

Exercise 1 (Inversion and Rejection)

1. Let $F_X(x) = \mathbb{P}(X \leq x)$ and $U \sim Unif[0, 1]$:

$$F_X(x) = 1 - e^{-\lambda(X-a)} \mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1 - U) = \lambda(x - a)$$

$$F_X^{-1}(U) = a - \frac{\ln(1 - U)}{\lambda}$$

To simulate X from U, just simulate value from U and substitute in the formula above.

2. Let $X = Y \mid a \leq Y \leq b$. First, let's show that $X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U)$:

$$\begin{aligned} \mathbb{P}(X \leq x) &= \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a) + U[F_Y(b) - F_Y(a)]) \leq x) \\ &= \mathbb{P}(F_Y(a) + U[F_Y(b) - F_Y(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} \end{aligned}$$

Note that since $x \in [a, b]$:

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b) - F_Y(a)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on $\geq a$, we first generate $U \sim Unif[0, 1]$, then, for $Y \sim Expo(\lambda)$:

$$F_Y(y) = 1 - e^{-\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1 - U)}{\lambda}$$

$$X = \frac{-\ln(1 - (1 - U)F_Y(a) + U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1 - U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let $q \sim Expo(\lambda)$, and $\pi(x) = \lambda e^{-\lambda(x-a)} \mathbb{I}_{x \geq a}$:

Note that $M = \max_x \pi(x)/q(x) = e^{\lambda a}$, since $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{-\lambda x}} = e^{\lambda a}$

\therefore

In the rejection method, we sample $x_i \sim q$, $u \sim Unif[0, 1]$, then we accept a sample x_i if $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$.

Hence,

- If $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$ is rejected;

- If $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$ is accepted;

Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

Finally, the expected number of trials is equal to $M = e^{\lambda a}$. Therefore, for $a \gg 1/\lambda$, the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.

Exercise 2 (Rejection)

1. Let A be the event where the value is accepted at some point, while A_b is accepted at step (b) and A_c is accepted at step (c):

In step (b) we have:

$$\mathbb{P}(x \in A_b) = \frac{h(x)}{M\tilde{q}(x)}$$

In step (c) we have:

$$\mathbb{P}(x \in A_c) = \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)}$$

Since step (b) is independent of (c),

$$\begin{aligned} \mathbb{P}(x \in A) &= \mathbb{P}(x \in A_b \cup x \in A_c) = \\ &= \mathbb{P}(x \in A_b) + \mathbb{P}(x \in A_c) - \mathbb{P}(x \in A_b \cap x \in A_c) = \\ &= \frac{h(x)}{M\tilde{q}(x)} + \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} + \frac{h(x)}{M\tilde{q}(x)} \cdot \frac{\tilde{\pi}(x) - h(x)}{M\tilde{q}(x) - h(x)} = \frac{\tilde{\pi}(x)}{M\tilde{q}(x)} \end{aligned}$$

2. Let B be an arbitrary event.

$$\mathbb{P}(X \in B \mid X \in A) = \mathbb{P}(X \in B \cap X \in A) / \mathbb{P}(X \in A) \therefore$$

$$\mathbb{P}(X \in B \cap X \in A) = \int_{\mathcal{X}} \int_0^1 \mathbb{I}_B(x) \mathbb{I}\left(u \leq \frac{\tilde{\pi}(x)}{M\tilde{q}(x)}\right) q(x) du dx$$

$$\mathbb{P}(X \in B \cap X \in A) = \int_B \frac{\tilde{\pi}(x)}{M\tilde{q}(x)} \tilde{q}(x) \cdot Z_q^{-1} dx$$

$$\mathbb{P}(X \in B \cap X \in A) = \frac{\pi(B) \cdot Z_\pi}{M \cdot Z_q}$$

Finally,

$$\mathbb{P}(X \in A) = \frac{Z_\pi}{M \cdot Z_q} \therefore \mathbb{P}(X \in B \mid X \in A) = \frac{\pi(B) Z_\pi}{M Z_q} \cdot \frac{M Z_q}{Z_\pi} = \pi(B)$$

3. We want to show that:

$$\mathbb{P}(\text{Step (c) is necessary}) = 1 - \frac{\int_{\mathcal{X}} h(x) dx}{M Z_q}$$

First, note that $\mathbb{P}(\text{Step (c) is necessary}) = \mathbb{P}(X \text{ not accepted in step(b)})$, hence:

$$\mathbb{P}(X \text{ not accepted in step(b)}) = 1 - \mathbb{P}(X \in A_b) = 1 - \mathbb{P}\left(U \leq \frac{h(X)}{M\tilde{q}(X)}\right)$$

$$\begin{aligned}\mathbb{P}(X \in A_b) &= \mathbb{P}(X \in \mathcal{X} \cap X \in A_b) = \\ &= \int_{\mathcal{X}} \int_0^1 \mathbb{I}_{\mathcal{X}}(x) \mathbb{I}\left(u \leq \frac{h(x)}{M\tilde{q}(x)}\right) du dx = \\ &= \int_{\mathcal{X}} \frac{h(x)\tilde{q}(x)}{M\tilde{q}(x)Z_q} dx = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q} \therefore \\ \mathbb{P}(\text{Step(c) is necessary}) &= 1 - \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q}\end{aligned}$$

4. We want to calculate the probability of not having to evaluate $\tilde{p}i(x)$, which is equal to the probability of accepting the sample in step (b).

