Exercise 1 (Kalma Filter)

1. $$f(X_t \mid X_{t-1}) \sim N(\phi X_{t-1}, \sigma_v^2)$$
$$g(Y_t \mid X_t) \sim N(\phi X_t, \sigma_w^2)$$

2. Note that $X_{t+1} = \phi X_t + V_t$. Then:
$$X_{t+1} \mid Y_{1:t} = \phi X_t + V_t \mid Y_{1:t}$$
Therefore, since $X_t \mid Y_{1:t} \sim N(m_{t|t}, \sigma^2_{t|t})$ and $V_t \sim N(0, \sigma_v^2)$, we have the sum of normals, hence:
$$X_{t+1} \mid Y_{1:t} \sim N(m_{t|t}, \sigma_v^2 + \phi^2 \sigma^2_{t|t})$$

3. Note that:
$$p(X_{t=1} \mid Y_{1:t}, Y_{t+1}) \propto N(X_{t+1}, \sigma_w^2) N(m_{t+1}, \sigma^2_{t+1})$$
The update of normal distribution by normal distribution is given by:
$$\mu_1 = \frac{y_{t+1}\sigma_w^{-2} + m_{t+1}\sigma^2_{t+1}}{\sigma_w^{-2} + \sigma_{t+1}^{-2}} \qquad \tau_1^2 = (\sigma_w^{-2} + \sigma_v^{-2})^{-1}$$
Hence, $p(x_{t+1} \mid y_{1:t}) = N(\mu_1, \tau_1^2) = N(m_{t+1|t}, \sigma^2_{t+1|t})$.

4. Note that:
$$Y_{t+1} \mid Y_{1:t} = X_{t+1} + W_{t+1} \mid Y_{1:t} = N(\mu_1, \tau_1) + N(0, \sigma_w^2)$$
$$Y_{t+1} \mid Y_{1:t} = N(\mu_1, \sigma_w^2 + \tau_1^2)$$

Exercise 2 (SIS filter)

1. v_t approximate $p(x_t \mid y_{0:t})$. Since this is the filtering distribution, that is why it is called a filter.

2.

$$p_{Y_0}(y_0) = \int p_{Y_0,X_0}(y_0, x_0)dx_0 = \int p_{Y_0|X_0}(y_0 \mid x_0)p(x_0)dx_0 = \int g(y_0 \mid x_0)\mu(x_0)dx_0$$

   And,

$$p_{Y_0,X_0}(y_0, x_0) = \int p_{Y_0,X_0,X_0,Y_1}(y_0, x_0, x_1, y_1)dx_0dx_1 =$$

$$\int p_{Y_1|X_0,X_0,Y_1}(y_1 \mid x_0, x_1, y_0)p_{X_1|Y_0,X_0}(x_1 \mid x_0, y_0)p_{Y_0|X_0}(y_0 \mid x_0)p_{X_0}(x_0)dx_0dx_1 =$$

$$\int g(y_1 \mid x_1)f(x_1 \mid x_0)g(y_0 \mid x_0)\mu(x_0)dx_0dx_1$$

3. First, $p_{Y_0}(y_0) = E_\mu[g(Y_0 \mid X_0)] = E_{X_0}[g(Y_0 \mid X_0)]$, hence an unbiased estimator is $\frac{\sum_{i=1}^n g(y_0|X_0^{(i)})}{n}$, which is an estimation for the expected value.
   Now, $p_{Y_0,Y_1}(y_0, y_1) = E_{X_0,X_1}[g(y_1 \mid x_1)g(y_0 \mid x_0)]$, hence we have the unbiased estimator equal to $\frac{\sum_{i=1}^n g(y_0|X_0^{(i)})g(y_1|X_1^{(i)})}{n}$.

4.

5.

6.

Simulation question (Reversible jump MCMC)

The model samples

$$\pi(\theta \mid k=1) = exp(-\theta^2/2) \qquad \pi(\theta \mid k=2) = exp(-(\theta_1^2+\theta_2^2)/2)$$

The analytical rate of visits of $\frac{k=2}{k=1}$ is given by $\sqrt{2\pi} \approx 2.51$.

