Topology

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1 Introduction

This chapter draws heavily from **?**].

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2.1 Functions

Definition 2.1 (Functions). A function f from a set X to Y is an assignment of an element of Y to each element of X, denoted by $f : X \to Y$. The set X is called domain, and Y the codomain. The set $f(X) := \{f(x) : x \in X\}$ is the image of f.

A partial function from X to Y is a function from a subset $S \subset X$, i.e. a partial function from X is not defined for every element of X. This type of object appears very often in Computation Theory, where partial functions usually take the role of a program, and when the program halts, this can be thought as the partial function not being defined for that specific entry of the domain. In this area, it's common to make the distinction of partial functions and complete functions, where the notion of a complete function is what we are calling a function.

Definition 2.2 (Injective, Surjective, Bijective). A function $f : X \to Y$ is:

(i) Injective if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$;

(ii) Surjective (onto) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$;

(iii) Bijective (one-to-one) if f is both injective and surjective.

Definition 2.3 (Inverse Image). Let f be a function from X to Y. For every subset $B \subset Y$, the inverse image (preimage) of B with respect to f is

$$f^{-1}(B) := \{x \in X : f(x) \in B\}. \tag{1}$$

The inverse image has some nice properties that will be useful in many subsequent proofs.

Proposition 2.4 (Properties of Inverse Images). Let $f : X \to Y$ and $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family of subsets of Y. We have the following identities:

(i) Union and Intersection:

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda), \quad f^{-1}\left(\bigcap_{\lambda \in \Lambda} U_\lambda\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(U_\lambda).$$

(ii) Complement:
$$(f^{-1}(U))^c = f^{-1}(U^c).$$

(iii) Set difference and symmetric difference:
$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V),$$
$$f^{-1}(U \Delta V) = f^{-1}(U) \Delta f^{-1}(V).$$

Proof. Just use the definition. $\qquad\square$

2.2 Order and Equivalence Relations

Definition 2.5 (Relation). A relation on a set X is a set $R \subset A \times A$.

This notion of a relation is very common in mathematics, and appears implicitly in many concepts.

Definition 2.6 (Total Order Relation). A total order relation on a set X is denoted by $<$ with:

(i) If $x, y \in X$ then either $x < y, x = y$ or $x > y$;

(ii) $x < y, y < z \implies x < z$.

A tuple $(X, <)$ is called an ordered set.

Definition 2.7 (Partial Order Relation). An order relation $\prec$ on a set X is called a strict partial order if:

(i) The relation $x \prec x$ is never true;

(ii) If $x \prec y, y \prec z$ then $x \prec z$.

A tuple $(X, <)$ is called an ordered set.

Note that in the partial order relation, not every element of X must be comparable.

Definition 2.8 (Well-Ordered Set). A set X with a total order relation $<$ is said to be well-ordered if every nonempty subset of A has a smallest element.

Note that $\mathbb{N}$ is well-ordered. $\mathbb{N} \times \mathbb{N}$ can also be well-ordered if we consider the dictionary order.

Definition 2.9 (Dictionary Order Relation). Let $(A, <_A)$ and $(B, <_B)$ be two ordered sets. For $A \times B$, the dictionary order relation is $<_D$ where for $x = (x_1, x_2) \in A \times B$ and $y = (y_1, y_2) \in A \times B$, we say that $x <_D y$ if $x_1 <_A x_2$, or, if $x_1 = x_2$, then $y_1 <_B y_2$. In words, the dictionary order sorts first by the first order relation and then by the second.

Definition 2.10 (Equivalence Relation). Let X be a set. The symbol $\sim$ is a set on $X \times X$, where for $x, y \in X$, then $x \sim y$ means that $(x, y) \in \sim$. Hence, we say that $\sim$ is an *equivalence relation* on X, if $\sim$ satisfies the following properties:

1. *reflexive:* $\forall x \in X, x \sim x$;

2. *symmetric:* if $x \sim y$, then $y \sim x$;

3. *transitive:* if $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 2.11 (Equivalence Class). Given $x \in X$, the *equivalence class* of x with respect to an *equivalence relation* $\sim$ is

$$[x] := \{y \in X : y \sim x\}. \tag{2}$$

One very useful property of equivalence classes is that they partition the set in which they are defined. For example, consider an equivalent relation $\sim$ on X. Every element $x \in X$ belongs to the equivalence class $[x]_\sim$, and each class is either the same of disjoint. We can easily show this, just note that [1]

$$[x] \neq [x'] \iff \text{w.l.g. } \exists y \in [x], y \notin [x'] \iff y \not\sim y', \forall y' \in [x']$$
$$\iff [x] \cap [x'] = \varnothing.$$

