Notation: here is a brief summary of the notation used in this worksheet.

- $p(X = x)$ is equal to the probability density function;
- Capital letters such as $X$ stand for the random variable.

## Exercise 1 (Inversion and Rejection)

1. Let $F_X(x) = \mathbb{P}(X \leq x)$ and $U \sim Unif[0, 1]$:

$$F_X(x) = 1 - e^{-\lambda(X-a)}\mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1 - U) = \lambda(x - a)$$

$$F_X^{-1}(U) = a - \frac{-\ln(1 - U)}{\lambda}$$

To simulate X from U, just simulate value from U and substitute in the formula above.

2. Let $X = Y \mid a \leq Y \leq b$. First, let's show that $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$:

$$\mathbb{P}(X \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U)+F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)+U[F_Y(b)-F_X(a)]) \leq x)$$

$$= \mathbb{P}(F_Y(a)+U[F_Y(b)-F_X(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}$$

Note that since $x \in [a, b]$:

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b) - F_Y(a)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on $\geq a$, we first generate $U \sim Unif[0, 1]$, then, for $Y \sim Expo(\lambda)$:

$$F_Y(y) = 1 - e^{\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1 - U)}{\lambda}$$

$$X = \frac{-\ln(1 - (1 - U)F_Y(a) + U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1 - U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let $q \sim Expo(\lambda)$, and $\pi(x) = \lambda e^{-\lambda(x-a)}\mathbb{I}_{x\geq a}$:

   Note that $M = max_x \pi(x)/q(x) = e^{\lambda a}$, since $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{\lambda(x)}} = e^{\lambda a}$

$$\therefore$$

   In the rejection method, we sample $x_i \sim q$, $u \sim Unif[0,1]$ ,then we accept a sample $x_i$ if $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$.

   Hence,

   - If $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$ is rejected;
   - If $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$ is accepted;

   Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

   Finally, the expected number of trials is equal to $M = e^{\lambda a}$. Therefore, for $a \gg 1/\lambda$, the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.

# Exercise 2 (Rejection)

1. Let $A$ be the event where the value is accepted at some point, while $A_b$ is accepted at step (b) and $A_c$ is accepted at step (c):

   In step (b) we have:
$$\mathbb{P}(X \in A_b) = \frac{h(x)}{M\tilde{q}(x)}$$
   In step (c) we have:
$$\mathbb{P}(X \in A_c) = \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)}$$
   Since step (b) is independent of (c),
$$\mathbb{P}(X \in A) = \mathbb{P}(X \in A_b \cup X \in A_c) =$$
$$= \mathbb{P}(X \in A_b) + \mathbb{P}(X \in A_b) - \mathbb{P}(X \in A_b \cap X \in A_c) =$$
$$= \frac{h(x)}{M\tilde{q}(x)} + \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} + \frac{h(x)}{M\tilde{q}(x)} \cdot \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} = \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}$$

2. Let B be an arbitrary event.
$$\mathbb{P}(X \in B \mid X \in A) = \mathbb{P}(X \in B \cap X \in A)/\mathbb{P}(X \in A) \therefore$$
$$\mathbb{P}(X \in B \cap X \in A) = \int_{\chi} \int_0^1 \mathbb{I}_B(x)\mathbb{I}\left(u \leq \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\right) dudx$$
$$\mathbb{P}(X \in B \cap X \in A) = \int_B \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\tilde{q}(x) \cdot Z_q^{-1} dx$$
$$\mathbb{P}(X \in B \cap X \in A) = \frac{\pi(B) \cdot Z_\pi}{M \cdot Z_q}$$
   Finally,
$$\mathbb{P}(X \in A) = \frac{Z_\pi}{M \cdot Z_q} \therefore \mathbb{P}(X \in B \mid X \in A) = \frac{\pi(B) Z_\pi}{M Z_q} \cdot \frac{M Z_q}{Z_\pi} = \pi(B)$$
   .