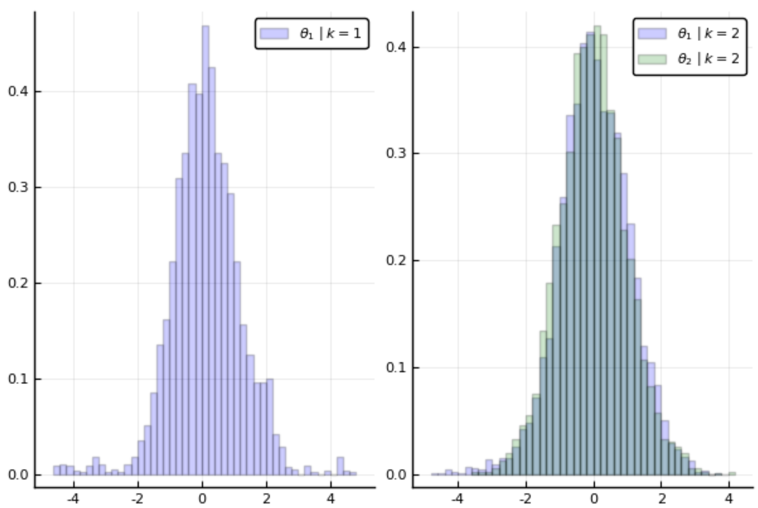

Figure 1: Distribution of θ for each model. The jump probability used was 0.1. The distribution for u was a standard Cauchy distribution. The proportion of visits found is 2.47.

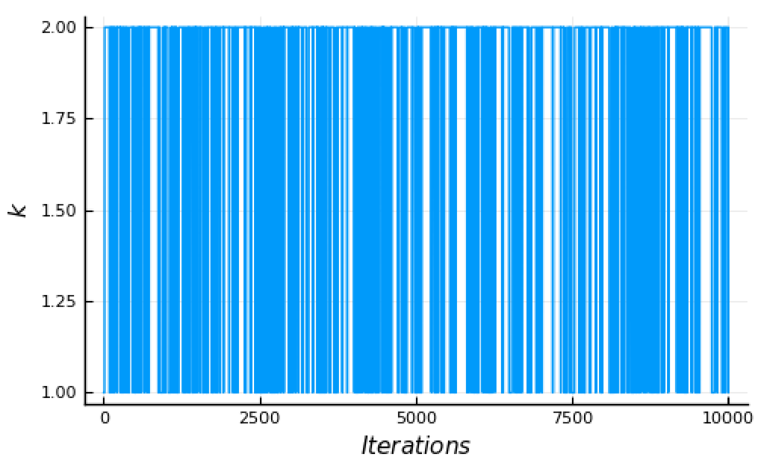

Figure 2: Trace plot of the model sampled. It shows that both models are well sampled.

Simulation question (Reversible jump MCMC)

The target distribution is $\pi(x) \approx exp(-10(x-1)^2)$. The tempered distributions are shown in the graph below.

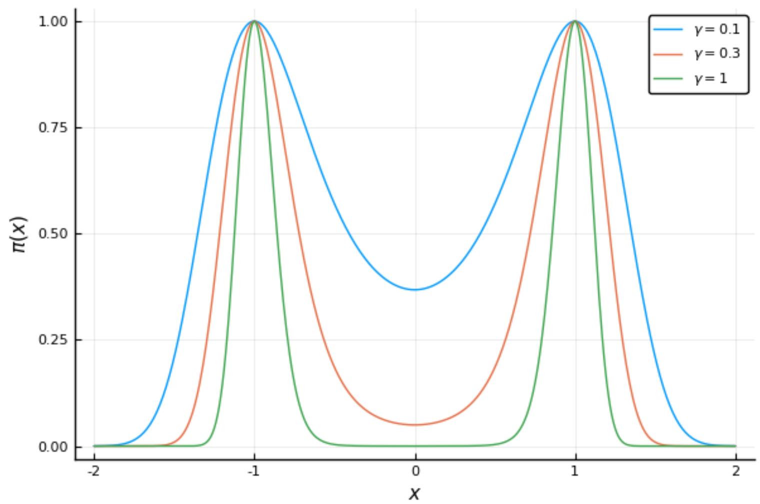

Figure 3: Tempered distributions.

Looking at the trace plot, for the distributions with higher γ we won't get a very good mixture. This problem is not present when γ is lower.

