

Notation: here is a brief summary of the notation used in this worksheet.

- $p(X = x)$  is equal to the probability density function;
- Capital letters such as  $X$  stand for the random variable.

## Exercise 1 (Inversion and Rejection)

1. Let  $F_X(x) = \mathbb{P}(X \leq x)$  and  $U \sim \text{Unif}[0, 1]$ :

$$F_X(x) = 1 - e^{-\lambda(x-a)} \mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1 - U) = \lambda(x - a)$$

$$F_X^{-1}(U) = a - \frac{\ln(1 - U)}{\lambda}$$

To simulate  $X$  from  $U$ , just simulate value from  $U$  and substitute in the formula above.

2. Let  $X = Y \mid a \leq Y \leq b$ . First, let's show that  $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$ :

$$\begin{aligned} \mathbb{P}(X \leq x) &= \mathbb{P}(F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a) + U[F_Y(b) - F_Y(a)]) \leq x) \\ &= \mathbb{P}(F_Y(a) + U[F_Y(b) - F_Y(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} \end{aligned}$$

Note that since  $x \in [a, b]$ :

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b) - F_Y(a)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on  $\geq a$ , we first generate  $U \sim \text{Unif}[0, 1]$ , then, for  $Y \sim \text{Expo}(\lambda)$ :

$$F_Y(y) = 1 - e^{-\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1 - U)}{\lambda}$$

$$X = \frac{-\ln(1 - (1 - U)F_Y(a) + U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1 - U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let  $q \sim \text{Expo}(\lambda)$ , and  $\pi(x) = \lambda e^{-\lambda(x-a)} \mathbb{I}_{x \geq a}$ :

Note that  $M = \max_x \pi(x)/q(x) = e^{\lambda a}$ , since  $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{-\lambda x}} = e^{\lambda a}$

$\therefore$

In the rejection method, we sample  $x_i \sim q$ ,  $u \sim \text{Unif}[0, 1]$ , then we accept a sample  $x_i$  if  $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$ .

Hence,

- If  $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$  is rejected;
- If  $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$  is accepted;

Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

Finally, the expected number of trials is equal to  $M = e^{\lambda a}$ . Therefore, for  $a \gg 1/\lambda$ , the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.

**Exercise 2 (Rejection)**

1. Let  $A$  be the event where the value is accepted at some point, while  $A_b$  is accepted at step (b) and  $A_c$  is accepted at step (c):

In step (b) we have:

$$\mathbb{P}(x \in A_b) = \frac{h(x)}{M\tilde{q}(x)}$$

In step (c) we have:

$$\mathbb{P}(x \in A_c) = \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)}$$

Since step (b) is independent of (c),

$$\begin{aligned} \mathbb{P}(x \in A) &= \mathbb{P}(x \in A_b \cup x \in A_c) = \\ &= \mathbb{P}(x \in A_b) + \mathbb{P}(x \in A_c) - \mathbb{P}(x \in A_b \cap x \in A_c) = \\ &= \frac{h(x)}{M\tilde{q}(x)} + \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} + \frac{h(x)}{M\tilde{q}(x)} \cdot \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} = \frac{\tilde{\pi}(x)}{M\tilde{q}(x)} \end{aligned}$$

2. Let  $B$  be an arbitrary event.

$$\mathbb{P}(X \in B \mid X \in A) = \mathbb{P}(X \in B \cap X \in A) / \mathbb{P}(X \in A) \therefore$$

$$\mathbb{P}(X \in B \cap X \in A) = \int_{\mathcal{X}} \int_0^1 \mathbb{I}_B(x) \mathbb{I}\left(u \leq \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\right) q(x) du dx$$

$$\mathbb{P}(X \in B \cap X \in A) = \int_B \frac{\tilde{\pi}(x)}{M\tilde{q}(x)} \tilde{q}(x) \cdot Z_q^{-1} dx$$

$$\mathbb{P}(X \in B \cap X \in A) = \frac{\pi(B) \cdot Z_\pi}{M \cdot Z_q}$$

Finally,

$$\mathbb{P}(X \in A) = \frac{Z_\pi}{M \cdot Z_q} \therefore \mathbb{P}(X \in B \mid X \in A) = \frac{\pi(B) Z_\pi}{M Z_q} \cdot \frac{M Z_q}{Z_\pi} = \pi(B)$$

3. We want to show that:

$$\mathbb{P}(\text{Step (c) is necessary}) = 1 - \frac{\int_{\mathcal{X}} h(x) dx}{M Z_q}$$

