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Worksheet 1

Computational Statistics
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Notation: here is a brief summary of the notation used in this worksheet.

- $p(X = x)$ is equal to the probability density function;
- Capital letters such as X stand for the random variable.

Exercise 1 (Inversion and Rejection)

1. Let $F_X(x) = \mathbb{P}(X \leq x)$ and $U \sim Unif[0,1]$:

$$F_X(x) = 1 - e^{-\lambda(X-a)}\mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1-U) = \lambda(x-a)$$

$$F_X^{-1}(U) = a - \frac{-\ln(1-U)}{\lambda}$$

To simulate X from U, just simulate value from U and substitute in the formula above.

2. Let $X = Y \mid a \leq Y \leq b$. First, let's show that $X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$:

$$\mathbb{P}(X \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U)+F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)+U[F_Y(b)-F_X(a)]) \leq x)$$

$$= \mathbb{P}(F_Y(a)+U[F_Y(b)-F_X(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}$$

Note that since $x \in [a, b]$:

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b) - F_Y(a)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on $\geq a$, we first generate $U \sim Unif[0,1]$, then, for $Y \sim Expo(\lambda)$:

$$F_Y(y) = 1 - e^{\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1-U)}{\lambda}$$

$$X = \frac{-\ln(1-(1-U)F_Y(a)+U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1-U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let $q \sim Expo(\lambda)$, and $\pi(x) = \lambda e^{-\lambda(x-a)}\mathbb{I}_{x\geq a}$:

   Note that $M = max_x \pi(x)/q(x) = e^{\lambda a}$, since $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{\lambda(x)}} = e^{\lambda a}$

   $$\therefore$$

   In the rejection method, we sample $x_i \sim q$, $u \sim Unif[0,1]$,then we accept a sample x_i if $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$.

   Hence,

   - If $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$ is rejected;
   - If $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$ is accepted;

   Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

   Finally, the expected number of trials is equal to $M = e^{\lambda a}$. Therefore, for $a \gg 1/\lambda$, the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.

Exercise 2 (Rejection)

1. Let A be the event where the value is accepted at some point, while A_b is accepted at step (b) and A_c is accepted at step (c):

   In step (b) we have:
   $$\mathbb{P}(x \in A_b) = \frac{h(x)}{M\tilde{q}(x)}$$
   In step (c) we have:
   $$\mathbb{P}(x \in A_c) = \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)}$$
   Since step (b) is independent of (c),
   $$\mathbb{P}(x \in A) = \mathbb{P}(x \in A_b \cup x \in A_c) =$$
   $$= \mathbb{P}(x \in A_b) + \mathbb{P}(x \in A_b) - \mathbb{P}(x \in A_b \cap x \in A_c) =$$
   $$= \frac{h(x)}{M\tilde{q}(x)} + \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} + \frac{h(x)}{M\tilde{q}(x)} \cdot \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} = \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}$$

2. Let B be an arbitrary event.
   $$\mathbb{P}(X \in B \mid X \in A) = \mathbb{P}(X \in B \cap X \in A)/\mathbb{P}(X \in A) \therefore$$
   $$\mathbb{P}(X \in B \cap X \in A) = \int_{\chi}\int_0^1 \mathbb{I}_B(x)\mathbb{I}\left(u \leq \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\right) q(x) du dx$$
   $$\mathbb{P}(X \in B \cap X \in A) = \int_B \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\tilde{q}(x) \cdot Z_q^{-1} dx$$
   $$\mathbb{P}(X \in B \cap X \in A) = \frac{\pi(B) \cdot Z_\pi}{M \cdot Z_q}$$
   Finally,
   $$\mathbb{P}(X \in A) = \frac{Z_\pi}{M \cdot Z_q} \therefore \mathbb{P}(X \in B \mid X \in A) = \frac{\pi(B)Z_\pi}{MZ_q} \cdot \frac{MZ_q}{Z_\pi} = \pi(B)$$
   .

3. We want to show that:
   $$\mathbb{P}(\text{Step (c) is necessery}) = 1 - \frac{\int_\chi h(x)dx}{MZ_q}$$
   First, note that $\mathbb{P}$(Step (c) is necessery) $= \mathbb{P}$(X not accepted in step(b)), hence:

$$\mathbb{P}(\text{X not accepted in step(b)}) = 1 - \mathbb{P}(X \in A_b) = 1 - \mathbb{P}\left(U \leq \frac{h(X)}{M\tilde{q}(X)}\right)$$

$$\mathbb{P}(X \in A_b) = \mathbb{P}(X \in \chi \cap X \in A_b) =$$
$$\int_\chi \int_0^1 \mathbb{I}_\chi(x)\mathbb{I}\left(u \leq \frac{h(x)}{M\tilde{q}(x)}\right) dudx =$$
$$\int_\chi \frac{h(x)\tilde{q}(x)}{M\tilde{q}(x)Z_q}dx = \frac{\int_\chi h(x)dx}{MZ_q} \therefore$$
$$\mathbb{P}(\text{Step(c) is necessery}) = 1 - \frac{\int_\chi h(x)dx}{MZ_q}$$

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