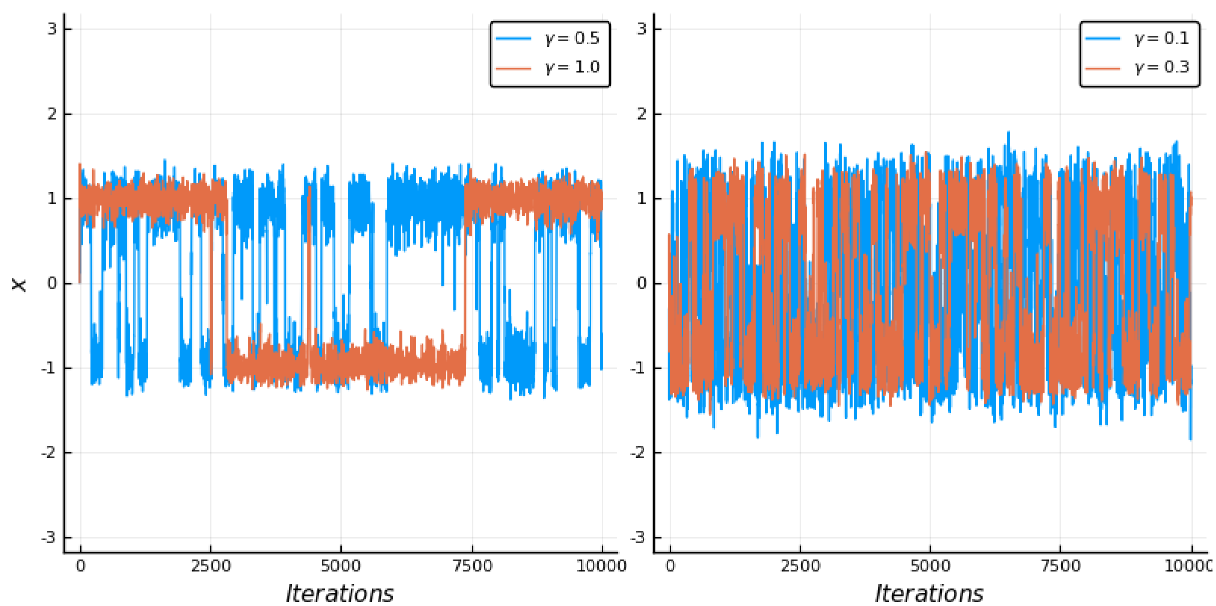

Figure 4: Trace plot for different values of tempering.

Implementing the *parallel tempering*, this problem is solved as can be seen in the figure below.

