The Ornstein Uhlenbeck Process
And its Applications to Pairs Trading

David Alber[a], Daniel Brus[b]

[a]*Department of Physics, Stockholm University, 106 91 Stockholm, Sweden*
[b]*Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden*

Abstract

Ornstein-Uhlenbeck (OU) processes are widely used in mathematical finanice due to the mean reversion related qualities. In this project an overview of OU processes is given, including a formal solution and a parameter study. Also an OU process model is presented for a pair of stocks including a way to infer the model parameters from the historical data of the time series. A novel proof regarding another model is proposed. Finally, a simple algorithm for pairs trading shows how pairs trading could be used for two S&P500 traded stocks and proofs that the strategy is successful.

Keywords: Stochastic differential equations, Ornstein-Ulenbeck Process, Pairs Trading, S&P500

1. Introduction

Mean reversion processes are widely observed in finance, in spread options, pairs trading and volatility models [1], [2]. Its properties are well understood by the studies of the Ornstein-Uhlenbeck (OU) process. In the framework of this project we want to present a formal solution of the Ornstein-Uhlenbeck process and discuss its guiding parameters and dynamics, sec. 2. In addition we will discuss an application, that takes advantage of this study: Pairs Trading. This is a popular investment strategy among hedge funds and investment banks. In order to justify the validity of a Ornstein-Uhlenbeck model, a way to calibrate the process for a pair of two suitable stocks is presented, sec. 3. In particular we consider the Hurst Exponent and co-intgration as indicators if two stocks are well described by an OU process and hence suitable for the trading strategy of pairs trading. This links the mathematical model to real world phenomenon at the stock market. Further we provide through an exemplary model simulation insights about how pairs trading could be applied to gain profits from the stock market, sec. 6. Given some real data of stocks for which a pairs trade is feasible, the model under consideration is tested. Also we evaluate how much one could have profited by applying the pairs trading strategy in question. The incentive for our work is [3], which is still in proceedings. In particular we tackle the deployment of the model for real stock data, which is the authors suggestions for further work.

Email addresses: `alber.p.david@gmail.com` (David Alber), `brus.daniel@gmail.com` (Daniel Brus)

2. A formal solution of the Ornstein Uhlnbeck process

An Ornstein Uhlenbeck process is governed by the following stochastic differential equation:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \tag{1}$$

where dW_t denotes to a Wiener process, σ, $\theta > 0$ are constants and μ is a so called drift term. For a given initial value X_0 the equation can be formally solved by making use of Ito's lemma:

$$\begin{aligned} G_t &= X_t e^{\theta t} \\ dG_t &= \partial_t G_t dt + \partial_X G_t dX + \partial_{XX} G_t (dX)^2 \\ X_t &= X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta(s-t)} dW(s) \end{aligned} \tag{2}$$

This (2) is the sum of deterministic terms and an integral of a deterministic function with respect to a Wiener process with normally distributed increments. Hence the distribution is normal. This result yields the conditional expectation value and the variance:

$$\begin{aligned} &E[X_t|X_0] = e^{-\theta t}(X_0 - \mu) + \mu \xrightarrow{t\to\infty} \mu \\ &Var[X_t|X_0] = E[X_t^2] - E[X_t]^2 \\ &= E[\sigma^2 \left(\int_0^t e^{\theta(s-t)} dW(s)\right)^2] = E[\sigma^2 \int_0^t e^{2\theta(s-t)} ds] \\ &= \frac{\sigma^2}{2\theta}(1 - e^{2\theta t}) \xrightarrow{t\to\infty} \frac{\sigma^2}{2\theta} \end{aligned} \tag{3}$$

The analysis shows that asymtotically $X_t \sim \mathcal{N}(\mu, \frac{\sigma^2}{2\theta})$. An interesting property of the Ornstein-Uhlnbeck process is, that it incorporates mean reversion. This means, that if

