Qualitative Theory of Ordinary Differential Equations

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December 14, 2021

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Notes for the 2021 PhD course of Ordinary Differential Equations at FGV/EMAp. The main reference used was Doering and Lopes [1].

1 Introduction

1.1 Initial Definitions

Definition 1.1 (ODE). An Ordinary Differential Equation is an expression

$$\frac{d^k x(t)}{dt^k} = x^{(k)}(t) = F(t, x, x', ..., x^{(k-1)}),$$

where $k, d \in \mathbb{N}, t \in \mathbb{R}, x \in \mathbb{R}^d$ and $F : U \to \mathbb{R}^d$ is a continuous function defined on an open set $U \subset \mathbb{R}^{1+d\cdot k}$. We call k the *order* and d the *dimension.*

If we take the canonical coordinates of, say, $\mathbb{R}^n$, then

$$x(t) = (x_1(t), x_2(t), ..., x_n(t)), \quad \text{and } f(t,x) = (f_1(t,x), f_2(t,x), ..., f_n(t,x))$$

can instead be written as a system of differential equations

$$\begin{cases} x_1'(t) = f_1(t, x_1(t), ..., x_n(t)), \\ x_2'(t) = f_2(t, x_1(t), ..., x_n(t)), \\ \vdots \\ x_n'(t) = f_n(t, x_1(t), ..., x_n(t)). \end{cases} \tag{1}$$

Definition 1.2 (Solution to ODE). A solution to an ODE $F : U \to \mathbb{R}^{1+d\cdot k}$ is a function $\gamma : I \to \mathbb{R}^d$ such that

1. $\gamma \in C^k$;
2. $I \subset R$ is an open interval such that $\forall t \in I$, $(t, \gamma(t), \gamma'(t), ..., \gamma^{(k-1)}(t)) \subset U$;
3. $\frac{d^k\gamma(t)}{dt^k} = \gamma^{(k)}(t) = F(t, \gamma'(t), ..., \gamma^{(k-1)}(t)),\ \forall t \in I$.

1.2 Classification

There are many ways in which we can "classify" an ODE. We say that an ODE is *normal* if we can explicitly write x', i.e. $x' = f(t, x)$. In these notes, we are mainly interested in this type of ODE which are easier to work with.

If $x' = f(t, x) = f(x)$, then we say that our equation is *autonomous*, meaning that is does not depend on time. Although, it can be shown that any non-autonomous ODE can be written as an autonomous ODE. Consider that $x'(t) = f(x, t)$ and make $X = (t, x)$ with $F(X) = (1, f(t, x))$. Thus,

$$X'(u) = F(X) = (1, f(t(u), x(u))).$$

Note that if $X(u) = (t(u), x(u))$ is a solution to the above ODE, then for $t(t_0) = t_0$, we have that $t'(u) = 1 \implies t(u) = u$, hence $x'(u) = f(u, x(u))$.

A similar thing can be said about the order of an ODE. Every second order (or more) ODE can be transformed into a first order problem. Let

$$\begin{cases} y'' = g(t, y, y'), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases},$$

such that $g : U \subset \mathbb{R}^3 \to \mathbb{R}$. Then, define $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Hence, we now have a system of first order differential equations

$$\begin{cases} x_1' = x_2, \\ x_2' = g(t, y, y'), \\ x_1(t_0) = y_0, \\ x_2(t_0) = y_1. \end{cases}$$

Note that we can do the same procedure if the ODE has a larger order. Thus, every ODE can be represented by a first order ODE.

Hence, we can restrict our study to the case of first order autonomous equations.

2 Existence and Uniqueness

Let's begin by proving that indeed there exists solutions to Ordinary Differential Equations, and that, under stronger assumptions, we can even show that such solutions are unique.

Remember that we say that a function $f : X \to Y$ between two metric spaces is *Lipschitz* if there exists $C > 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2), \quad \forall x_1, x_2 \in X,$$

where C is called the Lipschitz constant. Now, an analogous definition are the *locally Lipschitz* functions. We say that the function is locally Lipschitz if for every $x_0 \in X$, there exists an open set $U \in X$ and a constant $C_{x_0} > 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq C_{x_0} d_X(x_1, x_2), \quad \forall x_1, x_2 \in U.$$

Let $F : \mathbb{R}^k \to \mathbb{R}^n \in C^1$. By the Mean Value Inequality, we have that

$$\|F(t, x_1) - F(t, x_2)\| \leq \|x_1 - x_2\| M.$$

2.1 Picard's Theorem

Consider that we wish to solve the initial value problem

$$x'(t) = f(t, x), \quad x(t_0) = x_0,$$

where $x : \mathbb{R} \to \mathbb{R}^n$, thus, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Since x is differentiable, then $x(t)$ is a continuous path, hence, in the compact set $I = (t_0, T)$, the function is uniformly continuous and we can apply the Fundamental Theorem of Calculus to say that

$$L_x(t) := x_0 + \int_{t_0}^{t} x'(s) ds = x(t).$$

References

[1] Claus Ivo Doering and Artur O Lopes. *Equações diferenciais ordinárias.* Number 517.2 DOE. 2008.