Exercise 1 (Kalma Filter)

1.
$$f(X_t \mid X_{t-1}) \sim N(\phi X_{t-1}, \sigma_v^2)$$
$$g(Y_t \mid X_t) \sim N(\phi X_t, \sigma_w^2)$$

2. Note that $X_{t+1} = \phi X_t + V_t$. Then:

$$X_{t+1} \mid Y_{1:t} = \phi X_t + V_t \mid Y_{1:t}$$

Therefore, since $X_t \mid Y_{1:t} \sim N(m_{t|t}, \sigma^2_{t|t})$ and $V_t \sim N(0, \sigma_v^2)$, we have the sum of normals, hence:
$$X_{t+1} \mid Y_{1:t} \sim N(m_{t|t}, \sigma_v^2 + \phi^2 \sigma^2_{t|t})$$

3. Note that:
$$p(X_{t=1} \mid Y_{1:t}, Y_{t+1}) \propto N(X_{t+1}, \sigma_w^2) N(m_{t+1}, \sigma^2_{t+1})$$

The update of normal distribution by normal distribution is given by:

$$\mu_1 = \frac{y_{t+1}\sigma_w^{-2} + m_{t+1}\sigma^2_{t+1}}{\sigma_w^{-2} + \sigma^{-2}_{t+1}} \qquad \tau_1^2 = (\sigma_w^{-2} + \sigma_v^{-2})^{-1}$$

Hence, $p(x_{t+1} \mid y_{1:t}) = N(\mu_1, \tau_1^2) = N(m_{t+1|t}, \sigma^2_{t+1|t})$.

4. Note that:
$$Y_{t+1} \mid Y_{1:t} = X_{t+1} + W_{t+1} \mid Y_{1:t} = N(\mu_1, \tau_1) + N(0, \sigma_w^2)$$

$$Y_{t+1} \mid Y_{1:t} = N(\mu_1, \sigma_w^2 + \tau_1^2)$$