Notation: here is a brief summary of the notation used in this worksheet.

- $p(X = x)$ is equal to the probability density function;
- Capital letters such as X stand for the random variable.

Exercise 1 (Inversion and Rejection)

1. Let $F_X(x) = \mathbb{P}(X \leq x)$ and $U \sim Unif[0, 1]$:

$$F_X(x) = 1 - e^{-\lambda(X-a)}\mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1 - U) = \lambda(x - a)$$

$$F_X^{-1}(U) = a - \frac{\ln(1 - U)}{\lambda}$$

To simulate X from U, just simulate value from U and substitute in the formula above.

2. Let $X = Y \mid a \leq Y \leq b$. First, let's show that $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$:

$$\mathbb{P}(X \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U)+F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)+U[F_Y(b)-F_X(a)]) \leq x)$$

$$= \mathbb{P}(F_Y(a)+U[F_Y(b)-F_X(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}$$

Note that since $x \in [a, b]$:

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b) - F_Y(a)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on $\geq a$, we first generate $U \sim Unif[0, 1]$, then, for $Y \sim Expo(\lambda)$:

$$F_Y(y) = 1 - e^{\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1 - U)}{\lambda}$$

$$X = \frac{-\ln(1 - (1 - U)F_Y(a) + U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1 - U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let $q \sim Expo(\lambda)$, and $\pi(x) = \lambda e^{-\lambda(x-a)}\mathbb{I}_{x\geq a}$:

   Note that $M = max_x \pi(x)/q(x) = e^{\lambda a}$, since $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{\lambda(x)}} = e^{\lambda a}$

   $$\therefore$$

   In the rejection method, we sample $x_i \sim q$, $u \sim Unif[0,1]$,then we accept a sample x_i if $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$.

   Hence,

   - If $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$ is rejected;
   - If $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$ is accepted;

   Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

   Finally, the expected number of trials is equal to $M = e^{\lambda a}$. Therefore, for $a \gg 1/\lambda$, the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.

Exercise 2 (Rejection)

1. Let A be the event where the value is accepted at some point, while A_b is accepted at step (b) and A_c is accepted at step (c):

   In step (b) we have:
$$\mathbb{P}(x \in A_b) = \frac{h(x)}{M\tilde{q}(x)}$$
   In step (c) we have:
$$\mathbb{P}(x \in A_c) = \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)}$$
   Since step (b) is independent of (c),
$$\mathbb{P}(x \in A) = \mathbb{P}(x \in A_b \cup x \in A_c) =$$
$$= \mathbb{P}(x \in A_b) + \mathbb{P}(x \in A_b) - \mathbb{P}(x \in A_b \cap x \in A_c) =$$
$$= \frac{h(x)}{M\tilde{q}(x)} + \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} + \frac{h(x)}{M\tilde{q}(x)} \cdot \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} = \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}$$

2. Let B be an arbitrary event.
$$\mathbb{P}(X \in B \mid X \in A) = \mathbb{P}(X \in B \cap X \in A)/\mathbb{P}(X \in A) \therefore$$
$$\mathbb{P}(X \in B \cap X \in A) = \int_\chi \int_0^1 \mathbb{I}_B(x)\mathbb{I}\left(u \leq \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\right) q(x)dudx$$
$$\mathbb{P}(X \in B \cap X \in A) = \int_B \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\tilde{q}(x) \cdot Z_q^{-1}dx$$
$$\mathbb{P}(X \in B \cap X \in A) = \frac{\pi(B) \cdot Z_\pi}{M \cdot Z_q}$$
   Finally,
$$\mathbb{P}(X \in A) = \frac{Z_\pi}{M \cdot Z_q} \therefore \mathbb{P}(X \in B \mid X \in A) = \frac{\pi(B)Z_\pi}{MZ_q} \cdot \frac{MZ_q}{Z_\pi} = \pi(B)$$
   .

3. We want to show that:
$$\mathbb{P}(\text{Step (c) is necessery}) = 1 - \frac{\int_\chi h(x)dx}{MZ_q}$$
   First, note that $\mathbb{P}$(Step (c) is necessery) = $\mathbb{P}$(X not accepted in step(b)), hence:

$$\mathbb{P}(\text{X not accepted in step(b)}) = 1 - \mathbb{P}(X \in A_b) = 1 - \mathbb{P}\left(U \leq \frac{h(X)}{M\tilde{q}(X)}\right)$$

$$\mathbb{P}(X \in A_b) = \mathbb{P}(X \in \chi \cap X \in A_b) =$$
$$\int_\chi \int_0^1 \mathbb{I}_\chi(x)\mathbb{I}\left(u \leq \frac{h(x)}{M\tilde{q}(x)}\right) du dx =$$
$$\int_\chi \frac{h(x)\tilde{q}(x)}{M\tilde{q}(x)Z_q} dx = \frac{\int_\chi h(x)dx}{MZ_q} \therefore$$
$$\mathbb{P}(\text{Step(c) is necessery}) = 1 - \frac{\int_\chi h(x)dx}{MZ_q}$$

.

