Report homework 1
Network Dynamics and Learning

obtained final results compared with
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1 Exercise 1

1.1 Point A

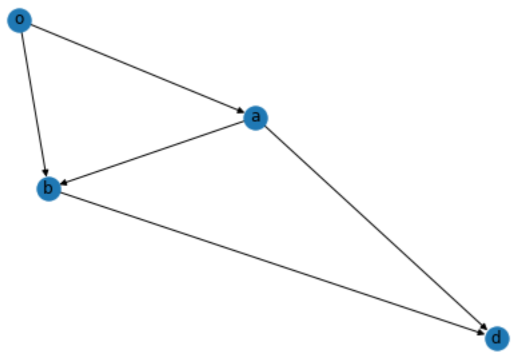

Figure 1: Graph Exercise 1

Given the graph portrayed in figure 1, the aim is to find the infimum of total capacity that needs to be removed for no feasible unitary flows from o to d to exist. So, to catch it the min cut max flow theorem comes to rescue.
The first step consists of determining all the possible cuts on the studied network:

1. μ_1={ø}
2. μ_2={ø,a}
3. μ_3={ø,b}
4. μ_4={ø,a,b}

Then, as second step the total capacity on all these cuts must be determined.

1. C_{μ_1}=$C_{e_1} + C_{e_2} = 3 + 2 = 5$
2. C_{μ_2}=$C_{e_2} + C_{e_3} + Ce_4 = 2 + 2 + 3 = 7$
3. C_{μ_3}=$C_{e_1} + C_{e_5} = 3 + 2 = 5$
4. C_{μ_4}=$C_{e_4} + C_{e_5} = 3 + 2 = 5$

As a result the min cut capacities C^* are C_{μ_1}, C_{μ_3} and C_{μ_4} are the bottlenecks. Finally, by applying the already mentioned min cut max flow theorem, the maximum flow that can be sent from o to d is $\tau^*_{o,d} = C_{\mu_1} = C_{\mu_3} = C_{\mu_4} = 5$.

1.2 Point B

Now there are two additional capacity units to allocate on graph 1 in view to maximize the feasible throughput. Note from section 1.1 that three cuts' capacities are equal in value, so the idea is to add one unit to the "most representative" links in the computation of the min cuts. Let's define S the set of the lowest min cuts capacities, S= { C_{μ_1}, C_{μ_3}, C_{μ_4}}$\dot{\text{N}}$ow, the numerosity in S of all cuts must be computed.

1. $C_{e_1} = 2$ in S
2. $C_{e_2} = 1$ in S
3. $C_{e_3} = 0$ in S
4. $C_{e_4} = 1$ in S
5. $C_{e_5} = 2$ in S

2 results to be the maximum in terms of numerosity in S, so to maximize the throughput a capacity unit must be added to e_1 and the other one to e_5.

1.3 Point C

Differently from point 1.2, all the links' capacities are considered to be infinite. The aim is to compute the Wardrop equilibrium. As a definition, a flow is a Wardrop equilibrium if the flow z_i along path i is non-zero only if the delay along this path is smaller or equal than the delay among all other paths. The delay on all edges are given, and so firstly it is necessary to underline all the possible paths.

- P(1) = { o,a,d }
- P(2) = { o,b,d }
- P(3) = { o,a,b,d }

Secondly, all the delays along the already found path are reported below.

- delay(P(1)) = $d_1 f(1) + d_4 f(4) = (z_1 + z_3 + 1) + (5z_1 + 1)$
- delay(P(2)) = $d_2 f(2) + d_5 f(5) = (5z_2 + 1) + (z_2 + z_3 + 1)$

- delay(P(3)) $= d_1 f(1) + d_3 f(3) + d_5 f(5) = (z_1 + z_3 + 1) + (1) + (z_2 + z_3 + 1)$

Now there all the elements needed to apply the definition of Wardrop equilibrium. But figure 1 infers that there is an horizontal symmetry that leads to conclude that delayP(1) = delayP(2) and as a consequence $z_1 = z_2$. In addition, the flow is unitary, so there is a constraint between the flows of this form $z_1 + z_2 + z_3 = 1$. This will be exploited during the computation.

