Stochastic Calculus and Quantiative Finance

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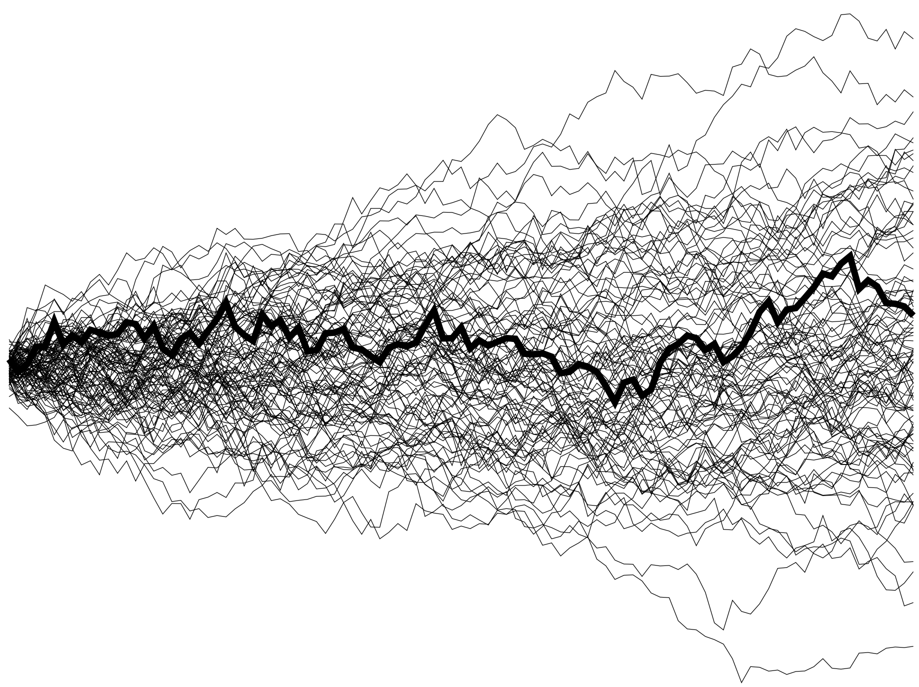

Contents

List of Definitions

List of Theorems

These notes are based on the lecture notes for the PhD course of Quantitative Finance at EMAp by David Evangelista, and on Baldi [2].

1 Stochastic Processes

1.1 Initial Definitions

We start by formalizing the notion of Stochastic Processes and filtration.

Definition 1.1 (Filtration). Let $\mathcal{F}$ be a σ-algebra and $T \subset \mathbb{R}_+$. We say that $(\mathcal{F}_t)_{t\in T}$ is a filtration, if it is an increasing family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$ for $s, t \in T$.

Definition 1.2 (Adaptation). A family $(X_t)_{t\in T}$ of random variables on $(\Omega, \mathcal{F})$ and taking value on $(E, \mathcal{E})$ is called adapted to the filtration $(\mathcal{F}_t)_{t\in T}$ if, for every $t \in T$, X_t is $\mathcal{F}_t$ measurable, i.e. for every $t \in T$, $X_t^{-1}(A) \in \mathcal{F}_t$ for any $A \in \mathcal{E}$.

Definition 1.3 (Natural Filtration). The natural filtration is $(\mathcal{G}_t)_t$ where $\mathcal{G}_t = \sigma(X_s, s \leq t)$, i.e. the filtration of the smallest σ-algebras such that X is adapted.

Definition 1.4 (Augumented Natural Filtration). The augumented natural filtration is $(\overline{\mathcal{G}}_t)_t$ where $\overline{\mathcal{G}}_t = \sigma(\mathcal{G}_t, \mathcal{N})$, i.e. the smallest σ-algebra containing the natural filtration and all null sets of $\mathcal{F}$.

Definition 1.5 (Stochastic Process). An Stochastic Process is a quintuple

$$(\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T}),$$

where $(\Omega, \mathcal{F}, P)$ is a probability space, $T \subset \mathbb{R}_+$, $(\mathcal{F}_t)_{t\in T}$ is a filtration of $\mathcal{F}$ and $(X_t)_{t\in T}$ is a family of random variables adapted to $(\mathcal{F}_t)_{t\in T}$.

Note that for brevity we usually say that $X = (\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$ is the process, instead of writing down the whole quintuple.

