

# Probability & Optimal Transport

Davi Sales Barreira

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## Contents

<b>1</b>	<b>introduction</b>	<b>1</b>
<b>2</b>	<b>Optimal Transport and the Wasserstein Distance</b>	<b>2</b>
<b>3</b>	<b>Transport Inequalities and Concentration</b>	<b>3</b>

### Abstract

The main goal of these notes is to present an introduction to the transport inequalities, which consist of methods of using Optimal Transport Theory to obtain concentration inequalities in high-dimensional probability. Along the way, as new concepts are presented, some side-lining will be done, with the aim to explore some of these new concepts, before diving back in the proof of the inequalities. The core of these notes are base on the excellent paper "Transport Inequalities. A Survey" by Gozlan and Léonard [2010].

## 1 introduction

Given two probability distributions  $\mu, \nu$ , there are many situations where one is interested in defining a way of measuring the distance between them. The Wasserstein distance is a metric that arises from the idea of optimal transport, and which has being gaining attention in Statistics and Machine Learning. One prominent example is the so called Wasserstein Generative Adversarial Network (WGAN), which uses this metric to evaluate how well the model generated distribution approximates the "real" distribution of the data. As will be shown shortly, the Wasserstein metric has several advantages compared to other metrics.

There are several ways of defining distances between two probability measures. Let's assume that  $\nu, \mu$  are defined on  $(\Omega, \mathcal{F})$  and that  $\nu \ll \lambda, \mu \ll \lambda$ , for  $\lambda$  representing the Lebesgue measure. Below we present some example of distances:

$$\text{Total Variation : } \|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)| = \frac{1}{2} \int_{\Omega} \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda$$

$$\text{Hellinger : } \sqrt{\int_{\Omega} \left( \sqrt{\frac{d\mu}{d\lambda}} - \sqrt{\frac{d\nu}{d\lambda}} \right)^2 d\lambda}$$

$$L_2 : \int_{\Omega} \left( \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right)^2 d\lambda$$

As pointed out by Wasserman [2018] in his lecture notes, although such distances are useful, there are drawbacks:

- If one distribution is discrete and the other is continuous, they cannot be compared. If  $X \sim U(0, 1)$  and  $Y$  is uniform on  $\{0, 1/N, 2/N, \dots, 1\}$ , although this distributions are very similar, their total variation is 1 (which is the maximum value). The Wasserstein distance is  $1/N$ , which is reasonable.
- These distances ignore the "geometry of the underlying space", while the Wasserstein distance preserves it, as shown in Figure 1.
- When "averaging" different distributions, one might be interested in obtaining a similar distribution, avoiding smoothing. This can be done using the Wasserstein barycenter, as shown in Figure 2.

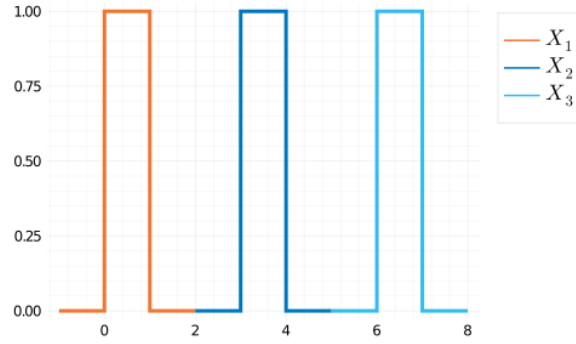


Figure 1: Each pair has the same distance in  $L_2$ , Hellinger and TV. But using Wasserstein,  $X_1$  is closer to  $X_2$  than to  $X_3$ .

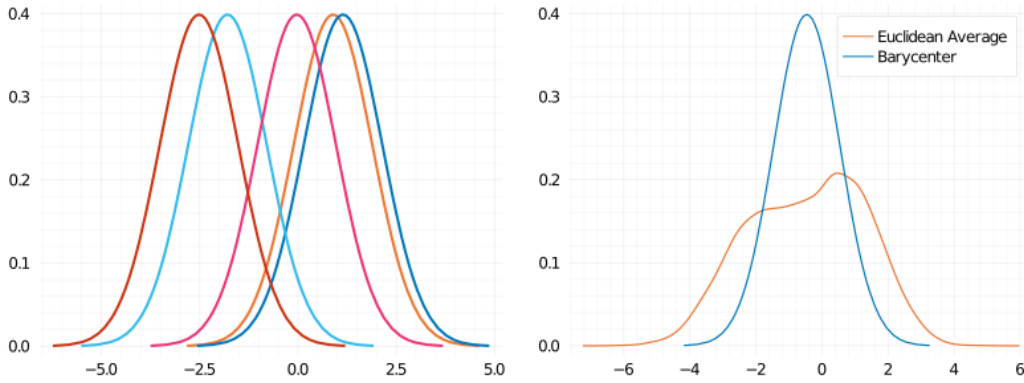


Figure 2: In the left you have different distributions, and in the right, there is a comparison between averaging these distribution versus finding the Wasserstein barycenter.

## 2 Optimal Transport and the Wasserstein Distance

Let  $\mathcal{X}$  be a polish space, and define a function  $c : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ , where  $c$  is a lower semicontinuous function and  $\mu, \nu \in P()$  (this means that  $\mu$  and  $\nu$  are probability measures on  $X$ ). Function  $c$  is usually called the *cost function*. The Wasserstein distance arises from the Monge-Kantorovich optimal transport problem, which seeks to find a map  $\pi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ , that transport  $\mu$  to  $\nu$  with minimum cost.