First, we know that:

$$\mathbb{P}(X \text{ is accepted in step(b)}) = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q}$$

Note that $h(x) \geq 0$, therefore $1 - x^2/2 \geq 0 \therefore -\sqrt{2} \leq x \leq \sqrt{2}$.

Hence, $h(x) = 0, \forall x \notin [-\sqrt{2}, \sqrt{2}]$.

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} h(x) dx &= \int_{-\sqrt{2}}^{\sqrt{2}} 1 - \frac{x^2}{2} dx = \frac{4\sqrt{2}}{3} \\ \int_{-\infty}^{\infty} e^{-|x|} dx &= Z_q = 2 \int_0^{\infty} e^{-x} dx = 2(-[e^{-\infty} - e^0]) = 2\end{aligned}$$

$$M = \sup_{x \in \mathbb{R}} \frac{\tilde{\pi}(x)}{\tilde{q}(x)} = \sup_{x \in \mathbb{R}} \frac{e^{-x^2/2}}{e^{-|x|}} = \sup_{x \in \mathbb{R}} e^{-x^2/2+|x|}$$

For $x \geq 0$, we have $\frac{d}{dx}(x - x^2/2) = 1 - x = 0 \implies x = 1$.

For $x < 0$, we have $\frac{d}{dx}(-x - x^2/2) = -1 - x = 0 \implies x = -1$. Hence, $M = \sqrt{e}$.

Finally, we have:

$$\mathbb{P}(X \text{ is accepted in step(b)}) = \frac{\int_{\mathcal{X}} h(x) dx}{MZ_q} = \frac{4\sqrt{2}}{3 \cdot 2 \cdot \sqrt{e}} = \frac{2\sqrt{2e^{-1}}}{3}$$

It can be beneficial to use this algorithm instead of the standard rejection sampling procedure because this algorithm can be more computationally efficient.

Exercise 3 (Transformation)

1. Let $V = \arctan(U_2/U_1)$ and $Y = U_1^2 + U_2^2 \leq 1$. We want to show that:

$$p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

First, note that $p_{Y,V}(y, \theta) = p_{U_1,U_2}(u_1, u_2) \left| \begin{array}{cc} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{array} \right|$

$$\left| \begin{array}{cc} \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial \theta} \\ \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial \theta} \end{array} \right| = \left| \begin{array}{cc} \frac{\partial y}{\partial u_1} & \frac{\partial \theta}{\partial u_1} \\ \frac{\partial y}{\partial u_2} & \frac{\partial \theta}{\partial u_2} \end{array} \right|^{-1} = \left| \begin{array}{cc} 2u_1 & \frac{1}{1+(u_1/u_2)^2} \cdot \frac{-1u_2}{u_1^2} \\ 2u_2 & \frac{1}{1+(u_1/u_2)^2} \cdot \frac{1}{u_1} \end{array} \right|^{-1} = \frac{2}{1+(u_2/u_1)^2} + \frac{2(u_2/u_1)^2}{1+(u_2/u_1)^2} =$$

$$= \frac{2}{1+\theta^2} + \frac{2(1+\theta^2)}{1+\theta^2} = 2$$

$$p_{Y,V}(y, \theta) = p_{U_1,U_2}(u_1, u_2) \cdot \frac{1}{2} = p_{U_1,U_2}(\sqrt{y}\cos(\theta), \sqrt{y}\sin(\theta)) \cdot \frac{1}{2}$$

Note that $p_{U_1,U_2}(\sqrt{y}\cos(\theta), \sqrt{y}\sin(\theta))$ is a uniform distribution over a circle of radius $\sqrt{y} \leq 1$, therefore, it's normalizing constant is equal to πy , which is the area of the circle of radius \sqrt{y} . With that we can write:

$$p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$$

2. We have shown that $p_{Y,V}(y, \theta) = \mathbb{I}_{[0,1]}(y) \frac{\mathbb{I}_{[0,2\pi]}(\theta)}{2\pi}$. Since we can factor the functions of Y and V , this means that they are independent and that $Y \sim \text{Unif}[0, 1]$, $V \sim \text{Unif}[0, 2\pi]$.

Note that for $Z = \sqrt{-2\log(Y)}$:

$$X_1 = Z \frac{U_1}{\sqrt{Y}} = Z \cos(V), X_2 = Z \frac{U_2}{\sqrt{Y}} = Z \sin(V)$$

Therefore, we get the Box-Muller algorithm, hence the proof proceeds accordingly to show that X_1 and X_2 are independent standard normal distributions.

