Geometric Algebra

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1 Brief Note on Algebra with Category Theory

Let's start by presenting some definitions from Algebra.

For an introduction to Category Theory, check the other notes.

1.1 Initial Definitions for Groups

Definition 1.1 (Groups). Consider the triple $(G, \cdot, e)$, where G is a set, $\cdot : G \times G \to G$ is the product mapping and $e \in G$ is the identity element. This triple is a group if:

1. (Associativity): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in G$;
2. (Identity): $a \cdot e = e \cdot a = a$ for every $a \in G$;
3. (Inverse): For every $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = a$;

When there is no ambiguity, we call the set G a group omitting the product and neutral element.

Whenever it's note ambiguous, we omit the product operator, thus, $g \cdot h \equiv gh$.

Definition 1.2 (Abelian Group). A group $(G, \cdot, e)$ is *Abelian* if besides the group properties (i.e. associativity, identity and inverse) it's also commutative, i.e. $a \cdot b = b \cdot a$ for every $a, b \in G$.

Example 1.1. Note that $(\mathbb{R}, +, 0)$ is an Abelian Group. In this case, a^{-1} is usally denoted as $-a$. The triple $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is also an Abelian Group.

An example of non-Abelian group would be the set of invertible matrices from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $\cdot$ as matrix composition, e.g. $A \cdot B = AB$. Since every matrix considered is invertible and we have the identity matrix as our identity element, then we indeed have a non-Abelian group, since the matrix product is not commutative.

Proposition 1.3 (Group Cancellation). Let $(G, \cdot)$ be a group. Therefore:

$$fa = ha \implies f = h, \qquad af = ah \implies f = h$$

Proof. If $fa = ha$, then $faa^{-1} = ha^a - 1 \implies f = h$. □

Definition 1.4 (Subgroup Generated). Let $(G, \cdot, e)$ be a group. We say that $S \subset G$ is a subgroup of G if $(S, \cdot, e)$ is a group. For $A \subset G$, $\mathrm{Gr}(A)$ is called the subgroup generated by A, and it's the smallest subgroup of G containing A, i.e. $\cap_{\alpha \in \Gamma} S_\alpha$ where $\{S_\alpha\}_{\alpha \in \Gamma}$ are all the sets that are subgroups of G. It's easy to prove that such set is indeed a subgroup.

For a singleton $\{g\}$, we define $\mathrm{Gr}(g) := \{g^n \ : \ n \in \mathbb{Z}\}$, where $g^0 = e$, and g^n is the product of n copies of g, while g^{-n} is the product of n copies of $-g$.

Definition 1.5 (Cyclic Group). If a group G is equal to $\mathrm{Gr}(g)$ for some $g \in G$, then we say that G is cyclic.

Definition 1.6 (Order of Groups). The order of a finite group G is the number of elements of G. An element $g \in G$ has *finite order* if $g^n = e$ for $n \in \mathbb{N}$. The order of g is then the smallest n such that $g^n = e$.

Definition 1.7 (Homomorphism and Isomorphism). Let $(G, \cdot_G, e_G)$ and $(H, \cdot_H, e_H)$ be two groups. A function $\theta : G \to H$ is a homomorphism between G and H if $\theta(g_1 \cdot_G g_2) = \theta(g_1) \cdot_H \theta(g_2)$ for every $g_1, g_2 \in G$.

If θ is bijective, then we say that θ is an isomorphism.

Definition 1.8 (Normal / Self-conjugate). Let K be a subgroup of G. We say that K is *normal*, or *self-conjugate*, if $gkg^{-1} \in K$ for every $g \in G$.

1.2 Groups and Category Theory

Remember that in Category Theory we have a notion of isomorphism that generalizes set isomorphism (i.e. bijective function between sets).

Definition 1.9 (Automorphism). Let A be an object of a category $\mathcal{C}$. An automorphism is an isomorphism from A to itself. The set[1] of automorphism of A is denoted by $\mathrm{Aut}_{\mathcal{C}}(A)$.

Definition 1.10 (Grupoid and Groups). A groupoid is a category where every morphism is an isomorphism. Hence, a group is a groupoid category with a single object G. We denote **Grp** as the catogory of groups. In similar fashion, we can define **Ab** as the category of abelian groups, where the only difference is that the objects are abelian groups.

Note that this definition is equivalent to our definition of a group in algebraic terms. Why? Because every morphism is equivalent to an element of G, and the morphism composition does the part of the product operator. Also, note that every category has an identity morphism, thus, $id_G \equiv e$ our neutral element. Since every morphism is an isomorphism, this means that for every $g \in Hom(G, G)$, there is a $g^{-1} \in Hom(G, G)$ such that $g \circ g^{-1} = id_G = e$.

In pure categorical terms. Let $\mathcal{C}$ be a locally small category and $G \in \mathcal{C}$, i.e. an object of $\mathcal{C}$.

[1] Remember that a $Hom(A, A)$ is guaranteed to be a set if the category is locally small.

1.3 Rings and Modules

Let's begin by remembering the concept of a monoid. A monoid $(M, \cdot, e)$ is a set M, together with the binary operator $\cdot : M \times M \to M$ and the identity element e. Besides, $\cdot$ is associative.

Definition 1.11 (Ring). A ring $(R, \cdot, +)$ is an abelian group $(R, +)$ together with a monoid $(R, \cdot)$, with the property of distributivity, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for every $a, b, c \in R$.

