Geometric Algebra

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Contents

List of Definitions

List of Theorems

List of Examples

1 Brief Note on Algebra

Let's start by presenting some definitions from Algebra.

Definition 1.1 (Groups). Consider the triple $(G, \odot, e)$, where G is a set, $\odot : G \times G \to G$ is the product mapping and $e \in G$ is the identity element. This triple is a group if:

1. (Associativity): $a \odot (b \odot c) = (a \odot b) \odot c$ for every $a, b, c \in G$;
2. (Identity): $a \odot e = e \odot a = a$ for every $a \in G$;
3. (Inverse): For every $a \in G$ there exists $a^{-1} \in G$ such that $a \odot a^{-1} = a^{-1} \odot a = a$;

When there is no ambiguity, we call the set G a group omitting the product and neutral element.

Definition 1.2 (Abelian Group). A group $(G, \odot, e)$ is *Abelian* if besides the group properties (i.e. associativity, identity and inverse) it's also commutative, i.e. $a \odot b = b \odot a$ for every $a, b \in G$.

Example 1.1. Note that $(\mathbb{R}, +, 0)$ is an Abelian Group. In this case, a^{-1} is usally denoted as $-a$. The triple $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is also an Abelian Group.

An example of non-Abelian group would be the set of invertible matrices from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $\odot$ as matrix composition, e.g. $A \odot B = AB$. Since every matrix considered is invertible and we have the identity matrix as our identity element, then we indeed have a non-Abelian group, since the matrix product is not commutative.

Definition 1.3 (Subgroup Generated). Let $(G, \odot, e)$ be a group. We say that $S \subset G$ is a subgroup of G if $(S, \odot, e)$ is a group. For $A \subset G$, $\mathrm{Gr}(A)$ is called the subgroup generated by A, and it's the smallest subgroup of G containing A, i.e. $\cap_{\alpha \in \Gamma} S_\alpha$ where $\{S_\alpha\}_{\alpha \in \Gamma}$ are all the sets that are subgroups of G. It's easy to prove that such set is indeed a subgroup.

For a singleton $\{g\}$, we define $\mathrm{Gr}(g) := \{g^n \ : \ n \in \mathbb{Z}\}$, where $g^0 = e$, and g^n is the product of n copies of g, while g^{-n} is the product of n copies of $-g$.

Definition 1.4 (Cyclic Group). If a group G is equal to $\mathrm{Gr}(g)$ for some $g \in G$, then we say that G is cyclic.

Definition 1.5 (Order of Group). The order of a group G is the number of elements of G.

Definition 1.6 (Homomorphism and Isomorphism). Let $(G, \odot_G, e_G)$ and $(H, \odot_H, e_H)$ be two groups. A function $\theta : G \to H$ is a homomorphism between G and H if $\theta(g_1 \odot_G g_2) = \theta(g_1) \odot_H \theta(g_2)$ for every $g_1, g_2 \in G$.

If θ is bijective, then we say that θ is an isomorphism.

References