$$\begin{gathered}
z_1 > 0 \to delayP(1) \leq delayP(2) \;\; \& \;\; delayP(1) \leq delayP(3) \\
(z_1 + z_3 + 1) + (5z_1 + 1) \leq (z_1 + z_3 + 1) + (1) + (z_2 + z_3 + 1) \\
5z_1 \leq 1 + z_2 + 1 - z_1 - z_2 \\
6z_1 \leq 2 \\
z_1 \leq \frac{1}{3}
\end{gathered}$$

$$\begin{gathered}
z_2 > 0 \to delayP(2) \leq delayP(1) \;\; \& \;\; delayP(2) \leq delayP(3) \\
(z_2 + z_3 + 1) + (5z_2 + 1) \leq (z_1 + z_3 + 1) + (1) + (z_2 + z_3 + 1) \\
5z_2 \leq 1 + z_1 + 1 - z_1 - z_2 \\
6z_2 \leq 2 \\
z_2 \leq \frac{1}{3}
\end{gathered}$$

Since, $z_1 = z_2$, they must be equal to $\frac{1}{3}$. As a consequence, from the constraint of the problem, $z_3 = 1 - z_1 - z2 = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$. Finally, the flow vector at Wardrop results to be $f_w = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.

1.4 Point D

Now, it is provided a solution to the computation of the social optimum on the network 1, which is defined in equation 1.

$$\sum_{e \in \epsilon} c_e f(e) = f_e delay(f_e) = f_1 * (f_1 + 1) + f_2 * (5f_2 + 1) + f_3 + f_4 * (5f_4 + 1) + f_5 * (f_5 + 1) \quad (1)$$

In computation 1, $f_1 = z_1 + z_3$, $f_2 = z_2$, $f_3 = z_3$, $f_4 = z_1$ and $f_5 = z_2 + z_3$. The social optimum is obtained by minimizing the cost function 1 and by substituting the f_i with their z_i.

$$\begin{gathered}
f_1 * (f_1 + 1) + f_2 * (5f_2 + 1) + f_3 + f_4 * (5f_4 + 1) + f_5 * (f_5 + 1) \\
= f_1^2 + f_1 + 5f_2^2 + f_2 + f_3 + 5f_4^2 + f_4 + f_5^2 + f_5 \\
= (z_1 + z_3)^2 + z_1 + z_3 + 5z_2^2 + z_2 + z_3 + 5z_1^2 + (z_2 + z_3)^2 + z_2 + z_3 \\
= z_1^2 + z_3^2 + 2z_1 z_3 + z_1 + z_3 + 5z_2^2 + z_2 + z_3 + 5z_1^2 + z_2^2 + z_3^2 + 2z_2 z_3 + z_2 + z_3 \\
= 6z_1^2 + 6z_2^2 + 2z_3^2 + 2z_1 z_3 + 2z_2 z_3 + 2z_1 + 2z_2 + 3z_3 \\
= 6z_1^2 + 6z_2^2 + 2(1 + z_1^2 + z_2^2 - 2z_1 - 2z_2 + 2z_1 z_2) + 2z_1 - 2z_1^2 - 2z_1 z_2 + z_2 - 2z_1 z_2 - 2z_2^2 + 2z_1 + 2z_2 + 3 - 3z_1 - 3z_2 \\
cost = 6z_1^2 + 6z_2^2 - 3z_1 - 3z_2 + 5
\end{gathered}$$

The cost function must be minimized exploiting partial derivatives. Let's denote it with c.

$$\begin{cases} \frac{\partial c}{\partial z_1} = 12z_1 - 3 = 0 \\ \frac{\partial c}{\partial z_2} = 12z_2 - 3 = 0 \end{cases} \quad \begin{cases} 12z_1 = 3 \\ 12z_2 = 3 \end{cases} \qquad\qquad \begin{cases} z_1 = \frac{1}{4} \\ z_2 = \frac{1}{4} \end{cases}$$

From the constraint $z_1+z_2+z_3 = 1$, $z_3 = 1-\frac{1}{4}-\frac{1}{4} = \frac{1}{2}$. The resulting flow vector at social optimum is $f^* = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4})$.