One usually denotes the identity of $(R, +)$ by 0_R and the identity of $(R, \cdot)$ by 1_R. The reason is clear, since these are the corresponding identities for the usual sum and multiplication of numbers.

Based on this definition, one can prove that:

Proposition 1.12. Let $(R, \cdot, +)$ be a ring. Therefore:

$$0 \cdot r = 0 = r \cdot 0.$$

Note that in this definition, the $+$ operator has much more stated properties, e.g. there are inverse elements, there is commutativity. The $\cdot$ has more freedom. For example, we are not requiring for an inverse to exist, and neither commutativity. Which leads to the following definition.

Definition 1.13 (Commutative Ring). A ring $(R, \cdot, +)$ is commutative if $a \cdot b = b \cdot a$ for every $a, b \in R$.

Now, we want to slowly increment the properties of these algebraic concepts in order to construct our usual suspects, e.g. $\mathbb{N}$ $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$.

Definition 1.14 (Zero-Divisor). Let $(R, \cdot, +)$ be a ring. We say that $a \in R$ is a left-zero-divisor if there exists $b \neq 0 \in R$ such that $ab = 0$. Analogously, we define a right-zero-divisor.

Note that $0 \in R$ is a zero-divisor of every ring R **with the exception** of the *zero-ring* case. The zero-ring is the ring where R is a singleton set. Hence, since there is only one element, there is no element such that $ab = 0$ for $b \neq 0$, since no such b exists.

Example 1.2 ($\mathbb{Z} \setminus n\mathbb{Z}$). Let n be a positive integer.

Definition 1.15 (R-Module). An abelian group $(M, \oplus)$ is called a module over a ring $(R, +, \cdot)$ if there is a map (often called scalar multiplication) where:

$$* : R \times M \to M,$$

such that for all $r, r' \in R$ and $m, m' \in M$ we have

(i) $0_R * m = 0_M$;

(ii) $1_R * m = m$;

(iii) $(r + r') * m = r * m \oplus r' * m$;

(iv) $r * (m \oplus m') = r * m \oplus r * m'$;

(v) $(r \cdot r') * m = r * (r' * m)$.

We also call this an R-Module M.

Definition 1.16 (R-Algebra). An R-Algebra M is an R-Module M together with a bilinear map $M \times M \to M$.

Note that a vector space V over $\mathbb{R}$ with an inner product is an example of R-algebra.

2 Tensors and Vectors

Definition 2.1 (Vector Space). A vector space is a module over a field R, i.e. an R-module where R is a field. Note, for an abelian group $(\mathbf{V}, \oplus)$ and a field R, we have the vector space $(\mathbf{V}, R)$. In order to reduce the amount of writing, we call $\mathbf{V}$ the vector space, which implies that there is an underlying field R and the existence of an scalar product.

The tensor product of a vector

3 On How to Construct Different Algebras

This section is more informal, and is used to give a better intution of how to construct differnt algebras. This is mainly based on Vaz Jr and da Rocha Jr [2].

Consider a vector space $\mathbf{V}$. To define an algebra in a vector space, we have to define a bilinear product between the vectors. A possible product is the inner product. Yet, there are many other possibilities. One of them is the tensor product.

The tensor algebra $T(\mathbf{V})$ is the vector space $\mathbf{V}$ together with the tensor product $\otimes : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$. The tensor algebra has the "free" algebra flavor, meaning, it's "largest" algebra one can construct from $\mathbf{V}$. Thus, all other algebras on $\mathbf{V}$ are quotients of $T(\mathbf{V})$, i.e. we can construct the other bilinear products by introducing equivalence relations.

Thus, the tensor algebra $T(\mathbf{V})$ basis consists of all possible finite combinations of $\mathbf{u}$ and $\mathbf{v}$, where the tensor product of k vectors defines a k-vector in a vector space T^k.

$$T = \bigoplus_{k=0}^{\infty} T^k.$$

For example, suppose that $\mathbf{V}$ has basis $\{\mathbf{u}, \mathbf{v}\}$.

1. $T^0 := \{\mathbf{1}\}$;
2. $T^1 := \{\mathbf{u}, \mathbf{v}\}$
3. $T^2 := \{\mathbf{u} \otimes \mathbf{u}, \mathbf{v} \otimes \mathbf{v}, \mathbf{u} \otimes \mathbf{v}, \mathbf{v} \otimes \mathbf{u}\}$
4. $T^3 := \{\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}, \mathbf{u}\otimes\mathbf{u}\otimes\mathbf{v}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{u}, \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{u}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{u}, \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}\}$
5. etc.

As we've said, the other algebras on $\mathbf{V}$ can be constructed from $T(\mathbf{V})$. One example is the exterior algebra. To construct it, just impose the following equivalence relation, for every $\mathbf{v} \in \mathbf{V}$,

$$\mathbf{v} \otimes \mathbf{v} \cong 0.$$

Note that this condition implies that $\mathbf{v} \otimes \mathbf{u} = -\mathbf{u} \otimes \mathbf{v}$. This follows from

$$(\mathbf{u} + \mathbf{v}) \otimes (\mathbf{u} + \mathbf{v}) = \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} \cong 0.$$

When considering the exterior algebra, we change the product notation from $\otimes$ to $\wedge$. In the exterior algebra, the number of possible combinations of the basis vectors is finite. For example, for a vector space of dimension 2, i.e. basis $\{\mathbf{u}, \mathbf{v}\}$, we have

1. $\bigwedge^0 : \{\mathbf{1}\}$;
2. $\bigwedge^1 : \{\mathbf{u}, \mathbf{v}\}$;
3. $\bigwedge^2 : \{\mathbf{u} \wedge \mathbf{v}\}$.