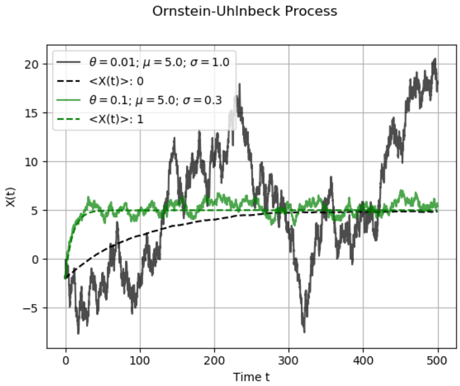

Figure 1: Two Ornstein Uhlenbeck processes (full lines) with differently chosen parameters θ, σ and common parameter μ. The expectation value (dashed lines) $E[X_t] \approx < X_t >$ is the average over 10000 processes. It converges to μ. The initial value is set to $X_0 = -2$.

X_t is high or low, it will tend to be pulled back towards its asymptotic expectation value $E[X_t] = \mu$ at a rate of θ [1]. The half life is defined as the expected time to return half way to its mean. Using the formal solution of the Ornstein-Uhlnbeck process (2) the half life time $t_{\frac{1}{2}}$ can be computed as follows:

$$E[X_{t_{\frac{1}{2}}}] - \mu = \frac{X_0 - \mu}{2} = e^{-\theta t_{\frac{1}{2}}}(X_0 - \mu) \tag{4}$$
$$t_{\frac{1}{2}} = \frac{ln(2)}{\theta}$$

2.1. *Exploring the parameters of the Ornstein-Uhlnbeck Process*

This section is dedicated to explore the parameters θ, μ and σ of the Ornstein-Uhlnbeck process, see sec. 2. The stochastic process is simulated with a Forward Euler Scheme with $\Delta t = t_{n+1} - t_n$ and $\Delta W \sim \mathcal{N}(0, \Delta t)$:

$$X(t_{n+1}) = X(t_n) + \theta(\mu - X(t_n))\Delta t + \sigma \Delta W \tag{5}$$

Figure 1 gives a graphical intuition how the parameters in eq. (1) determine the structure of the process. As the analysis of eq. (3) reveals, the expectation value converges to the mean μ, independent of the choice of the initial value X_0. The time scale at which it converges is determined by the parameter θ and the relationship (4). The size of the noise or variance is determined by σ.

One of the main properties of this process is that it is stationary, i.e: the process has a tendency to reverts to its mean.

3. Stock data and the OU Process

Having analyzed the Ornstein-Uhlenbeck process from a theoretical angle, we now show that the defining characteristics of such a process can be found in the daily variation of stock prices. As a measure of how much this mean reverting property is present in stocks, we use two methods, namely the Hurst exponent and Co-integration.

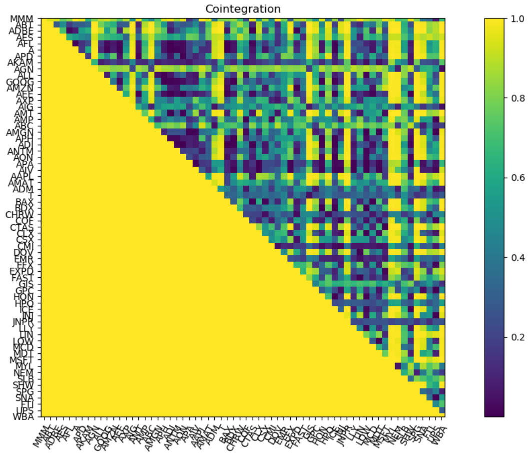

Figure 2: Map of co-integration for 64 S&P500 stock pairs. The color bar indicates a p-value, evaluated with the augmented Engle-Granger two-step cointegration test [5]. The null hypothesis for this p-value is, that the the two time series closing values are not co-integrated.

