Convex Analysis

Davi Sales Barreira

January 11, 2022

Contents

List of Definitions

List of Theorems

Notes mostly based on the summer course of Interactive Methods for Solving Structured Optimization Problems ministered by professor José Yunier Bello Cruz.

1 Initial Definitions

We use H to represent Hilbert spaces.

Let's assume that we are working in a topological space X that is Hausdorff.

Definition 1.1 (Standard for Convex Analysis). Let $f : X \to \mathbb{R} \cup \{+\infty\}$.

1. $\text{dom} f := \{x \in X : f(x) < +\infty\}$;
2. $\text{epi} f := \{(x, \xi) \in X \times \mathbb{R} : f(x) \leq \xi\}$;
3. $\text{lev}_{\leq \xi} f := \{x \in X : f(x) \leq \xi\}$;

Definition 1.2 (Lower Semi-Continuity). A function $f : X \to \overline{\mathbb{R}}$ is lower semi-continuous (l.s.c) if

$$\forall x \in X,\ f(x) \leq \liminf_{n \to +\infty} f(x_n) \tag{1}$$

It can be shown that f is l.s.c $\iff$ $\text{epi} f$ is closed in $X \times \mathbb{R}$.

Definition 1.3 (Chebyshev Set). If every point $x \in H$ has exactly one projection onto C, then C is a Chebyshev set.

Theorem 1.4. Let C be a nonempty, closed and convex set of H. Then, C is Chebyshev set and $\forall x$ and $p \in H$,

$$p = P_C(x) \iff p \in C \text{ and } \forall y \in C, \langle y - p, x - p \rangle \leq 0.$$

Theorem 1.5. Let C be a convex set of H. Then, C is closed $\iff$ C is weakly closed.

Proof. $\Longleftarrow$) We know that every set closed in the weak topology is closed in the strong topology.

$\Longrightarrow$) Take a weakly convergent sequence $(x_n) \subset C$. But,

$$\langle x_n - P_C(x), x - P_C(x) \rangle \leq 0 \rightharpoonup \langle x - P_C(x), x - P_C(x) \rangle \leq 0.$$

Since $\langle x - P_C(x), x - P_C(x) \rangle = \|x - P_C(x)\| \leq 0 \implies x = P_c(x)$. Thus, $x \in C$.

□

Proposition 1.6. Let C be a convex set in H. Then,

1. $\overline{C}$ is convex;
2. int$C$ is convex.

Definition 1.7. Let C and $D \subset H$. Then, C and D are separated if there exists $h \in H \setminus \{0\}$ such that

$$\sup_{y \in C} \langle y, h \rangle \leq \inf_{z \in D} \langle z, h \rangle.$$

Theorem 1.8. Let C be a nonempty convex closed subset of H and $x \in H \setminus C$. Then, x is strongly separated from C.

Proof. Take $h := x - P_C(x)$. Then,

$$0 \geq \langle y - P_c(x), x - P_c(x) = \langle y - x + h, h \rangle = \langle y - x, h \rangle + \|h\|^2.$$

Therefore,

$$\langle y - x, h \rangle \leq -\|h\|^2 < 0 = \inf \langle D - x, h \rangle.$$

Which means that

$$\sup \langle C - x, h \rangle < \inf \langle D - x, h \rangle,$$

where $D = \{x\}$.

□

Corollary 1.9. Let C and D be nonempty closed convex sets such that $C \cap D = \varnothing$ and D is bounded. Then, C and D are strongly separated.

Proof. We'll show that $C \setminus D$ is convex and closed. The convexity can be easily verified. For closedness, take a convergent sequence $(z_n) \subset C \setminus D$, with $z_n = x_n - y_n$ where $x_n \in C$ and $y_n \in C$, and assume that $z_n \to z$. Now, since D is bounded, then (y_n) has a weakly convergent subsequence $y_{n_k} \rightharpoonup y \in D$.

Since $z_n \to z$, then $z_{n_k} \to z$ and $\lim_{k \to \infty} x_{n_k} - y_{n_k} = \lim_{k \to \infty} x_{n_k} - y$. Hence, $\lim_k x_{n_k} = z + y \in C \implies z \in C \setminus D$. □

Definition 1.10 (Convex Function). A function $f : H \to \overline{\mathbb{R}}$ is convex if it's epigraph is a convex set.

$$\Gamma(H) := \{\text{Set of convex function with non empty domain}\}$$

Moreover, f is said to be proper if dom$f \neq \varnothing$, and we define

$$\Gamma_0(H) := \{\text{Set of convex, proper and l.s.c functions}\}$$

Note that a convex function will always be continuous only in the interior of the domain. Hence, the "l.s.c" condition in Γ_0 is not redundant.

Theorem 1.11. A function f is convex if and only if for every $x \in \text{dom} f$, and $\alpha \in [0, 1]$ we have for $y \in \text{dom} f$

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

Theorem 1.12. Let $f \in \Gamma_0(H)$ and $x \in \text{int dom} f$. Then, there exists a continuous affine minorant h of f such that $h(x) = f(x)$. In other words, $\exists u \in H$, for all $y \in H$ such that

$$f(y) \geq f(x) + \langle u, y - x\rangle.$$

Definition 1.13 (Subgradient for Convex Functions).

$$\partial f(x) := \{u \in H : f(y) \geq f(x) + \langle u, y - x\rangle, \forall y \in H\}$$

Proposition 1.14. Let $f \in \Gamma(H)$. Then, every local minimizer of f is a minimizer.

Proposition 1.15. Let $f \in \Gamma_0(H)$, then f is weakly l.s.c.

References