Geometric Algebra

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1 Brief Note on Algebra with Category Theory

This is based partially on Garling [2] and Aluffi [1] (for the Category Theory). Let's start by presenting some definitions from Algebra.

For an introduction to Category Theory, check the other notes.

1.1 Initial Definitions for Groups

Definition 1.1 (Groups). Consider the triple $(G, \odot, e)$, where G is a set, $\odot : G \times G \to G$ is the product mapping and $e \in G$ is the identity element. This triple is a group if:

1. (Associativity): $a \odot (b \odot c) = (a \odot b) \odot c$ for every $a, b, c \in G$;
2. (Identity): $a \odot e = e \odot a = a$ for every $a \in G$;
3. (Inverse): For every $a \in G$ there exists $a^{-1} \in G$ such that $a \odot a^{-1} = a^{-1} \odot a = a$;

When there is no ambiguity, we call the set G a group omitting the product and neutral element.

Whenever it's note ambiguous, we omit the product operator, thus, $g \odot h \equiv gh$.

Definition 1.2 (Abelian Group). A group $(G, \odot, e)$ is *Abelian* if besides the group properties (i.e. associativity, identity and inverse) it's also commutative, i.e. $a \odot b = b \odot a$ for every $a, b \in G$.

Example 1.1. Note that $(\mathbb{R}, +, 0)$ is an Abelian Group. In this case, a^{-1} is usally denoted as $-a$. The triple $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is also an Abelian Group.

An example of non-Abelian group would be the set of invertible matrices from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $\odot$ as matrix composition, e.g. $A \odot B = AB$. Since every matrix considered is invertible and we have the identity matrix as our identity element, then we indeed have a non-Abelian group, since the matrix product is not commutative.

Definition 1.3 (Subgroup Generated). Let $(G, \odot, e)$ be a group. We say that $S \subset G$ is a subgroup of G if $(S, \odot, e)$ is a group. For $A \subset G$, $\text{Gr}(A)$ is called the subgroup generated by A, and it's the smallest subgroup of G containing A, i.e. $\cap_{\alpha \in \Gamma} S_\alpha$ where $\{S_\alpha\}_{\alpha \in \Gamma}$ are all the sets that are subgroups of G. It's easy to prove that such set is indeed a subgroup.

For a singleton $\{g\}$, we define $\text{Gr}(g) := \{g^n \ : \ n \in \mathbb{Z}\}$, where $g^0 = e$, and g^n is the product of n copies of g, while g^{-n} is the product of n copies of $-g$.

Definition 1.4 (Cyclic Group). If a group G is equal to $\text{Gr}(g)$ for some $g \in G$, then we say that G is cyclic.

Definition 1.5 (Order of Group). The order of a group G is the number of elements of G.

Definition 1.6 (Homomorphism and Isomorphism). Let $(G, \odot_G, e_G)$ and $(H, \odot_H, e_H)$ be two groups. A function $\theta : G \to H$ is a homomorphism between G and H if $\theta(g_1 \odot_G g_2) = \theta(g_1) \odot_H \theta(g_2)$ for every $g_1, g_2 \in G$.

If θ is bijective, then we say that θ is an isomorphism.

1.2 Groups in Category Theory

Remember that in category theory we have a notion of isomorphism that generalizes set isomorphism (i.e. bijective function between sets).

Definition 1.7 (Grupoid and Groups). A groupoid is a category where every morphism is an isomorphism. Hence, a group is a groupoid category with a single object G.

Note that this definition is equivalent to our definition of a group in algebraic terms. Why? Because every morphism is equivalent to an element of G, and the morphism composition does the part of the product operator. Also, note that every category has an identity morphism, thus, $id_G \equiv e$ our neutral element. Since every morphism is an isomorphism, this means that for every $g \in Hom(G, G)$, there is a $g^{-1} \in Hom(G, G)$ such that $g \circ g^{-1} = id_G = e$.

2 Geometric Algebra

Let's start the formal definition of Geometric Algebra, which is also known as Clifford Algebra.

Definition 2.1 (Quadratic Form and Quadratic Space). Let E be a real vector space. A quadratic form on E is a function $q : E \to \mathbb{R}$ such that $q(x) = b(x, x)$ for all $x \in E$, where b is a symmetric bilinear form on E.

We call the tuple (E, q) a quadratic space. The set $Q(E)$ is composed by all quadratic forms on E and it's a linear subspace of the space of linear real-valued functions on E.

Proposition 2.2. Given two different bilinear forms b_1, b_2, they induce different quadratic forms.

Proof. For $q(x) = b(x, x)$, then

$$q(x + y) = b(x + y, x + y) = q(x) + q(y) + 2b(x, y).$$

Thus, we have

$$b(x, y) = \frac{1}{2} \left(q(x + y) - q(x) - q(y) \right).$$

Note that

$$\begin{aligned} q(x - y) &= b(x - y, x - y) = q(x) + q(y) - 2b(x, y) \\ b(x, y) &= \frac{1}{2} \left(-q(x - y) + q(x) + q(y) \right) \end{aligned}$$

Hence, summing both equations we get $b(x, y) = \frac{1}{4}(q(x+y) - q(x-y))$. If there c is another bilinear form such that $c(x, y) \neq b(x, y)$ for some x and y, then the quadratic form induced by c is different than q, i.e. $q_c(x + y) - q_c(x - y) \neq q(x + y) - q(x - y) \implies q_c \neq q$. $\square$

References

[1] Paolo Aluffi. *Algebra: chapter 0*, volume 104. American Mathematical Soc., 2021.

[2] David JH Garling. *Clifford algebras: an introduction*, volume 78. Cambridge University Press, 2011.