Definition 1.6 (Continuous Process). We say that a process is continuous (or almost surely continuous) if for every ω (or almost every ω) we have that if $t_n \to t$, then $X_{t_n}(\omega) \to X_t(\omega)$.

Definition 1.7 (Finite-dimensional distributions). Consider the stochastic process $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, (X_t)_{t\in T}, P)$ defined on $(E, \mathcal{B}(E))$. The family of probability measures

$$\{\mu_{(t_1,...,t_k)}(\cdot) := P((X_{t_1}, ..., X_{t_k}) \in \cdot) : \ (t_1, ..., t_k) \in T^k, k \in \mathbb{N}\}$$

is called the *finite-dimensional distribution family.*

Definition 1.8 (Equivalent Processes). Two stochastic processes

$$(\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T}), \quad (\Omega, \mathcal{F}, P, (X'_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$$

are said to be equivalent if they have the same finite-dimensional distribution family, i.e., for any finite set of $\{t_1, ..., t_n\} \subset T$, $(X_{t_1}, ..., X_{t_n})$ and $(X'_{t_1}, ..., X'_{t_n})$ have the same law, i.e.

$$P((X_{t_1}, ..., X_{t_n})^{-1}(B)) = P((X'_{t_1}, ..., X'_{t_n})^{-1}(B)), \quad \forall B \in \mathcal{E},$$

where we assumed that both X_t and X'_t take values on $(E, \mathcal{E})$.

Definition 1.9 (Modification). We say that a process is a *modification* of another if

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T}) = (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$$

and, for every $t \in T$, we have that $X_t = X'_t$ P-a.s.

Definition 1.10 (Indistinguishable). Two stochastic processes

$$(\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T}), \quad (\Omega, \mathcal{F}, P, (X'_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$$

are said to be indistinguishable if $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T}) = (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ and

$$P(X_t = X'_t, \forall t \in T) = 1.$$

Proposition 1.11 (Indistinguishable $\subset$ Modification $\subset$ Equivalent). Let $(\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$, $(\Omega, \mathcal{F}, P, (X'_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$ be two stochastic processes with $X_t : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$, $X'_t : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$. Then Indistinguishable $\implies$ Modification $\implies$ Equivalent, but the converse is not true.

Proof. **(i)** Consider that the processes X and X' are indistinguishable. To prove that they are modifications, we need to prove that for each $t \in T$, there exists a set A_t such that $X_t(\omega) = X'_t(\omega)$ for every $\omega \in A_t$. But, since they are indistinguishable, then just take the set $\{\omega \in \Omega : X_t = X'_t \forall t \in T\}$, thus, it's proved.

Now, assume that they are modifications. Take $t_1, ..., t_n \in T$. We know that $\exists A_1, ..., A_n$ such that $P(A_1) = ... = P(A_n) = 1$ and $X_{t_i}(\omega) = X'_{t_i}(\omega)$ for every $\omega \in A_i$. Also, since every $P(A_i) = 1$, then $P(A_1 \cap ... \cap A_n) = 1$, and for every $B \in \mathcal{E}$,

$$\begin{aligned} P((X_{t_1}, ..., X_{t_n})^{-1}(B)) &= P(A_1 \cap ... \cap A_n \cap (X_{t_1}, ..., X_{t_n})^{-1}(B)) \\ &= P(A_1 \cap ... \cap A_n \cap (X'_{t_1}, ..., X'_{t_n})^{-1}(B)) \\ &= P((X'_{t_1}, ..., X'_{t_n})^{-1}(B)) \end{aligned}$$

which is true since both processes are equal in $A_1 \cap ... \cap A_n$.

Next, let's show that the converse is not true. For equivalent processes, just take $X_t(\cdot) \sim N(0,1)$ and $X'_t = -X_t$. Thus, $X'_t(\cdot) \sim N(0,1)$, which means that they are equivalent, but $P(X_t = X'_t) = P(X_t = 0) = 0$. Hence, equivalent $\not\Longrightarrow$ modification.

Finally, consider $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $P = \lambda$ (Lebesgue measure). Then, let

$$X_t(\omega) = \mathbb{1}_{\{\omega\}}(t), \quad X'_t(\omega) = 0.$$

This processes are modificationsl, since for a $t \in T$, we have

$$P(X_t(\omega) = 0) = 1 = P(X'_t = 0).$$

But, they are not Indistinguishable, since

$$P(X_t = X'_t \forall t \in T) = 0,$$

because for every $\omega \in \Omega$, there exists one $t \in T$ such that $\omega = t$ implying $X_t(t) = 1$.