1.5 Point E

The price of anarchy is defined as the ratio between the average delay at Wardrop equilibrium and the average delay at social optimum. So it is needed to find both the average delays following equation 2 and by substituting the flow vectors' elements determined in point 1.3 and 1.4.

$$(f_1 + 1)f_1 + (5f_2 + 1)f_2 + 1 * f_3 + (5f_4 + 1)f_4 + (f_5 + 1)f_5 \tag{2}$$

Wardrop equilibrium average delay results to be equal to $\frac{13}{3}$, while social optimum average delay is $\frac{34}{8}$.

As a consequence, poA=$\frac{\frac{13}{3}}{\frac{34}{8}} \sim 1,1$.

1.6 Point F

As section 1.5 infers, the price of anarchy is higher than one, bringing it to 1 correspond to the introduction of tolls on the delay functions $d_e(x)$ able to make the Wardrop equilibrium equal to the social optimum, which cannot be modified. The vector of tolls is defined in formula 3, where f^* is the vector solving the social optimum assignment problem.

$$w_e^* = c_e^{'} - d_e(f_e^*) = f_e^* d_e^{'}(f_e^*) \tag{3}$$

So, the tolls are obtained by the following computation:

$$w_1 = f_1^* d'[x + 1] = \frac{3}{4} * 1 = \frac{3}{4}$$
$$w_2 = f_2^* d'[5x + 1] = \frac{1}{4} * 5 = \frac{5}{4}$$
$$w_3 = f_3^* d'[1] = \frac{1}{2} * 0 = 0$$
$$w_4 = f_4^* d'[5x + 1] = \frac{3}{4} * 5 = \frac{5}{4}$$
$$w_5 = f_5^* d'[x + 1] = \frac{3}{4} * 1 = \frac{3}{4}$$

In a more compact way, the vector of tolls can be written as $w_e^* = (\frac{3}{4}, \frac{5}{4}, 0, \frac{5}{4}, \frac{3}{4})$.

2 Exercise 2

2.1 Point A

W=$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, w = W $\mathbb{1}$=$\begin{bmatrix} a+1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, D = diag(w) = $\begin{bmatrix} a+1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$P = D^{-1}W = \begin{bmatrix} \frac{a}{a+1} & \frac{1}{a+1} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \text{ L} = \text{D}-\text{W} = \begin{bmatrix} a+1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

2.2 Point B and C

First of all the considered graph is strongly connected. In addition to that, the graph is aperiodic, since there is a self-loop on node 1. Moreover, also when a is equal to 0 the graph remains aperiodic, since node 1 has always period 1. These observations let to conclude that $\lim_{t\to\infty} x(t)$ converges for every $a \geq 0$.

2.3 Point D

The aim is to find all the elements populating equation 4.

$$\lim_{t\to\infty} x(t) = \mathbb{1}\pi^{'} x(0) \tag{4}$$

Now, given a equal to 0, w turns into $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and initial conditions x(0) = (-1, 1, -1, 1). Since the graph is balanced, w results to be the eigenvector of the P matrix, as a consequence, by applying Perron-Frobenius theorem, the invariant probability distribution is obtained by normalizing w. In fact, $\pi^{'} = \frac{w}{|w|}$.

$$\pi^{'} = (\frac{1}{6}, \frac{2}{6}, \frac{1}{6}, \frac{2}{6})$$
$$\lim_{t\to\infty} x(t) = (\frac{1}{6}, \frac{2}{6}, \frac{1}{6}, \frac{2}{6})^{'} * (-1, 1, -1, 1)$$
$$= [(\frac{1}{6} * (-1)) + (\frac{2}{6} * 1) + (\frac{1}{6} * (-1)) + (\frac{2}{6} * 1)]$$
$$= \frac{-1}{6} + \frac{2}{6} + \frac{-1}{6} + \frac{2}{6}$$
$$= \frac{1}{3}\mathbb{1}$$

So, $\lim_{t\to\infty} x(t) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2.4 Point E

The goal is to test if there exists an $a \geq 0$ for which $\lim_{t\to\infty} x_1(t) \leq 0$. It is needed to compute in a parametric form the invariant distribution π.

