Geometric Algebra

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1 Brief Note on Algebra with Category Theory

Let's start by presenting some definitions from Algebra.

For an introduction to Category Theory, check the other notes.

1.1 Initial Definitions for Groups

Definition 1.1 (Groups). Consider the triple $(G, \cdot, e)$, where G is a set, $\cdot : G \times G \to G$ is the product mapping and $e \in G$ is the identity element. This triple is a group if:

1. (Associativity): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in G$;
2. (Identity): $a \cdot e = e \cdot a = a$ for every $a \in G$;
3. (Inverse): For every $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = a$;

When there is no ambiguity, we call the set G a group omitting the product and neutral element.

Whenever it's note ambiguous, we omit the product operator, thus, $g \cdot h \equiv gh$.

Definition 1.2 (Abelian Group). A group $(G, \cdot, e)$ is *Abelian* if besides the group properties (i.e. associativity, identity and inverse) it's also commutative, i.e. $a \cdot b = b \cdot a$ for every $a, b \in G$.

Example 1.1. Note that $(\mathbb{R}, +, 0)$ is an Abelian Group. In this case, a^{-1} is usally denoted as $-a$. The triple $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is also an Abelian Group.

An example of non-Abelian group would be the set of invertible matrices from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $\cdot$ as matrix composition, e.g. $A \cdot B = AB$. Since every matrix considered is invertible and we have the identity matrix as our identity element, then we indeed have a non-Abelian group, since the matrix product is not commutative.

Proposition 1.3 (Group Cancellation). Let $(G, \cdot)$ be a group. Therefore:

$$fa = ha \implies f = h, \qquad af = ah \implies f = h$$

Proof. If $fa = ha$, then $faa^{-1} = ha^a - 1 \implies f = h$. ☐

Definition 1.4 (Subgroup Generated). Let $(G, \cdot, e)$ be a group. We say that $S \subset G$ is a subgroup of G if $(S, \cdot, e)$ is a group. For $A \subset G$, $\mathrm{Gr}(A)$ is called the subgroup generated by A, and it's the smallest subgroup of G containing A, i.e. $\cap_{\alpha \in \Gamma} S_\alpha$ where $\{S_\alpha\}_{\alpha \in \Gamma}$ are all the sets that are subgroups of G. It's easy to prove that such set is indeed a subgroup.

For a singleton $\{g\}$, we define $\mathrm{Gr}(g) := \{g^n \ : \ n \in \mathbb{Z}\}$, where $g^0 = e$, and g^n is the product of n copies of g, while g^{-n} is the product of n copies of $-g$.

Definition 1.5 (Cyclic Group). If a group G is equal to $\mathrm{Gr}(g)$ for some $g \in G$, then we say that G is cyclic.

Definition 1.6 (Order of Groups). The order of a finite group G is the number of elements of G. An element $g \in G$ has *finite order* if $g^n = e$ for $n \in \mathbb{N}$. The order of g is then the smallest n such that $g^n = e$.

Definition 1.7 (Homomorphism and Isomorphism). Let $(G, \cdot_G, e_G)$ and $(H, \cdot_H, e_H)$ be two groups. A function $\theta : G \to H$ is a homomorphism between G and H if $\theta(g_1 \cdot_G g_2) = \theta(g_1) \cdot_H \theta(g_2)$ for every $g_1, g_2 \in G$.

If θ is bijective, then we say that θ is an isomorphism.

Definition 1.8 (Normal / Self-conjugate). Let K be a subgroup of G. We say that K is *normal*, or *self-conjugate*, if $gkg^{-1} \in K$ for every $g \in G$.

1.2 Groups and Category Theory

Remember that in Category Theory we have a notion of isomorphism that generalizes set isomorphism (i.e. bijective function between sets).

Definition 1.9 (Automorphism). Let A be an object of a category $\mathcal{C}$. An automorphism is an isomorphism from A to itself. The set[1] of automorphism of A is denoted by $\mathrm{Aut}_{\mathcal{C}}(A)$.

Definition 1.10 (Grupoid and Groups). A groupoid is a category where every morphism is an isomorphism. Hence, a group is a groupoid category with a single object G. We denote **Grp** as the catogory of groups. In similar fashion, we can define **Ab** as the category of abelian groups, where the only difference is that the objects are abelian groups.

Note that this definition is equivalent to our definition of a group in algebraic terms. Why? Because every morphism is equivalent to an element of G, and the morphism composition does the part of the product operator. Also, note that every category has an identity morphism, thus, $id_G \equiv e$ our neutral element. Since every morphism is an isomorphism, this means that for every $g \in Hom(G, G)$, there is a $g^{-1} \in Hom(G, G)$ such that $g \circ g^{-1} = id_G = e$.

In pure categorical terms. Let $\mathcal{C}$ be a locally small category and $G \in \mathcal{C}$, i.e. an object of $\mathcal{C}$.

[1] Remember that a $Hom(A, A)$ is guaranteed to be a set if the category is locally small.