**Definition 2.1.** Coupling A probability measure  $\pi \in P(\mathcal{X} \times \mathcal{Y})$  is called a coupling of  $\mu \in P(\mathcal{X})$  and  $\nu \in P(\mathcal{Y})$  if it's marginal distributions  $\pi_1$  and  $\pi_2$  are  $\mu$  and  $\nu$ .

$$(MK) \quad \text{Minimize } \pi \in P(\mathcal{X}^2) \mapsto \int_{\mathcal{X}^2} c(x, y) d\pi(x, y) \text{ subject to } \pi \text{ coupling of } (\nu, \mu)$$

From solving the above problem, one obtains the optimal transport cost, given by:

$$T_c(\nu, \mu) := \inf \left\{ \int_{\mathcal{X}^2} c(x, y) d\pi(x, y) ; \pi_1 = \nu, \pi_2 = \mu \right\} \quad (1)$$

Now, given a metric space  $(\mathcal{X}, d)$ , we say that  $f : \mathcal{X} \rightarrow \mathbb{R}$  is  $L$ -Lipschitz if

$$|f(x) - f(y)| \leq Ld(x, y) \quad \forall x, y \in \mathcal{X} \quad (2)$$

We will use  $\|f\|_{Lip}$  to denote the smallest  $L$  for which this inequality holds. Also, denote  $P_1(\mathcal{X})$  the space of probability measures with finite first moment.

Therefore, for  $\mu, \nu \in P_1(\mathcal{X})$ , the Wasserstein distance is then defined as

$$W_d(\nu, \mu) := \sup_{\|f\|_{Lip} \leq 1} \left| \int f d\nu - \int f d\mu \right|$$

A classical results in the optimal transport theory, called *Kantorovich-Rubinstein duality* (which will not be proved here), guarantees the equivalence between the Wasserstein distance and the optimal transport cost. Hence, by setting  $c = d^p$  where  $d$  is a metric on  $\mathcal{X}$  and  $p \geq 1$ ,

$$W_{d^p}(\nu, \mu) := T_{d^p}(\nu, \mu)^{\frac{1}{p}} \quad (3)$$

When the context allows, we will write  $W_p$  instead of  $W_{d^p}$ . Note that depending on the reference, the Wasserstein distance might be first defined as a transport cost. A simple proof for this duality in the discrete setting is present in the lecture notes by van Handel [2014].

Note that  $W_p$  are proper metrics (positive, symmetric and satisfy the triangle inequality) in the space of probability measures. Also, the spaces with  $p$ -th moments finite, the metric space  $(P_p(\mathcal{X}), W_p)$  is complete and separable if so is  $\mathcal{X}$  [Panaretos and Zemel, 2019]. Hence, the Wasserstein distance has the interesting features already presented, such as incorporating the geometry of the "ground space", and allowing the comparison between discrete and continuous distributions. Another interesting aspect, is that convergence of r.v  $X_n$  to  $X$  in  $W_p$  distance is equivalent to convergence in distribution with  $E|X_n|^p \rightarrow E|X|^p$  [Panaretos and Zemel, 2019].

### 3 Transport Inequalities and Concentration

In the literature, there are a range of different methods for obtaining inequalities concerning the phenomenon of concentration of measure. The use of transport inequalities is a more recent endeavor and usually not present in more basic texts concerning high-dimensional probability, such as the book by Vershynin [2018]. The relation of optimal transport to concentration inequalities was first pointed out by Marton [1986], and advanced through the nineties, with advances in optimal transport theory. Take a look at Gozlan and Léonard [2010] for a more complete overview on the topic.

**Definition 3.1** (Transport Inequalities). Let  $J(\cdot | \mu) : P(\mathcal{X})$  and  $\alpha : [0, \infty) \rightarrow [0, \infty)$  an increasing function with  $\alpha(0) = 0$ . One says that  $\mu \in P(\mathcal{X})$  satisfy the transport inequality  $\alpha(T_c) \leq J$  if

$$\alpha(T_c(\nu, \mu)) \leq J(\nu | \mu), \quad \forall \nu \in P(\mathcal{X})$$

**Definition 3.2** (Relative Entropy).

$$H(\nu | \mu) = \begin{cases} \int_{\mathcal{X}} \log\left(\frac{d\nu}{d\mu}\right) d\nu, & \text{if } \nu \ll \mu \\ +\infty, & \text{otherwise} \end{cases}$$

When  $J = H$ , one also calls these transport-entropy inequalities. Among the family of transport inequalities, two classical ones are the  $\mathbf{T}_1$  and  $\mathbf{T}_2$ . We say that  $\mu \in P_p(\mathcal{X}) := \{\nu \in P(\mathcal{X}) : \int d(x_o, \cdot)^p d\nu < \infty\}$  satisfies  $\mathbf{T}_p(C)$ , for  $C > 0$  if

$$W_p^2(\nu, \mu) \leq CH(\nu \mid \mu) \quad (4)$$

Note that  $\mathbf{T}_1(C)$  is weaker than  $\mathbf{T}_2(C)$ , because, if  $\mu$  satisfies  $\mathbf{T}_2(C)$ , then

$$\begin{aligned} W_2^2(\nu, \mu) \leq CH(\nu \mid \mu) &\implies C^{-1}(T_{d^2}(\nu, \mu)^{1/2})^2 \leq H \\ &\xRightarrow{\text{Jensen}} C^{-1}(T_d)^2(\nu, \mu) \leq C^{-1}T_{d^2}(\nu, \mu) \leq H(\nu \mid \mu) \end{aligned}$$

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