2.3 A bit of Algebra

Definition 2.12 (Ring). A ring is a triple $(R, +, \cdot)$ where R is a set with at least two elements, and that contains two operations called addition $(+)$ and multiplication $(\cdot)$, where they satisfy the following conditions:

Addition:

(i) If $x, y \in F$, then $x + y \in F$;

(ii) $x + y = y + x$;

(iii) $(x + y) + z = x + (y + z)$;

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[1]In the proof below there is a small technical issue where the first *iff* should actually be $\exists y \in [x] : y \notin [x'] \lor \exists y' \in [x'] : \notin [x]$. But we omitted this "or" condition for brevity, and used instead the "w.g.l" which stands for "without loss of generality".

(iv) There exists a null element $0 \in F$ such that $0 + x = x$ for every $x \in F$;

(v) For every $x \in F$ there exists a $-x \in F$ such that $x + (-x) = 0$.

Multiplication:

(i) If $x, y \in F$, then $x \cdot y \in F$;

(ii) $x \cdot y = y \cdot x$;

(iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(iv) There exists an identity element $1 \in F$ such that $1 \cdot x = x$ for every $x \in F$;

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(i) Multiplication is distributive in terms of addition, i.e. $x \cdot (y + z) = x \cdot y + x \cdot z$.

If the commutative property for multiplication is not satisfied, we have a *non-commutative ring.*

Definition 2.13 (Field). A field is a ring $(F, +, \cdot)$ with the extra condition that every $x \neq 0 \in F$ has an inverse element $1/x \in F$ such that $x \cdot (1/x) = 1$.

If we add an order relation to F, then $(F, +, \cdot, <)$ is an ordered field.

Example 2.1. Note that $\mathbb{Z}$ is a ring, since the usual sum and multiplication satisfies all the properties. Yet, it's not a field, since the inverse elements are not part of $\mathbb{Z}$. It's easy to show that both $\mathbb{Q}$ and $\mathbb{R}$ are fields. More surprisingly, the space $\mathbb{Z}_p$, known as the modulo of p where p is a prime number, is also a field.

2.4   Axiom of Choice and Zorn's Lemma

From **?**]. Consider the following. We define a set A to be infinite if it is not finite, where a set A is finite if there exists a bijection $f : A \to \{1, ..., n\}$ for an $n \in \mathbb{N}$. Hence, an infinite set is one where there is no such bijection. Now, we wish to prove that there exists a function $f : \mathbb{N} \to A$ such that f as an injective function. The question is then, how do we construct a candidate?

A simple procedure is to consider the following. We pick a $a_1 \in A$ and make $f(1) = a_1$. For $f(2)$, we pick an element from $A \setminus \{1\}$. For $f(3)$ we pick from $A \setminus \{1, 2\}$. And so on. It's clear that such function would be injective and defined for every element of $\mathbb{N}$. But we can feel that something is off. We are not explicitly constructing our function. What is going on here?

This strange procedure of affirming that we could construct such function without ever specifying exactly how we select the elements is only possible if we consider the "controversial" **Axiom of Choice**.

Definition 2.14 (Axiom of Choice). Version from **?**]. Given as collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C such that C has one element of each set of $\mathcal{A}$, i.e. for $A \in \mathcal{A}$, $C \cap A$ has only one element.

The use of the Axiom of Choice leads to the proof of the so called Well-ordering theorem, proved by Zermelo in 1904.

Theorem 2.15 (Well-ordering theorem). Every set A has an order relation such that A is well-ordered.

3 Basics on Point-Set Topology

Let's now give a brief overview on some general concepts related to point-set topology.

Definition 3.1 (Topological Space). (X, τ) is a topological space where τ is the collection of open sets in X, such that:

(i) $X \in \tau$;

(ii) Se $A_1, ..., A_n \in \tau \implies \cap_{i=1}^n A_i \in \tau$;

(iii) Se $A_\alpha \in \tau$ for any $\alpha \in \Lambda \implies \cup_{\alpha \in \Lambda} A_\alpha \in \tau$;

τ is the topology of X.

Definition 3.2 (Hausdorff Space). A topological space (X, τ) is said to be Hausdorf if

4 Metric Spaces

Definition 4.1 (Metric). Let X be a space. A function $d : X \times X \to [0, +\infty)$ is called a metric if

(i) $d(x, y) = 0 \iff x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

If d satisfies only (i) and (ii) then it's called a pseudo-metric.

Definition 4.2 (Metric Space). A metric space is the tuple (X, d), where d is the metric over X, e.g. $X = \mathbb{R}$ and $d(x, y) = |x - y|$.