1.3 Rings and Modules

Let's begin by remembering the concept of a monoid. A monoid $(M, \cdot, e)$ is a set M, together with the binary operator $\cdot : M \times M \to M$ and the identity element e. Besides, $\cdot$ is associative.

Definition 1.11 (Ring). A ring $(R, \cdot, +)$ is an abelian group $(R, +)$ together with a monoid $(R, \cdot)$, with the property of distributivity, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for every $a, b, c \in R$.

One usually denotes the identity of $(R, +)$ by 0_R and the identity of $(R, \cdot)$ by 1_R. The reason is clear, since these are the corresponding identities for the usual sum and multiplication of numbers.

Based on this definition, one can prove that:

Proposition 1.12. Let $(R, \cdot, +)$ be a ring. Therefore:

$$0 \cdot r = 0 = r \cdot 0.$$

Note that in this definition, the $+$ operator has much more stated properties, e.g. there are inverse elements, there is commutativity. The $\cdot$ has more freedom. For example, we are not requiring for an inverse to exist, and neither commutativity. Which leads to the following definition.

Definition 1.13 (Commutative Ring). A ring $(R, \cdot, +)$ is commutative if $a \cdot b = b \cdot a$ for every $a, b \in R$.

Now, we want to slowly increment the properties of these algebraic concepts in order to construct our usual suspects, e.g. $\mathbb{N}$ $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$.

Definition 1.14 (Zero-Divisor). Let $(R, \cdot, +)$ be a ring. We say that $a \in R$ is a left-zero-divisor if there exists $b \neq 0 \in R$ such that $ab = 0$. Analogously, we define a right-zero-divisor.

Note that $0 \in R$ is a zero-divisor of every ring R **with the exception** of the *zero-ring* case. The zero-ring is the ring where R is a singleton set. Hence, since there is only one element, there is no element such that $ab = 0$ for $b \neq 0$, since no such b exists.

Example 1.2 ($\mathbb{Z} \setminus n\mathbb{Z}$). Let n be a positive integer.

Definition 1.15 (R-Module). An abelian group $(M, \oplus)$ is called a module over a ring $(R, +, \cdot)$ if there is a map (often called scalar multiplication) where:

$$* : R \times M \to M,$$

such that for all $r, r' \in R$ and $m, m' \in M$ we have

(i) $0_R * m = 0_M$;

(ii) $1_R * m = m$;

(iii) $(r + r') * m = r * m \oplus r' * m$;

(iv) $r * (m \oplus m') = r * m \oplus r * m'$;

(v) $(r \cdot r') * m = r * (r' * m)$.

We also call this an R-Module M.

Definition 1.16 (R-Algebra). An R-Algebra M is an R-Module M together with a bilinear map $M \times M \to M$.

Note that a vector space V over $\mathbb{R}$ with an inner product is an example of R-algebra.

2 Tensors and Vectors

Definition 2.1 (Vector Space). A vector space is a module over a field R, i.e. an R-module where R is a field. Note, for an abelian group $(\mathbf{V}, \oplus)$ and a field R, we have the vector space $(\mathbf{V}, R)$. In order to reduce the amount of writing, we call $\mathbf{V}$ the vector space, which implies that there is an underlying field R and the existence of an scalar product.

The tensor product of a vector

3 On How to Construct Different Algebras

This section is more informal, and is used to give a better intution of how to construct differnt algebras. This is mainly based on Vaz Jr and da Rocha Jr [1].

Consider a vector space $\mathbf{V}$. To define an algebra in a vector space, we have to define a bilinear product between the vectors. A possible product is the inner product. Yet, there are many other possibilities. One of them is the tensor product.

The tensor algebra $T(\mathbf{V})$ is the vector space $\mathbf{V}$ together with the tensor product $\otimes : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$. The tensor algebra has the "free" algebra flavor, meaning, it's "largest" algebra one can construct from $\mathbf{V}$. Thus, all other algebras on $\mathbf{V}$ are quotients of $T(\mathbf{V})$, i.e. we can construct the other bilinear products by introducing equivalence relations.

Thus, the tensor algebra $T(\mathbf{V})$ basis consists of all possible finite combinations of $\mathbf{u}$ and $\mathbf{v}$, where the tensor product of k vectors defines a k-vector in a vector space T^k.

$$T = \bigoplus_{k=0}^{\infty} T^k.$$

For example, suppose that $\mathbf{V}$ has basis $\{\mathbf{u}, \mathbf{v}\}$.

1. $T^0 := \{\mathbf{1}\}$;
2. $T^1 := \{\mathbf{u}, \mathbf{v}\}$
3. $T^2 := \{\mathbf{u} \otimes \mathbf{u}, \mathbf{v} \otimes \mathbf{v}, \mathbf{u} \otimes \mathbf{v}, \mathbf{v} \otimes \mathbf{u}\}$
4. $T^3 := \{\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}, \mathbf{u}\otimes\mathbf{u}\otimes\mathbf{v}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{u}, \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{u}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{u}, \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}\}$
5. etc.