$$\pi' = (\frac{a+1}{6+a}, \frac{2}{6+a}, \frac{1}{6+a}, \frac{2}{6+a})$$

$$\lim_{t\to\infty} x_1(t) = (\frac{a+1}{6+a}, \frac{2}{6+a}, \frac{1}{6+a}, \frac{2}{6+a})' * (-1, 1, -1, 1)$$

$$= [(\frac{(a+1)}{6+a} * (-1)) + (\frac{2}{6+a} * 1) + (\frac{1}{6+a} * (-1)) + (\frac{2}{6+a} * 1)]$$

$$= \frac{-1}{6+a} + \frac{2}{6+a} + \frac{-1}{6+a} + \frac{2}{6+a}$$

$$= \frac{2-a}{6+a}$$

The already retrieved quantity must be set ≤ 0. As a result, to satisfy the problem's request, $a \geq 2$.

2.5 Point F

The initial conditions $x_i(0)$ are all considered random variables, where $i = 1, 2, 3, 4$. The expected value and variance of all random variables are know and equal respectively to 0 and 1 for all $x_i(0)$. The aim is to find the a minimizing the variance of $\lim_{t\to\infty} x_1(t)$. Point 2.4 already gave the definition of $\lim_{t\to\infty} x_1(t)$, for which results $\lim_{t\to\infty} x_1(t) = [\frac{a+1}{6+a} * x_1 + \frac{2}{6+a} * x_2 + \frac{1}{6+a} * x_3 + \frac{2}{6+a} * x_4]$. But, now it is requested the variance of this value : $var(\lim_{t\to\infty} x_1(t)) = var(\frac{a+1}{6+a} * x_1) + var(\frac{2}{6+a} * x_2) + var(\frac{1}{6+a} * x_3) + var(\frac{2}{6+a} * x_4)$. Now the statistical definition of a random variable's variance multiplied by a number $var(ax) = a^2 * var(x)$ is exploited. Then, the variance of each random variable is equal to 1 and it will be substituted in the following computation.

$$var(\frac{a+1}{6+a} * x_1) + var(\frac{2}{6+a} * x_2) + var(\frac{1}{6+a} * x_3) + var(\frac{2}{6+a} * x_4)$$

$$= \frac{(a+1)^2}{(6+a)^2} var(x_1) + \frac{4}{(6+a)^2} var(x_2) + \frac{1}{(6+a)^2} var(x_3) + \frac{4}{(6+a)^2} var(x_4)$$

$$= \frac{(a+1)^2 + 4 + 1 + 4}{(6+a)^2}$$

$$= \frac{(a+1)^2 + 9}{(6+a)^2}$$

$$obj = \frac{a^2 + 2a + 10}{(6+a)^2}$$

At this step, the obtained formula must be minimized, so the first derivative must be kept equal to 0.

$$obj' = \frac{(2a+2)(a+6)^2 - 2(a+6)(a^2 + 2a + 10)}{(a+6)^4}$$

$$= \frac{2a^2 + 12a + 2a + 12 - 2a^2 - 4a - 20}{(a+6)^3}$$

$$= \frac{10a - 8}{(a+6)^3}$$

Finally, the obtained quantity must be set to 0 to catch a, which results to be $\frac{4}{5}$. On top of that, $\frac{4}{5}$ is a minimum, because from there the function will increase.

3 Exercise 3

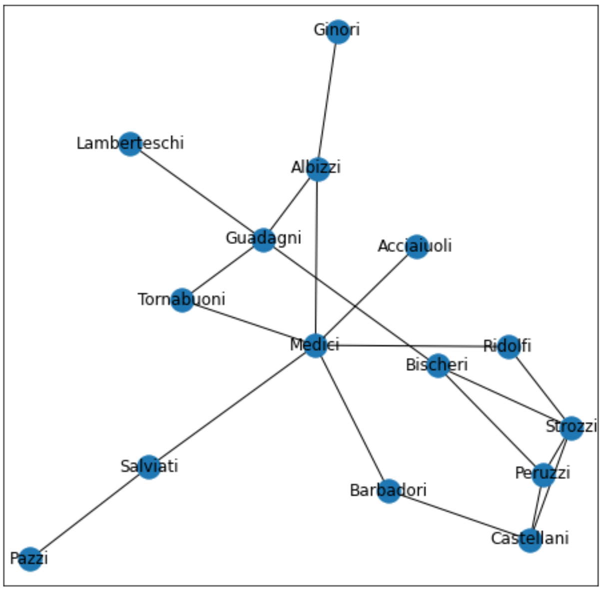

Figure 2: Florence family graph

3.1 Point A

Graph 2 is strongly connected and aperiodic, so it can be said that equality 5 holds.