First, note that  $\mathbb{P}(\text{Step (c) is necessary}) = \mathbb{P}(X \text{ not accepted in step(b)})$ , hence:

$$\mathbb{P}(X \text{ not accepted in step(b)}) = 1 - \mathbb{P}(X \in A_b) = 1 - \mathbb{P}\left(U \leq \frac{h(X)}{M\tilde{q}(X)}\right)$$

$$\begin{aligned}\mathbb{P}(X \in A_b) &= \mathbb{P}(X \in \mathcal{X} \cap X \in A_b) = \\ &= \int_{\mathcal{X}} \int_0^1 \mathbb{I}_{\mathcal{X}}(x) \mathbb{I}\left(u \leq \frac{h(x)}{M\tilde{q}(x)}\right) du dx = \\ &= \int_{\mathcal{X}} \frac{h(x)\tilde{q}(x)}{M\tilde{q}(x)Z_q} dx = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q} \therefore \\ \mathbb{P}(\text{Step(c) is necessary}) &= 1 - \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q}\end{aligned}$$

4. We want to calculate the probability of not having to evaluate  $\tilde{p}i(x)$ , which is equal to the probability of accepting the sample in step (b).

First, we know that:

$$\mathbb{P}(X \text{ is accepted in step(b)}) = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q}$$

Note that  $h(x) \geq 0$ , therefore  $1 - x^2/2 \geq 0 \therefore -\sqrt{2} \leq x \leq \sqrt{2}$ .

Hence,  $h(x) = 0, \forall x \notin [-\sqrt{2}, \sqrt{2}]$ .

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} h(x) dx &= \int_{-\sqrt{2}}^{\sqrt{2}} 1 - \frac{x^2}{2} dx = \frac{4\sqrt{2}}{3} \\ \int_{-\infty}^{\infty} e^{-|x|} dx &= Z_q = 2 \int_0^{\infty} e^{-x} dx = 2(-[e^{-\infty} - e^0]) = 2\end{aligned}$$

$$M = \sup_{x \in \mathbb{R}} \frac{\tilde{\pi}(x)}{\tilde{q}(x)} = \sup_{x \in \mathbb{R}} \frac{e^{-x^2/2}}{e^{-|x|}} = \sup_{x \in \mathbb{R}} e^{-x^2/2+|x|}$$

For  $x \geq 0$ , we have  $\frac{d}{dx}(x - x^2/2) = 1 - x = 0 \implies x = 1$ .

For  $x < 0$ , we have  $\frac{d}{dx}(-x - x^2/2) = -1 - x = 0 \implies x = -1$ . Hence,  $M = \sqrt{e}$ .

Finally, we have:

$$\mathbb{P}(X \text{ is accepted in step(b)}) = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q} = \frac{4\sqrt{2}}{3 \cdot 2 \cdot \sqrt{e}} = \frac{2\sqrt{2e^{-1}}}{3}$$

It can be beneficial to use this algorithm instead of the standard rejection sampling procedure because this algorithm can be more computationally efficient.

**Exercise 3 (Transformation)**

1. Let  $V = \arctan(U_2/U_1)$  and  $Y = U_1^2 + U_2^2 \leq 1$ . We want to show that:

$$p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

First, note that  $p_{Y,V}(y, \theta) = p_{U_1,U_2}(u_1, u_2) \left| \begin{array}{cc} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{array} \right|$

$$\left| \begin{array}{cc} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{array} \right| = \left| \begin{array}{cc} \frac{\partial y}{\partial u_1} & \frac{\partial \theta}{\partial u_1} \\ \frac{\partial y}{\partial u_2} & \frac{\partial \theta}{\partial u_2} \end{array} \right|^{-1} = \left| \begin{array}{cc} 2u_1 & \frac{1}{1+(u_1/u_2)^2} \cdot \frac{-1u_2}{u_1^2} \\ 2u_2 & \frac{1}{1+(u_1/u_2)^2} \cdot \frac{1}{u_1} \end{array} \right|^{-1} = \frac{2}{1+(u_2/u_1)^2} + \frac{2(u_2/u_1)^2}{1+(u_2/u_1)^2} =$$

$$= \frac{2}{1+\theta^2} + \frac{2(1+\theta^2)}{1+\theta^2} = 2$$

$$p_{Y,V}(y, \theta) = p_{U_1,U_2}(u_1, u_2) \cdot \frac{1}{2} = p_{U_1,U_2}(\sqrt{y}\cos(\theta), \sqrt{y}\sin(\theta)) \cdot \frac{1}{2}$$

Note that  $p_{U_1,U_2}(\sqrt{y}\cos(\theta), \sqrt{y}\sin(\theta))$  is a uniform distribution over a circle of radius  $\sqrt{y} \leq 1$ , therefore, it's normalizing constant is equal to  $\pi y$ , which is the area of the circle of radius  $\sqrt{y}$ . With that we can write:

$$p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

2. We have shown that  $p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$ . Since we can factor the functions of  $Y$  and  $V$ , this means that they are independent and that  $Y \sim \text{Unif}[0, 1]$ ,  $V \sim \text{Unif}[0, 2\pi]$ .

