Exercise 1 (Monte Carlo for Gaussians)

1. Let's prove that $E[\phi(X)] = E[\phi(X+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)]$.

$$E[\phi(X+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)] = \int_{\mathbb{R}^d} \phi(x+\theta)exp(\frac{-1}{2}\theta^T\theta - \theta^T X)\pi(x)dx_1...dx_d =$$

$$= \int_{\mathbb{R}^d} \phi(x+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right) exp(-x^T x/2)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

$$\int_{\mathbb{R}^d} \phi(x+\theta)exp\left(\frac{-1}{2}(x-\theta)^T(x-\theta)\right)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d$$

Finally, making $x - \theta = y$,

$$\int_{\mathbb{R}^d} \phi(y)exp\left(\frac{-1}{2}(y)^T(y)\right)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d = E[\phi(Y)]$$

$\square$

2. Let's show that

$$\sigma^2(\theta) = E\left[\phi^2(X)exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] - (E[\phi(X)]^2$$

Note that, using the result in the previous item we have:

$$\sigma^2(\theta) = V\left[\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right] =$$

$$= E\left[\left(\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right)^2\right] - E\left[\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right]^2 =$$

$$= E\left[\left(\phi(X+\theta)exp\left(\frac{-1}{2}\theta^T\theta - \theta^T X\right)\right)^2\right] - E\left[\phi(X)\right]^2$$

Now, let's rearrange the first term in the variance.

$$\sigma^2(\theta) = \int_{\mathbb{R}^d} \phi(x+\theta)^2 exp\left(-\theta^T\theta - 2\theta^T X\right) exp(-x^T x/2)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

Make $X + \theta = Y$, then:

$$\int_{\mathbb{R}^d} \phi(y)^2 exp\left(-\theta^T\theta - 2\theta^T(y-\theta)\right) exp(-(y-\theta)^T(y-\theta)/2)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

$$= \int_{\mathbb{R}^d} \phi(y)^2 exp\left(\frac{1}{2}(y-\theta)^T(y-\theta) - \frac{y^Ty}{2}\right) exp\left(\frac{-y^Ty}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx_1...dx_d =$$

$$= E\left[\phi^2(X)exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right]$$

Therefore,

$$\sigma^2(\theta) = E\left[\phi^2(X)exp\left(\frac{-1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] - (E[\phi(X)]^2$$

□

3. Let's calculate $\nabla^2\sigma^2(\theta) = H(\theta)$.

$$\frac{\partial\sigma^2(\theta)}{\partial\theta_i} = \frac{E[\phi(X)^2 exp(\frac{-X^TX+(X-\theta)^T(X-\theta)}{2})]}{\partial\theta_i} =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx)\frac{\partial}{\partial\theta_i}exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i)exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx$$

We calculated the gradient, let's now calculate the second derivative. First the diagonal.

$$\frac{\partial}{\partial\theta_i}\int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i)exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$

$$= E[\phi(X)^2] + \int_\chi \phi(x)^2 exp(-x^Tx)exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)(x_i-\theta_i)(x_i-\theta_i)dx$$

Now the rest:

$$\frac{\partial}{\partial\theta_j}\int_\chi \phi(x)^2 exp(-x^Tx)(\theta_i - x_i)exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)\frac{1}{(\sqrt{2\pi})^d}dx =$$

$$= \int_\chi \phi(x)^2 exp(-x^Tx)exp\left(\frac{(x-\theta)^T(x-\theta)}{2}\right)(x_i-\theta_i)(x_j-\theta_j)dx$$

□