Approximate Bayesian Computation

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Introduction

Objective

The objective of this presentation is to give an overview of the Approximate Bayesian Computation (ABC) algorithm through the replication of the paper **Approximate Bayesian computational methods** by Marin et al. (2012).

The paper talks about different variants of ABC by estimating the posterior of Moving Average models.

Motivation

ABC methods are known as likelihood-free techniques, thus are a useful approach in problems that the likelihood is intractable, e.g., likelihood not available in closed form, or likelihood too expensive to calculate.

- Coalecent models in population genetics (Tavaré et al., 1997);
- Species dynamics (Jabot and Lohier, 2016);
- Real-world model of HIV transmission (McKinley et al., 2018).

Motivation

In some settings where we have latent variables, the likelihood is expressed as:

$$\ell(\boldsymbol{\theta} \mid \boldsymbol{y}) = \int \ell^*(\boldsymbol{\theta} \mid \boldsymbol{y}, \boldsymbol{u}) d\boldsymbol{u}$$

Hence, $\boldsymbol{y}$ is observed and $\boldsymbol{u}$ is latent and $\boldsymbol{\theta}$ is the parameter of interest.

Original Algorithm

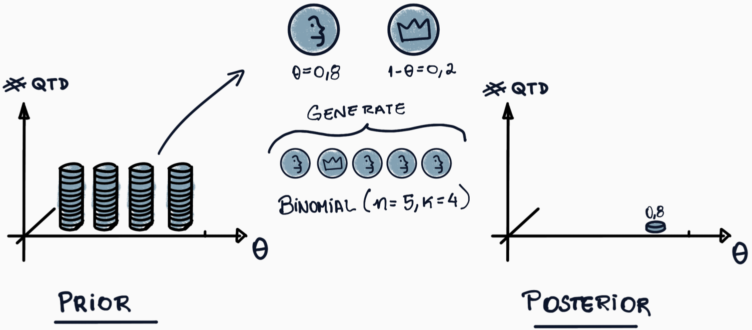
Rubin (1984) described the ABC algorithm as a thought experiment to explain how to sample from a posterior distribution. Tavaré et al. (1997) is usually considered the paper responsible for the proposing ABC for infering the posterior distribution.

Algorithm 1: Original ABC method

for *i=1 to N* **do**
    repeat
        Sample $\boldsymbol{\theta}' \sim \pi(\cdot)$
        Generate $\boldsymbol{z} \sim p(\cdot \mid \boldsymbol{\theta}')$
    until $\boldsymbol{y} = \boldsymbol{z}$;
end

Original Algorithm

Below we have an schematic drawing with an example of the ABC method for Beta/Binomial model.



Original Algorithm

The proof that the algorithm indeed results in an iid sample from the posterior is shown below. Let $\boldsymbol{y}$ be the observed, $\boldsymbol{\theta}$ the parameter of interest and $\boldsymbol{z}$ the generated samples.

$$p(\boldsymbol{\theta}_i) \propto \sum_{\boldsymbol{z} \in \mathbb{D}} \pi(\boldsymbol{\theta}_i) p(\boldsymbol{z} \mid \boldsymbol{\theta}_i) \mathbb{I}_{\boldsymbol{y}}(\boldsymbol{z}) = \pi(\boldsymbol{\theta}_i) p(\boldsymbol{y} \mid \boldsymbol{\theta}_i) \propto \pi(\boldsymbol{\theta}_i \mid \boldsymbol{y})$$

Original Algorithm

Pritchard et al. (1999) extended the original algorithm to the case of continuos sample spaces.

Algorithm 2: ABC method for discrete and continuous distributions

for *i=1 to N* **do**
  repeat
    Sample $\boldsymbol{\theta}' \sim \pi(\cdot)$
    Generate $\boldsymbol{z} \sim p(\cdot \mid \boldsymbol{\theta}')$
  until $\rho[\eta(\boldsymbol{y}), \eta(\boldsymbol{z})] \leq \epsilon$;
end

- η: function defining a statistic (e.g. the mean),
- ρ: a distance function,
- ϵ: acceptance tolerance.

Original Algorithm

For this ABC algorithm, instead of the actual posterior, we get

$$\pi_\epsilon(\boldsymbol{\theta} \mid \boldsymbol{y}) = \frac{\pi(\boldsymbol{\theta}) p(\boldsymbol{z} \mid \boldsymbol{\theta}) \mathbb{I}_{A_{\epsilon,\boldsymbol{y}}}(\boldsymbol{z})}{\int_{A_{\epsilon,\boldsymbol{y}} \times \boldsymbol{\theta}} \pi(\boldsymbol{\theta}) p(\boldsymbol{z} \mid \boldsymbol{\theta}) d\boldsymbol{z} d\boldsymbol{\theta}}$$

Where, $A_{\epsilon,\boldsymbol{y}} = \{\boldsymbol{z} \in \mathbb{D} \mid \rho[\eta(\boldsymbol{z}), \eta(\boldsymbol{y}) \leq \epsilon].\}$

Hence, for a tolerance (ϵ) "small enough", we expect a good approximation.

$$\pi_\epsilon(\boldsymbol{\theta} \mid \boldsymbol{y}) = \int \pi_\epsilon(\boldsymbol{\theta}, \boldsymbol{z} \mid \boldsymbol{y}) d\boldsymbol{z} \approx \pi(\boldsymbol{\theta} \mid \boldsymbol{y})$$

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References ii

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