$\square$

Proposition 1.12. Let $(\Omega, \mathcal{F}, P, (X_t)_{t \in T}, (\mathcal{F}_t)_{t \in T})$, $(\Omega, \mathcal{F}, P, (X'_t)_{t \in T}, (\mathcal{F}_t)_{t \in T})$ be two stochastic processes with $X_t : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$, $X'_t : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$, and T is an interval of $\mathbb{R}_+$. Then if the processes are (a.s) continuous and modifications of one another, then they are indistinguishable.

Proof. Let $D = \mathbb{Q} \cap T$. This set is dense in T and for each $t_k \in D$, there exists an $A_k \subset \Omega$ such that $P(A_k) = 1$ and $X_{t_k}(\omega) = X'_{t_k}(\omega)$ for $\omega \in D$. Since $X_\cdot(\omega)$ is continuous a.s, then there exists a set $F \subset \Omega$ such that $P(F) = 1$ and for $t_n \to t$ we have $X_{t_n}(\omega) \to X_t(\omega)$.

Now, for a $t \in T$, there exists a sequence $(t_n) \subset D$ such that $t_n \to t$. Note that for each t_n there is a sequence of A_n with $X_{t_n} = X'_{t_n}$ in each A_n. Consider the set

$$A := \limsup A_n = \cap_{k=1}^{\infty} \cup_{n \geq k} A_n.$$

If $\omega \in A$, then there exists a subsequence $t_{n_k} \to t$ such that $X'_{t_{n_k}}(\omega) = X_{t_{n_k}}(\omega) \to X_t(\omega) = X'_t(\omega)$.

If $\omega \notin A$, then $\exists N \in \mathbb{N}$ such that $n \geq N \implies \omega \notin A_n$, which means that $\omega \in \cap_{n \geq N} A_n^c$. Since $P(A_n) = 1 \forall n \in N$, then $P(A_n^c) = 0 \implies P(\cap_{n \geq N} A_n^c) = 0$. Therefore, we conclude that $P(X_t(\omega) \neq X'_t(\omega)) = 0$ □

1.2 Measurability

Definition 1.13 (Measurable Process). We say that a process X is measurable if $X : T \times \Omega \to E$ is measurable with respect to $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}) \to (E, \mathcal{B}(E))$. We are implicitly assuming the Borel σ-algebra. Also, note that the domain is defined in the product measure space $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F})$.

Definition 1.14 (Progressively Measurable). We say that a process X is *progressively measurable* if for every $u \in T$ we have that $X : [0, u] \times \Omega \to E$ is measurable with respect to $([0, u] \times \Omega, \mathcal{B}([0, u]) \otimes \mathcal{F}) \to (E, B(E))$.

Theorem 1.15. If a process X is right-continuous, then it is progressively measurable.

Proof. Just note that for every $u \in T$, the process X restricted to $[0, u]$ can be approximated by a sequence of piece-wise processes $X^{(n)}$ that are progressively measurable. Also, since X is right-continuous, then for $s_n \downarrow s$, we have $X^{(n)}_{s_n} \to X_s$ (here a sort of diagonal argument is used, where the limit is taken at the sequence of times and the sequence of approximation jointly). Since X is the limit of a progressively measurable proces, it will also be progressively measurable. Look Baldi [2] Proposition 2.1 for a more detailed proof. □

Proposition 1.16. Let $X = (\Omega, \mathcal{F}, P, (X_t)_{t \in T}, (\mathcal{F}_t)_{t \in T})$ be right-continuous process. Then, X is progressively measurable.

Proof. Sketch. For every $u \in T$ and $s \in [0, u]$, construct a sequence of piece-wise processes X_s^n converging to X_s from the right. Since for each n, the process X_s^n is measurable $([0, u] \times \Omega, \mathcal{B}([0, u]) \otimes \mathcal{F}_u)$, we know that the limit of piece-wise ("simple") processes will also be, hence X_s is measurable for any u and thus progressively measurable.

For complete proof, look Baldi [2] Proposition 2.1. □

Definition 1.17 (Right-continuous Filtration). Let $(\mathcal{F}_t)_t$ be a filtration, with $\mathcal{F}_{t+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. It's clear that $\mathcal{F}_t \subset \mathcal{F}_{t+}$ and that $\mathcal{F}_{t+}$ is a σ-algebra.