Using a similar idea, we arrive at the Geometric (Clifford) Algebra. Instead of the equality to zero as in the exterior algebra, we use:

$$\mathbf{v} \otimes \mathbf{v} - B(\mathbf{v}, \mathbf{v}) \cong 0,$$

where B is a symmetric bilinear form. Note that, this definition actually defines a family of algebras (one for each possible B), where the exterior algebra is one of them (just use $B(x, x) = 0$).

A real symmetric bilinear form can be completely characterized by what is called a signature. For a vector space of dimension n, the signature of B is a triple (p, q, z), where p is the number of positive eigenvalues, q is the number of negative eigenvalues and z is the number of eigenvalues equal to zero. Thus, $p + q + z = n$. Remember that eigenvalues are a "fundamental way" of characterizing a transformation.

4 Quadratic Forms

Let's start the formal definition of Geometric Algebra, which is also known as Clifford Algebra.

Definition 4.1 (Quadratic Form and Quadratic Space). Let E be a real vector space. A quadratic form on E is a function $q : E \to \mathbb{R}$ such that $q(x) = b(x, x)$ for all $x \in E$, where b is a symmetric bilinear form on E. We say that b is the associated bilinear form.

We call the tuple (E, q) a quadratic space. The set $Q(E)$ is composed by all quadratic forms on E and it's a linear subspace of the space of linear real-valued functions on E.

Proposition 4.2. Given two different bilinear forms b_1, b_2, they induce different quadratic forms.

Proof. For $q(x) = b(x, x)$, then

$$q(x + y) = b(x + y, x + y) = q(x) + q(y) + 2b(x, y).$$

Thus, we have

$$b(x, y) = \frac{1}{2}\left(q(x + y) - q(x) - q(y)\right).$$

Note that

$$\begin{aligned} q(x - y) = b(x - y, x - y) &= q(x) + q(y) - 2b(x, y) \\ b(x, y) &= \frac{1}{2}\left(-q(x - y) + q(x) + q(y)\right) \end{aligned}$$

Hence, summing both equations we get $b(x, y) = \frac{1}{4}(q(x+y) - q(x-y))$. If there c is another bilinear form such that $c(x, y) \neq b(x, y)$ for some x and y, then the quadratic form induced by c is different than q, i.e. $q_c(x + y) - q_c(x - y) \neq q(x + y) - q(x - y) \implies q_c \neq q$. □

Definition 4.3 (Regular Quadratic Space). Given a quadratic space (E, q), we say that this space is regular if q is regular, i.e. if the associated bilinear form b is invertible (non-singular).

5 First Concepts in Geometric Algebra

As we've already pointed out, the Geometric Algebra over a vector space $\mathbf{V}$ is defined the tensor algebra $T(\mathbf{V})$ with the equivalence relation:

$$\mathbf{v} \otimes \mathbf{v} - B(\mathbf{v}, \mathbf{v}) \cong 0,$$

where B is a symmetric bilinear form. This bilinear form represents the metric of the geometric space, and is fully characterized by what we called a signature. The famous Euclidean space of $\mathbb{R}^n$ is the vector space with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} I \mathbf{v}$ where I is the identity matrix. Thus, the signature of the Euclidean space is $(3, 0, 0)$.

Geometric Algebra is about working with "vector spaces" together with a special product operator, called geometric product.

While Linear Algebra deals with vectors, Geometric Algebra deals with multivectors. A multivector is a generalization of a vector. While vectors are only "arrows", a multivector can be an arrow, but it can also be a plane, a volume, etc.

5.1 Outer (Exterior) Product

Let V be a vector space. Then, a blade $\mathbf{B}$ is just a vector subspace of V, i.e. if $\mathbf{v}_1, \mathbf{v}_2 \in B$, then for any scalars α, β, we have $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \in B$, which implies that $\mathbf{B}$ is also a vector space.

We want to somehow manipulate these blades (subspaces), and do algebra with them, similar to how we do algebra with vectors (e.g. we can sum vectors and so on). One way to do this is to describe how these subspaces are generated. Hence, we introduce the **outer (exterior) product**. This product is denoted by the wedge operator $\wedge$.

The idea behind the outer product is that for two vectors $\mathbf{v}, \mathbf{u} \in V$, $\mathbf{v} \wedge \mathbf{u}$ represents the weighted subspace generated by these two vectors. These subspaces are "weighted" because $\alpha \mathbf{v} \wedge \mathbf{u}$ generates the same subspace, where α is a scalar.

Since the outer product defines vector subspaces, it must be:

- Associative: $\mathbf{v} \wedge (\mathbf{u} \wedge \mathbf{w}) = (\mathbf{v} \wedge \mathbf{u}) \wedge \mathbf{w}$,
- Commutative with the scalars: $\mathbf{v} \wedge \alpha \mathbf{u} = \alpha(\mathbf{v} \wedge \mathbf{u})$,
- Distributive: $v \wedge (\mathbf{u} + \mathbf{w}) = \mathbf{v} \wedge \mathbf{u} + \mathbf{v} \wedge \mathbf{w}$.

