

**Exercise 1 (Gibbs Sampler)**

1. First, let  $X' = X^{(t)}$ ,  $X = X^{(t-1)}$ ,  $Y' = Y^{(t)}$  and  $Y = Y^{(t-1)}$ . Then:

$$K^S((x, y), (x', y')) = \pi_{Y|X}(y' | x) \pi_{X|Y}(x' | y')$$

Then, to show that it is not reversible:

$$\pi(x, y) K((x, y), (x', y')) = \pi(x, y) \pi(y' | x) \pi(x' | y')$$

$$\pi(x', y') K((x', y'), (x, y)) = \pi(x', y') \pi(y | x') \pi(x | y)$$

$$\frac{\pi(x, y) K((x, y), (x', y'))}{\pi(x', y') K((x', y'), (x, y))} = \frac{\pi(x, y) \pi(y' | x) \pi(x' | y')}{\pi(x', y') \pi(y | x') \pi(x | y)} = \frac{\pi(y) \pi(y' | x)}{\pi(y') \pi(y | x')} \neq 1$$

Therefore, it is not reversible. □

2. First, the kernel expression:

$$K(x, x') = \int \pi(y' | x) \pi(x' | y') dy'$$

Now, let's show that it is  $\pi_X$ -reversible.

$$\begin{aligned} \pi(x') K(x', x) &= \pi(x') \int \pi(y | x') \pi(x | y) dy = \int \pi(x') \frac{\pi(y, x')}{\pi(x')} \pi(x | y) dy = \\ &= \int \pi(y, x') \pi(x | y) dy = \int \pi(y, x') \frac{\pi(x, y)}{\pi(y)} dy = \int \pi(x, y) \frac{\pi(x', y)}{\pi(y)} dy = \\ &= \pi(x) \int \pi(y | x) \pi(x' | y) dy = \pi(x) K(x, x') \end{aligned}$$

□

3. First, the kernel expression is:

$$K^R((x, y), (x', y')) = \pi(y' | x) \pi(x' | y') 0.5 + \pi(x' | y) \pi(y' | x') 0.5$$

Note that it is half the density of sampling first from  $y$  plus half the density of sampling first from  $x$ .

Now, let's show that it is reversible:

$$\begin{aligned} \frac{\pi(x, y) [\pi(y' | x) \pi(x' | y') 0.5 + \pi(x' | y) \pi(y' | x') 0.5]}{\pi(x', y') [\pi(y | x') \pi(x | y') 0.5 + \pi(x | y') \pi(y | x) 0.5]} &= \\ &= \frac{\frac{\pi(y' | x)}{\pi(y')} + \frac{\pi(x' | y)}{\pi(x')}}{\frac{\pi(y | x')}{\pi(y)} + \frac{\pi(x | y')}{\pi(x)}} = 1 \end{aligned}$$

□

## **Exercise 2 (Metropolis-within-Gibbs)**

1.