$$\lim_{t\to\infty} x(t) = \mathbb{1}\pi^{'}x(0) \tag{5}$$

At this step, the invariant distribution must be computed by following relation $\pi^{'} = \frac{w}{|w|}$, because of Perron-Frobenius theorem.
$\pi = (\frac{1}{38}, \frac{3}{38}, \frac{3}{38}, \frac{4}{38}, \frac{3}{38}, \frac{4}{38}, \frac{2}{38}, \frac{2}{38}, \frac{3}{38}, \frac{1}{38}, \frac{2}{38}, \frac{6}{38}, \frac{1}{38}, \frac{2}{38}, \frac{1}{38})$, while the initial condition vector $x(0) = (0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$. Finally, the limit calculation is obtained $\lim_{t\to\infty} x(t) = \pi^{'}x(0) = [\frac{4}{38} * (-1)] + [\frac{6}{38} * (1)] = \frac{-4}{38} + \frac{6}{38} = \frac{2}{38} = \frac{1}{19}\mathbb{1}$.

3.2 Point B

Making reference to graph 2, a simulation of average dynamics evolution is performed. The number of iterations selected is 50. All implementation's details and comments are in the uploaded file "exercise3.ipynb".

The set S of stubborn nodes considered is S={ Strozzi, Medici }. Moreover, their initial immutable opinions are $x_{Strozzi} = -1$ and $x_{Medici} = 1$. Picture 3 infers how fast the regular nodes (the other families) converges to a final opinion, depending on the influencing position of the two stubborn nodes. As you can see, 50 is a reasonable choice for the number of iterations because all the families keep and maintain an opinion. Practically, the opinions in the negative part follow node "Strozzi", while in the positive part there are families supporting "Medici" family. On top of that, "Ridolfi" family is neutral, in fact it keeps position to 0 and it maintains it until the end.