Lastly, if we want to somehow encode the orientation of the subspace (e.g. the orientation of a plane), we have to add one last property:

- Antisymmetry: $\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$, which implies $\mathbf{v} \wedge \mathbf{v} = 0$.

Hence, if these vectors are colinear (i.e. $\mathbf{u} = \alpha\mathbf{v}$ for some scalar α), we want the outer product to be zero.

We can use this outer product to generalize the idea of a vector, and define k-vectors. The k represents the "grade"of the multivector, which is the same as the dimension of the spanned subspace. A k-vector is the linear combination of *simple* k-vectors, also known as k-blades. A k-blade is a k-vector that can be written as the outer product of 1-vectors.

For example, consider a 4 dimensional vector space V. In this vector space, we define the base vectors to be $\mathbf{e}_1, ..., \mathbf{e}_4$. Thus, an example of a 2-blade would be $2\mathbf{e}_1 \wedge \mathbf{e}_2$, and an example of a 2-vector (bivector) would be $1\mathbf{e}_3 \wedge \mathbf{e}_4 + 2\mathbf{e}_1 \wedge \mathbf{e}_2$. Note that our bivector cannot be factored in terms of a wedge product. To see that, consider instead the following bivcecto $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 \wedge \mathbf{e}_3$. This one can be written as $(\mathbf{e}_1 - \mathbf{e}_3) \wedge \mathbf{e}_2$, thus, it's a 2-blade.

Once we've defined what a blade and a multivector is, we can define the multivector space. Let V be a finite dimensional vector space with base $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$. The multivector space of this vector space is denoted by $\bigwedge V$, and connsists of the linear combination of all multivectors. The **base blades** of this space are

$$\bigcup_{k=0}^{n} \{\mathbf{e}_{i_1} \wedge ... \wedge \mathbf{e}_{i_k} \ : \ 1 \leq i_1 \leq ... \leq i_k\}$$

Consider the case of the vector space of 3 dimensions. We have:

$$\begin{aligned}
\bigwedge^0 V &= 1 \\
\bigwedge^1 V &= \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \\
\bigwedge^2 V &= \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3 \\
\bigwedge^3 V &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3,
\end{aligned}$$

where $\bigwedge^k V$ is the k-blade space. Note that a multivector is any linear combination of such elements multiplied by a scalar, e.g. $\alpha + \beta\mathbf{e}_1 + \gamma\mathbf{e}_2 \wedge \mathbf{e}_3$.

The 0-blade is called scalar, and the k-th blade is called pseudo-scalar.

Finally, we can define the outer product as:

$$\wedge : \bigwedge^r V \times \bigwedge^s V \to \bigwedge^{r+s} V,$$

such that the properties we've listed before hold (associativity, commutativity, distributivity).

Definition 5.1 (Grade). Given a blade $B \in \bigwedge^k V$, the grade of B is equal to k.

Note that every blade has a grade, yet, an arbitrary multivector might have many grades, e.g. $1 + e_1 + e_1 \wedge e_3$.

5.2 Geometric Product

We've introduced the outer product, but we still haven't talked about the Geometric Product, which is the real star of the show. Consider a vector space V with an inner product. Let $\mathbf{a}$ be a known vector and α a known scalar such that $\mathbf{v} \cdot \mathbf{a} = \alpha$. Can we obtain $\mathbf{v}$ from this equation? The answer is no, as there are many possible $\mathbf{v}$ that satisfy this condition. This means that the inner product is not invertible, and a similar exercise can be done to show that the outer product is also not invertible.

Thus comes the geometric product. Let's denote the geometric product by just justaposing symbols, e.g $\mathbf{ab}$ is the geometric product of $\mathbf{a}$ with $\mathbf{b}$. The idea here is to define a product that is invertible, i.e. $\mathbf{av} = \alpha \implies \mathbf{v} = \alpha \mathbf{a}^{-1}$, where $\exists ! \mathbf{a}^{-1} \; : \; \mathbf{aa}^{-1} = \mathbf{1}$.

So what is this product like? For vectors it's actually quite simple,

$$\mathbf{av} = \mathbf{a} \cdot \mathbf{v} + \mathbf{a} \wedge \mathbf{v}$$

Note that the result of the geometric product is a multivector with a scalar $\mathbf{a} \cdot \mathbf{v}$ and a bivector $\mathbf{a} \wedge \mathbf{v}$. It can then be shown that this product is associative, distributive, linear, invertible, but **not commutative**.

Consider the two dimensional Euclidean space. Note that $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$, and that

$$(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1.$$

This fact shows how the imaginary number appears in geometry. The square of our blade base $\mathbf{e}_1\mathbf{e}_2$ is negative under the geometric product.

What about the inverse? It's clear from the defintion that the inverse of a vector $\mathbf{v}$ is just $\frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$, hence

$$\mathbf{vv}^{-1} = \mathbf{v}\left(\frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) = \frac{\mathbf{vv}}{\mathbf{v} \cdot \mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \wedge \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = 1.$$

Note that the inverse of a vector only exists if it's norm is non-null.

Our definition of the Geomtric Product was done only for vectors. We want to extend it to multivectors. Thus, our product must be defined on $\bigwedge V \times \bigwedge V \to \bigwedge V$.

