Worksheet 1

Notation: here is a brief summary of the notation used in this worksheet.

- $p(X = x)$ is equal to the probability density function;
- Capital letters such as X stand for the random variable.

Exercise 1 (Inversion and Rejection)

1. Let $F_X(x) = \mathbb{P}(X \leq x)$ and $U \sim Unif[0,1]$:

$$F_X(x) = 1 - e^{-\lambda(X-a)}\mathbb{I}_{\{X \geq a\}} = U$$

$$-\ln(1-U) = \lambda(x-a)$$

$$F_X^{-1}(U) = a - \frac{-\ln(1-U)}{\lambda}$$

To simulate X from U, just simulate value from U and substitute in the formula above.

2. Let $X = Y \mid a \leq Y \leq b$. First, let's show that $X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$:

$$\mathbb{P}(X \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U)+F_Y(b)U) \leq x) = \mathbb{P}(F_Y^{-1}(F_Y(a)+U[F_Y(b)-F_X(a)]) \leq x)$$

$$= \mathbb{P}(F_Y(a)+U[F_Y(b)-F_X(a)] \leq F_Y(x)) = \mathbb{P}\left(U \leq \frac{F_Y(x)-F_Y(a)}{F_Y(b)-F_Y(a)}\right) = \frac{F_Y(x)-F_Y(a)}{F_Y(b)-F_Y(a)}$$

Note that since $x \in [a,b]$:

$$\mathbb{P}(Y \leq x \mid a \leq Y \leq b) = \frac{\mathbb{P}(Y \leq x, a \leq Y \leq b)}{\mathbb{P}(a \leq Y \leq b)} = \frac{\mathbb{P}(a \leq Y \leq x)}{F_Y(b)-F_Y(a)} = \frac{F_Y(x)-F_Y(a)}{F_Y(b)-F_Y(a)} = \mathbb{P}(X \leq x)$$

Now that we proved the above relation, to simulate an exponential conditioned on $\geq a$, we first generate $U \sim Unif[0,1]$, then, for $Y \sim Expo(\lambda)$:

$$F_Y(y) = 1 - e^{\lambda y} \therefore F_Y^{-1}(U) = \frac{-\ln(1-U)}{\lambda}$$

$$X = \frac{-\ln(1-(1-U)F_Y(a)+U)}{\lambda} = \frac{-\ln(e^{-\lambda a} + U \cdot e^{-\lambda a})}{\lambda} = a - \frac{\ln(1-U)}{\lambda}$$

The formula yields the same solution as the one obtained using inversion.

3. Let $q \sim Expo(\lambda)$, and $\pi(x) = \lambda e^{-\lambda(x-a)}\mathbb{I}_{x\geq a}$:

   Note that $M = max_x \pi(x)/q(x) = e^{\lambda a}$, since $\pi(x)/q(x) = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{\lambda(x)}} = e^{\lambda a}$

   $$\therefore$$

   In the rejection method, we sample $x_i \sim q$, $u \sim Unif[0,1]$,then we accept a sample x_i if $u_i \leq \frac{\pi(x_i)}{Mq(x_i)}$.

   Hence,

   - If $x \leq a \implies \pi(x) = 0 \implies u \leq 0 \therefore x_i$ is rejected;
   - If $x > a \implies \pi(x) = 1 \implies u \leq 1 \therefore x_i$ is accepted;

   Which is the same procedure described in the question, implying that it is equal to the rejection algorithm.

   Finally, the expected number of trials is equal to $M = e^{\lambda a}$. Therefore, for $a \gg 1/\lambda$, the expected number of trials becomes very large (greater computational cost), while this problem doesn't happen with inversion, since every sample is used.