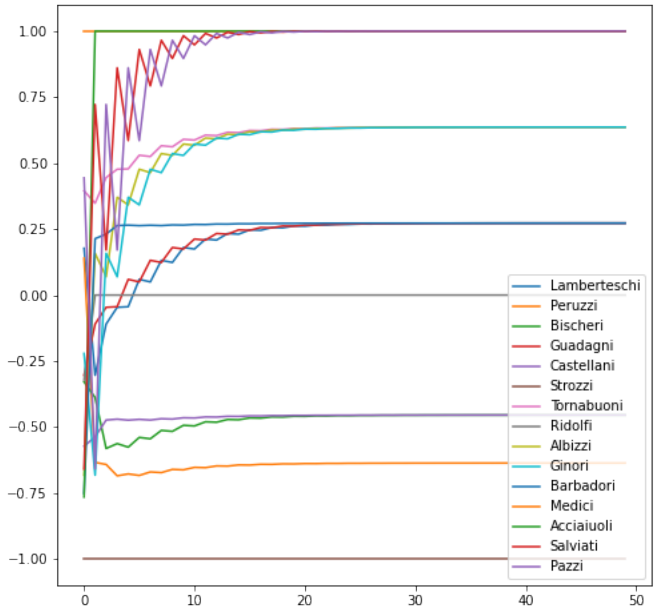

Figure 3: evolution of average dynamics

3.3 Point C

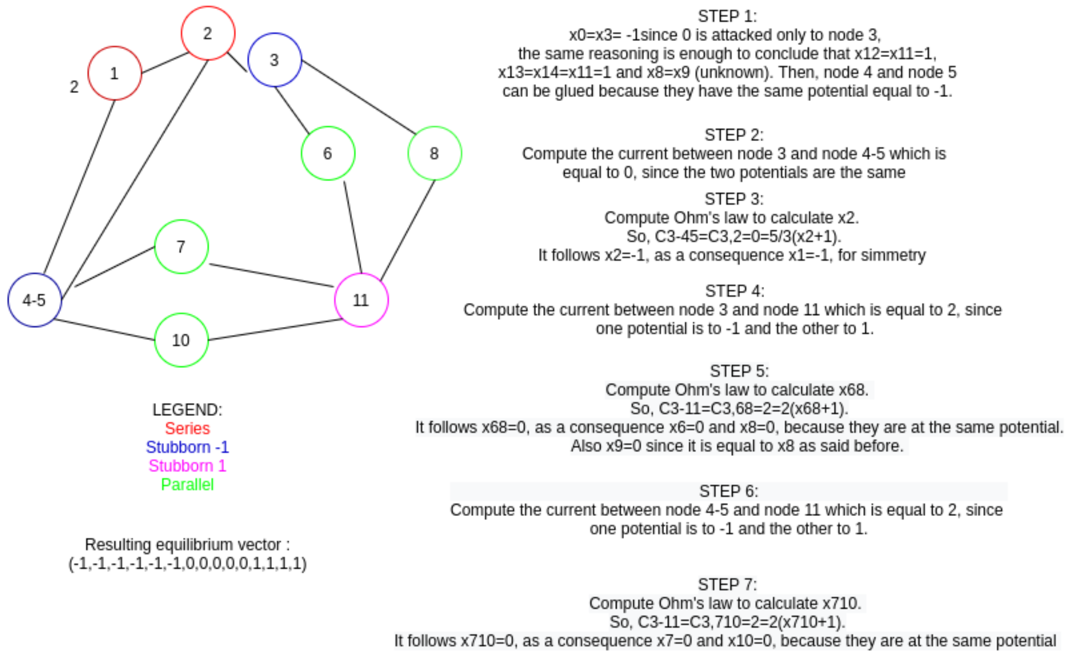

Figure 4: Resolution with electric networks

3.4 Point D

Average dynamics gives a new definition of centrality. Starting from this concept, the Pagerank algorithm for centrality is revisited and implemented. Image 5 offers a quick view of the "importance" of each family through the dimension and the brightness of the dot plots. As you can infer, "Medici" and "Guadagni" have the highest centrality.
All the code and comments are inserted in the uploaded file "exercise3.ipynb".