Although our original construction gave an explicit formula for the geomtric product, the strategy to define it in multivector space is different. Instead of a formula, we postulate the desired properties, which are:

- Scalars are commutative, and their geometric product is equal to their prdocut;
- Vector Square is just $\mathbf{v}^2 = \mathbf{v}\mathbf{v} = \mathbf{v} \cdot \mathbf{v}$;
- Distributive and Linear for multivectors, i.e. $A(B + C) = AB + C$ and $(B + C)A = BA + CA$;
- Associative for multivectors, i.e. $A(BC) = (AB)C$.

Note that we do note enforce commutativeness.

From these properties, we can prove, for example, that for base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ we have $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$.

5.3 More Operators on Multivectors

We've introduced the geometric product, which is the king of Geometric Algebra. From this product we can construct other operators which can be very useful. For example, we can define the left contraction, which can be seen as a generalization of the inner product for multivectors.

Definition 5.2 (Left Contraction - according to Dorst). The left contraction is a bilinear operator

$$\rfloor : \bigwedge^{k} V \times \bigwedge^{l} V \to \bigwedge^{k-l} V,$$

with the following properties:

- $\alpha \rfloor \mathbf{B} = \alpha \mathbf{B}$;
- $\mathbf{B} \rfloor \alpha = 0$ if $\text{grade}(\mathbf{B}) > 0 > 0 > 0 > 0$;
- $\mathbf{a} \rfloor \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$;
- $\mathbf{a} \rfloor (\mathbf{B} \wedge \mathbf{C}) = (a \rfloor \mathbf{B}) \wedge \mathbf{C} + (-1)^{\text{grade}(\mathbf{B})} \mathbf{B} \wedge (\mathbf{a} \rfloor \mathbf{C})$;
- $(\mathbf{A} \wedge \mathbf{B}) \rfloor \mathbf{C} = \mathbf{A} \rfloor (\mathbf{B} \rfloor \mathbf{C})$.

Where α is a scalar, $\mathbf{a}, \mathbf{b}$ are vectors, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are blades.

In an analogous way, we have the right contraction.

Definition 5.3 (Grade Involution - Dorst).

$$\hat{\mathbf{B}} = (-1)^{\text{grade}(\mathbf{B})}\mathbf{B}.$$

One can show that:

$$\mathbf{aB} = \mathbf{a} \rfloor \mathbf{B} + \mathbf{a} \wedge \mathbf{B},$$

where **a** is a vector and **B** is a blade.

Definition 5.4 (Reversion - Dorst).

$$\tilde{\mathbf{B}} = (-1)^{\text{grade}(\mathbf{B})(\text{grade}(\mathbf{B})-1)/2}\mathbf{B}.$$

Here is an example, let $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Then, $\tilde{\mathbf{B}} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{B}$. Both the **reverse** and **involution** can be extended to any multivector by applying them to each grade. For example:

$$\begin{aligned} X = \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2 &\implies \hat{X} = (-1)^1\mathbf{e}_1 + (-1)^2\mathbf{e}_1 \wedge \mathbf{e}_2 = -1\mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2 \\ &\implies \tilde{X} = \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_1 = \mathbf{e}_1 - \mathbf{e}_2 \wedge \mathbf{e}_1. \end{aligned}$$

Another clever way of obtaining these operators is by simply using the grade selection $\langle\rangle_k$. This operation consists of spitting out the corresponding value of the k-th grade of a multi-vector. For example, let $x = 1 + 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_1 \wedge \mathbf{e}_2$, then

$$\begin{aligned} \langle x\rangle_0 &= 1 \\ \langle x\rangle_1 &= 2\mathbf{e}_1 + 3\mathbf{e}_2 \\ \langle x\rangle_2 &= 3\mathbf{e}_1 \wedge \mathbf{e}_2. \end{aligned}$$

Using this operation, we have:

$$\begin{aligned} \mathbf{A}_k \wedge \mathbf{B}_l &\equiv \langle \mathbf{A}_k\mathbf{B}_l\rangle_{k+l} \\ \mathbf{A}_k \rfloor \mathbf{B}_l &\equiv \langle \mathbf{A}_k\mathbf{B}_l\rangle_{l-k} \\ \mathbf{A}_k \lfloor \mathbf{B}_l &\equiv \langle \mathbf{A}_k\mathbf{B}_l\rangle_{k-l} \\ \mathbf{A}_k * \mathbf{B}_l &\equiv \langle \mathbf{A}_k\mathbf{B}_l\rangle_0 \end{aligned}$$

The last operation is called **scalar product**.

Another operation that comes often is dualization, which is defined as an mapping $^* : \bigwedge^k V \to \bigwedge^{\dim(V)-k} V$. We can compute dualization with the following formula:

$$\mathbf{A}^* = \mathbf{A} \rfloor \mathbf{I}^{-1},$$

where $\mathbf{A}$ is a blade, and $\mathbf{I}$ is the pseudo-vector of the multivector space.

The norm squared of a vector is simply $\mathbf{a} \cdot \mathbf{a}$, where the inner product is a property of the underlying vector space. For a blade, this norm can be computed as:

$$||\mathbf{B}||^2 = \mathbf{B} * \tilde{\mathbf{B}}.$$

One can check that this satisfies the properties of a norm.