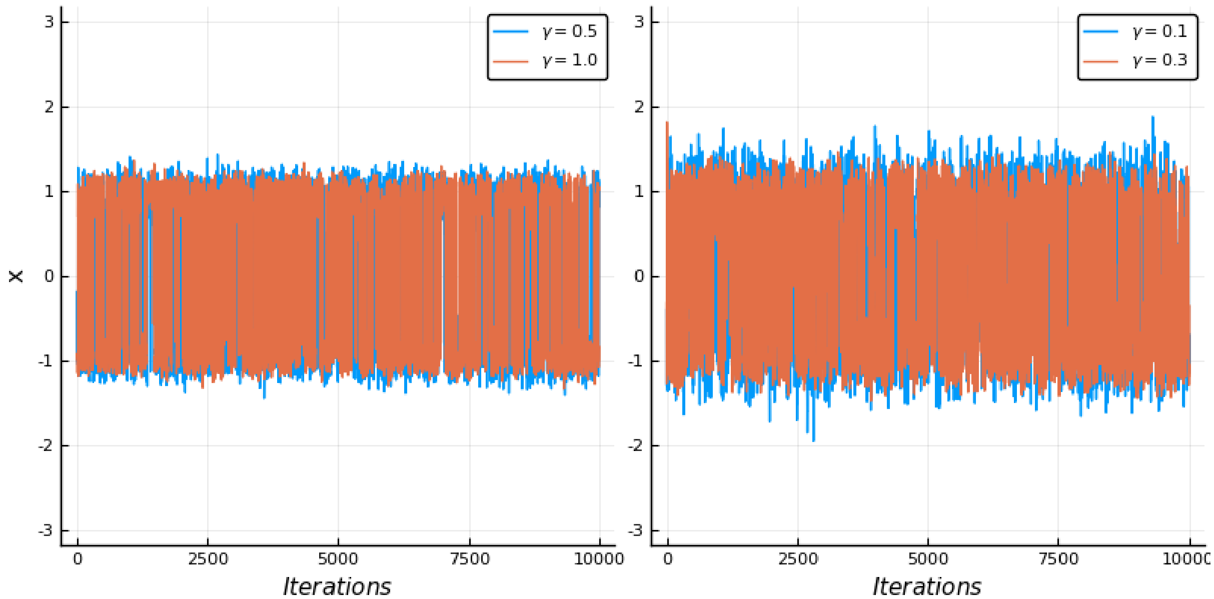

Figure 5: Trace plot for the *parallel tempering* model.

Simulation question (linear Gaussian model - SIS and SIR)

The model is

$$X_t = \phi X_{t-1} + \sigma_V V_t, \qquad Y_T = X_t + \sigma_W W_t, \qquad \phi = 0.95, \quad \sigma_V = 1, \quad \sigma_W = 1$$

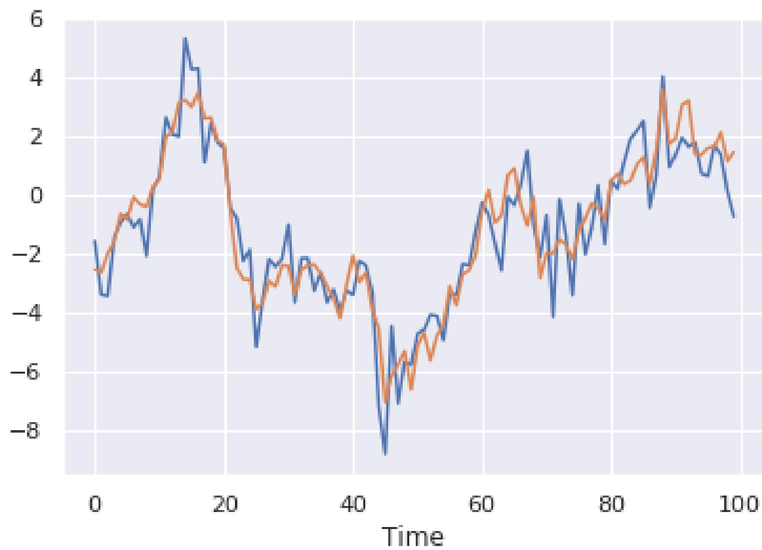

Figure 6: $T = 100$ simulated observations for the Linear Gaussian model.

Using the prior as a proposal, we implement the sequential importance sampling algorithm. The results are shown below.

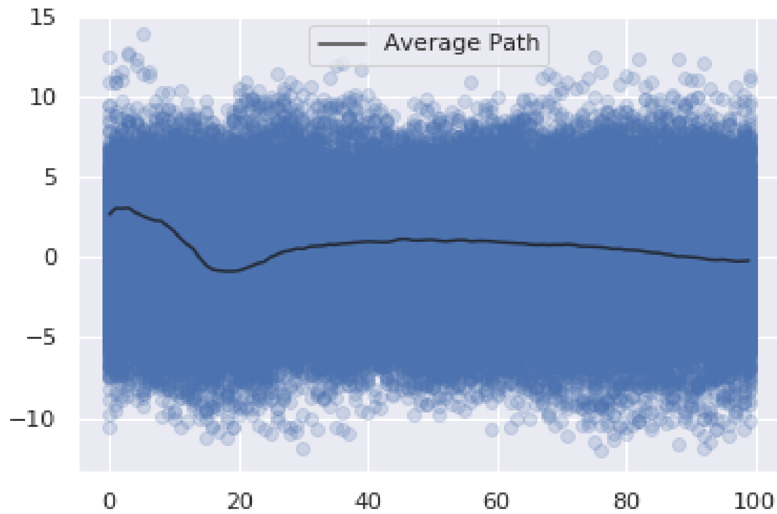

Figure 7: Simulated particles for N = 1000 using SIS.