3. In this approach it is not necessary to calculate the trigonometric functions (cosine and sine) which are computationally expensive.

Exercise 4 (Transformation)

1. Let
- $W = V/U$
- .

$$p_{W,U}(w, u) = p_{U,V}(u, v) \left| \begin{array}{cc} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial u} \end{array} \right|$$

$$p_{U,V}(u, v) \left| \begin{array}{cc} \frac{-u}{w} & 1 \\ u & w \end{array} \right| = p_{W,U}(w, u) = p_{U,V}(u, v) \cdot 2u$$

$$p_W(w) = \int_{\mathbb{U}} p_{W,U}(w, u) du = 2 \int_{\mathbb{U}} u \cdot p_{U,V}(u, v) du$$

Note that $p_{U,V}(u, v) = \frac{\mathbb{I}_G(u, v)}{Z_G}$, hence:

$$p_W(w) = \frac{2}{Z_G} \int_{\mathbb{U}} u \cdot \mathbb{I}_G(u, v) du = \frac{2}{Z_G} \int_0^{\sqrt{\tilde{\pi}(w)}} u du = \frac{2}{Z_G} [u^2/2]_0^{\sqrt{\tilde{\pi}(w)}} = \frac{\tilde{\pi}(w)}{Z_G} = \frac{\tilde{\pi}(w)}{Z_{\pi}}$$

2. Let
- $R = \left(\left[0, \sup_x \sqrt{\tilde{\pi}(x)} \right], \left[\inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)} \right] \right)$
- .

Since $G = \{(u, v) \mid 0 \leq u \leq \sqrt{\tilde{\pi}(v/u)}\}$, it's evident that

$$\inf_G u = 0, \sup_G u = \sup_x \sqrt{\tilde{\pi}(x)} \implies u \in [0, \sup_x \sqrt{\tilde{\pi}(x)}]$$

Note that, for $x = v/u$:

$$0 \leq u \leq \sqrt{\tilde{\pi}(v/u)} \implies 0 \leq v/x \leq \sqrt{\tilde{\pi}(v/u)} \therefore 0 \leq v \leq x \sqrt{\tilde{\pi}(v/u)}, \forall x \neq 0$$

Then, similarly to u , we have:

$$\inf_G v = \inf_x x \sqrt{\tilde{\pi}(x)}, \sup_G v = \sup_x x \sqrt{\tilde{\pi}(x)} \implies v \in [\inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)}]$$

We conclude that $G \subset R$.

To sample uniformly in G , first:

- Generate $u_i \sim \text{Unif}[0, \sup_x \sqrt{\tilde{\pi}(x)}]$ and
- Generate $v_i \sim \text{Unif}[\inf_x x \sqrt{\tilde{\pi}(x)}, \sup_x x \sqrt{\tilde{\pi}(x)}]$

Then, accept (u_i, v_i) if $0 \leq u_i \leq \sqrt{\tilde{\pi}(v_i/u_i)}$, where $\tilde{\pi}(x)$ is a probability density function.

3. Let $\tilde{\pi}(x) = \exp(-x^2/2)$.

$$\sup_{x \in \mathbb{R}} \exp(-x^2/2) = \exp(0) = 1$$

$$\inf_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = 0$$

$$\sup_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = \sup_{x \in \mathbb{R}} x \cdot \exp(-x^2/4)$$

Note that:

$$\frac{d}{dx} x \cdot \exp(-x^2/4) = \exp(-x^2/4) - \frac{x^2}{2} \exp(-x^2/4) = 0 \implies x = \pm\sqrt{2}$$

$$\sup_{x \in \mathbb{R}} x \sqrt{\exp(-x^2/2)} = \sqrt{2 \exp(-2/2)} = \sqrt{2 \exp(-1)}$$

$$U \sim \text{Unif}[0, 1], V \sim \text{Unif}[0, \sqrt{2 \exp(-1)}]$$

Therefore, generate samples from U and V . For the sample (u, v) , if $0 \leq u \leq \sqrt{2 \cdot \exp(-v^2/u^2)}$, then accept $x = v/u$, which will be a sample from $\tilde{\pi}(x)$.

Exercise 5 (Rejection and Importance Sampling)

1. Let $A = \text{sample is accepted in step (a)}$.

$$q^*(x) = p(X = x \mid X \in A) = \frac{P(X \in A \mid X = x)p(X = x)}{p(X \in A)} = \frac{P(U \leq \min\{1, \frac{\pi(x)}{q(x) \cdot c}\} \mid X = x)q(x)}{P(X \in A)}$$

$$P(X \in A) = \int_{\mathcal{X}} P(X \in A \mid X = x)q(x)dx = \int_{\mathcal{X}} \min\{1, \frac{\pi(x)}{q(x) \cdot c}\}q(x)dx = Z_c$$