5.4 Inverting Multivectors

We've shown how to invert a vector. This operation can be useful in different situation. For example, if a vector $\mathbf{a}$ is proportional to $\mathbf{b}$, what is the vector $\mathbf{x}$ that is proportional in the same way to a vector $\mathbf{c}$? This is just:

$$\frac{\mathbf{a}}{\mathbf{b}} = \frac{\mathbf{x}}{\mathbf{c}} \implies \mathbf{x} = \frac{\mathbf{a}}{\mathbf{b}}\mathbf{c} = \mathbf{a}\mathbf{b}^{-1}\mathbf{c}$$

We know how to find $\mathbf{b}^{-1}$, but what about a more generic multivector? Again, the inverse only exists for multivector with $x^2 \neq 0$.

For blades, the formula for the inverse is almost the same as for vectors:

$$\mathbf{B}^{-1} = \frac{\mathbf{B}}{\mathbf{B} * \mathbf{B}} = \frac{\tilde{\mathbf{B}}}{\mathbf{B} * \tilde{\mathbf{B}}} = \frac{\tilde{\mathbf{B}}}{||\mathbf{B}||^2}.$$

The last point we make is that division is not commutative. Consider for example:

$$\mathbf{x}\mathbf{a}\mathbf{a}^{-1} = (\mathbf{x} \cdot \mathbf{a} + \mathbf{x} \wedge \mathbf{a})\mathbf{a}^{-1} = (\mathbf{x} \cdot \mathbf{a})\mathbf{a}^{-1} + (\mathbf{x} \wedge \mathbf{a})\mathbf{a}^{-1}.$$

If we change the order we have:

$$\mathbf{a}^{-1}\mathbf{x}\mathbf{a} = \frac{1}{\mathbf{a}\mathbf{a}}\mathbf{a}\mathbf{x}\mathbf{a} = \mathbf{a}\mathbf{x}\mathbf{a}\frac{1}{\mathbf{a}\mathbf{a}} = (\mathbf{a} \cdot \mathbf{x})\mathbf{a}^{-1} + (\mathbf{a} \wedge \mathbf{x})\mathbf{a}^{-1} = (\mathbf{x} \cdot \mathbf{a})\mathbf{a}^{-1} - (\mathbf{x} \wedge \mathbf{a})\mathbf{a}^{-1}.$$

Note that this sandwishing of $\mathbf{a}$ and $\mathbf{a}^{-1}$ results in a reflection of $\mathbf{x}$ around the vector $\mathbf{a}$. In the next section we explore more how these transformations are done in Geometric Algebra.

5.5 Clifford Algebra versus Geometric Algebra

The distinction between Clifford Algebra and Geometric Algebra is not generally accepted, yet, it's posed by Leo Dorst. The ideas is that, while Clifford Algebra considers all possible multivectors of a multivector space $\bigwedge V$, the Geometric Algebra only allows objects that are constructed via either geometric product of scalars, vectors, dual scalars and dual vectors. Hence, there might exist multivectors in a Clifford Algebra that cannot be constructed in Geometric Algebra.

6 Transformations and Versors

When applying transformations to our geometric primitives, we want to ensure that the algebraic structure is preserved. What does this means? Simply that for two multivectors A, B and a transformation $\mathcal{T}$ we have that:

$$\mathcal{T}(A \circ B) = \mathcal{T}(A) \circ \mathcal{T}(B).$$

In the equation above, $\circ$ is any product that can be constructed via the geometric product, for example:

$$\mathcal{T}(A \wedge B) = \mathcal{T}(A) \wedge \mathcal{T}(B),$$

$$\mathcal{T}(A \cdot B) = \mathcal{T}(A) \cdot \mathcal{T}(B),$$

$$\mathcal{T}(AB) = \mathcal{T}(A)\mathcal{T}(B),$$

Now, a k-versor $\mathcal{V}$ is defined by the geometric product of k invertible vectors. This means that $\mathcal{V} := v_k ... v_2 v_1$.

6.1 Reflections

The reflection around a **a** line can be performed via

$$\mathbf{a} X \mathbf{a}^{-1},$$

where X is a multivector (i.e. it can be a scalar, a vector,a blade, or a multivector with multiple grades). We've actually shown this transformation for vectors, but one can apply induction in order to prove it's validity to any blade, and then, through linearity, show it's validity to any multivector.

We know that a vector **a** is the dual representation of a hyperplane, i.e. if we consider **a** to be the vector normal to a plane, then any point **x** that falls into this plane satisfies the condition that $\mathbf{a} \cdot \mathbf{x} = 0$. This condition of null inner product is the dual expression. It provides a way of checking whether a point belongs to the plane, but it does not provide an expression for the actual points in the plane.

The reflection formula for the hyperplane represented as the dual to **a** is

$$\mathbf{a}\widehat{\mathbf{X}}\mathbf{a}^{-1}.$$

Note that this formula only applies to blade, i.e. $\mathbf{X}$ is a blade. The reason is clear, the involution formula is $\widehat{\mathbf{X}} = (-1)^{\text{grade}(\mathbf{X})}\mathbf{X}$, and the grade operator is only well defined for blades.