3.1. *Co-integration*

Co-integration is a well known phenomenon in economic time series. This feature is observed for example in the behaviour of interest rates, on assets of different maturities, spot and future price in commodity markets and the value of sales and production costs of an industry [4]. A formal definition can be formulated as follows:

Two non stationary time series A_t, B_t are co-integrated if there is some $\gamma \in \mathbb{R}$ s.t $A_t - \gamma B_t$ is stationary [3]. This means that the ratio between the two series is mean reverting. Thus the definition above is equivalent with saying two series A_t, B_t are co-integrated if $A_t = \gamma B_t + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

Statistical testing of two time series reveals if the ratio between them carries the property of co-integration. The null hypothesis is that the ratio between the two series is not co integrated. More precisely, we analyse the time series of closing values from 01.01.2010 to 01.01.2020 of 64 traded stocks from the S&P 500. Statistical testing with the augmented Engle-Granger two-step cointegration test [5] lets us construct this map 2.

The p-values in fig. 2 are symmetric. The 64 tested time series, allow 2016 different unique combinations. Our analysis reveals that from the 64 tested stocks the p-value of 154 pair combinations have a p-value below the significance level of 0.5%. We conclude that these are stocks which may be well described by an OU process as they show properties of mean reversion. Another measurement of how well historical data of two traded stocks is described by a mean reverting OU process is the Hurst exponent. Which we will discuss in the next subsection 3.2.

3.2. *Hurst exponent*

The Hurst exponent is a measure for long term memory of time series. A stationary price series will diffuse more

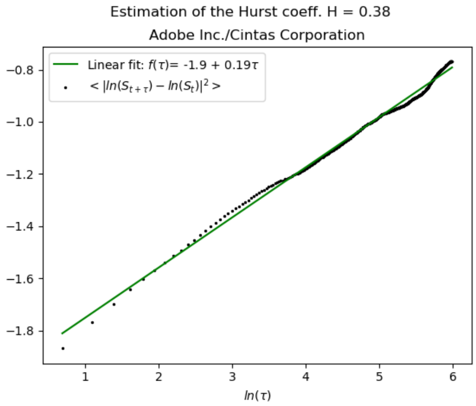

Figure 3: Inference of the Hurst exponent for the ratio between the closing values of Adobe Inc. - Cintas Corporation from 01.01.2010-01.01.2020. The graph shows the logarithmic values. A polynomial of order 1 accurately fits the evaluated logarithmic data of quadratic variation. The Hurst exponent is twice the coefficient of the linear fit.

slowly than Geometric Brownian Motion (GBM). Assuming a time series of stock values behaves like GBM, the quadratic variation of the logarithmic value S from time t to $t + \tau$ is described by the following relationship:

$$[S_t] = E[|ln(S_{t+\tau}) - ln(S_t)|^2] = \sigma^2\tau \sim \tau = \tau^{2H} \qquad (6)$$

In eq. (6) $\sim$ indicates proportional to. For the equation to hold the Hurst exponent H has to be $\frac{1}{2}$. Analysing a given set of time series we can differentiate between three cases:

- $H > 0.5$: process is trending
- $H = 0.5$: process is releated to GBM
- $H < 0.5$: process is mean reverting

Recall that we are interested in the ratio between the closing value of two stocks and seek a OU description of this ratio. Thus let the time series in eq. (6) under consideration be the ratio of the closing values of two stocks $S_t = \frac{X_t}{Y_t}$. In particular we analyse a selection of the pairs that have already shown to have co-integration properties, see fig. 2. Figure 3 depicts the inference of the Hurst exponent. The quadratic variation of the logarithmic values of the time series S_t is evaluated for different lags $\Delta t = \tau$. On a logarithmic scale, a polynomial of order 1 fits the data. Its coefficient times two is the Hurst exponent. Figure 2 shows that the pair Adobe Inc. - Cintas Corporation shows to have properties of co-integration and figure 3 shows that the ratio of this pair also shows to have mean reverting properties, since its Hurst exponent is $H = 0.38$. This is below the critical limit of GBM processes of 0.5.

