Category Theory

Davi Sales Barreira

February 6, 2022

Contents

List of Definitions

List of Theorems

Notes mostly based on Ribeiro [3], Bradley et al. [1] and Milewski [2].

1 What are Categories?

The study of Category Theory enables us to view Mathematics from a vantage point, and better understand how the different areas are connected. For example, it might not always be clear which properties are *topological*, and which aren't. By looking at the subject from the distance (via Category Theory), we get a a glimpse at the connections (and disconnections) between fields.

Definition 1.1 (Category). A category $\mathcal{C} = \langle Ob_{\mathcal{C}}, Mor_{\mathcal{C}} \rangle$ is a collection of objects $Ob_{\mathcal{C}}$ and morphisms $Mor_{\mathcal{C}}$ satisfying the following conditions:

(i) Every morphism $f \in Mor_{\mathcal{C}}$ is associated to two objects $X, Y \in Ob_{\mathcal{C}}$ which is represented by $f : X \to Y$ or $X \xrightarrow{f} Y$, where $dom(f) = X$ is called the domain of f and $cod(f) = Y$ is the codomain. Moreover, we define $Mor_{\mathcal{C}}(X, Y)$ as

$$Mor_{\mathcal{C}}(X, Y) := \{f \in Mor_{\mathcal{C}} \ : \ X \in dom(f), \ Y \in cod(f)\};$$

(ii) For any three objects $X, Y, Z \in Ob_{\mathcal{C}}$, there exists a composition operator

$$\circ : Mor_{\mathcal{C}}(X, Y) \times Mor_{\mathcal{C}}(Y, Z) \to Mor_{\mathcal{C}}(X, Z),$$

(iii) For each object $X \in Ob_{\mathcal{C}}$ there exists a morfism $id_X \in Mor_{\mathcal{C}}(A, A)$ called the identity.

The composition operator must have the following properties:

(p.1) *Associative*: for every $f \in Mor_{\mathcal{C}}(A, B), g \in Mor_{\mathcal{C}}(B, C), h \in Mor_{\mathcal{C}}(C, D)$ then

$$h \circ (gcircf) = (h \circ g) \circ f.$$

(p.2) For any $f \in Mor_{\mathcal{C}}(X, Y)$, $g \in Mor_{\mathcal{C}}(Y, X)$,

$$f \circ id_X = f, \quad id_A \circ g = g.$$

There are many ways to refers to the set of morphisms $Mor_{\mathcal{C}}(X, Y)$, such as $\mathcal{C}(X, Y)$ or $\hom_{\mathcal{C}}(X, Y)$. The reason for this is that this set is sometimes called hom-set. In this notes, we'll use either $Mor_{\mathcal{C}}(X, Y)$ or $\mathcal{C}(X, Y)$ when there is no ambiguity.

Definition 1.2 (Categorical Isomorphism). Let $\mathcal{C}$ be a category with $X, Y \in Ob_{\mathcal{C}}$ and $f \in Mor_{\mathcal{C}}(X, Y)$.

(i) We say that f is *left invertible* if there exists $g \in Mor_{\mathcal{C}}(Y, X)$ such that $g \circ f = id_X$;

(ii) We say that f is *right invertible* if there exists $h \in Mor_{\mathcal{C}}(Y, X)$ such that $f \circ h = id_Y$;

(iii) We say that f is invertible if it's both left and right invertible.

When an invertible morphism exists between X and Y, we say that they are isomorphic.

Note that when f is invertible, the morphism that inverts f is unique with the left and right inverses coinciding, since $g \circ id_Y = g \circ f \circ h = id_X \circ h = h$.

References

[1] Tai-Danae Bradley, Tyler Bryson, and John Terilla. *Topology: A Categorical Approach*. MIT Press, 2020.

[2] Bartosz Milewski. *Category theory for programmers*. Blurb, 2018.

[3] Maico Ribeiro. *Teoria das Categorias para Matemáticos. Uma breve introdução*. 05 2020. ISBN 9786599039515.