Hence,

$$q^*(x) = \frac{\min\{1, \frac{\pi(x)}{q(x) \cdot c}\}q(x)}{Z_c} = \frac{\left(P(U \leq 1) \cdot \mathbb{I}_{\{\frac{w(x)}{c} \geq 1\}}(x) + \frac{\pi(x)}{q(x)c} \mathbb{I}_{\{\frac{w(x)}{c} < 1\}}(x)\right)q(x)}{Z_c}$$

$$q^*(x) = \frac{\mathbb{I}_{\{\frac{w(x)}{c} \geq 1\}}(x) \cdot q(x) + \frac{\pi(x)}{c} \mathbb{I}_{\{\frac{w(x)}{c} < 1\}}(x)}{Z_c}$$

2. For $w^*(x) = \pi(x)/q^*(x)$, we want to show that:

$$E_{q^*}([w^*(X)]^2) = Z_c E_q(\max\{w(X), c\}w(X))$$

First:

$$E_{q^*}([w^*(X)]^2) = \int_{\mathcal{X}} w(x)^2 \cdot q^*(x)dx = \int_{\mathcal{X}} \frac{\pi(x)^2}{q^*(x)}dx = \int_{\mathcal{X}} \frac{\pi(x)w(x)Z_c}{\min\{1, w(x)/c\}}dx$$

$$E_q(\max\{w(X), c\}w(X)) = \int_{\mathcal{X}} w(x)q(x)\max\{w(X), c\}dx = \int_{\mathcal{X}} \pi(x)\max\{w(X), c\}dx$$

Now, note that:

- If $w(x)/c < 1 \therefore w(x)/\min\{1, w(x)/c\} = w(x)c/w(x) = c \therefore \max\{w(X), c\} = c$
- If $w(x)/c \geq 1 \therefore w(x)/\min\{1, w(x)/c\} = w(x) \cdot 1 = w(x) \therefore \max\{w(X), c\} = w(x)$

Hence,

$$w(x)/\min\{1, w(x)/c\} = \max\{w(X), c\}$$

With that, we conclude:

$$E_{q^*}([w^*(X)]^2) = Z_c \int_{\mathcal{X}} \frac{\pi(x)w(x)}{\min\{1, w(x)/c\}}dx = Z_c \int_{\mathcal{X}} \pi(x)\max\{w(X), c\}dx$$

$$E_{q^*}([w^*(X)]^2) = Z_c E_q(\max\{w(X), c\}w(X))$$

3. Couldn't solve.

4. We want to show that $\mathbb{V}_{q^*}(w^*(X)) \leq \mathbb{V}_q(w(X))$.

We will use the results from items (2) and (3), even though item (3) was not properly proven.

First, using (3) we have:

$$\begin{aligned} E_q(w(X)^2) &= \frac{1}{c} E_q(w(X)^2 \cdot c) \geq \frac{1}{c} E_q(\max\{w(X), c\} w(X)) E_q(\min\{w(x), c\}) \\ E_q(w(X)^2) &\geq \frac{1}{c} \frac{E_{q^*}(w(X)^2) E_q(\min\{w(x), c\})}{Z_c} \end{aligned}$$

Note that:

- If $w(X) > c$, then $\min\{w(X), c\}/c = w(X)/c$;
- If $w(X) \leq c$, then $\min\{w(X), c\}/c = 1$.

Hence, $\min\{w(X), c\}/c = \min\{1, w(x)/c\}$, which implies:

$$\frac{E_q(\min\{w(x), c\})}{c} = \int_{\mathcal{X}} \min\{w(x), c\} \cdot \frac{q(x)}{c} dx = \int_{\mathcal{X}} \min\{1, w(x)/c\} \cdot q(x) dx = Z_c$$

Therefore:

$$E_q(w(X)^2) \geq E_{q^*}(w(X)^2)$$

Exercise 6 (Rejection and Importance Sampling)

1.

$$\begin{aligned}
\tilde{\pi}_X(x) &= \int_0^1 \tilde{\pi}_{X,U}(x, u) p_U(u) du = \int_0^1 \tilde{\pi}_{X,U}(x, u) du = \\
&= \int_0^1 (\mathbb{I}_{[0, w(x)/M]}(u) \cdot Mq(x) + \mathbb{I}_{[w(x)/M, 1]}(u) \cdot 0) du = \int_0^{w(x)/M} Mq(x) du = \\
&= Mq(x) \cdot w(x)/M = \pi(x)
\end{aligned}$$

2. Let \hat{I}_n be the estimate using important sampling, and \hat{I}_r be the one using rejection sampling.

$$\begin{aligned}
I &= \int_0^1 \int_{\mathcal{X}} \tilde{\pi}_{X,U}(x, u) dx du = \frac{\int_0^1 \int_{\mathcal{X}} \tilde{\pi}_{X,U}(x, u) dx du}{\int_0^1 \int_{\mathcal{X}} \tilde{w}(x, u) \tilde{q}(x, u) dx du} = \frac{E_q[\phi(x) \tilde{w}(x, u)]}{E_{\tilde{q}}[\tilde{w}(x, u)]} \implies \\
\hat{I}_n &= \frac{\sum_{i=1}^n \phi(x_i) \tilde{w}(x_i, u_i)}{\sum_{i=1}^n \tilde{w}(x_i, u_i)} = \frac{\sum_{i=1}^n \phi(x_i) \tilde{\pi}(x_i, u_i) / \tilde{q}(x_i, u_i)}{\sum_{i=1}^n \tilde{\pi}(x_i, u_i) / \tilde{q}(x_i, u_i)} =
\end{aligned}$$