4. One Process Model

The following model is proposed in [3]. This model assumes a faily stable stock X, with predictable behaviour with respect to external factors captured in F_t.

$$X_t = x \exp\{F_t\} \qquad (7)$$

$$Y_t = kX_t \exp\left\{\tilde{Y}_t\right\} \qquad (8)$$

Taking $y = kx$ to be representative of the historical price ratio, we obtain:

$$Y_t = y \exp\left\{\tilde{Y}_t + F_t\right\}, \quad t \geq 0 \qquad (9)$$

Such that:

$$\frac{Y_t}{kX_t} = \exp\left\{\tilde{Y}_t\right\} \qquad (10)$$

$$\tilde{Y}_t = \ln\frac{Y_t}{kX_t} \qquad (11)$$

Where we use the OU process to model $\tilde{Y}_t$:

$$\Delta\tilde{Y}_t = -\theta\tilde{Y}_t\Delta t + \sigma\Delta W_t \qquad (12)$$

In this model we are free to fit the parameters θ and σ. We note that from historical data of Y_t and X_t and a k of our choice we can calculate the implied historical $\Delta\tilde{Y}_t$:

$$Y_{t+1} = y \exp\left\{\tilde{Y}_{t+1} + F_{t+1}\right\} \qquad (13)$$

$$\frac{Y_{t+1}}{Y_t} = \exp\left\{\Delta\tilde{Y}_t + \Delta F_t\right\} \qquad (14)$$

$$\frac{\Delta\tilde{Y}_t}{\tilde{Y}_t} = \exp\left\{\Delta\tilde{Y}_t + \Delta F_t\right\} - 1 \qquad (15)$$

$$\frac{\Delta\tilde{X}_t}{\tilde{X}_t} = \exp\{\Delta F_t\} - 1 \qquad (16)$$

$$\Delta\tilde{Y}_t = \ln(\frac{\Delta Y_t}{Y_t} + 1) - \ln(\frac{\Delta X_t}{X_t} + 1) \qquad (17)$$

Using the historical values of $\Delta\tilde{Y}_t$ and $\tilde{Y}_t$ we can choose the parameters which best fit the observed data.

4.1. Inferring parameters from the One Process Model

As we have seen in sec. 3 the ratio of two stocks can be reasonably well described by an OU process. Now assuming that a One Process Model, sec. 4 holds true we now want to infer the guiding parameters θ and σ, see eq. (12) for the given historical data X_t, Y_t of the closing values of two stocks. Note that from eq. (10), (13) the values for $\tilde{Y}_t$ and $\Delta\tilde{Y}_t$ in eq. (12) can be evaluated and thus the task of inferring the guiding parameters for a description of an OU process comes down to solving a maximal likelihood estimation for the parameter θ w.r.t a linear model $y = \theta x + \epsilon$, with Gaussian distributed noise $\epsilon = \sigma\Delta W \sim \mathcal{N}(\mu, \sigma^2\Delta t) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}}e^{-\frac{(x-\mu)^2}{2\sigma^2\Delta t}}$. For the sake of simplicity we rename the variables $y = \Delta\tilde{Y}_t$, $x = \tilde{Y}_t\Delta t$

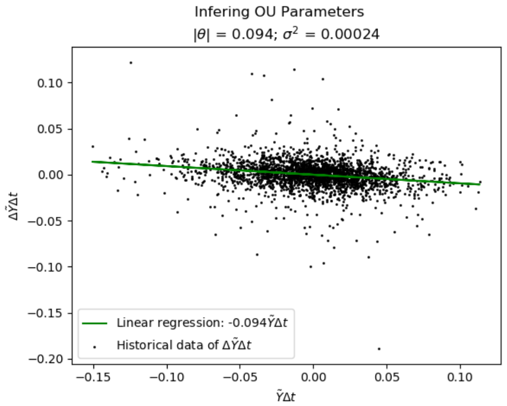

Figure 4: OU maximum likelihood parameter inference according to a One Process Model (10), (13) from historical stock data of Adobe Inc. and Cintas Corporation.

