Chapter 1 - Review

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July 7, 2021

1 Starting with Cauchy

In these notes, we consider all a_i, b_j for $i, j \in \mathbb{N}$ to be real values. For a vector space V over $\mathbb{R}$, the inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ has the following properties:

(i) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and equal to 0 iff $\mathbf{v} = 0$;

(ii) $\langle \alpha\mathbf{v} + \mathbf{w}, \mathbf{v} \rangle = \alpha\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$;

(iii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

Theorem 1.1. (Cauchy-Schwarz Inequality *Finite* terms)

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2). \tag{1}$$

Proof. Using induction.

$$(a_1 - b_1)^2 \geq 0 \implies a_1^2 + b_1^2 \geq 2a_1 b_1.$$

Suppose that for n:

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}.$$

$$\sum_{i=1}^n a_i b_i + a_{n+1}b_{n+1} \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} + a_{n+1}b_{n+1}.$$

Apply Cauchy-Schwarz to $c_1 := \sqrt{\sum_{i=1}^n a_i^2}$, $d_1 := \sqrt{\sum_{i=1}^n b_i^2}$, $c_2 := a_{n+1}$ and $d_2 := b_{n+1}$, hence:

$$c_1 d_1 + c_2 d_2 \leq \sqrt{c_1^2 + c_2^2}\sqrt{d_1^2 + d_2^2} = \sqrt{\sum_{i=1}^n a_i^2 + a_{n+1}^2}\sqrt{\sum_{i=1}^n b_i^2 + b_{n+1}^2}.$$

□

Theorem 1.2. (Cauchy-Schwarz Inequality *Infinite* terms)

$$\left(\sum_{i=1}^{\infty} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{\infty} a_i^2\right)\left(\sum_{i=1}^{\infty} b_i^2\right). \tag{2}$$

Proof. First, note that $(a+b)^2 \geq 0$, $(a-b)^2 \geq 0$, then $a^2+b^2 \geq 2|ab|$. Hence,

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{\infty} a_i^2 + \sum_{i=1}^{\infty} b_i^2 \geq \lim_{n\to\infty} \sum_{i=1}^{n} 2|a_i b_i| = \sum_{i=1}^{\infty} 2|a_i b_i|.$$

Since the inequality above is valid for any $a_i, b_i \in \mathbb{R}$, make:

$$\hat{a}_i := \frac{a_i}{\sqrt{\sum^{\infty} a_i^2}}, \quad \hat{b}_i := \frac{b_i}{\sqrt{\sum^{\infty} b_i^2}}.$$

Note that

$$\sum_{i=1}^{\infty} \hat{a}_i^2 = \frac{\sum_{i=1}^{\infty} a_i^2}{\sum_{i=1}^{\infty} a_i^2} = 1, \quad \sum_{i=1}^{\infty} \hat{b}_i^2 = \frac{\sum_{i=1}^{\infty} b_i^2}{\sum_{i=1}^{\infty} b_i^2} = 1.$$

Therefore,

$$2 = \sum_{i=1}^{\infty} \hat{a}_i^2 + \sum_{i=1}^{\infty} \hat{b}_i^2 \geq \sum_{i=1}^{\infty} 2|\hat{a}_i \hat{b}_i| \implies 1 \geq \sum_{i=1}^{\infty} |\hat{a}_i \hat{b}_i| = \frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i^2 \sum_{i=1}^{\infty} b_i^2}.$$

$\square$

Corollary 1.1. (Sharpness of Cauchy-Schwarz Inequality The Cauchy-Schwarz inequality is sharp iff $a_k = \lambda b_k$ for every $k \in \mathbb{N}$, where

$$\lambda = \frac{\sqrt{\sum_{i=1}^{\infty} a_i^2}}{\sqrt{\sum_{i=1}^{\infty} b_i^2}}. \tag{3}$$

Proof. Using the previous demonstration, we can see that

$$\begin{aligned}(\hat{a}_k - \hat{b}_k)^2 = 0 \iff \hat{a}_k = \hat{b}_k \iff \frac{a_k}{\sqrt{\sum_{i=1}^{\infty} a_i^2}} &= \frac{b_k}{\sqrt{\sum_{i=1}^{\infty} b_i^2}} \\ \iff a_k &= \frac{\sqrt{\sum_{i=1}^{\infty} a_i^2}}{\sqrt{\sum_{i=1}^{\infty} b_i^2}} b_k.\end{aligned}$$

$\square$