

Exercise 1 (Monte Carlo for Gaussians)

1. Let's prove that $E[\phi(X)] = E[\phi(X + \theta) \exp(\frac{-1}{2}\theta^T \theta - \theta^T X)]$.

$$\begin{aligned} E[\phi(X + \theta) \exp(\frac{-1}{2}\theta^T \theta - \theta^T X)] &= \int_{\mathbb{R}^d} \phi(x + \theta) \exp(\frac{-1}{2}\theta^T \theta - \theta^T X) \pi(x) dx_1 \dots dx_d = \\ &= \int_{\mathbb{R}^d} \phi(x + \theta) \exp\left(\frac{-1}{2}\theta^T \theta - \theta^T X\right) \exp(-x^T x/2) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d = \\ &= \int_{\mathbb{R}^d} \phi(x + \theta) \exp\left(\frac{-1}{2}(x - \theta)^T (x - \theta)\right) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d \end{aligned}$$

Finally, making $x - \theta = y$,

$$\int_{\mathbb{R}^d} \phi(y) \exp\left(\frac{-1}{2}(y)^T (y)\right) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d = E[\phi(Y)]$$

□

2. Let's show that

$$\sigma^2(\theta) = E\left[\phi^2(X) \exp\left(\frac{-1}{2}X^T X + \frac{1}{2}(X - \theta)^T (X - \theta)\right)\right] - (E[\phi(X)])^2$$

Note that, using the result in the previous item we have:

$$\begin{aligned} \sigma^2(\theta) &= V\left[\phi(X + \theta) \exp\left(\frac{-1}{2}\theta^T \theta - \theta^T X\right)\right] = \\ &= E\left[\left(\phi(X + \theta) \exp\left(\frac{-1}{2}\theta^T \theta - \theta^T X\right)\right)^2\right] - E\left[\phi(X + \theta) \exp\left(\frac{-1}{2}\theta^T \theta - \theta^T X\right)\right]^2 = \\ &= E\left[\left(\phi(X + \theta) \exp\left(\frac{-1}{2}\theta^T \theta - \theta^T X\right)\right)^2\right] - E[\phi(X)]^2 \end{aligned}$$

Now, let's rearrange the first term in the variance.

$$\sigma^2(\theta) = \int_{\mathbb{R}^d} \phi(x + \theta)^2 \exp(-\theta^T \theta - 2\theta^T X) \exp(-x^T x/2) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d =$$

Make $X + \theta = Y$, then:

$$\begin{aligned}
& \int_{\mathbb{R}^d} \phi(y)^2 \exp(-\theta^T \theta - 2\theta^T(y - \theta)) \exp(-(y - \theta)^T(y - \theta)/2) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d = \\
& = \int_{\mathbb{R}^d} \phi(y)^2 \exp\left(\frac{1}{2}(y - \theta)^T(y - \theta) - \frac{y^T y}{2}\right) \exp\left(\frac{-y^T y}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx_1 \dots dx_d = \\
& = E\left[\phi^2(X) \exp\left(\frac{-1}{2}X^T X + \frac{1}{2}(X - \theta)^T(X - \theta)\right)\right]
\end{aligned}$$

Therefore,

$$\sigma^2(\theta) = E\left[\phi^2(X) \exp\left(\frac{-1}{2}X^T X + \frac{1}{2}(X - \theta)^T(X - \theta)\right)\right] - (E[\phi(X)])^2$$

□

3.