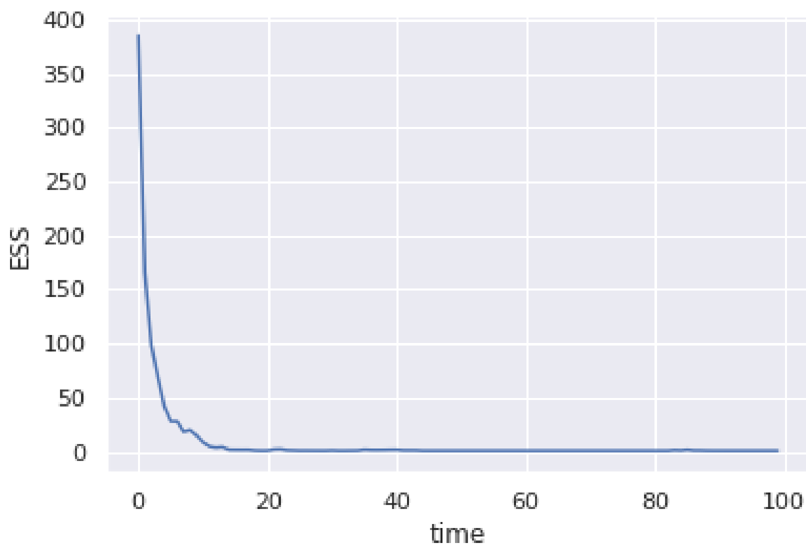

Figure 8: Effective sample size.

Looking at the effective sample size we immediatly notice the problem with the SIS algorithm. The effective sample size quickly drops.

Now let's show the SIR algorithm and how this problem is solved.

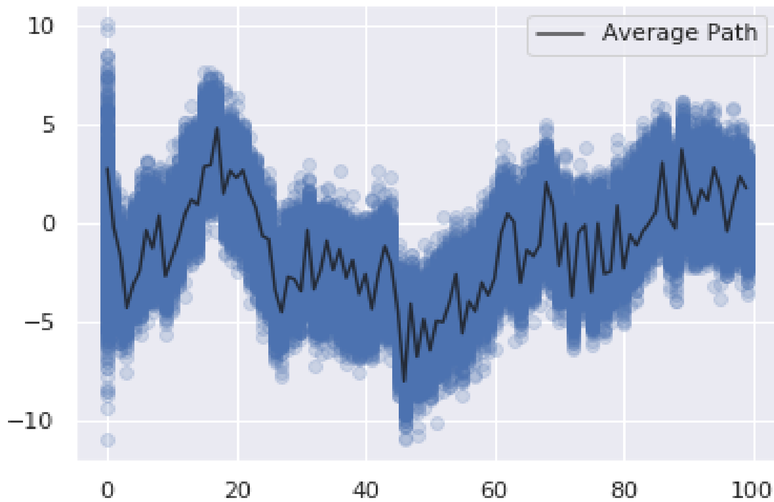

Figure 9: Simulated particles for N = 1000 using SIR.

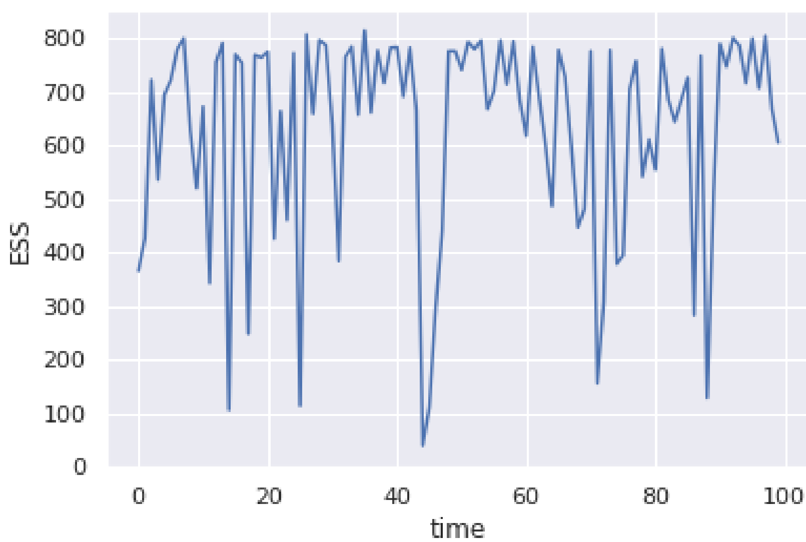

Figure 10: Effective sample size.