Let n be the number of trials in the simulation, n_a be the number of accepted samples from these trials, and $x^{(a)}$ the accepted samples.

$$= \frac{\sum_{i=1}^n \phi(x_i) \cdot \mathbb{I}(u)_{[0, w(x)/M]} \cdot Mq(x) / (q(x) \cdot \mathbb{I}(u)_{[0, 1]})}{\sum_{i=1}^n \mathbb{I}(u)_{[0, w(x)/M]} \frac{Mq(x)}{q(x) \mathbb{I}(u)_{[0, 1]}}} = \frac{\sum_{i=1}^n \phi(x_i) I(u)_{[0, w(x)/M]}}{\sum_{i=1}^n I(u)_{[0, w(x)/M]}}$$

In the rejection sampling we have:

$$\hat{I}_r = \frac{\sum_{i=1}^{n_a} \phi(x_i^{(a)})}{n_a}$$

Finally, note that:

$$\begin{aligned}
\sum_{i=1}^n I(u)_{[0, w(x)/M]} &= n_a \\
\sum_{i=1}^n \phi(x_i) I(u)_{[0, w(x)/M]} &= \sum_{i=1}^{n_a} \phi(x_i^{(a)})
\end{aligned}$$

Therefore, $\hat{I}_n = \hat{I}_r$.

3. We want to show that $\mathbb{V}_q(w(X)) \leq \mathbb{V}_{\tilde{q}_{X,U}}(\tilde{w}(X, U))$.

Note that:

$$\begin{aligned} E_q[w(X)] &= \int_{\mathcal{X}} \pi(x)q(x)/q(x)dx = 1 \therefore \\ V_q(w(X)) &= E_q[w(X)^2] - E_q[w(X)]^2 = E_q[w(X)^2] - 1 \\ E_{\tilde{q}}[\tilde{w}(X, U)] &= \int_{\mathcal{X}} \int_0^1 \tilde{\pi}(x, u)\tilde{q}(x, u)/\tilde{q}(x, u)dudx = 1 \therefore \\ V_{\tilde{q}}(\tilde{w}(X, U)) &= E_{\tilde{q}}[\tilde{w}(X, U)^2] - 1 \end{aligned}$$

Hence, we only need to show that $E_q[w(X)^2] \leq E_{\tilde{q}}[\tilde{w}(X, U)^2]$.

$$\begin{aligned} E_{\tilde{q}}[\tilde{w}(X, U)^2] &= \int_0^1 \int_{\mathcal{X}} \frac{\tilde{\pi}(x, u)^2}{\tilde{q}(x, u)} dx du = \int_0^1 \int_{\mathcal{X}} \frac{M^2 q(x)^2 \mathbb{I}(u)_{[0, w(x)/M]} dx du}{q(x)} dx du = \\ &= M^2 \int_{\mathcal{X}} \int_0^{w(x)/M} q(x) du dx = M^2 \int_{\mathcal{X}} \frac{w(x)q(x)}{M} dx = M \cdot E_q[w(X)] = M \end{aligned}$$

Finally, using rejection we know that:

$$Mq(x) \geq \pi(x) \implies Mq(x)w(x) \geq \pi(x)w(x) \implies M \int_{\mathcal{X}} q(x)w(x)dx \geq \int_{\mathcal{X}} \pi(x)w(x)dx$$

Which means that $\mathbb{V}_q(w(X)) \leq \mathbb{V}_{\tilde{q}_{X,U}}(\tilde{w}(X, U))$.

4. Couldn't solve.

Simulation Question (Rejection)

The simulations were implemented in Julia 1.05. The code was omitted when the question didn't specifically request it. Still, the code for all simulations are in the Appendix.