6.2 Rotations

A rotation cane be constructed from an even number of reflections. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors with an angle $\phi/2$ from $\mathbf{a}$ to $\mathbf{b}$. Both vectors forma subspace given by $\mathbf{a} \wedge \mathbf{b}$. We can then rotate our vector $\mathbf{x}$ by an angle ϕ by applying the reflection on $\mathbf{a}$ followed by $\mathbf{b}$.

$$\mathbf{b}(\mathbf{a}\mathbf{x}\mathbf{a}^{-1})\mathbf{b}^{-1}.$$

Remember that $\mathbf{a}^{-1}\mathbf{x}\mathbf{a}\mathbf{a}^{-1}$, thus

$$\mathbf{b}\mathbf{a}^{-1}\mathbf{x}\mathbf{a}\mathbf{b}^{-1}.$$

Next, note that:$(\mathbf{b}/\mathbf{a})^{-1} = \mathbf{a}/\mathbf{b}$, because

$$(\mathbf{b}/\mathbf{a})(\mathbf{a}/\mathbf{b}) = 1 = (\mathbf{b}/\mathbf{a})(\mathbf{b}/\mathbf{a})^{-1}.$$

and the inverse is unique.

Now, $\mathbf{a}\mathbf{b}^{-1} = \frac{\mathbf{ab}}{||\mathbf{b}||}$

Hence, denoting $R = \mathbf{b}\mathbf{a}^{-1}$ we can write the rotation by $R\mathbf{x}R^{-1}$, which is again a sandwiching of multivectors.

Note that this transformation is well defined for *any* multivector, and not only vector. Hence:

$$RXR^{-1}$$

can be used to rotate planes, volumes or some other multivector.

While the reflection transformation depends on the grade of the object to be reflected, the rotation does not, thus can be applied to any multivector.

6.3 Projection

Another important transformation is a projection. Consider two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. We obtain the subspace defined by them using $\mathbf{b}_1 \wedge \mathbf{b}_2 = \mathbf{B}_{\langle 2 \rangle}$.

Now, suppose we want to project a third vector **a** on this subspace. How can we do this? We could start by applying the left contraction $\rfloor$. Yet, the left contraction does more than simply project, it also rotates the projected vector. Look the image below (taken from the book ”Álgebra Geométrica e Aplicações” by Fernandes, Lavor and Neto). When we do $\mathbf{a} \rfloor \mathbf{B}_{\langle 2 \rangle} = \mathbf{B}_{\langle 2 \rangle}$, we get the vector **c** in the drawing.

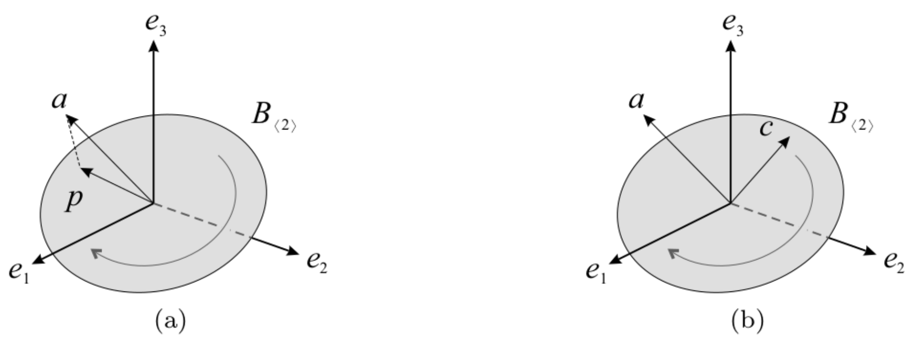

Figure 1: Image from Dorst et al. [1] showcasing blade projection.

But we want to get **p**. How do we do this? We need to apply the left contraction again, but using it on c with the same subspace but with the inverse rotation. We can do this by using the actual inverse blade, i.e.

$$\mathbf{p} = c \rfloor \left(\mathbf{B}_{\langle 2 \rangle}\right)^{-1}.$$

Thus, the projection formula becomes:

$$\mathrm{Proj}_{\mathbf{B}}(\mathbf{x}) = \left(\mathbf{x} \rfloor \mathbf{B}_{\langle 2 \rangle}\right) \rfloor \left(\mathbf{B}_{\langle 2 \rangle}\right)^{-1}$$

This formula works fine even for arbitrary blades, **but only in Euclidean spaces**, i.e. spaces with the Euclidean metric. And we are actually going to work on spaces that are not Euclidean (e.g. Conformal space). So, instead of this formula for projection, we do:

$$\mathrm{Proj}_{\mathbf{B}}(\mathbf{A}) = \left(\mathbf{A} \rfloor \left(\mathbf{B}_{\langle 2 \rangle}\right)^{-1}\right) \rfloor \mathbf{B}_{\langle 2 \rangle},$$

where **A** is a blade.

First, note that exchanging the order of the inverse does not alter our previous argument of how to obtain the projection. The only difference is that we start rotating in the other direction. Yet, this other formulation will come in hand when we are in spaces other than Euclidean, where we'll have multivectors with null norms, which means that they are not invertible. Yet, we can still do an operation akin to the inverse, and by finishing with the left contraction on $\mathbf{B}$, we guarantee that our projected blade will be on the correct subspace.

6.4 Rotors

Consider our rotation multivector $R = \mathbf{b}/\mathbf{a}$. Depending on the norms of the vectors, our multivector R will have a norm different than 1. If we normalize it, we get what is called a **rotor**.

