Exercise 1 (Gibbs Sampler)

1. First, let $X' = X^{(t)}$, $X = X^{(t-1)}$, $Y' = Y^{(t)}$ and $Y = Y^{(t-1)}$. Then:

$$K^S((x,y),(x',y')) = \pi_{Y|X}(y' \mid x)\pi_{X|Y}(x' \mid y')$$

Then, to show that it is not reversible:

$$\pi(x,y)K((x,y),(x',y')) = \pi(x,y)\pi(y' \mid x)\pi(x' \mid y')$$
$$\pi(x',y')K((x',y'),(x,y)) = \pi(x',y')\pi(y \mid x')\pi(x \mid y)$$
$$\therefore$$
$$\frac{\pi(x,y)K((x,y),(x',y'))}{\pi(x',y')K((x',y'),(x,y))} = \frac{\pi(x,y)\pi(y' \mid x)\pi(x' \mid y')}{\pi(x',y')\pi(y \mid x')\pi(x \mid y)} = \frac{\pi(y)\pi(y' \mid x)}{\pi(y')\pi(y \mid x')} \neq 1$$

Therefore, it is not reversible. □

2. First, the kernel expression:

$$K(x,x') = \int \pi(y' \mid x)\pi(x' \mid y')dy'$$

Now, let's show that it is π_X-reversible.

$$\pi(x')K(x',x) = \pi(x')\int \pi(y \mid x')\pi(x \mid y)dy = \int \pi(x')\frac{\pi(y,x')}{\pi(x')}\pi(x \mid y)dy =$$
$$= \int \pi(y,x')\pi(x \mid y)dy = \int \pi(y,x')\frac{\pi(x,y)}{\pi(y)}dy = \int \pi(x,y)\frac{\pi(x',y)}{\pi(y)}dy =$$
$$= \pi(x)\int \pi(y \mid x)\pi(x' \mid y)dy = \pi(x)K(x,x')$$

□

3. First, the kernel expression is:

$$K^R((x,y),(x',y')) = \pi(y' \mid x)\pi(x' \mid y')0.5 + \pi(x' \mid y)\pi(y' \mid x')0.5$$

Note that it is half the density of sampling first from y plus half the density of sampling first from x.

Now, let's show that it is reversible:

$$\frac{\pi(x,y)[\pi(y' \mid x)\pi(x' \mid y')0.5 + \pi(x' \mid y)\pi(y' \mid x')0.5]}{\pi(x',y')[\pi(y \mid x')\pi(x \mid y')0.5 + \pi(x \mid y')\pi(y \mid x)0.5]} =$$
$$= \frac{\frac{\pi(y'|x)}{\pi(y')} + \frac{\pi(x'|y)}{\pi(x')}}{\frac{\pi(y|x')}{\pi(y)} + \frac{\pi(x|y')}{\pi(x)}} = 1$$

□

Exercise 2 (Metropolis-within-Gibbs)

1. $$q(x_1' \mid x_1, x_2)$$