Note that for  $Z = \sqrt{-2\log(Y)}$ :

$$X_1 = Z \frac{U_1}{\sqrt{Y}} = Z \cos(V), X_2 = Z \frac{U_2}{\sqrt{Y}} = Z \sin(V)$$

Therefore, we get the Box-Muller algorithm, hence the proof proceeds accordingly to show that  $X_1$  and  $X_2$  are independent standard normal distributions.

3. In this approach it is not necessary to calculate the trigonometric functions (cosine and sine) which are computationally expensive.

**Exercise 4 (Transformation)**

1. Let
- $W = V/U$
- .

$$p_{W,U}(w, u) = p_{U,V}(u, v) \left| \begin{array}{cc} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial u} \end{array} \right|$$

$$p_{U,V}(u, v) \left| \begin{array}{cc} -\frac{u}{w} & 1 \\ u & w \end{array} \right| = p_{W,U}(w, u) = p_{U,V}(u, v) \cdot 2u$$

$$p_W(w) = \int_{\mathbb{U}} p_{W,U}(w, u) du = 2 \int_{\mathbb{U}} u \cdot p_{U,V}(u, v) du$$

Note that  $p_{U,V}(u, v) = \frac{\mathbb{I}_G(u, v)}{Z_G}$ , hence:

$$p_W(w) = \frac{2}{Z_G} \int_{\mathbb{U}} u \cdot \mathbb{I}_G(u, v) du = \frac{2}{Z_G} \int_0^{\sqrt{\tilde{\pi}(w)}} u du = \frac{2}{Z_G} [u^2/2]_0^{\sqrt{\tilde{\pi}(w)}} = \frac{\tilde{\pi}(w)}{Z_G} = \frac{\tilde{\pi}(w)}{Z_{\pi}}$$

2. Let
- $R = \left( \left[ 0, \sup_x \sqrt{\tilde{\pi}(x)} \right], \left[ \inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)} \right] \right)$
- .

Since  $G = \{(u, v) \mid 0 \leq u \leq \sqrt{\tilde{\pi}(v/u)}\}$ , it's evident that

$$\inf_G u = 0, \sup_G u = \sup_x \sqrt{\tilde{\pi}(x)} \implies u \in [0, \sup_x \sqrt{\tilde{\pi}(x)}]$$

Note that, for  $x = v/u$ :

$$0 \leq u \leq \sqrt{\tilde{\pi}(v/u)} \implies 0 \leq v/x \leq \sqrt{\tilde{\pi}(v/u)} \therefore 0 \leq v \leq x \sqrt{\tilde{\pi}(v/u)}, \forall x \neq 0$$

Then, similarly to  $u$ , we have:

$$\inf_G v = \inf_x x \sqrt{\tilde{\pi}(x)}, \sup_G v = \sup_x x \sqrt{\tilde{\pi}(x)} \implies v \in [\inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)}]$$

We conclude that  $G \subset R$ .

To sample uniformly in  $G$ , first:

- Generate  $u_i \sim \text{Unif}[0, \sup_x \sqrt{\tilde{\pi}(x)}]$  and
- Generate  $v_i \sim \text{Unif}[\inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)}]$

Then, accept  $(u_i, v_i)$  if  $0 \leq u_i \leq \sqrt{\tilde{\pi}(v_i/u_i)}$ , where  $\tilde{\pi}(x)$  is a probability density function.

3. Let  $\tilde{\pi}(x) = \exp(-x^2/2)$ .

$$\sup_{x \in \mathbb{R}} \exp(-x^2/2) = \exp(0) = 1$$

$$\inf_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = 0$$

$$\sup_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = \sup_{x \in \mathbb{R}} x \cdot \exp(-x^2/4)$$

Note that:

$$\frac{d}{dx} x \cdot \exp(-x^2/4) = \exp(-x^2/4) - \frac{x^2}{2} \exp(-x^2/4) = 0 \implies x = \pm\sqrt{2}$$

$$\sup_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = \sqrt{2 \exp(-2/2)} = \sqrt{2 \exp(-1)}$$

$$U \sim \text{Unif}[0, 1], V \sim \text{Unif}[0, \sqrt{2 \exp(-1)}]$$

Therefore, generate samples from  $U$  and  $V$ . For the sample  $(u, v)$ , if  $0 \leq u \leq \sqrt{2 \cdot \exp(-v^2/u^2)}$ , then accept  $x = v/u$ , which will be a sample from  $\tilde{\pi}(x)$ .