Exercise 1 (Gibbs Sampler)

1. First, let $X' = X^{(t)}$, $X = X^{(t-1)}$, $Y' = Y^{(t)}$ and $Y = Y^{(t-1)}$. Then:

$$K^S((x,y),(x',y')) = \pi_{Y|X}(y' \mid x)\pi_{X|Y}(x' \mid y')$$

Then, to show that it is not reversible:

$$\pi(x,y)K((x,y),(x',y')) = \pi(x,y)\pi(y' \mid x)\pi(x' \mid y')$$

$$\pi(x',y')K((x',y'),(x,y)) = \pi(x',y')\pi(y \mid x')\pi(x \mid y)$$

$$\therefore$$

$$\frac{\pi(x,y)K((x,y),(x',y'))}{\pi(x',y')K((x',y'),(x,y))} = \frac{\pi(x,y)\pi(y' \mid x)\pi(x' \mid y')}{\pi(x',y')\pi(y \mid x')\pi(x \mid y)} = \frac{\pi(y)\pi(y' \mid x)}{\pi(y')\pi(y \mid x')} \neq 1$$

Therefore, it is not reversible. $\square$

2. First, the kernel expression:

$$K(x,x') = \int \pi(y' \mid x)\pi(x' \mid y')dy'$$

Now, let's show that it is π_X-reversible.

$$\pi(x')K(x',x) = \pi(x')\int \pi(y \mid x')\pi(x \mid y)dy = \int \pi(x')\frac{\pi(y,x')}{\pi(x')}\pi(x \mid y)dy =$$

$$= \int \pi(y,x')\pi(x \mid y)dy = \int \pi(y,x')\frac{\pi(x,y)}{\pi(y)}dy = \int \pi(x,y)\frac{\pi(x',y)}{\pi(y)}dy =$$

$$= \pi(x)\int \pi(y \mid x)\pi(x' \mid y)dy = \pi(x)K(x,x')$$

$\square$