1. Estimating π .

$$\pi = 3.14844$$

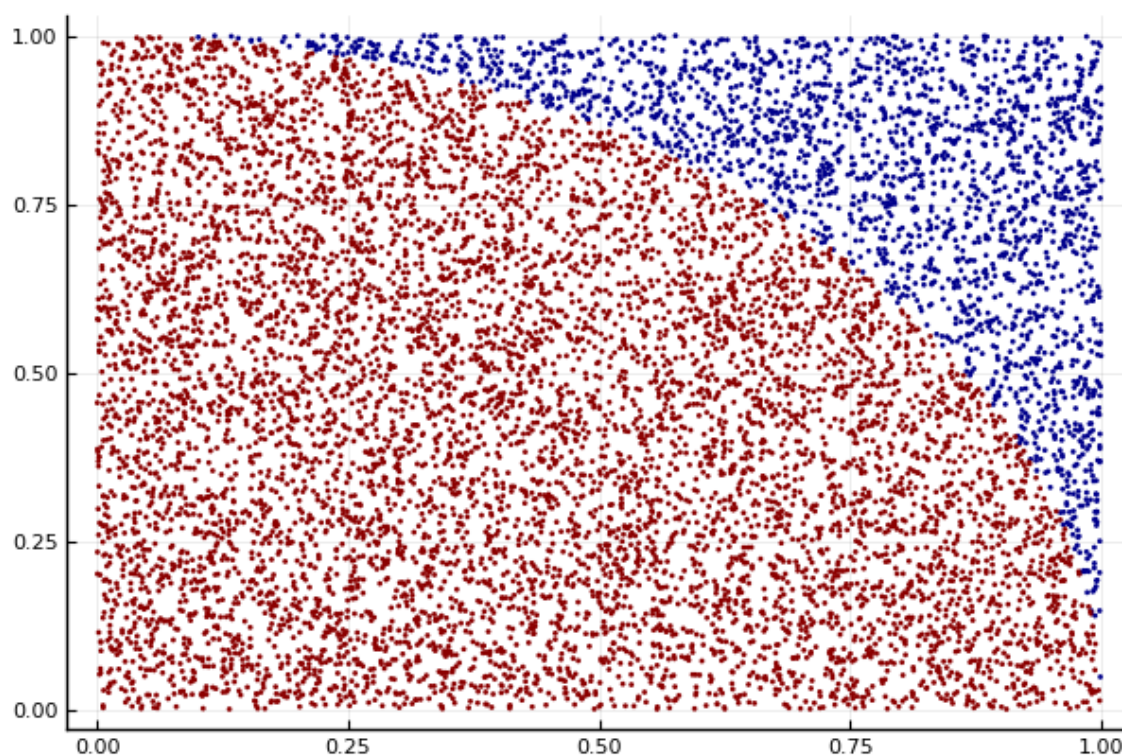


Figure 1: Blue dots outside of circle and red dots inside

2. Implementation of function for simulating Box-Muller.

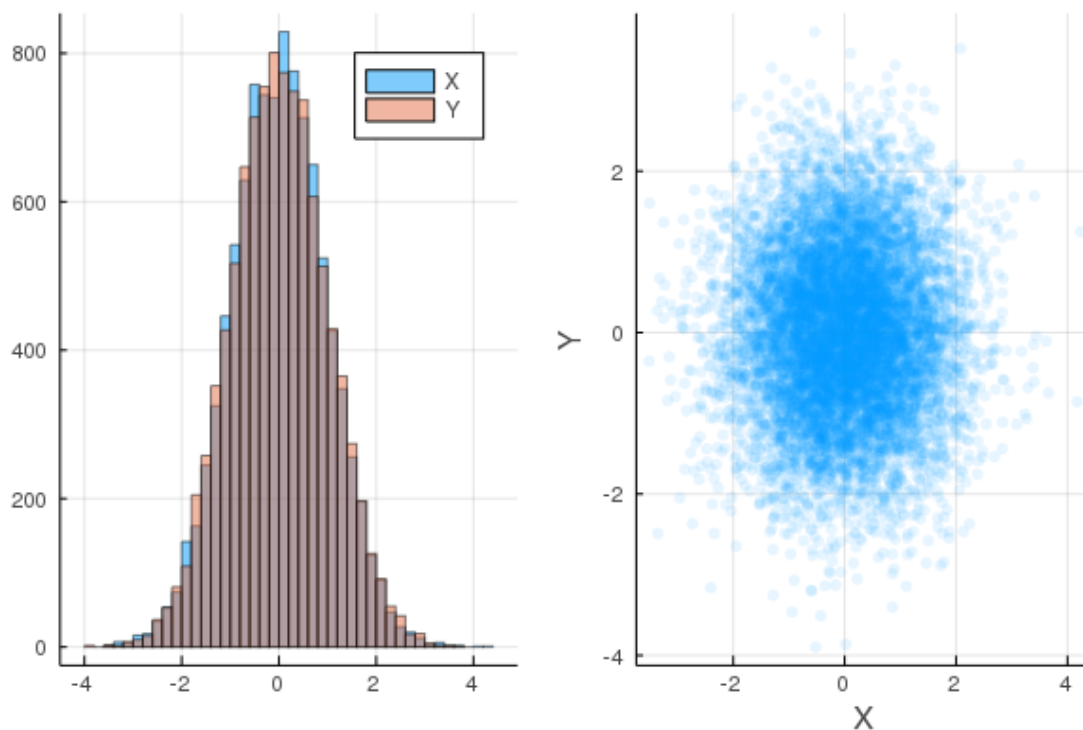


Figure 2: In the left the histograms for the distributions and in the right a scatter plot to highlight that X and Y are independent

```
1 ##### Simulation Exercise 2 #####
2
3 function box_muller(N=100)
4     R = (-2*log.(rand(N))).^(1/2)
5     v = 2*pi*rand(N)
6     X = R.*cos.(v)
7     Y = R.*sin.(v)
8     return X,Y
9 end
```

