Quantum Advantage for Pathfinding in Regular Toroidal Sunflower Graphs

Master thesis

presented by

David Jesús Árbol Guerrero

Internship carried out at

University of Toronto

affiliation

under the supervision of

Nathan Wiebe

Master degree of Physics

Erasmus Mundus QUARMEN

Toronto, Canada - August 2024

Statement of Originality

Mr. David Jesús Árbol Guerrero

I hereby declare that the work presented as my Master's Thesis, corresponding to the academic year 2024-2025, is original in the sense that I have not used any sources without proper citation.

In Toronto, on September 8, 2025

Signed: David Jesús Árbol Guerrero

Abstract

One of the main objectives in quantum computing is to tackle computational problems with a superpolynomial quantum speedup. Realizing such an advantage requires not only identifying tasks that exhibit quantum advantage, but also understanding and leveraging the specific quantum resources responsible for it. In this work, we sought to build upon the exponential speedup achieved by Li and Tong [LT24], who presented a quantum algorithm for efficiently finding a path between two marked vertices in the regular sunflower graph. To extend their result, we introduced a new class of graphs, the toroidal sunflower graphs, which generalise the original structure by embedding it into a toroidal topology. We find show that this structure can be exploited to lead to a polynomial quantum advantage relative to classical approaches and provide an explicit quantum algorithm for finding the entry and exit vertices on the internal torus.

Contents

1 Introduction **5**

2 The Oracle Models **7**

3 Graph Definition **9**

4 Adjacency Matrix in the Supervertex Space **14**

5 Spectral Properties of the Effective Hamiltonian **19**

5.1 Eigenvalues and Eigenvectors 19

5.2 The Nullspace 22

5.3 Spectral Gap of the Adjacency Matrix 28

5.3.1 Preliminary Results 28

5.3.2 Case 1: Non-zero Eigenvalue 32

5.3.3 Case 2: Eigenvalue Equals Zero 33

5.3.4 General Result for the Spectral Gap of the Adjacency Matrix 33

6 Overlap of the Projected Entry Vertex with the 0-eigenspace **34**

7 The Algorithm **40**

8 The Classical Lower Bound **51**

9 Conclusions **55**

Appendix. Coupon Collector's Problem 56

Chapter 1

Introduction

Quantum computing is a new computational paradigm that leverages the principles of quantum mechanics to approach problems in fundamentally different ways. This makes quantum algorithms particularly well-suited for certain tasks where they outperform classical algorithms, while being inefficient for others that are easily solvable by classical means. For this reason, finding algorithms presenting a superpolynomial advantage for some problem is critical while the technology for building quantum processors is developed. However, this is a tough daunting tasks due to the counterintuitive nature of quantum mechanics. There are a handful of fundamental quantum algorithms over which new quantum algorithms are built.

In this context, finding quantum algorithms that provide an exponential advantage over their classical counterparts is particularly challenging. Graph theory has emerged as a promising avenue for achieving such speedups, due to the ability to define families of graphs whose size grows exponentially with certain parameters. A paradigmatic example is the *welded tree* graph, which has been used to demonstrate exponential quantum advantage.

In [CCD$^+$03], Childs *et al.* introduce the welded tree graph as two binary trees of equal height connected by a random cycle that alternates between the leaves of each tree. The roots of the trees are referred to as the *entrance* (s) and *exit* (s') vertices, which are distinguished from the rest by having degree two. The total size of the graph grows exponentially with the height of the trees, and traversing it classically is exponentially hard. However, due to the existence of an equitable partition that includes $\{s\}$ and $\{s'\}$, a continuous-time quantum walk can efficiently traverse the graph.