in the following calculations for the maximum likelihood estimate for θ. The set of all $[x_1, \dots, x_N] = \mathcal{X}$

$$\text{Consider the model: } y_i = \theta x_i + \epsilon_i \tag{18}$$

$$f(y_i) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-\frac{(y_i-\theta x_i)^2}{2\sigma^2\Delta t}}$$

$$L(\theta,\sigma^2) = \Pi_{x\in\mathcal{X}} f(y_i) = \Pi_{x\in\mathcal{X}} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-\frac{(y_i-\theta x_i)^2}{2\sigma^2\Delta t}}$$

$$l(\theta,\sigma^2) = ln(L(\theta,\sigma^2)) = \frac{N ln(2\pi\sigma^2\Delta t)}{2} + \frac{\sum_{x\in\mathcal{X}}(y_i-\theta x_i)^2}{2\sigma^2\Delta t}$$

$$\frac{\partial l(\theta,\sigma^2)}{\partial\theta} = 0 \Rightarrow \theta = \frac{\sum_{x\in\mathcal{X}} y_i x_i}{\sum_{x\in\mathcal{X}} x_i^2}$$

$$\frac{\partial l(\theta,\sigma^2)}{\partial\sigma^2} = 0 \Rightarrow \sigma^2 = \frac{1}{N}\sum_{x\in\mathcal{X}}(y_i-\theta x_i)^2$$

Assuming a One Process Model, the calculation of the defining parameters θ and σ for given data X_t and Y_t comes down to a simple calculation (18). We use two stocks (Adobe Inc., Cintas Corporation), that have shown to have properties of mean reversion 2, 3, to demonstrate the practical validity of the theoretical estimations. Figure 4 shows a graphical depiction of the parameter inference, which comes down to a linear regression task.

In fig. 5 we can see that the previously estimated parameters from historical data, see fig. 4 are a good description of the historical data.

We conclude, that the ratio between two stocks is well described by the mean reverting OU process. Assuming a One Process Model, inferring the guiding parameters for an OU description of this ratio comes down to a linear regression task. They can be inferred by maximum likelihood estimation.

5. Two Process Model

In the conclusion of the paper [3], which is still in proceedings, the author suggests a Two Process Model to

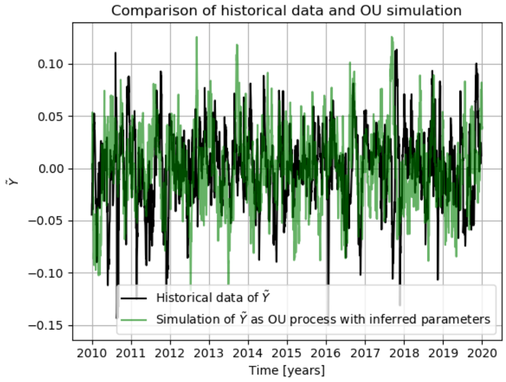

Figure 5: The ratio between the two stocks (Adobe Inc. and Cintas Corporation) is well described by a OU process. Comparing the historical ratio (10) to a OU simulation with the ML inferred parameters $\theta = 0.093$, $\sigma^2 = 0.00024$ shows that the two stochastic processes are similar. The time increment $\Delta t = 1$ corresponds to one day.

be a more accurate model for the behaviour of two cointegrated stocks. In this section a proof is presented that such a model cannot be more accurate than the One Process Model under reasonable assumptions.

5.1. The Model

We model the prices of two stocks X_t and Y_t as exponentials of common outside market forces F_t and stochastic processes $\tilde{X}_t$ and $\tilde{Y}_t$ fluctuating about some fair values x and y.

$$\begin{aligned} X_t &= x \exp\left\{F_t + \tilde{X}_t\right\} \\ Y_t &= y \exp\left\{F_t + \tilde{Y}_t\right\} \end{aligned} \tag{19}$$

Furthermore, we model the fluctuating terms as a general two dimensional OU-process because of their mean reverting nature.

$$\begin{bmatrix}\Delta\tilde{X}_t \\ \Delta\tilde{Y}_t\end{bmatrix} = -\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\tilde{X}_t \\ \tilde{Y}_t\end{bmatrix}\Delta t + \begin{bmatrix}\sigma_1^2 & \alpha\sigma_1\sigma_2 \\ \alpha\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\begin{bmatrix}\Delta W_t^1 \\ \Delta W_t^2\end{bmatrix}$$