3. Genetic Linkage model, generating histograms.

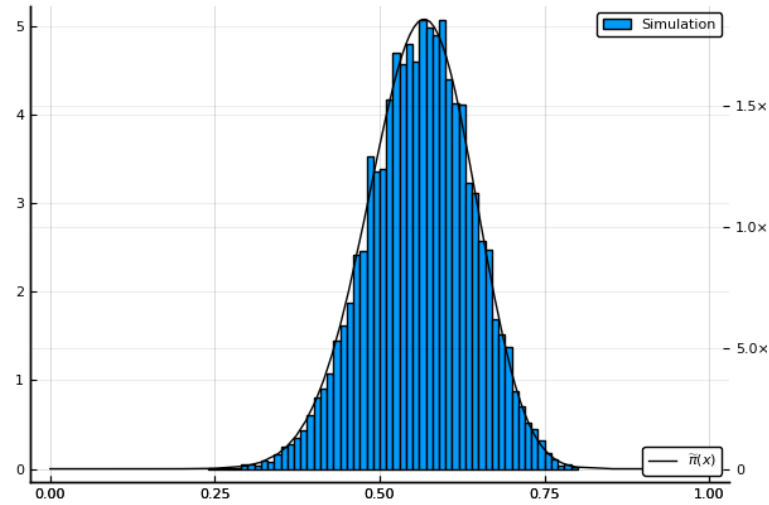
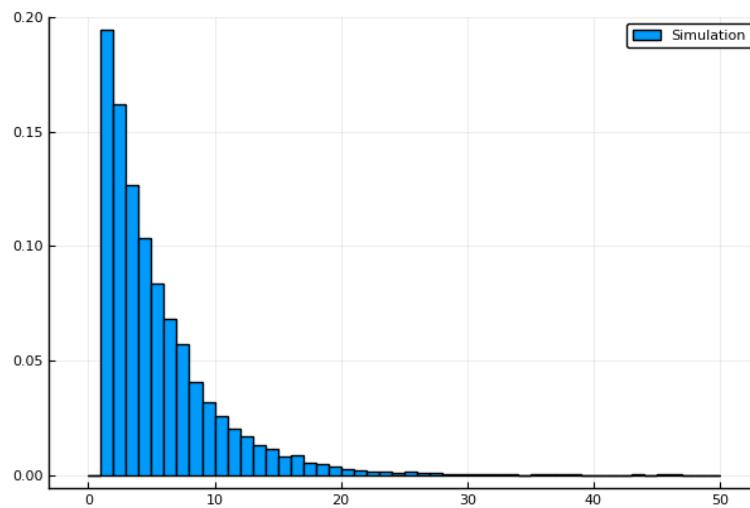
Figure 3: Histogram for 10.000 samples from $p(\theta \mid y_1, \dots, y_4)$ obtained by rejection sampling

Figure 4: Histogram of waiting time distribution before acceptance

4. Implementation of function for Gaussian Mixture.

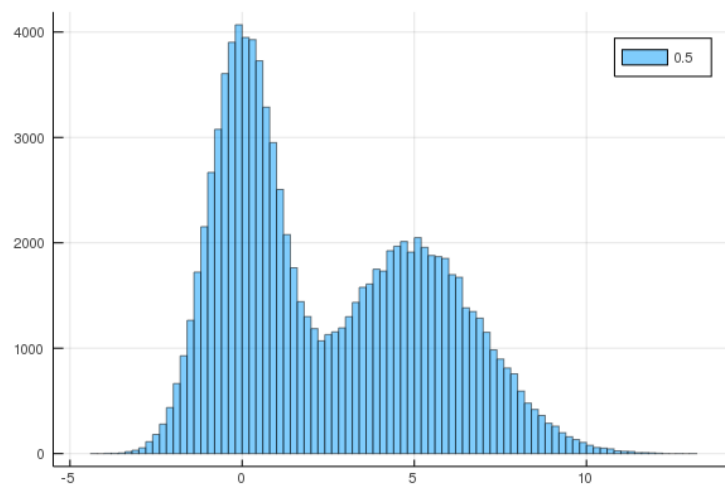


Figure 5: Histogram of mixture of Gaussians $N(0, 1)$ and $N(5, 2)$, with mixture parameter of 0.5

```
1 ##### Simulation Exercise 4 #####
2
3 function mixture(N=1000,mix_rate=0.5,mu1=0,sigma1=1,mu2=0,sigma2=1)
4     W = rand(N) .<= (1-mix_rate)
5     X,Y = box_muller(N)
6     X = sigma1.*X .+ mu1
7     Y = sigma2.*Y .+ mu2
8     Mix = (X).*(W*1) + Y.*(.!W*1)
9     return Mix
10 end
```

5. Let $h(x) = \cos(50x) + \sin(20x)$

- The exact answer for the integral of $h(x)$ between 0 and 1 is 0.02435;
- Using rejection sampling with an uniform distribution, we are able to get an error of roughly 4.31% for a sample of size 1.000.000;
- Using an importance sampling with $q \sim \text{Beta}(2, 2)$ and a sample size of 1.000.000, we get an error of roughly 0.62%.

