Exercise 1 (Gibbs Sampler)

1. First, let $X' = X^{(t)}$, $X = X^{(t-1)}$, $Y' = Y^{(t)}$ and $Y = Y^{(t-1)}$. Then:

$$K^S((x,y),(x',y')) = \pi_{Y|X}(y' \mid x)\pi_{X|Y}(x' \mid y')$$

Then, to show that it is not reversible:

$$\pi(x,y)K((x,y),(x',y')) = \pi(x,y)\pi(y' \mid x)\pi(x' \mid y')$$
$$\pi(x',y')K((x',y'),(x,y)) = \pi(x',y')\pi(y \mid x')\pi(x \mid y)$$
$$\therefore$$
$$\frac{\pi(x,y)K((x,y),(x',y'))}{\pi(x',y')K((x',y'),(x,y))} = \frac{\pi(x,y)\pi(y' \mid x)\pi(x' \mid y')}{\pi(x',y')\pi(y \mid x')\pi(x \mid y)} = \frac{\pi(y)\pi(y' \mid x)}{\pi(y')\pi(y \mid x')} \neq 1$$

Therefore, it is not reversible. □

2. First, the kernel expression:

$$K(x,x') = \int \pi(y' \mid x)\pi(x' \mid y')dy'$$

Now, let's show that it is π_X-reversible.

$$\pi(x')K(x',x) = \pi(x')\int \pi(y \mid x')\pi(x \mid y)dy = \int \pi(x')\frac{\pi(y,x')}{\pi(x')}\pi(x \mid y)dy =$$
$$= \int \pi(y,x')\pi(x \mid y)dy = \int \pi(y,x')\frac{\pi(x,y)}{\pi(y)}dy = \int \pi(x,y)\frac{\pi(x',y)}{\pi(y)}dy =$$
$$= \pi(x)\int \pi(y \mid x)\pi(x' \mid y)dy = \pi(x)K(x,x')$$

□

3. First, the kernel expression is:

$$K^R((x,y),(x',y')) = \pi(y' \mid x)\pi(x' \mid y')0.5 + \pi(x' \mid y)\pi(y' \mid x')0.5$$

Note that it is half the density of sampling first from y plus half the density of sampling first from x.

Now, let's show that it is reversible:

$$\frac{\pi(x,y)[\pi(y' \mid x)\pi(x' \mid y')0.5 + \pi(x' \mid y)\pi(y' \mid x')0.5]}{\pi(x',y')[\pi(y \mid x')\pi(x \mid y')0.5 + \pi(x \mid y')\pi(y \mid x)0.5]} =$$
$$= \frac{\frac{\pi(y'|x)}{\pi(y')} + \frac{\pi(x'|y)}{\pi(x')}}{\frac{\pi(y|x')}{\pi(y)} + \frac{\pi(x|y')}{\pi(x)}} = 1$$

□

Exercise 2 (Metropolis-within-Gibbs)

1. Note that:

$$\alpha(X_1 \mid X_1^{(t-1)}, X_2^{(t-2)}) = min\left\{1, \frac{\pi(X_1', X_2^{(t-1)})\pi(X_1^{(t-1)} \mid X_2^{(t-1)})}{\pi(X_1^{(t-1)}, X_2^{(t-1)})\pi(X_1' \mid X_2^{(t-1)})}\right\} = min\{1, 1\}$$

   Therefore, we get a systematic scan Gibbs sampler, where one samples $X_1^t \sim \pi(\cdot \mid X_2^{(t-1)})$, then we accept, since $\alpha = 1$, and finally sample $X_2^t \sim \pi(\cdot \mid X_1^{(t)})$. $\square$

2. First, let's write the kernel. Since we only accept or reject the variabel X_1, the kernel is the M-H kernel multiplied by the probability density function of $\pi_{X_2|X_1}(X_2 \mid X_1)$. Let $X_1^t, X_2^t = Y_1, Y_2$:

$$K((x_1, x_2), (y_1, y_2)) = (q(y_1 \mid x_1, x_2)\alpha(y_1 \mid x_1, x_2)) + (1 - a(y_1 \mid x_1, x_2))\delta_{y_1}(x_1)\pi_{(Y_2|Y_1}(y_2 \mid y_1)$$

   Note that $\alpha = 1$. With that, we show that the kernel is invarant:

$$\int\int K((x_1, x_2), (y_1, y_2))\pi(x_1, x_2)dx_1dx_2 = \int\int \pi(y_1 \mid x_2)\pi(y_2 \mid y_1)\pi(x_1, x_2)dx_1dx_2 =$$

$$= \int \pi(y_1 \mid x_2)\pi(y_2 \mid y_1)\pi(x_2)dx_2 = \int \pi(y_1, x_2)\pi(y_2 \mid y_1)dx_2 = \pi(y_1, y_2)$$

$\square$

Exercise 3 (Metropolis-Hastings and Gibbs Sampler)

1. Let's show that the chain is reversible. If $x = y$, it is trivially reversible. If $x \neq y$, then:

$$T(x,y)\pi(x) = \alpha(x,y)q(x,y)\pi(x) = \frac{\gamma(x,y)}{\pi(x)q(x,y)}q(x,y)\pi(x) = \gamma(y,x) =$$

$$= \alpha(y,x)q(y,x)\pi(y) = T(y,x)\pi(y)$$

$\square$

2. First, let's verify that it is the M-H algorithm:

$$\alpha = \frac{\gamma(x,y)}{\pi(x)q(x,y)} = \frac{min\{\pi(x)q(x,y), \pi(y)q(y,x)\}}{\pi(x)q(x,y)} = min\left\{1, \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)}\right\}$$

Now, let's give the Barker acceptance ratio:

$$\alpha(x,y) = \frac{\pi(x)q(x,y)\pi(y)q(y,x)}{\pi(x)q(x,y)+\pi(y)q(y,x)} = \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)+\pi(y)q(y,x)}$$

$\square$

3. Let's consider $x \neq y$. Note that:

$$\frac{1}{\pi(x)q(x,y)} \geq \frac{1}{\pi(x)q(x,y)+\pi(y)q(y,x)}$$

$$\therefore$$

$$\frac{\pi(y)q(y,x)q(x,y)}{\pi(x)q(x,y)} \geq \frac{q(x,y)\pi(y)q(y,x)}{\pi(x)q(x,y)+\pi(y)q(y,x)}$$

Finally, if $\frac{\pi(y)q(y,x)q(x,y)}{\pi(x)q(x,y)} \leq 1$, then:

$$min\left\{1, \frac{\pi(y)q(y,x)q(x,y)}{\pi(x)q(x,y)}\right\} \geq \frac{q(x,y)\pi(y)q(y,x)}{\pi(x)q(x,y)+\pi(y)q(y,x)}$$

We conclude that the M-H algorithm provides estimators with lower asymptotic variance than Barker's algorithm.