5.2. Constraints

Suppose the existence of some unknown ratio k which represents the implicit fair value ratio of the two assets: $y = kx$. Then, when the current ratio $k_t = Y_t/X_t$ trades at that ratio k, e.g. $k_t = k$, the fair value normalised ratio of the assets must equal one:

$$\frac{Y_t}{kX_t} = \exp\left\{\tilde{Y}_t - \tilde{X}_t\right\} = 1 \tag{20}$$

This implies that at the fair ratio k, $\tilde{Y}_t = \tilde{X}_t$. If the Two Process Model is to model mean reversion towards the fair value ratio well, a reasonable assumption is that the drift of the fair value normalised k_t should be zero when it is trading at this unknown fair value. Noting the OU-process equality discerned above, we obtain the constraint:

$$\begin{aligned}
&\mathbb{E}[\Delta(Y_t/kX_t)|\tilde{Y}_t = \tilde{X}_t] = 0 \\
&\mathbb{E}[\exp\{\tilde{Y}_{t+1} - \tilde{X}_{t+1}\} - \exp\{\tilde{Y}_t - \tilde{X}_t\}|\tilde{Y}_t = \tilde{X}_t] = 0 \\
&\mathbb{E}[\exp\{(\tilde{Y}_t + \Delta\tilde{Y}_t) - (\tilde{X}_t + \Delta\tilde{X}_t)\} - 1] = 0 \\
&\mathbb{E}[\exp\{\Delta\tilde{Y}_t - \Delta\tilde{X}_t\}] = 1
\end{aligned} \tag{21}$$

Furthermore, from our general OU-process, we know:

$$\Delta\tilde{Y}_t - \Delta\tilde{X}_t = (a_{11} - a_{21})\tilde{X}_t\Delta t + (a_{12} - a_{22})\tilde{Y}_t\Delta t + \Delta\overline{W}_t \tag{22}$$

Where all randomness is captured in the normally distributed random variable $\Delta\overline{W}_t$:

$$\begin{aligned}
&\Delta\overline{W}_t = (\alpha\sigma_1\sigma_2 - \sigma_1^2)\Delta W_t^1 + (\sigma_2^2 - \alpha\sigma_1\sigma_2)\Delta W_t^2 \\
&\Delta\overline{W}_t \sim \mathcal{N}(0, [(\alpha\sigma_1\sigma_2 - \sigma_1^2)^2 + (\sigma_2^2 - \alpha\sigma_1\sigma_2)^2]\Delta t)
\end{aligned} \tag{23}$$

From $\Delta\overline{W}_t$ we derive the expected value of the log-normal distributed random variable exponential:

$$\mathbb{E}[\exp\{\Delta\overline{W}_t\}] = \exp\{\sigma_{\overline{W}}^2/2\} \tag{24}$$

Noting again that $\tilde{Y}_t = \tilde{X}_t$ when $k_t = k$, defining $\overline{a} := (a_{11} - a_{21} + a_{12} - a_{22})$ and substituting (24), we derive the expected value of the exponential of (22):

$$\mathbb{E}[\exp\{\Delta\tilde{Y}_t - \Delta\tilde{X}_t\}] = \exp\{\overline{a}\tilde{X}_t\Delta t\}\exp\{\sigma_{\overline{W}}^2/2\} \tag{25}$$

Which we have shown to be required to equal 1 under reasonable assumptions. The constraint of general parameters is then given by the form:

$$\exp\{\overline{a}\tilde{X}_t\Delta t\}\exp\{\sigma_{\overline{W}}^2/2\} = 1 \ \forall\tilde{X}_t,\ \Delta t > 0 \tag{26}$$

Which clearly implies $\overline{a} = \sigma_{\overline{W}}^2/2 = 0$.

