Geometric Measure Theory

Davi Sales Barreira

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The notes are based on Evans and Garzepy [1]. Note that in Geometric Measure Theory there are different standards that might be confusing to someone that has taken an introductory course in Measure Theory. For example, it is common to refer to "outer measures" as "measures", and thus, define them in the power set instead of σ-algebra. I'll try to keep the distinction clear.

1 Classical Measure Theory, briefly

Before going into Geometric Measure Theory, I'll present the construction of measures with the "classical" approach of using Caratheodory's Extension Theorem.

1.1 Constructing Measures

This is based on the lectures from the Master's course of Measure and Integration at IMPA 2015.

Definition 1.1 (Semi-Algebra). Consider a space Ω, and $\mathcal{S} \subset 2^{\Omega}$. We say that $\mathcal{S}$ is a semi-algebra if

(i) $\Omega \in \mathcal{S}$;

(ii) $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$;

(iii) $A \in \mathcal{S} \implies A^c = B_1 \cup B_2 \cup ... \cup B_n$ where $B_i \cap B_j = \varnothing \forall i \neq j$.

Example 1.1. The set of $(a, b]$, $(a, +\infty)$ and $(-\infty, a)$ for $a, b \in \mathbb{R}$, is a semi-algebra.

Definition 1.2 (Algebra). Consider a space Ω, and $\mathcal{A} \subset 2^{\Omega}$. We say that $\mathcal{A}$ is an algebra if

(i) $\Omega \in \mathcal{S}$;

(ii) $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$;

(iii) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.

Note that since $A^c \cap B^c = A \cup B$, then the algebra is closed under finite unions.

Proposition 1.3. If $\mathcal{S}$ is a semi-algebra, then

$$\mathcal{A}(\mathcal{S}) := \{\cup_{j=1}^{n} E_j : \ E_j \in \mathcal{S}, E_j \cap E_i = \varnothing \text{ for } i \neq j\},$$

is an algebra. Moreover, $\mathcal{A}(\mathcal{S})$ is the smallest algebra containing $\mathcal{S}$, i.e. $\mathcal{A}(\mathcal{S}) = \cap_{\alpha \in I} \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha \supset \mathcal{S}$.

Proof. Let's prove the three properties of an algebra.

(i) $\Omega \in \mathcal{A}(\mathcal{S})$.

(ii) $A = \cup_{j=1}^{n} E_j, B = \cup_{k=1}^{m} F_k \implies A \cap B = cup_{j,k}(E_j \cap F_k) \in \mathcal{A}(S)$, since $E_j \cap F_k \in \mathcal{S}$;

(iii) $A \in \mathcal{A}(S) \implies A^c = \cap_{j=1}^{n}(E_j^c) = \cap_{j=1}^{n}(\cup_{k=1}^{m_j} F_{j,k}) = \cup_{k_1=1}^{n_1} \ldots \cup_{k_n=1}^{m_n} F_{1,k_1} \cap \ldots \cap F_{n,k_n} \in \mathcal{A}(\mathcal{S})$.

We proved that $\mathcal{A}(\mathcal{S})$ is an algebra. Also, since every element of $\mathcal{S}$ is in $\mathcal{A}(\mathcal{S})$, then $\mathcal{A}(\mathcal{S}) \supset \cap_{\alpha \in I} \mathcal{A}_\alpha$.

But, we can show that $\cap_{\alpha \in I} \mathcal{A}_\alpha$ defines an algebra that contains S, hence, it contains $E_j \in \mathcal{S}$ and it's closed under finite unions, hence $\cup_{i=1}^{n} E_i \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. Thus, $\mathcal{A}(\mathcal{S}) \subset \cap_{\alpha \in I} \mathcal{A}_\alpha$.

$\square$

Proposition 1.4 (Extending measures in semi-algebras to algebras). Let $\mu : \mathcal{S} \to [0, +\infty]$ where $\mathcal{S}$ is a semi-algebra, and $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ if $A_i \cap A_j = \varnothing$ for $i \neq j$ and any $A_1, \ldots, A_n \in \mathcal{S}$. We can then extend μ to $\bar{\mu} : \mathcal{A}(\mathcal{S}) \to [0, +\infty]$, such extension is unique and is still additive, i.e. $\bar{\mu}(\cup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \bar{\mu}(A_i)$ if $B_i \cap B_j = \varnothing$ for $i \neq j$ and every $B_1, \ldots, B_n \in \mathcal{A}(\mathcal{S})$.

