Real Analysis

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1 Number Systems - Real and Complex

1.1 Supremum and Infimum

Definition 1.1 (Ordered Set). Let S be a set. An order relation on S is denoted by $<$ with:

(i) If $x, y \in S$ then either $x < y, x = y$ or $x > y$;

(ii) If $x, y, z \in S$ then $x < y, y < z \implies x < z$.

A tuple $(S, <)$ is called an ordered set.

Definition 1.2 (Supremum and Infimum). For an ordered set $(S, <)$, we say that α is the supremum of $E \subset S$ ($\alpha = \sup_{x \in S} E$) if α is the least upper bound of S, i.e.

(i) For any $x \in E$, then $x \leq \alpha$;

(ii) If $y < \alpha$ then y is not an upper bound of E;

The Infimum is the analogous definition, but on the other direction.

Definition 1.3 (Least-upper-bound property). An ordered set $(S, <)$ is said to have the Least-upper-bound property if for any non-empty and bounded set $E \subset S$, then $\sup E$ exists and belongs to S.

Definition 1.4 (Ring). A ring is a triple $(R, +, \cdot)$ where R is a set with at least two elements, and that contains two operations called addition $(+)$ and multiplication $(\cdot)$, where they satisfy the following conditions:

Addtion:

(i) If $x, y \in F$, then $x + y \in F$;

(ii) $x + y = y + x$;

(iii) $(x + y) + z = x + (y + z)$;

(iv) There exists a null element $0 \in F$ such that $0 + x = x$ for every $x \in F$;

(v) For every $x \in F$ there exists a $-x \in F$ such that $x + (-x) = 0$.

Multiplication:

(i) If $x, y \in F$, then $x \cdot y \in F$;

(ii) $x \cdot y = y \cdot x$;

(iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(iv) There exists an identity element $1 \in F$ such that $1 \cdot x = x$ for every $x \in F$;

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(i) Multiplication is distributive in terms of addition, i.e. $x \cdot (y + z) = x \cdot y + x \cdot z$.

If the commutative property for multiplication is not satisfied, we have a *non-commutative ring.*

Definition 1.5 (Field). A field is a ring $(F, +, \cdot)$ with the extra condition that every $x \neq 0 \in F$ has an inverse element $1/x \in F$ such that $x \cdot (1/x) = 1$.

If we add an order relation to F, then $(F, +, \cdot, <)$ is an ordered field.

Example 1.1. Note that $\mathbb{Z}$ is a ring, since the usual sum and multiplication satisfies all the properties. Yet, it's not a field, since the inverse elements are not part of $\mathbb{Z}$. It's easy to show that both $\mathbb{Q}$ and $\mathbb{R}$ are fields. More surprisingly, the space $\mathbb{Z}_p$, known as the modulo of p where p is a prime number, is also a field.

1.2 Sequences and Limits

We'll construct the Real numbers using Cauchy sequences.

Definition 1.6 (Equivalence Relation). Let X be a set. The symbol $\sim$ is a set on $X \times X$, where for $x, y \in X$, then $x \sim y$ means that $(x, y) \in \sim$. Hence, we say that $\sim$ is an *equivalence relation* on X, if $\sim$ satisfies the following properties:

1. *reflexive:* $\forall x \in X, x \sim x$;

2. *symmetric:* if $x \sim y$, then $y \sim x$;

3. *transitive:* if $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.7 (Equivalence Class). Given $x \in X$, the *equivalence class* of x with respect to an *equivalence relation* $\sim$ is

$$[x] := \{y \in X : y \sim x\}. \tag{1}$$

1.3 Topology

Definition 1.8 (Topological Space and Open Sets). $(X, \mathcal{T})$ is a topological space where $\mathcal{T}$ is the collection of *open sets* in X, such that:

(i) $X \in \mathcal{T}$;

(ii) Se $A_1, ..., A_n \in \mathcal{T} \implies \cap_{i=1}^{n} A_i \in \mathcal{T}$;

(iii) Se $A_\alpha \in \mathcal{T}$ for any $\alpha \in \Lambda \implies \cup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$;

$\mathcal{T}$ is the topology of X.

Definition 1.9 (Closed Sets). Given a topologial space $(X, \mathcal{T})$, we say that a set $F \subset X$ is closed if $F^c \in \mathcal{T}$.

Definition 1.10 (Interior and Closure). Let $(X, \mathcal{T})$. Take a set $A \subset X$. The interior of A is the union of all open sets that are subsets of A, and it's denoted by A°. The closure of A is the intersection of all closed sets containing A, denoted by $\bar{A}$. E.i.

$$A^\circ := \bigcup_{U \subset A,\ U \in \mathcal{T}} U, \quad \bar{A} := \bigcap_{F \supset A,\ F^c \in \mathcal{T}} F \tag{2}$$

Definition 1.11 (Metric). Let X be a space. A function $d : X \times X \to [0, +\infty)$ is called a metric if

(i) $d(x, y) = 0 \iff x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

If d satisfies only (i) and (ii) then it's called a pseudo-metric.

Definition 1.12 (Metric Space). A metric space is the tuple (X, d), where d is the metric over X, e.g. $X = \mathbb{R}$ and $d(x, y) = |x - y|$.

Definition 1.13 (Open Ball). Let (X, d) be a metric space, and $x \in X, r > 0$. The open ball is

$$B_r(x) := \{y \in X : d(y, x) < r\}. \tag{3}$$

Definition 1.14 (Open Family Induced by d). Let (X, d) be a metric space. x is an interior point of $A \subset X$ if there exists an open ball $B_r(x) \subset A$. The set of interior points of A is denoted by A°. We can define a family of open sets by defining that A is open if $A = A^\circ$.

Hence, the open family $\mathcal{O}$ induced by d is the set

$$\mathcal{O} := \{A \in \mathcal{O} : \exists x \in A, r > 0 \text{ such that } B_r(x) \subset A\}. \tag{4}$$

Note that in this way, the notion of being an open set is directly related to the metric d, since it requires that in every open set there is an open ball inside, which was defined using d. One can check that defining an open family this way satisfies the definition **??**.

1.4 Construction of Real

Theorem 1.15 (Compactnes + Unique Subsequences). Let $x_n \in K \subset \mathbb{R}^n$, with K compact. If $\exists x$ such that if a subsequence converges, then it converges to x (i.e. all convergent subsequences have a unique limit point x), then $x_n \to x$.

Proof. Suppose that $x_n \nrightarrow x$. Then, $\exists \varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $n \geq N$ such that $d(x_n, x) \geq \varepsilon$. Now, construct a subsequence x_{n_k} such that $d(x_{n_k}, x) \geq \varepsilon$ for every x_{n_k}. Since $(x_{n_k}) \subset K$ compact, then there exists a subsequence $x_{n_{k_j}}$ that converges, therefore, $x_{n_{k_j}} \to x$, which is a contradiction, since $d(x_{n_{k_j}}, x) \geq \varepsilon$. $\square$

1.5 Differentiation

Theorem 1.16 (Mean Value Theorem Inequality). Let $\boldsymbol{f} : [a, b] \to \mathbb{R}^k$ with $\boldsymbol{f}$ differentiable in (a, b). Then, there exists $c \in (a, b)$ such that

$$|\boldsymbol{f}(b) - \boldsymbol{f}(a)| \leq (b - a)|\boldsymbol{f}'(x)|.$$

References