Recently, Jianqiang Li and Yu Tong, inspired by the work of Childs *et al.*, designed a quantum pathfinding algorithm on regular sunflower graphs that achieves exponential advantage [LT24]. They define this family of graphs consisting of an n-cycle whose vertices serve as the roots of $(d-1)$-ary trees. The leaves of each tree are connected to those of adjacent trees via random perfect matchings, with an equal number of edges connecting to the left and right neighbouring trees. The *entrance* vertex is a designated root, while the *exit* vertex is an unknown root located diametrically opposite in the cycle. This structure induces an equitable partition similar to that of the welded tree graph, which can be leveraged to construct the algorithm.

By using singular value transformation [GSLW19] to perform the projection, and fixed-point amplitude amplification to reduce the overhead, the authors efficiently project the entrance vertex onto the nullspace of the adjacency matrix. The resulting quantum state encodes sufficient information to identify a subgraph that contains both the entrance and exit vertices, as well as a path connecting them. This subgraph can then be explored efficiently using a classical Breadth-First Search algorithm.

In this work, we extend the result of Li and Tong to a general family of graphs: the *toroidal sunflower graphs* with the aim of understanding the form of quantum speedup that is attainable with quantum and classical algorithms. We provide a quantum algorithm that provides an exponential speedup with respect to a naïve classical walk but also provide a new classical walk algorithm relative to which our quantum algorithm provides a polynomial advantage in queries to the graph. Our generalization of the sunflower graphs graphs consist of a q-dimensional torus, where the cycle length in each dimension is specified by a set of integers $\{n_k\}_{k\in[q]}$. Each vertex in the torus is identified by a tuple $i = (i_1, \ldots, i_q)$, with $i_k \in [n_k]$ for all $k \in [q]$, and corresponds to the root of a tree of fixed height. The leaves of each tree are connected to those of neighbouring trees via random perfect matchings. In particular, for fixed values of all but one coordinate, the subgraph consisting of trees with roots $(i_1, \ldots, i_k, \ldots, i_q)$, where $i_k \in [n_k]$, and their connecting edges, forms a regular sunflower graph.

Chapter 2

The Oracle Models

Let $G = (V, E)$ be a multigraph, which is a graph that can contain multiple edges adjacent to the same two vertices, with $|V| = N$ vertices and bounded degree $d = \mathcal{O}(1)$. Each vertex is uniquely identified by a bit string of length $\mathfrak{n} = \lceil \log_2 N \rceil$. The goal of this chapter is to define a pathfinding problem on a graph and provide oracles that define the input to this problem. To that end, we rely on oracles that reveal only local information of the graph, as access to the global structure would render the problem trivial. Consequently, the problem must be approached as a random walk.

Local access is provided via both classical and quantum oracles, defined analogously to those in [LT24]. These are necessary to define the problem for both quantum and classical computers.

Definition 1 (The Classical Adjacency List Oracle [LT24])**.** *Let $G = (V, E)$ be an undirected multigraph of degree $d = \mathcal{O}(1)$, $V = \{1, \ldots, N\}$, $\mathfrak{n} = \lceil \log(N) \rceil$. For any $v \in V$ and $k \in \{1, 2, \ldots, d\}$, the variable k designates the k-th neighbour of v, using the natural order of bit-strings without counting the multiplicity. The following classical oracles are defined as follows*

i) $O_{G,1}(v, k)$: Returns the k-th neighbour of the vertex $v \in V$. If this neighbour does not exist, then $O_{G,1}(v, k) = k + 2^{\mathfrak{n}}$.

ii) $O_{G,2}(v, v')$: Returns the multiplicity of the edge $\{v, v'\}$ given that $v, v' \in V$.

Next we introduce our definitions of the quantum analogues of the above classical oracles.