We say that $(\mathcal{F}_t)_t$ is right-continuous if $\mathcal{F}_{t+} = \mathcal{F}_t$ for every t.

Definition 1.18 (Standard Process). A process $(\Omega, \mathcal{F}, P, (X_t)_{t\in T}, (\mathcal{F}_t)_{t\in T})$ is called *standard* if

1. the filtration is right-continuous;
2. for every t, $\mathcal{F}_t$ contains all negligible sets of $\mathcal{F}$.

1.3 Constructing Stochastic Processes

In this section we present two important results for the construction of stochastic processes, Kolmogorov's extension theorem [1], and Kolmogorov's continuity theorem. With these two theorems, we'll be able to construct the continuous Brownian Motion.

To prove Kolmogorov's theorems, we'll need to remember some results from Measure Theory.

Definition 1.19 (Algebra). A family $\mathcal{G} \subset 2^{\Omega}$ is an algebra if

(i) $\Omega \in \mathcal{G}$;

(ii) If $A \in \mathcal{G} \implies A^c \in \mathcal{G}$;

(iii) If $A_1, ..., A_n \in \mathcal{F} \implies \cup_{i=1}^n A_i \in \mathcal{G}$.

Theorem 1.20 (Caratheodory's Extension Theorem). Let $\mathcal{G} \subset \in^{\Omega}$ be an algebra in Ω, and $\mu : \mathcal{G} \to \mathbb{R}_+$. If for $A_1, ..., A_n, ... \in \mathcal{G}$ such that $A_i \cap A_j = \varnothing \forall i \neq j$, we have $\mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$, i.e. μ is σ-additive. Then, there exists an extension $\bar{\mu} : \sigma(\mathcal{G}) \to \mathbb{R}_+$ such that $\bar{\mu}(A) = \mu(A)$ for every $A \in \mathcal{G}$. Moreover, if μ is σ-finite, then $\bar{\mu}$ is unique.

Hence, when talking about probability measures, the Caratheodory Extension Theorem gives us a unique way to extend measures from algebras to σ-algebras. Also, if μ_1 and μ_2 are finite measures on $(E, \mathcal{E})$. If they agree on $\mathcal{G} \subset \mathcal{E}$ where $\mathcal{G}$ is an algebra, then by Caratheodory's Extension Theorem, $\mu_1 = \mu_2$ on $\mathcal{E}$.

Let $X : \Omega \to \mathbb{R}$ be a random variable with a given distribution μ. Remember that in order for μ to be the distribution of X, we assume that there exists a probability space $(\Omega, \mathcal{F}, P)$, such that $\mu(A) = P(\{\omega \in \Omega : X(\omega) \in A)$. The question is if this probability space indeed exists. It could be the case that for a given distribution μ (e.g. a Normal distribution), no possible probability measure P existed.

[1] This theorem is sometimes called Kolmogorov's existence theorem or consistency theorem.

For the example above, it's easy to prove that for any distribution r.v $X \sim \mu$, we do have the existance of an underlying probability space. Just consider $\Omega = \mathbb{R}$, with $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $P = \mu$, with $X(\omega) = \omega$. In other words, we just defined the probability space to be the same space where the distribution is defined.

Now, when talking about stochastic processes, the answer is not that simple. Consider $(X_t)_{t\in T}$ (e.g. $T = \mathbb{R}$, or $T = \mathbb{N}$, etc) and each $X_t : \Omega \to \mathbb{R}$ is a random variable (i.e. Borel measurable function) with distribution μ_t. In this case, what would be the underlying $(\Omega, \mathcal{F}, P)$? Does it exist?

Well, note that we didn't define how the X_t random variables relate to each other. Thus, we actually have many possible non-equivalent (1.8) stochastic process. For example, we could have one possible process where X_t are independent, and another where they have some sort of dependence. Hence, consider this following problem instead.

Let $(X_t)_{t\in T}$, where each $X_t : \Omega \to \mathbb{R}$ is a random variable with distribution μ_t, and for any $(t_1, ..., t_n) \in T^n$, the random vector $(X_{t_1}, ..., X_{t_n})$ has distribution $\mu_{(t_1,...,t_n)}$. Again we ask, what would be the underlying $(\Omega, \mathcal{F}, P)$?

