Exercise 1 (Monte Carlo for Gaussians)

1. Let's prove that $E[\phi(X)] = E[\phi(X+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)]$.

$$E[\phi(X+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)] = \int_{\mathbb{R}^d} \phi(x+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)\pi(x)dx_1...dx_d =$$

$$= \int_{\mathbb{R}^d} \phi(x+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right) exp(-x^T x/2)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

$$\int_{\mathbb{R}^d} \phi(x+\theta)exp\left(\frac{-1}{2}(x-\theta)^T(x-\theta)\right)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d$$

Finally, making $x - \theta = y$,

$$\int_{\mathbb{R}^d} \phi(y)exp\left(\frac{-1}{2}(y)^T(y)\right)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d = E[\phi(Y)]$$

□

2. Let's show that

$$\sigma^2(\theta) = E\left[\phi^2(X)exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] - (E[\phi(X)]^2$$

Note that, using the result in the previous item we have:

$$\sigma^2(\theta) = V\left[\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right] =$$

$$= E\left[\left(\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right)^2\right] - E\left[\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right]^2 =$$

$$= E\left[\left(\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right)^2\right] - E\left[\phi(X)\right]^2$$

Now, let's rearrange the first term in the variance.

$$\sigma^2(\theta) = \int_{\mathbb{R}^d} \phi(x+\theta)^2 exp\left(-\theta^T\theta - 2\theta^T X\right) exp(-x^T x/2)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

Make $X + \theta = Y$, then:

$$\int_{\mathbb{R}^d} \phi(y)^2 exp\left(-\theta^T\theta - 2\theta^T(y-\theta)\right) exp(-(y-\theta)^T(y-\theta)/2) \frac{1}{(\sqrt{2\pi})^d} dx_1 ... dx_d =$$

$$= \int_{\mathbb{R}^d} \phi(y)^2 exp\left(\frac{1}{2}(y-\theta)^T(y-\theta) - \frac{y^Ty}{2}\right) exp\left(\frac{-y^Ty}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx_1 ... dx_d =$$

$$= E\left[\phi^2(X) exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right]$$

Therefore,

$$\sigma^2(\theta) = E\left[\phi^2(X) exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] - (E[\phi(X)]^2$$

$\square$

3. Let's calculate $\nabla^2\sigma^2(\theta) = H(\theta)$.

$$\frac{\partial\sigma^2(\theta)}{\partial\theta_i} = \frac{E[\phi(X)^2 exp(\frac{-X^TX + (X-\theta)^T(X-\theta)}{2})]}{\partial\theta_i} =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx) \frac{\partial}{\partial\theta_i} exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i) exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx$$

We calculated the gradient, let's now calculate the second derivative. First the diagonal.

$$\frac{\partial}{\partial\theta_i} \int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i) exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx =$$

$$= E[\phi(X)^2] + \int_\chi \phi(x)^2 exp(-x^Tx) exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)(x_i-\theta_i)(x_i-\theta_i) \frac{1}{(\sqrt{2\pi})^d} dx$$

Now the rest:

$$\frac{\partial}{\partial\theta_j} \int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i) exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right) \frac{1}{(\sqrt{2\pi})^d} dx =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx) exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)(x_i-\theta_i)(x_j-\theta_j) \frac{1}{(\sqrt{2\pi})^d} dx$$

$\square$

4. We already know that the Hessian is positive definite. Hence, we only need to show that the derivative is equal to zero at θ^*.

$$\nabla\sigma^2(\theta) = \int_\chi \phi(x)^2 exp(-x^T x)(\theta - x) exp\left(\frac{-(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$
$$= \int_\chi \phi(x)^2 exp(-x^T x)(\theta - x) exp\left(\frac{-(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$
$$= \int_\chi \phi(x)^2 (\theta - x) exp\left(\frac{-x^T x}{2} - \theta^T x + \frac{-\theta^T\theta}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$
$$= \int_\chi \phi(x)^2 (\theta - x) exp(-\theta^T x) exp(-x^T x/2)\frac{exp(\theta^T\theta/2)}{(\sqrt{2\pi})^d}dx = 0$$

Finally, since the last term doens't depend on X, we can eliminate it, obtaining:

$$E[\phi(X)^2(\theta - X)exp(-\theta^T X)] = 0$$

$\square$