The Quest for Pi

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This article gives a brief history of the analysis and computation of the mathematical constant $\pi = 3.14159\ldots$, including a number of formulas that have been used to compute π through the ages. Some exciting recent developments are then discussed in some detail, including the recent computation of π to over six billion decimal digits using high-order convergent algorithms, and a newly discovered scheme that permits arbitrary individual hexadecimal digits of π to be computed.

For further details of the history of π up to about 1970, the reader is referred to Petr Beckmann's readable and entertaining book [3]. A listing of milestones in the history of the computation of π is given in Tables 1 and 2, which we believe to be more complete than other readily accessible sources.

The Ancients

In one of the earliest accounts (about 2000 B.C.) of π, the Babylonians used the approximation $3\frac{1}{8} = 3.125$. At this same time or earlier, according to an account in an ancient Egyptian document, Egyptians were assuming that a circle with diameter nine has the same area as a square of side eight, which implies $\pi = 256/81 = 3.1604\ldots$. Others of antiquity were content to use the simple approximation 3, as evidenced by the following passage from the Old Testament:

> Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (I Kings 7:23; see also 2 Chron. 4:2)

The first rigorous mathematical calculation of the value of π was due to Archimedes of Syracuse (~250 B.C.), who used a geometrical scheme based on inscribed and circumscribed polygons to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$, or in other words, $3.1408\ldots < \pi < 3.1428\ldots$ [11]. No one was able to improve on Archimedes's method for many centuries, although a number of persons used this general method to obtain more accurate approximations. For example, the astronomer Ptolemy, who lived in Alexandria in A.D. 150, used the value $3\frac{17}{120} = 3.141666\ldots$, and the fifth-century Chinese mathematician Tsu Chung-Chih used a variation of Archimedes's method to compute π correct to seven digits, a level not attained in Europe until the 1500s.

The Age of Newton

As in other fields of science and mathematics, progress in the quest for π in medieval times occurred mainly in the Islamic world. Al-Kashi of Samarkand computed π to 14 places in about 1430.

In the 1600s, with the discovery of calculus by Newton

TABLE 1. History of π Calculations (Pre-20th-Century)			
Babylonians	2000? B.C.E.	1	3.125 ($3\frac{1}{8}$)
Egyptians	2000? B.C.E.	0	3.16045 [$4(\frac{8}{9})^2$]
China	1200? B.C.E.	0	3
Bible (1 Kings 7:23)	550? B.C.E.	0	3
Archimedes	250? B.C.E.	3	3.1418 (ave.)
Hon Han Shu	A.D. 130	0	3.1622 (= $\sqrt{10}$?)
Ptolemy	150	3	3.14166
Chung Hing	250?	0	3.16227 ($\sqrt{10}$)
Wang Fau	250?	0	3.15555 ($\frac{142}{45}$)
Liu Hui	263	5	3.14159
Siddhanta	380	4	3.1416
Tsu Ch'ung Chi	480?	7	3.1415926
Aryabhata	499	4	3.14156
Brahmagupta	640?	0	3.162277 (= $\sqrt{10}$)
Al-Khowarizmi	800	4	3.1416
Fibonacci	1220	3	3.141818
Al-Kashi	1429	14	
Otho	1573	6	3.1415929
Viète	1593	9	3.1415926536 (ave.)
Romanus	1593	15	
Van Ceulen	1596	20	
Van Ceulen	1615	35	
Newton	1665	16	
Sharp	1699	71	
Seki	1700?	10	
Kamata	1730?	25	
Machin	1706	100	
De Lagny	1719	127	(112 correct)
Takebe	1723	41	
Matsunaga	1739	50	
Vega	1794	140	
Rutherford	1824	208	(152 correct)
Strassnitzky and Dase	1844	200	
Clausen	1847	248	
Lehmann	1853	261	
Rutherford	1853	440	
Shanks	1874	707	(527 correct)

TABLE 2. History of π Calculations (20th Century)		
Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and Wrench	Sep. 1947	808
Smith and Wrench	1949	1,120
Reitwiesner, *et al.* (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	4,194,288
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino, and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, *et al.*	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	June 1989	525,229,270
Kanada and Tamura	July 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	June 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938

and Leibniz, a number of substantially new formulas for π were discovered. One of them can be easily derived by recalling that

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$$

Substituting $x = 1$ gives the well-known Gregory–Leibniz formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

Regrettably, this series converges so slowly that hundreds of terms would be required to compute the numerical value of π to even two digits accuracy. However, by employing the trigonometric identity

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

(which follows from the addition formula for the tangent function), one obtains

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots\right)$$
$$+ \left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \cdots\right),$$

which converges much more rapidly. An even faster formula, due to Machin, can be obtained by employing the identity

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

in a similar way. Shanks used this scheme to compute π to 707 decimal digits accuracy in 1873. Alas, it was later found that this computation was in error after the 527th decimal place.