For this case, we can do the following. Make

$$\Omega = \mathbb{R}^T := \{f : T \to \mathbb{R}\},$$

i.e. $\mathbb{R}^T$ is the space of functions from T to $\mathbb{R}$. Hence, $\omega \in \Omega$ is a function $\omega : T \to \mathbb{R}$.

Again we consider $X(\omega) = \omega$ with $X_t(\omega) = \omega(t)$ [2]. Thus, X_t are the *canonical projections* onto t. For the σ-algebra, consider

$$\mathcal{F} = \mathcal{B}^T(\mathbb{R}) := \sigma\left(X_t : t \in T\right),$$

i.e. $\mathcal{F}$ is the smallest σ-algebra where every canonical projection is measurable.

Finally, there are only two things left to do: construct a probability measure P on $(\Omega, \mathcal{F})$, and show that it matches the finite-dimensional distribution family, i.e. for any $A \in \mathcal{B}(\mathbb{R}^n)$ and $t_1, ..., t_n \in T$,

$$P((X_{t_1}, ..., X_{t_n})^{-1}(A)) = P(\omega : (\omega(t_1), ..., \omega(t_n)) \in A) = \mu_{(t_1,...,t_n)}(A).$$

Unfortunately, we cannot always construct such probability measure. We can only construct such probability on $(\Omega, \mathcal{F})$ is the underlying space E

[2] We would need to prove that this function is indeed measurable. But this is indeed true, and the proof can be found in Proposition 6.3.3 from Athreya and Lahiri [1]

has some regularity, which will appear in Kolmogorov's Existence Theorem. Secondly, suppose that a probability P does exist. For it to match the finite-dimensional distributions, it is required that the finite-dimensional family be consistent, i.e. that it doesn't break the "laws of probability". We'll formalize this idea further. But first, let's exemplify how family of finite-dimensional distributions can be inconsistent, and thus impossible to have any probability measure matching it.

Example 1.1 (Non-Consistent Probability Family). Consider $T = \mathbb{N}$, $X_n \sim U[0, n]$, and the finite-dimensional distribution for $n_1, ..., n_k$ as

$$\mu_(n_1, ..., n_k) \sim U([0, n_k]^k).$$

Make $\Omega = \mathbb{R}^{\mathbb{N}}$ and $\mathcal{F} = \mathcal{B}^{\mathbb{N}}(\mathbb{R})$, with $X_n(\omega) = \omega(n)$. Note that since $T = \mathbb{N}$, then we can identify each function $f : \mathbb{N} \to \mathbb{R}$ with an infinite sequence $(f(1), f(2), ...)$. Thus, Ω is the space of infinite sequences, such that

$$X_n(\omega) = X_n(\omega_1, \omega_2, ..., \omega_n, ...) = \omega_n.$$

Let's now show that no probability measure satisfies the condition that

$$P((X_{n_1}, ..., X_{n_k})^{-1}(A)) = \mu_{n_1,...,n_k}(A).$$

For this, take the event $A = A_1 \times \mathbb{R}$, where $A_1 = [0, 1]$. Hence,

$$\begin{aligned} P((X_1, X_2)^{-1}(A)) &= P(\{\omega : \omega_1 \in A_1, \omega_2 \in \mathbb{R}\}) \\ &= P(\{\omega_1 \in [0, 1]\}) \sim U[0, 1] \implies P((X_1, X_2)^{-1}(A)) = 1. \end{aligned}$$

But,

$$\mu_{(1,2)} \sim U([0, 2] \times [0, 2]) \implies \mu_{(1,2)}(A) = \mu_{(1,2)}([0, 1] \times \mathbb{R}) = 1/2.$$

Therefore, $P((X_1, X_2)) \neq \mu_{(1,2)}$, showing that no possible P exists.

We can see that in this example the "past distribution" changed with future observations. Thus, for such odd processes, we cannot always guarantee existence.