5.3. Implications

We have shown that when the model is constrained to exhibit the desired characteristics, we require $(a_{11} - a_{21} + a_{12} - a_{22}) = 0$. We now investigate the implications of this constraint when $k_t \neq k$ and $\tilde{Y}_t \neq \tilde{X}_t$. Returning to our OU-process definition, defining $\Delta\tilde{D}_t = \Delta\tilde{Y}_t - \Delta\tilde{X}_t$, use $\tilde{Y}_t = \tilde{X}_t + \tilde{D}_t$ and noting the constraint we have proven above we obtain:

$$\begin{aligned}
\Delta\tilde{D}_t =& (a_{11} - a_{21})\tilde{X}_t\Delta t + (a_{12} - a_{22})\tilde{Y}_t\Delta t + \Delta\overline{W}_t \\
\Delta\tilde{D}_t =& (a_{12} - a_{22})\tilde{D}_t\Delta t + \Delta\overline{W}_t
\end{aligned} \tag{27}$$

Which we identify as a one dimensional OU-process when $a_{12} < a_{22}$. Furthermore, if $a_{12} >= a_{22}$ the process is not mean reverting, invalidating the model. Thus, under reasonable assumptions, the Two Process Model reduces to the One Process Model which implies it cannot model the underlying assets more accurately, as was to be shown.

6. Pairs Trading

Based on the aforementioned One Process Model we can set set up a simple trading algorithm based on the underlying implied OU process. We lay out such a model here and present results on historical data for different algorithm hyper-parameters in a parameter study.

6.1. Algorithm

From the deviation of the market ratio k_t to the historical average k, we can calculated the current value of the implied underlying OU-process, $\tilde{Y}$. As $\tilde{Y}$ increases, the probability distribution of $\Delta\tilde{Y}$ is shifted down due to the negative correlation of $\Delta\tilde{Y}$ with $\tilde{Y}$. This can be seen from the definition of the OU-process. Based on this knowledge a trading strategy can be devised: When $\tilde{Y}_t$ is positive, Y_t is overpriced with respect to X_t. Thus, at a certain threshold we short Y_t and long X_t. When the ratio shortens and crosses a lower threshold we close our position. Clearly, when $\tilde{Y}_t$ is negative we do the inverse. The algorithm performance is very sensitive to the thresholds chosen. If the threshold to entry is set too high very few trades can be carried out, and if the threshold is set too low more trades with a lower likelihood of success will be performed. Clearly, great care should be taken to minimize risk.

6.2. Parameter study

Maximizing profits shall not be the main achievement of this paper. We want to mention it here as a proof of concept and demonstrate the practical application of our model description to pairs trading. Still we want to present in this section an intuition on how our trading strategy performs on historical data with respect to the three free parameters. These are: the buying signal, the selling signal and the time window d for the calculation of the rolling mean. In general, there is no mathematical restriction on how these parameters shall be chosen. Our strategy to optimize hypothetical profits with our trading strategy is to subdivide the closing values of the stocks into a training and test set. The training set will consist of 0.7% of the original time series, i.e: closing values from 01.01.2010 - 01.01.2017. The test set is chosen to be the remaining values in the time series: 02.01.2017-01.01.2020.

To maximize hypothetical profits we use a numerical optimizer, namely the Nelder-Mead optimizer [6], which is easily accessible in Pythons scipy library. It yields the values of the three free parameters, for which the profits at the end of the simulation are the biggest.For the stocks in question (ADBE-CTAS), the optimization converges after 50 random walks to a buying signal of 0.016 and a selling signal of 0.104. The choice of these parameters indeed leads to a profit of 2388.1[EUR] in the training set. Note that the optimizers suggestion for the buying signal is smaller than the selling signal. This particular choice may be profitable for the particular

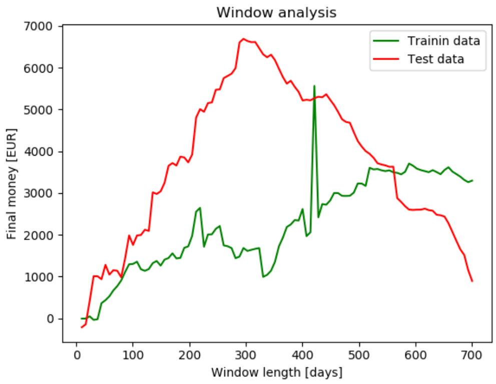

Figure 6: Profit analysis for the pairs trading model. The stocks under consideration are ADBE & CTAS for a time frame of 01.01.2010 - 01.01.2017 (training), 01.01.2017-01.01.2020 (test).

time series under consideration, but since it is not in accordance to our trading strategy we conclude, that a general purpose optimizer is of no practical use for our purpose. For further investigations we set these parameters to, buying signal = 0.05, selling signal = 0.01. This choice is in accordance with our theory and the model produces a comparable profit of 2211[EUR].