Proof. Consider $\mathcal{A}(\mathcal{S}) := \{\cup_{j=1}^{n} E_j : \ E_j \in \mathcal{S}, E_j \cap E_i = \varnothing \text{ for } i \neq j\}$ and define $\bar{\mu} = \sum_{i=1}^{n} \mu(E_i)$.

Note that this is well defined. Take $A \in \mathcal{A}(\mathcal{S})$, and suppose that $A = \cup_{i=1}^{n} A_i = \cup_{j=1}^{m} B_j$.

$$A = \cup_{i=1}^{n} A_i = \cup_{j=1}^{m} B_j \implies B_j = \cup_{i=1}^{n} A_i \cap B_j, A_i = \cup_{j=1}^{m} A_i \cap B_j.$$

Using the fact that above, we have that

$$\bar{\mu}(A) = \sum_{i=1}^{n} \mu(A_i) = \sum_{i,j} \bar{\mu}(A_i \cap B_j) = \sum_{j=1}^{m} \mu(B_j).$$

Moreover, $\bar{\mu}$ is additive by it's construction. Also, $\bar{\mu}(S) = \mu(S)$, if $\mathcal{S}$, again by the definition of $\bar{\mu}$.

Finally, it's unique, since if $\bar{\mu}_1(S) = \bar{\mu}_2(S)$ for every $S \in \mathcal{S}$. Then, for any $A \in \mathcal{A}(\mathcal{S})$, we have

$$A = \sum_{i=1}^{n} S_i \implies \sum_{i=1}^{n} \bar{\mu}_1(S_i) = \sum_{i=1}^{n} \bar{\mu}_2(S_i).$$

$\square$

Definition 1.5. A function $\mu : \mathcal{A} \to [0, +\infty]$ is σ-additive if for every enumerable disjoint collection $A_1, ..., A_n, ... \in \mathcal{A}$,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Lemma 1.6. Let $\mu : \mathcal{A} \to [0, +\infty]$, where $\mathcal{A}$ is an algebra. Then, μ is σ-additive if and only if μ is continuous from below, i.e. $A_n \supset A_{n+1}$ with $A_n \downarrow \varnothing$, then $\mu(A_n) \to 0$.

Lemma 1.7. Let $\mathcal{S}$ be a semi-algebra on 2^{Ω}. If $\mu : S \to [0, +\infty]$ is σ-additive then the extension $\bar{\mu} : \mathcal{A}(\mathcal{S}) \to [0, +\infty]$ is also σ-additive.

Definition 1.8 (σ-Algebra). Consider a space Ω, and $\mathcal{F} \subset 2^{\Omega}$. We say that $\mathcal{F}$ is a σ-algebra if

(i) $X \in \mathcal{F}$;

(ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;

(iii) If $A_n \in \mathcal{F}$ $\forall n \in \mathbb{N}$ then $\cup_{n \in \mathbb{N}} A_n \in \mathcal{F}$;

2 Basics of Geometric Measure Theory

2.1 Measures and Outer Measures

Definition 2.1 (Outer Measure). A mapping $\mu : 2^X \to [0, +\infty]$ is called an *outer measure* on X if

1. $\mu(\varnothing) = 0$;

Subadditive : $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$;

Definition 2.2. An outer measure μ on X restricted to a set $C \subset X$ is

$$(\mu \llcorner C)(A) := \mu(C \cap A).$$

Definition 2.3 (μ-Measurability). A set $A \subseteq X$ is said to be μ-measurable if for each $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c).$$

This concept of μ-measurability is a very telling difference when compared to how we talk in measurability in "regular" measure theory. Here a set is measurable in relation to a the measure, and not to the σ-algebra considered.

Theorem 2.4 (Properties of Outer Measures). Let μ be an outer measure on X. Then,

(i) If $A \subset B \subset X \implies \mu(A) \leq \mu(B)$;

(ii) A is μ-measurable if and only if $X \setminus A$ is μ-measurable;

(iii) $\varnothing$ and X are μ-measurable and every null set is μ-measurable;

(iv) Let $C \subset X$, then if A is μ-measurable then it is also $\mu \llcorner C$-measurable.

Note that this definition of measurability again shows some benefits. For example, we don't need to worry about null sets not in the σ-algebra.

Proof. (i) is clearly true. Now, if A is μ-measurable then

$\square$

References

[1] Lawrence C Evans and Ronald F Garzepy. *Measure theory and fine properties of functions.* Routledge, 2018.