Definition 1.21 (Rectangular and Cylindrical Events). When extending a measure, sometimes it's better to prove results in a smaller subset of events that generates the full σ-algebra. Two common examples are the family of rectangular events, and cylindrical events. Consider $\Omega = \mathbb{R}^T$ and $\mathcal{F} = \mathcal{B}^T(\mathbb{R})$. Then,

(i) Cylindrical events:

$$C := \{f : T \to \mathbb{R} :\ (f(t_1), ..., f(t_k)) \in \mathcal{B}(\mathbb{R}^k)\}$$

(ii) Rectangular events:

$$R := \{f : T \to \mathbb{R} :\ f(t_1) \in B_1, ..., f(t_k) \in B_k,\ B_i \in \mathcal{B}(\mathbb{B}), \forall i = 1, ..., k\}$$

Example 1.2. Example from Athreya and Lahiri [1]. Let $T = 1, 2, 3$ and $\Omega = \mathbb{R}^T = \mathbb{R}^3$, i.e. the set of all functions $f : \{1, 2, 3\} \to \mathbb{R}$ is equivalent to the set of all triples (x_1, x_2, x_3). To show that this is true, just note that every function is equivalent to the triple of it's values at 1,2 and 3, i.e. $(f(1), f(2), f(3)) = (x_1, x_2, x_3)$.

With that said, the set $C := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq 1\}$ is a cylindrical event (and also a cylinder) in $\mathbb{R}^3$.

Similarly, the set $R := \{x_1, x_2, x_3 : x_1 \in [0, 1], x_2 \in [0, 1]\}$ is a rectangular event.

As we pointed out, the existence of the stochastic process requires that there exists a probability measure P such that for any $n \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^n)$ and $t_1, ..., t_n \in T$, then

$$P((X_{t_1}, ..., X_{t_n})^{-1}(A)) = \mu_{(t_1,...,t_n)}(A). \tag{1}$$

We can show that if we assume that a probability P exists, then for it to match the finite-dimensional distribution family 1, it requires that for any $A = A_1 \times ... \times A_n \in \mathcal{B}(\mathbb{R}^n)$, the following two conditions are true:

- (C1) For a permutation π,

$$\mu_{(t_1,...,t_n)}(A_1 \times ... \times A_n) = \mu_{(t_{\pi 1},...,t_{\pi n})}(A_{\pi 1} \times ... \times A_{\pi n}), \tag{2}$$

- (C2)

$$\mu_{(t_1,...,t_n)}(A_1 \times ... \times A_n) = \mu_{(t_1,...,t_n,t_{n+1})}(A_1 \times ... \times A_n \times \mathbb{R}). \tag{3}$$

Proposition 1.22. Let P be a probability measure and X_t the coordinate projection random variables (i.e. $X_t(\omega) = \omega(t)$). If for any $n \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^n)$ and $t_1, ..., t_n \in T$, we have

$$P((X_{t_1}, ..., X_{t_n})^{-1}(A)) = \mu_{(t_1,...,t_n)}(A).$$

Then, (C1) and (C2) are true.

Proof. For (C1), note that for $A = A_1 \times ... \times A_n \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned}
\mu_{(t_1,...,t_n)}(A_1 \times ... \times A_n) &= P((X_{t_1}, ..., X_{t_n})^{-1}(A_1 \times ... \times A_n)) \\
&= P(X_{t_1}^{-1}(A_1), ..., X_{t_n}^{-1}(A_n)) \\
&= P(X_{t_{\pi 1}}^{-1}(A_{\pi 1}), ..., X_{t_n}^{-1}(A_{\pi n})) \\
&= \mu_{(t_{\pi 1},...,t_{\pi n})}(A_{\pi 1} \times ... \times A_{\pi n}).
\end{aligned}$$

For (C2), note that

$$\begin{aligned}
\mu_{(t_1,...,t_n,t_{n+1})}(A_1 \times ... \times A_n \times \mathbb{R}) &= P((X_{t_1}, ..., X_{t_n}, X_{t_{n+1}})^{-1}(A_1 \times ... \times A_n \times \mathbb{R})) \\
&= P(X_{t_1}^{-1}(A_1), ..., X_{t_n}^{-1}(A_n), X_{t_{n+1}}^{-1}(\mathbb{R})) \\
&= P(X_{t_1}^{-1}(A_1), ..., X_{t_n}^{-1}(A_n)) \\
&= \mu_{(t_1,...,t_n)}(A_1 \times ... \times A_n).
\end{aligned}$$

□

Conditions (C1) and (C2) are known as the consistency conditions for Kolmogorov's Existence Theorem. Yet, some books such as Baldi [2] use a different consistency condition, but which we can show to be equivalent to (C1) and (C2).