As we've said, the other algebras on $\mathbf{V}$ can be constructed from $T(\mathbf{V})$. One example is the exterior algebra. To construct it, just impose the following equivalence relation, for every $\mathbf{v} \in \mathbf{V}$,

$$\mathbf{v} \otimes \mathbf{v} \cong 0.$$

Note that this condition implies that $\mathbf{v} \otimes \mathbf{u} = -\mathbf{u} \otimes \mathbf{v}$. This follows from

$$(\mathbf{u} + \mathbf{v}) \otimes (\mathbf{u} + \mathbf{v}) = \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} \cong 0.$$

When considering the exterior algebra, we change the product notation from $\otimes$ to $\wedge$. In the exterior algebra, the number of possible combinations of the basis vectors is finite. For example, for a vector space of dimension 2, i.e. basis $\{\mathbf{u}, \mathbf{v}\}$, we have

1. $\bigwedge^0 : \{\mathbf{1}\}$;
2. $\bigwedge^1 : \{\mathbf{u}, \mathbf{v}\}$;
3. $\bigwedge^2 : \{\mathbf{u} \wedge \mathbf{v}\}$.

Using a similar idea, we arrive at the Geometric (Clifford) Algebra. Instead of the equality to zero as in the exterior algebra, we use:

$$\mathbf{v} \otimes \mathbf{v} - B(\mathbf{v}, \mathbf{v}) \cong 0,$$

where B is a symmetric bilinear form. Note that, this definition actually defines a family of algebras (one for each possible B), where the exterior algebra is one of them (just use $B(x, x) = 0$).

A real symmetric bilinear form can be completely characterized by what is called a signature. For a vector space of dimension n, the signature of B is a triple (p, q, z), where p is the number of positive eigenvalues, q is the number of negative eigenvalues and z is the number of eigenvalues equal to zero. Thus, $p + q + z = n$. Remember that eigenvalues are a "fundamental way" of characterizing a transformation.

4 Quadratic Forms

Let's start the formal definition of Geometric Algebra, which is also known as Clifford Algebra.

Definition 4.1 (Quadratic Form and Quadratic Space). Let E be a real vector space. A quadratic form on E is a function $q : E \to \mathbb{R}$ such that $q(x) = b(x, x)$ for all $x \in E$, where b is a symmetric bilinear form on E. We say that b is the associated bilinear form.

We call the tuple (E, q) a quadratic space. The set $Q(E)$ is composed by all quadratic forms on E and it's a linear subspace of the space of linear real-valued functions on E.

Proposition 4.2. Given two different bilinear forms b_1, b_2, they induce different quadratic forms.

Proof. For $q(x) = b(x, x)$, then

$$q(x + y) = b(x + y, x + y) = q(x) + q(y) + 2b(x, y).$$

Thus, we have

$$b(x, y) = \frac{1}{2}\left(q(x + y) - q(x) - q(y)\right).$$

Note that

$$\begin{aligned} q(x - y) &= b(x - y, x - y) = q(x) + q(y) - 2b(x, y) \\ b(x, y) &= \frac{1}{2}\left(-q(x - y) + q(x) + q(y)\right) \end{aligned}$$

Hence, summing both equations we get $b(x, y) = \frac{1}{4}(q(x+y) - q(x-y))$. If there c is another bilinear form such that $c(x, y) \neq b(x, y)$ for some x and y, then the quadratic form induced by c is different than q, i.e. $q_c(x + y) - q_c(x - y) \neq q(x + y) - q(x - y) \implies q_c \neq q$. □

Definition 4.3 (Regular Quadratic Space). Given a quadratic space (E, q), we say that this space is regular if q is regular, i.e. if the associated bilinear form b is invertible (non-singular).

5 Geometric Algebra

As we've already pointed out, the Geometric Algebra over a vector space $\mathbf{V}$ is defined the tensor algebra $T(\mathbf{V})$ with the equivalence relation:

$$\mathbf{v} \otimes \mathbf{v} - B(\mathbf{v}, \mathbf{v}) \cong 0,$$

where B is a symmetric bilinear form. This bilinear form represents the metric of the geometric space, and is fully characterized by what we called a signature. The famous Euclidean space of $\mathbb{R}^n$ is the vector space with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} I \mathbf{v}$ where I is the identity matrix. Thus, the signature of the Euclidean space is $(3, 0, 0)$.

Definition 5.1 (Versor). A k-versor is the geometric product of k invertible 1-vectors, e.g. $\mathcal{V} = v_k ... v_2 v_1$, where v_i^{-1} is defined for every $i \in \{1, ..., k\}$.

Note 5.1 (Composable Operations). Orthogonal transformations, which are defined by versors, preserve the structures under the geometric product. This means that we can easily compose them. Note, for two multivectors A and B, and a versor $\mathcal{V}$,

$$\mathcal{V}(A \circ B)\mathcal{V}^{-1} = \mathcal{V}A\mathcal{V}^{-1} \circ \mathcal{V}B\mathcal{V}^{-1},$$

where $\circ$ represents any product of the Geometric Algebra (e.g. the geometric product, duality, inversion, projection).

References

[1] Jayme Vaz Jr and Roldão da Rocha Jr. *An introduction to Clifford algebras and spinors.* Oxford University Press, 2016.