A more precise definition of a rotor is that a "A rotor R is the geometric product of an even number of unit vectors, such that $R\tilde{R} = 1$" - Leo Dorst For example, $R = \mathbf{b}_n\mathbf{a}_n$, where both are unit vectors, which means that $\mathbf{a}_n^{-1} = \mathbf{a}_n$. Therefore, we have

$$R[X] = RXR^{-1} = (\mathbf{b}_n\mathbf{a}_n)X(\mathbf{b}_n\mathbf{a}_n)^{-1} = RX\tilde{R}.$$

Our multivector is not normalized. We can do this by dividing it by it's norm, or by normalizing the vectors themselves. Note that we can construct our rotor by either normalizing the vectors or the multivector itself. For normalized vectors, we have $\mathbf{b}_n/\mathbf{a}_n = \mathbf{b}_n\mathbf{a}_n$. Remember the formula for the inverse:

$$X^{-1} = \frac{\tilde{X}}{||\mathbf{X}||^2}.$$

Since the norm is equal to 1, we have $X^{-1} = \tilde{X}$. Thus, for a rotor we have $R\tilde{R} = 1$

Note another important aspect of our rotors. The formula of a rotor can be written as:

$$R = \mathbf{b}_n\mathbf{a}_n = \mathbf{b}_n \cdot \mathbf{a}_n + \mathbf{b}_n \wedge \mathbf{a}_n = cos(\phi/2) - \mathbf{a}_n \wedge \mathbf{b}_n = cos(\phi/2) - \mathbf{I}sin(\phi/2),$$

where $\mathbf{I}$ is the unit blade of $\mathbf{a}_n \wedge \mathbf{b}_n$, i.e.

$$\mathbf{I} = \frac{\mathbf{a}_n \wedge \mathbf{b}_n}{||\mathbf{a}_n \wedge \mathbf{b}_n||},$$

and $\phi/2$ is the angle between $\mathbf{a}_n$ and $\mathbf{b}_n$.

6.5 Versors

As we've seen, transformations such as rotations and reflections are done via sandwiching a multivector. We can generalize these transformations by the fact that they are all comprised of the sandwiching of versors.

Definition 6.1 (Versor). A k-versor V is any multivector which can be described as the geometric product of k non-null vectors, i.e. $V = \mathbf{v}_k...\mathbf{v}_1$, where $\mathbf{v}_i^{-1}$ exists for every $i \in \{1, ..., k\}$.

Note that $VXV^{-1} = \mathbf{v}_k...\mathbf{v}_1 X(\mathbf{v}_k...\mathbf{v}_1)^{-1}$. This is almost the formula we had for applying multiple reflections. We can rearrange the terms in the right side to obtain:

$$(-1)^k \mathbf{v}_k...\mathbf{v}_1 X \mathbf{v}_1^{-1}...\mathbf{v}_k^{-1} = (-1)^k VXV^{-1}$$

which is the same as applying the reflection over $\mathbf{v}_1$ followed by the reflection over $\mathbf{v}_2$ until $\mathbf{v}_k$.

Hence, a versor can be **even** or **odd**, where the first is if the number k is even and the latter if the number k is odd. This gives us:

$$V[X] = VXV^{-1} \text{ if } V \text{ is even} V[X] = V\hat{X}V^{-1} \text{ if } V \text{ is odd}$$

A very important property of versors is that they preserve the geometric product and all the other operations that are built from it (e.g. outer prodct, left contraction, right contraction, scalar product). This is known as the covariance of the versors transformations:

$$V[AB] = V[A]V[B]V[A \cdot B] = V[A] \cdot V[B]V[A \wedge B] = V[A] \wedge V[B].$$

This is easy to show. If V is even, we have:

$$V[AB] = VABV^{-1} = VA(V^{-1}V)BV^{-1} = (VAV^{-1})(VBV^{-1}) = V[A]V[B].$$

If V is odd, we have:

$$V[AB] = V\widehat{AB}V^{-1} = V\hat{A}\hat{B}V^{-1} = V\hat{A}V^{-1}V\hat{B}V^{-1} = V[A]V[B].$$

We used the fact that $\widehat{AB} = \hat{A}\hat{B}$. Note that a rotor is also a versor, since $R = \mathbf{b}_n\mathbf{a}_n^{-1}$ which are both invertible vectors.

6.6 Blade Exponentiation

First, remember that we can write rotor as:

$$R_{\mathbf{I}\phi/2} = cos(\phi/2) - \mathbf{I}sin(\phi/2),$$

where $\mathbf{I}$ is the normalized subspace for the rotation angle. From this formula, we can define the rotor as an exponential, i.e.

$$R_{\mathbf{I}\phi/2} = cos(\phi/2) - \mathbf{I}sin(\phi/2) = e^{-\mathbf{I}\phi/2}.$$

We know how to sum and multiply blades. Thus, we can define it's exponential:

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} \tag{1}$$

$$\text{if } \mathbf{A}^2 = -\alpha^2 \implies \exp(\mathbf{a}) = cos\alpha + \frac{sin\alpha}{\alpha}\mathbf{A} \tag{2}$$

$$\text{if } \mathbf{A}^2 = 0 \implies \exp(\mathbf{a}) = 1 + \mathbf{A} \tag{3}$$

$$\text{if } \mathbf{A}^2 = \alpha^2 \implies \exp(\mathbf{a}) = cosh\alpha + \frac{sinh\alpha}{\alpha}. \tag{4}$$

References

[1] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric algebra for computer science: an object-oriented approach to geometry*. Elsevier, 2010.

[2] Jayme Vaz Jr and Roldão da Rocha Jr. *An introduction to Clifford algebras and spinors*. Oxford University Press, 2016.