- (C3) For any $n \in \mathbb{N}$, and any $t_1 < ... < t_n \in \mathbb{T}$

$$\begin{aligned}
&\mu_{(t_1,...,t_{i-1},t_{i+1},...t_n)}(A_1 \times ... \times A_{i-1} \times A_{i+1} \times .. \times A_n) \\
&= \mu_{(t_1,...,t_{i-1},t_i,t_{i+1},...t_n)}(A_1 \times ... \times A_{i-1} \times \mathbb{R} \times A_{i+1} \times .. \times A_n) \qquad (4)
\end{aligned}$$

Note that the significant difference between (C1)-(C2) to (C3) is the assumption that the $t_1, ..., t_n$ are ordered, which is assumed in (C3), but not in (C1)-(C2).

Although we stated this conditions using $\mathbb{R}$, they could have been postulated for any generic space E.

Proposition 1.23. Let P be a probability measure and X_t the coordinate projection random variables (i.e. $X_t(\omega) = \omega(t)$). Conditions (C1) and (C2) are true if and only if condition (C3) is true.

Proof. $\Longrightarrow$) For (C1) and (C2) true, just note that for $t_1 < ... < t_n$,

$$\begin{aligned}
\mu_{(t_1,...,t_{i-1},t_{i+1},...,t_n)}(A_1 \times ... \times A_n) &\underset{C2}{=} \mu_{(t_1,...,t_{i-1},t_{i+1},...,t_n,t_i)}(A_1 \times ... \times A_n \times \mathbb{R}) \\
&\underset{C1}{=} \mu_{(t_1,...,t_{i-1},t_i,t_{i+1},...,t_n)}(A_1 \times ... \times \mathbb{R} \times ... \times A_n)
\end{aligned}$$

$\Longleftarrow$) This implication is trickier [3]. Note that if we restrict the construction of our finite-dimensional distribution to ordered sets $t_1 < ... < t_n \in T$, i.e., for any ordered set $t_1 < ... < t_n$, $\mu_{(t_1,...,t_n)}(A) = P((X_{t_1}, ..., X_{t_n}) \in A)$. then if $s_1, ..., s_n$ is a permutation of $t_1, ..., t_n$, it's not clear what

$$\begin{aligned}\mu_{(t_1,...,t_n)}(A_1 \times ... \times A_n) &= P((X_{t_1}, ..., X_{t_n}) \in (A_1 \times ... \times A_n)) \\ &= P(X_{t_1} \in A_1..., X_{t_n} \in A_n).\end{aligned}$$

Since we are constructing our finite-dimensional dimensional distribution
and thus, for any permutation π such that $s_1 = t_{\pi 1}, ..., s_n = t_{\pi n}$, we have

$$\mu_{(s_1,...,s_n)}(A_{\pi 1} \times ... \times A_{\pi n}) = P(X_{s_1} \in A_{\pi 1} ... X_{s_n} \in A_{\pi n}).$$

Which we already proved that will imply both (C1) and (C2).

$\square$

Theorem 1.24 (Kolmogorov's Existence Theorem)). Let E be a Polish space (complete and separable metric space), with $(\mu_\pi)_\pi$ a family of finite-dimensional distributions on E satisfying the consistency condition (C3). Make $\Omega = E^T$ (the set of all continuous functions $f : T \to E$) and define $X_t(\omega) = \omega(t)$, with $\mathcal{F} = \sigma(X_t, t \in T)$ and natural filtration. Then, there exists on $(\Omega, \mathcal{F})$ a unique probability P such that the finite-dimensional distributions coincide with $(\mu_\pi)_\pi$.

[3] This is answer is adapted from Billingsley [3], Example 36.4

2 Itô Calculus

This chapter is based mostly on Baldi [2] and Oksendal [5].

2.1 Itô's Integral'

Let $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, P)$ be the continuous Brownian motion.

2.2 Itô's Lemma'

3 Basics of Financial Markets

This section is about the basics of the financial market, which is based on Cartea et al. [4].

4 Stochastic Optimal Control Application

This section is focused in collecting some results from Stochastic Optimal Control in order to solve problems in finance.

First, let's consider the problem which we are going to solve. We start with an amount of wealth denoted by $X_0 = X_0^b + \pi_0$, where X_0^b is the amount of money in the bank and π_0 is the amount of money in a risky asset. Now, as time passes, our wealth X_t will change,

5 Ergodicity Economics

This section is an extra and is based on Peters and Adamou [6]

References

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[6] Ole Peters and Alexander Adamou. Ergodicity economics, 2018.