

Exercise 1 (Gibbs Sampler)

1. First, let $X' = X^{(t)}$, $X = X^{(t-1)}$, $Y' = Y^{(t)}$ and $Y = Y^{(t-1)}$. Then:

$$K^S((x, y), (x', y')) = \pi_{Y|X}(y' | x) \pi_{X|Y}(x' | y')$$

Then, to show that it is not reversible:

$$\pi(x, y) K((x, y), (x', y')) = \pi(x, y) \pi(y' | x) \pi(x' | y')$$

$$\pi(x', y') K((x', y'), (x, y)) = \pi(x', y') \pi(y | x') \pi(x | y)$$

$$\frac{\pi(x, y) K((x, y), (x', y'))}{\pi(x', y') K((x', y'), (x, y))} = \frac{\pi(x, y) \pi(y' | x) \pi(x' | y')}{\pi(x', y') \pi(y | x') \pi(x | y)} = \frac{\pi(y) \pi(y' | x)}{\pi(y') \pi(y | x')} \neq 1$$

Therefore, it is not reversible. □

2. First, the kernel expression:

$$K(x, x') = \int \pi(y' | x) \pi(x' | y') dy'$$

Now, let's show that it is π_X -reversible.

$$\begin{aligned} \pi(x') K(x', x) &= \pi(x') \int \pi(y | x') \pi(x | y) dy = \int \pi(x') \frac{\pi(y, x')}{\pi(x')} \pi(x | y) dy = \\ &= \int \pi(y, x') \pi(x | y) dy = \int \pi(y, x') \frac{\pi(x, y)}{\pi(y)} dy = \int \pi(x, y) \frac{\pi(x', y)}{\pi(y)} dy = \\ &= \pi(x) \int \pi(y | x) \pi(x' | y) dy = \pi(x) K(x, x') \end{aligned}$$

□

3. First, the kernel expression is:

$$K^R((x, y), (x', y')) = \pi(y' | x) \pi(x' | y') 0.5 + \pi(x' | y) \pi(y' | x') 0.5$$

Note that it is half the density of sampling first from y plus half the density of sampling first from x .

Now, let's show that it is reversible:

$$\begin{aligned} \frac{\pi(x, y) [\pi(y' | x) \pi(x' | y') 0.5 + \pi(x' | y) \pi(y' | x') 0.5]}{\pi(x', y') [\pi(y | x') \pi(x | y') 0.5 + \pi(x | y') \pi(y | x) 0.5]} &= \\ &= \frac{\frac{\pi(y' | x)}{\pi(y')} + \frac{\pi(x' | y)}{\pi(x')}}{\frac{\pi(y | x')}{\pi(y)} + \frac{\pi(x | y')}{\pi(x)}} = 1 \end{aligned}$$

□

Exercise 2 (Metropolis-within-Gibbs)

1. Note that:

$$\alpha(X_1 \mid X_1^{(t-1)}, X_2^{(t-2)}) = \min \left\{ 1, \frac{\pi(X'_1, X_2^{(t-1)})\pi(X_1^{(t-1)} \mid X_2^{(t-1)})}{\pi(X_1^{(t-1)}, X_2^{(t-1)})\pi(X'_1 \mid X_2^{(t-1)})} \right\} = \min\{1, 1\}$$

Therefore, we get a systematic scan Gibbs sampler, where one samples $X_1^t \sim \pi(\cdot \mid X_2^{(t-1)})$, then we accept, since $\alpha = 1$, and finally sample $X_2^t \sim \pi(\cdot \mid X_1^t)$. \square

2. First, let's write the kernel. Since we only accept or reject the variable X_1 , the kernel is the M-H kernel multiplied by the probability density function of $\pi_{X_2|X_1}(X_2 \mid X_1)$. Let $X_1^t, X_2^t = Y_1, Y_2$:

$$K((x_1, x_2), (y_1, y_2)) = (q(y_1 \mid x_1, x_2)\alpha(y_1 \mid x_1, x_2)) + (1 - a(y_1 \mid x_1, x_2))\delta_{y_1}(x_1)\pi_{(Y_2|Y_1)}(y_2 \mid y_1)$$

Note that $\alpha = 1$. With that, we show that the kernel is invariant:

$$\begin{aligned} \int \int K((x_1, x_2), (y_1, y_2))\pi(x_1, x_2)dx_1dx_2 &= \int \int \pi(y_1 \mid x_2)\pi(y_2 \mid y_1)\pi(x_1, x_2)dx_1dx_2 = \\ &= \int \pi(y_1 \mid x_2)\pi(y_2 \mid y_1)\pi(x_2)dx_2 = \int \pi(y_1, x_2)\pi(y_2 \mid y_1)dx_2 = \pi(y_1, y_2) \end{aligned}$$

\square