4. We want to calculate the probability of not having to evaluate $\tilde{pi}(x)$, which is equal to the probability of accepting the sample in step (b).

   First, we know that:

$$\mathbb{P}(\text{X is accepted in step(b)}) = \frac{\int_\chi h(x)dx}{MZ_q}$$

   .

   Note that $h(x) \geq 0$, therefore $1 - x^2/2 \geq 0 \therefore -\sqrt{2} \leq x \leq \sqrt{2}$.
   Hence, $h(x) = 0, \forall x \notin [-\sqrt{2}, \sqrt{2}]$.

$$\int_{-\sqrt{2}}^{\sqrt{2}} h(x)dx = \int_{-\sqrt{2}}^{\sqrt{2}} 1 - \frac{x^2}{2} dx = \frac{4\sqrt{2}}{3}$$

$$\int_{-\infty}^{\infty} e^{-|x|}dx = Z_q = 2\int_0^{\infty} e^{-x}dx = 2(-[e^{-\infty} - e^0]) = 2$$

$$M = \sup_{x\in\mathbb{R}} \frac{\tilde{\pi}(x)}{\tilde{q}(x)} = \sup_{x\in\mathbb{R}} \frac{e^{-x^2/2}}{e^{-|x|}} = \sup_{x\in\mathbb{R}} e^{-x^2/2+|x|}$$

   For $x \geq 0$, we have $\frac{d}{dx}(x - x^2/2) = 1 - x = 0 \implies x = 1$.
   For $x < 0$, we have $\frac{d}{dx}(-x - x^2/2) = -1 - x = 0 \implies x = -1$. Hence, $M = \sqrt{e}$.
   Finally, we have:

$$\mathbb{P}(\text{X is accepted in step(b)}) = \frac{\int_\chi h(x)dx}{MZ_q} = \frac{4\sqrt{2}}{3 \cdot 2 \cdot \sqrt{e}} = \frac{2\sqrt{2e^{-1}}}{3}$$

   It can be beneficial to use this algorithm instead of the standard rejection sampling procedure because this algorithmcan be more computationally efficient.

Exercise 3 (Transformation)

1. Let $V = arctan(U_2/U_1)$ and $Y = U_1^2 + U_2^2 \leq 1$. We want to show that:

$$p_{Y,V}(y,\theta) = \mathbb{I}_{[0,1]}(y)\frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

First, note that $p_{Y,V}(y,\theta) = p_{U_1,U_2}(u_1,u_2)\begin{vmatrix} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{vmatrix}$

$$\begin{vmatrix} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial y}{\partial u_1} & \frac{\partial \theta}{\partial u_1} \\ \frac{\partial y}{\partial u_2} & \frac{\partial \theta}{\partial u_2} \end{vmatrix}^{-1} = \begin{vmatrix} 2u_1 & \frac{1}{1+(u_1/u_2)^2}\cdot\frac{-1u_2}{u_1^2} \\ 2u_2 & \frac{1}{1+(u_1/u_2)^2}\cdot\frac{1}{u_1^2} \end{vmatrix}^{-1} = \frac{2}{1+(u_2/u_1)^2} + \frac{2(u_2/u_1)^2}{1+(u_2/u_1)^2} =$$

$$= \frac{2}{1+\theta^2} + \frac{2(1+\theta^2)}{1+\theta^2} = 2$$

$$p_{Y,V}(y,\theta) = p_{U_1,U_2}(u_1,u_2)\cdot\frac{1}{2} = p_{U_1,U_2}(\sqrt{y}cos(\theta),\sqrt{y}sin(\theta))\cdot\frac{1}{2}$$

Note that $p_{U_1,U_2}(\sqrt{y}cos(\theta),\sqrt{y}sin(\theta))$ is a uniform distribution over a circle of radius $Y \leq 1$, therefore, it's normalizing constant is equal to pi, which is the area of the circumference of radius 1. With that we can write:

$$p_{Y,V}(y,\theta) = \mathbb{I}_{[0,1]}(y)\frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

2. We have shown that $p_{Y,V}(y,\theta) = \mathbb{I}_{[0,1]}(y)\frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$. Since we can factor the functions of Y and V, this means that they are independent and that $Y \sim Unif[0,1], V \sim Unif[0,2\pi]$.

