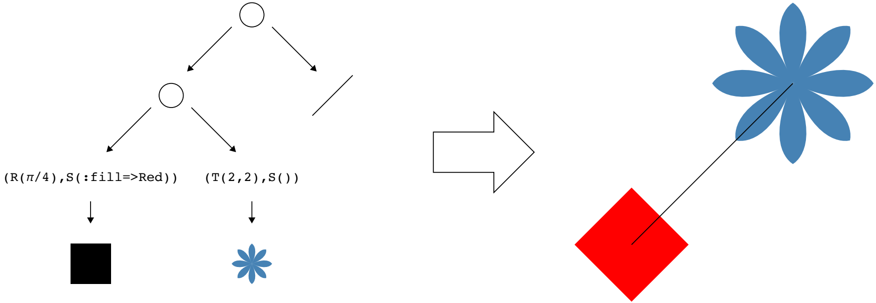
With all these concepts in mind, we can define **graphic** as a value of $\mathbb{T}\text{Mark}$, i.e. a tree where the leafs contain marks. Since each mark has a θ function, we can turn a $\mathbb{T}\text{Mark}$ into a $\mathbb{T}[\text{Prim}]$ using $\mu \circ \theta$.

Note that $\mathbb{T}\theta : \mathbb{T}\text{Mark} \to \mathbb{T}\mathbb{T}\text{Mark}$ and $\mu : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$.

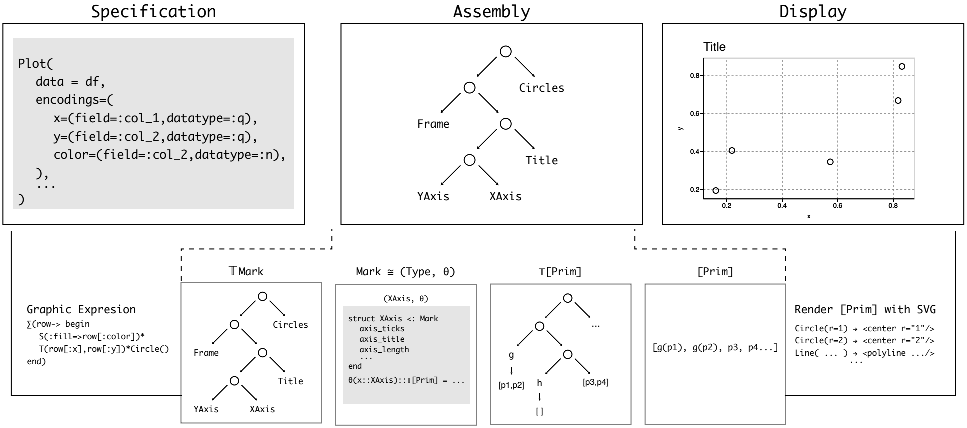
Data Visualization + Category Theory

Using $+$ as diagram composition and $*$ to apply transformations, we can construct values of $\mathbb{T}$Mark using a notation similar to mathematical expressions:

```
S(:fill=>:red)*R(π/4)*Square() + T(2,2)*Flower() + Line([0,0],[2,2])
```



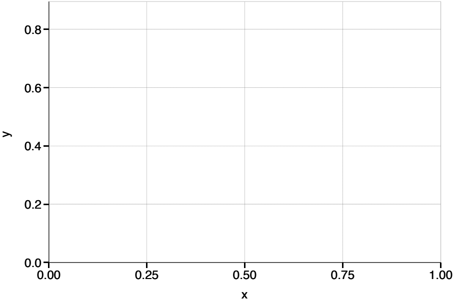
Data Visualization + Category Theory

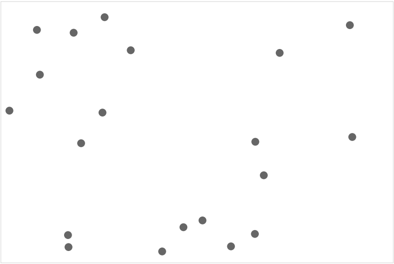


Data Visualization + Category Theory

Plot Specifications

$$\text{Plot} = \text{Guide} + \text{Graphic Expression} \circ \text{Scale}(\text{Data}, \text{Encodings})$$





References