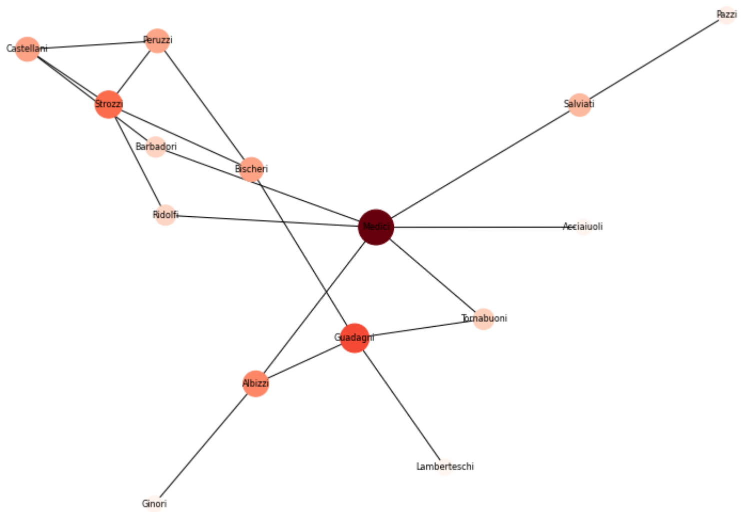

Figure 5: Centralities pagerank algorithm with average dynamics

4 Exercise 4

4.1 Point A

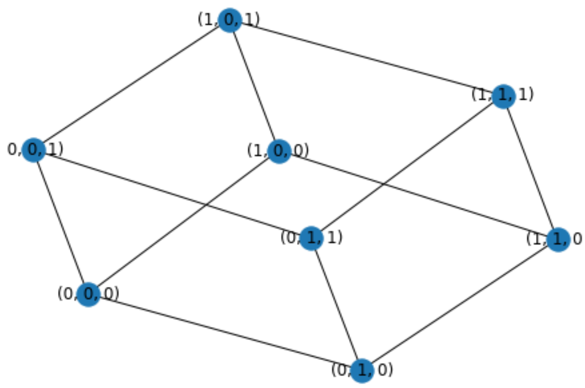

Figure 6: Cube graph

In the network portrayed in image 6 the node (0,0,0) is supposed to be a stubborn node with opinion $x_{0,0,0} = 0$. (0,0,0), due to its stubborn nature, will not change its opinion during time (in the experiment 50 iterations were considered). The aim of this numerical computation is to find the optimal position (i,j,k) of another stubborn node with opinion $x_{i,j,k} = 1$ in view to turn the asymptotic average opinion maximal. The code is in the uploaded file "exercise4.ipynb", with some comments. Picture 7 shows the optimal solutions compared to the dimension of the nodes. As you can infer, all nodes seem to be an optimal position. In fact, the results obtained are all equal around

50 percent. This is reasonable, because of the symmetry of the cube.

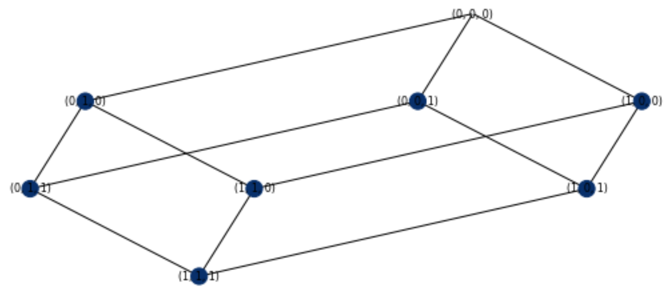

Figure 7: Cube optimal placement

4.2 Point B

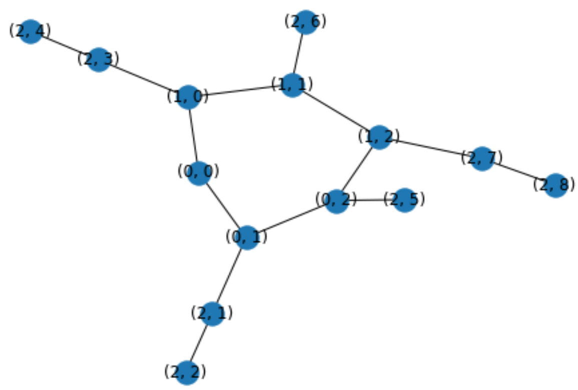

Figure 8: Second graph considered

In the network portrayed in image 6 the node (0,0,0) is supposed to be a stubborn node with opinion $x_{0,0} = 0$. (0,0), due to its stubborn nature, will not change its opinion during time (in the experiment 50 iterations were considered). The aim of this numerical computation is to find the optimal position (i,j,k) of another stubborn node with opinion $x_{i,j} = 1$ in view to turn the asymptotic average opinion maximal. The code is in the uploaded file "exercise4.ipynb", with some comments. Picture 9 shows the optimal solutions compared to the dimension of the nodes. In this case, due to absence of symmetry, the optimal placement results to be on one neighbor node from the initial stubborn node $x_{0,0}$. In terms of numeric ranking neighbors arrive to 57 percent of optimality.

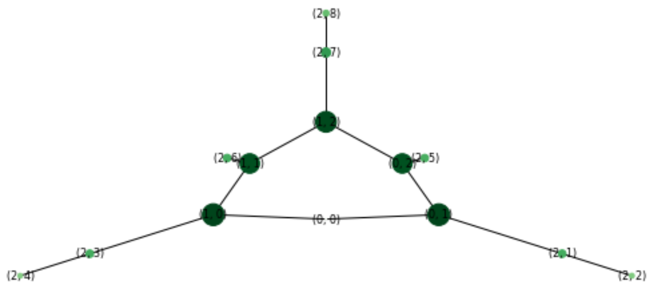

Figure 9: Second graph optimal placement