   Note that for $Z = \sqrt{-2\log(Y)}$:

$$X_1 = Z\frac{U_1}{\sqrt{Y}} = Zcos(V), X_2 = Z\frac{U_2}{\sqrt{Y}} = Zsin(V)$$

   Therefore, we get the Box-Muller algorithm, hence the proof proceeds accordingly to show that X_1 and X_2 are independent standard normal distributions.

3. In this approach it is not necessery to calculate the trigonometric functions (cossine and sine) which are computationally expensie.

Exercise 4 (Transformation)

1. Let $W = V/U$.

$$p_{W,U}(w,u) = p_{U,V}(u,v)\begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial u} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial u} \end{vmatrix}$$

$$p_{U,V}(u,v)\begin{vmatrix} \frac{-u}{w} & 1 \\ u & w \end{vmatrix} = p_{W,U}(w,u) = p_{U,V}(u,v)\cdot 2u$$

$$p_W(w) = \int_{\mathbb{U}} p_{W,U}(w,u)du = 2\int_{\mathbb{U}} u\cdot p_{U,V}(u,v)du$$

Note that $p_{U,V}(u,v) = \frac{\mathbb{I}_G(u,v)}{Z_G}$, hence:

$$p_W(w) = \frac{2}{Z_G}\int_{\mathbb{U}} u\cdot \mathbb{I}_G(u,v)du = \frac{2}{Z_G}\int_0^{\sqrt{\tilde{\pi}(w)}} udu = \frac{2}{Z_G}[u^2/2]\Big|_0^{\sqrt{\tilde{\pi}(w)}} = \frac{\tilde{\pi}(w)}{Z_G} = \frac{\tilde{\pi}(w)}{Z_\pi}$$

2. Let $R = \left(\left[0, \sup_x \sqrt{\tilde{\pi}(x)}\right], \left[\inf_x x\sqrt{\tilde{\pi}(x)}, \sup_x \sqrt{\tilde{\pi}(x)}\right]\right)$.

Since $G = \{(u,v) \mid 0 \le u \le \sqrt{\tilde{\pi}(v/u)}\}$, it's evident that

$$\inf_G u = 0, \sup_G u = \sup_x \sqrt{\tilde{\pi}(x)} \implies u \in [0, \sup_x \sqrt{\tilde{\pi}(x)}]$$

Note that, for $x = v/u$:

$$0 \le u \le \sqrt{\tilde{\pi}(v/u)} \implies 0 \le v/x \le \sqrt{\tilde{\pi}(v/u)} \therefore 0 \le v \le x\sqrt{\tilde{\pi}(v/u)}, \forall x \neq 0$$

Then, similarly to u, we have:

$$\inf_G v = \inf_x x\sqrt{\tilde{\pi}(x)}, \sup_G v = \sup_x x\sqrt{\tilde{\pi}(x)} \implies v \in [\inf_x x\sqrt{\tilde{\pi}(x)}, \sup_x x\sqrt{\tilde{\pi}(x)}]$$

We conclude that $G \subset R$.

To sample uniformly in G, first:

- Generate $u_i \sim Unif[0, \sup_x \sqrt{\tilde{\pi}(x)}]$ and
- Generate $v_i \sim Unif[\inf_x x\sqrt{\tilde{\pi}(x)}, \sup_x x\sqrt{\tilde{\pi}(x)}]$

Then, accept (u_i, v_i) if $0 \le u_i \le \sqrt{\tilde{\pi}(v_i/u_i)}$, where $\tilde{\pi}(x)$ is a probability density function.

3. Let $\tilde{\pi}(x) = exp(-x^2/2)$.

$$\sup_{x\in\mathbb{R}} exp(-x^2/2) = exp(0) = 1$$

$$\inf_{x\in\mathbb{R}} x\sqrt{exp(-x^2/2)} = 0$$

$$\sup_{x\in\mathbb{R}} x\sqrt{exp(-x^2/2)} = \sup_{x\in\mathbb{R}} x \cdot exp(-x^2/4)$$

Note that:

$$\frac{d}{dx} x \cdot exp(-x^2/4) = exp(-x^2/4) - \frac{x^2}{2} exp(-x^2/4) = 0 \implies x = \pm\sqrt{2}$$

$$\sup_{x\in\mathbb{R}} x\sqrt{exp(-x^2/2)} = \sqrt{2exp(-2/2)} = \sqrt{2exp(-1)}$$

$$U \sim Unif[0,1], V \sim Unif[0, \sqrt{2exp(-1)}]$$

Therefore, generate samples from U and V. For the sample (u, v), if $0 \leq u \leq \sqrt{2 \cdot exp(-v^2/u^2)}$, then accept $x = v/u$, which will be a sample from $\tilde{\pi}(x)$.