Appendix (Simulation code)

1. Estimating π .

```
1 ##### Simulation Exercise 1 #####
2
3 using Plots; pyplot()
4
5 # x^2 + y^2 <= r^2
6 function calculate_pi(N = 1000)
7     S = 0
8     x, y = rand(N), rand(N)
9     c = zeros(N)
10    for i in 1:N
11        if x[i]^2+y[i]^2 <= 1
12            S+=1
13            c[i] = 1
14        end
15    end
16    pi_hat = 4*S/N
17    return pi_hat,x,y,c
18 end
19
20 N = [100,1_000,10_000,100_000]
21 v,x,y,c = calculate_pi(10_000)
22
23 p = scatter(x,y,zcolor=c,markersize=2,markerstrokewidth = 0,c=:bluesreds,fmt
24           =:png,legend=false)
25 savefig(p,./images/Ex1.png)
```

2. Box-Muller.

```
1 ##### Simulation Exercise 2 #####
2
3 function box_muller(N=100)
4     R = (-2*log.(rand(N))).^(1/2)
5     v = 2*\pi*rand(N)
6     X = R.*cos.(v)
7     Y = R.*sin.(v)
8     return X,Y
9 end
```

3. Genetic Linkage model.

```
1 ##### Simulation Exercise 3 #####
2
3 using Plots; pyplot()
4 using SymPy
5 using LaTeXStrings
6
7 x = symbols(x, real=true)
8 ex = ((2+x)^y[1])*((1-x)^(y[2]+y[3]))*x^y[4]
9 solution = solve(diff(ex))
10 argmle = SymPy.N(solution[5])
11 mle = round(SymPy.N(ex.subs(x,argmle)))
12
13 function rejection_sampling(N=1000)
14     post(Îŷ,y) = ((2+Îŷ)^y[1])*((1-Îŷ)^(y[2]+y[3]))*Îŷ^y[4]
15     y = [69,9,11,11]
16     size = 0
17     sample = zeros(N)
18     T = zeros(0)
19     t = 0
20     while size < N
21         x = rand()
22         u = rand()
23         if u <= post(x,y)/mle
24             size+=1
25             sample[size] = x
26             append!(T , t)
27             t = 0
28         end
29         t += 1
30
31     end
32     return sample, T
33 end
34
35
36 sample, T = rejection_sampling(10_000)
```

4. Gaussian Mixture.

```
1 ##### Simulation Exercise 4 #####
2
3 function mixture(N=1000,mix_rate=0.5,mu1=0,sigma1=1,mu2=0,sigma2=1)
4     W    = rand(N) .<= (1-mix_rate)
5     X,Y = box_muller(N)
6     X    = sigma1.*X .+ mu1
7     Y    = sigma2.*Y .+ mu2
8     Mix = (X).*(W*1) + Y.*(.!W*1)
9     return Mix
10 end
```

5. Rejection and Important Sampling.

```
1 ##### Simulation Exercise 5 #####
2
3 using Plots, Distributions, StatsPlots
4 pyplot()
5
6
7 # Item 1 – Exact answer
8 x = symbols(x)
9 eq = cos(50*x)+sin(20*x)
10 I = integrate(eq,(x,0,1))
11 Iv = SymPy.N(I)
12 println(I, = ,Iv)
13
14 # Item 2 and 3 – Rejection Sampling
15 function rejection_sampling(N=1000,M=4)
16     f(x) = cos(50*x)+sin(20*x)+2
17     size = 0
18     sample = zeros(N)
19     T = zeros(0)
20     t = 0
21     while size < N
22         x = rand()
23         u = rand()
24         if u <= f(x)/M
25             size+=1
26             sample[size] = x
27             append!(T , t)
28             t = 0
29         end
30         t += 1
31
32     end
33     return sample, T
34 end
35
36 erro = 0
37 for i = 1:10
38     N = 1_000_000
39     sample, T = rejection_sampling.(N)
40     erro += abs(N / sum(T)*4 - 2 - Iv)/Iv*100
41 end
42
43 error = erro/10
```

```
44  
45  
46 # Item 4 – Importance sampling  
47  
48 f(x) = cos(50*x)+sin(20*x)  
49 error = 0  
50 for i = 1:10  
51     N = 1_000_000  
52     sample = rand(Distributions.Beta(2,2),N)  
53     q(x)    = Distributions.pdf(Beta(2,2),x)  
54     w      = exp(-log.(q.(sample)))  
55     error += abs(sum(f.(sample)'* w)/N - Iv)*100  
56 end  
57  
58 error = error / 10
```