Analysing profits for the time window d fig. 6 reveals that profitable time windows for the training set $d \geq 550$ days may not lead to a comparably good test score. We like to use this analysis to make our reader aware of the pitfall of over fitting a model. The model parameters are tuned to deliver successful results on the training data but deliver a poor score on the test data. The best parameter choice for the training data does not uncover any general trend and can not be used for further investigations. However setting $100 \leq d \leq 300$ seems to be a reasonable choice as test and training analysis scores are high.

6.3. Concluding remarks

Because the trading strategy essentially works by assuming a predictable and therefore tradeable market event when implied OU process drifted far away from zero, it was thought that models with a stronger (more negative) correlation between $\Delta\tilde{Y}$ and $\tilde{Y}$ would perform better. To this end, the authors attempted to vary k so that the linear least squares fit parameter θ was maximised (in a negative sense). However, this model did not perform well at all. No explanation has been found for this phenomena, and it could pose an interesting avenue for further study.

7. Conclusion

This project shows a formal solution of the Ornstein-Uhlenbeck process, a stationary stochastic process with the key property of being mean reversive. Also in finance we can observe time series that carry this property. More precisely, statistical testing for co-integration and the Hurst exponent of 64 S&P500 trade stocks, justifies an OU description of the ration between a subset of the tested stocks and links the theoretical description to a practical phenomenon. We follow the authors [3] suggestion for a OU description of stock values, i.e: the One Process Model. Further more the authors suggestion of a second descriptive model of stock prices (Two Process Model) is shown to be equally descriptive as the One Process Model. Now, since a OU description of the ratio between stock values is justified and described by a suitable model, we infer the parameters for an OU description of the ratio between two well suited stocks with maximum likelihood estimators. This means, that given the historical data of stock values we can find an OU description of the data, that qualitatively agrees with the historical data. We formulate an algorithm that executes the pairs trading strategy on the historical data of two stocks. We demonstrate that this strategy exploits the previously formulated model description of stocks as OU process. Hypothetical profits of 2211[EUR] over a time period of 10 years proofs the concept of the used strategy. Even though these results sound promising for a trading strategy with minimal risk, the authors of this project want to warn the reader with the discussion about parameter study about careless use of the suggested strategy. Since a general purpose optimiser to infer the most profitable choice for the free trading parameters (buying-, selling threshold, time window) fails, we recommend to manually tune these parameters and find a suiting choice when comparing the trading strategy on a training and test set. We conclude that the analysis of time series data remains to be a challenging task. Still, an Ornstein-Uhlenbeck description between the ratio of two stocks seems to be promising because the pairs trading strategy, describing the ratio between two stocks as a stationary stochastic process, shows to be profitable.

.

References

[1] O.Vasicek, An equilibrium characterization of the terms structure, Journal of Financial Economics (5) (1977) 177–188.

[2] R. Engle, C. Granger (Eds.), Long-Run Economic Relationships: Readings in Cointegration, Oxford University Press, 1991.

[3] S.Rampertshammer, An ornstein-uhlenbeck framework for pairs trading, 2007.

[4] V.Fasen, Statistical estimation of multivariate ornstein–uhlenbeck processes and applications to co-integration, Journal of Econometrics 172 (2013) 325–337.

[5] J. J.Perktold, S.Seabold, statsmodels.tsa.stattools.coint. URL https://www.statsmodels.org/stable/generated/statsmodels.tsa.stattools.coint.html

[6] F. Gao, L. Han, Implementing the nelder-mead simplex algorithm with adaptive parameters, Computational Optimization and Applications 51 (2012) 259–277.