Exercise 5 (Rejection and Importance Sampling)

1. Let $A =$ sample is accepted in step (a).

$$q^*(x) = p(X = x \mid X \in A) = \frac{P(X \in A \mid X = x)p(X = x)}{p(X \in A)} = \frac{P(U \leq min\{1, \frac{\pi(x)}{q(x)\cdot c}\} \mid X = x)q(x)}{P(X \in A)}$$

$$P(X \in A) = \int_\chi P(X \in A \mid X = x)q(x)dx = \int_\chi min\{1, \frac{\pi(x)}{q(x)\cdot c}\}q(x)dx = Z_c$$

Hence,

$$q^*(x) = \frac{min\{1, \frac{\pi(x)}{q(x)\cdot c}\}q(x)}{Z_c} = \frac{\left(P(U \leq 1)\cdot \mathbb{I}_{\{\frac{w(x)}{c}\geq 1\}}(x) + \frac{\pi(x)}{q(x)c}\mathbb{I}_{\{\frac{w(x)}{c}<1\}}(x)\right)q(x)}{Z_c}$$

$$q^*(x) = \frac{\mathbb{I}_{\{\frac{w(x)}{c}\geq 1\}}(x)\cdot q(x) + \frac{\pi(x)}{c}\mathbb{I}_{\{\frac{w(x)}{c}<1\}}(x)}{Z_c}$$

2. For $w * (x) = \pi(x)/q^*(x)$, we want to show that:

$$E_{q^*}([w^*(X)]^2) = Z_c E_q(max\{w(X), c\}w(X))$$

First:

$$E_{q*}([w^*(X)]^2]) = \int_\chi w(x)^2 \cdot q^*(x)dx = \int_\chi \frac{\pi(x)^2}{q*(x)}dx = \int_\chi \frac{\pi(x)w(x)Z_c}{min\{1, w(x)/c\}}dx$$

$$E_q(max\{w(X), c\}w(X)) = \int_\chi w(x)q(x)max\{w(X), c\}dx = \int_\chi \pi(x)max\{w(X), c\}dx$$

Now, note that:

- If $w(x)/c < 1 \therefore w(x)/min\{1, w(x)/c\} = w(x)c/w(x) = c \therefore max\{w(X), c\} = c$
- If $w(x)/c \geq 1 \therefore w(x)/min\{1, w(x)/c\} = w(x)\cdot 1 = w(x) \therefore max\{w(X), c\} = w(x)$

Hence,

$$w(x)/min\{1, w(x)/c\} = max\{w(X), c\}$$

With that, we conclude:

$$E_{q*}([w^*(X)]^2]) = Z_c \int_\chi \frac{\pi(x)w(x)}{min\{1, w(x)/c\}}dx = Z_c \int_\chi \pi(x)max\{w(X), c\}dx$$

$$E_{q*}([w^*(X)]^2]) = Z_c E_q(max\{w(X), c\}w(X))$$

3. Couldn't solve.

4. We want to show that $\mathbb{V}_{q^*}(w^*(X)) \leq \mathbb{V}_q(w(X))$.

   We will use the results from items (2) and (3), eventhough item (3) was not properly proven.

   First, using (3) we have:

$$E_q(w(X)^2) = \frac{1}{c}E_q(w(X)^2 \cdot c) \geq \frac{1}{c}E_q(max\{w(X), c\})w(X))E_q(min\{w(x), c\})$$

$$E_q(w(X)^2) \geq \frac{1}{c}\frac{E_{q^*}(w(X)^2)E_q(min\{w(x), c\})}{Z_c}$$

   Note that:

   - If $w(X) > c$, then $min\{w(X), c\}/c = w(X)/c$;
   - If $w(X) \leq c$, then $min\{w(X), c\}/c = 1$.

   Hence, $min\{w(X), c\}/c = min\{1, w(x)/c\}$, which implie:

$$\frac{E_q(min\{w(x), c\})}{c} = \int_{\chi} min\{w(x), c\} \cdot \frac{q(x)}{c}dx = \int_{\chi} min\{1, w(x)/c\} \cdot q(x)dx = Z_c$$

   Therefore:

$$E_q(w(X)^2) \geq E_{q^*}(w(X)^2)$$