Definition 2 (The Quantum Oracles [LT24])**.** *Let $G = (V, E)$ be an undirected multigraph of degree $d = \mathcal{O}(1)$, $V = \{1, \ldots, N - 1\}$, $\mathfrak{n} = \lceil \log_2(N) \rceil$. The quantum oracles $O^Q_{G,1}$, $O^Q_{G,2}$ are the unitary operators defined as follows: for any $v, v' \in V$, $k \in \{0, 1 \ldots, N - 1\}$,*

$$O^Q_{G,1} |v, k, c\rangle = |v, k, c \oplus O_{G,1}(v, k)\rangle , \quad O^Q_{G,2} |v, v', c\rangle = |v, v', c \oplus O_{G,2}(v, v')\rangle , \tag{2.1}$$

where $O_{G,1}$, $O_{G,2}$ are defined in Definition 1, c is a bit-string in $\{1, \ldots, \mathfrak{n}\}$, and $\oplus$ denotes bit-wise addition modulo 2.

The adjacency matrix is the standard representation of a graph and serves as a suitable formalism for describing and analysing random walks. In the quantum formalism, we will identify a qubit with a vertex and define a quantum random walk by implementing the adjacency matrix operator. However, to implement non-unitary operators in a quantum processor we need the block encoding algorithm. We define a block encoding, which is a means of expressing an arbitrary matrix as a sub-block of a larger unitary matrix.

Definition 3 (Block Encoding)**.** *Let A be a n-qubit operator. Any unitary matrix U_A acting on $m+n$ qubits is said to be a (α, m, ϵ)-block encoding of A on registers β_1 and β_2 if*

$$||A - \alpha(\langle 0^m|_{\beta_1} \otimes I_{\beta_2})U_A(|0^m\rangle_{\beta_1} \otimes I_{\beta_2})|| \leq \epsilon, \tag{2.2}$$

where α is a subnormalisation factor.

Using this formalism, let $|\psi\rangle$ be an n-qubit state. Then,

$$U_A(|0^m\rangle_{\beta_1} \otimes |\psi\rangle) = |0^m\rangle \otimes A/\alpha\,|\psi\rangle + |\perp\rangle\,, \tag{2.3}$$

where

$$(|0^m\rangle\,\langle 0^m| \otimes I_n)\,|\perp\rangle = 0. \tag{2.4}$$

That means that measuring $|0^m\rangle$ in the first register projects the second register onto $A\alpha\,|\psi\rangle$, and this outcome ocurrs with probability

$$p(0^m) = \frac{1}{\alpha^2}\,\|A\,|\psi\rangle\|^2\,. \tag{2.5}$$

The following Lemma, taken from [LT24], presents a method to construct the adjacency matrix using the quantum oracles. It specifies the query complexity, as well as the number of additional elementary gates and ancilla qubits required for its implementation.

Lemma 4 (Block Encoding of an Undirected Multigraph Adjacency Matrix [LT24])**.** *Let $G = (V, E)$ be an undirected multigraph of degree d, $V = \{0, \ldots, N-1\}$, and $\mathfrak{n} = \lceil \log_2(N) \rceil$. Assume that its adjacency matrix is sparse. Then, a $(d^2, \mathfrak{n}+3, \epsilon_A)$-block encoding of the adjacency matrix A of the multigraph can be constructed using $\mathcal{O}(1)$ queries to the adjacency list oracle in Definition 2, $\mathcal{O}(\mathfrak{n} + \log^{2.5}(d/\epsilon_A))$ additional elementary gates, and $\mathcal{O}(log^{2.5}(d/\epsilon_A))$ ancilla qubits.*

Proof. See [LT24], Theorem 2.12. □

In order to find a path between the entrance and exit vertices we need to have a method for computing whether a vertex is either the entrance or exit. We define such an oracle in the classical case below. The quantum case is simply the reversible implementation of this oracle.

Definition 5 (Exit Oracle)**.** *Given a hidden vertex for the exit vertex, $s' \in V$, the classical exit oracle is defined to be:*

$$f_{s'}(v) = \begin{cases} 1 & \textit{if } v = s', \\ 0 & \textit{otherwise}, \end{cases} \quad \forall\ v \in V. \tag{2.6}$$

This work addresses the problem of finding a path between two designated vertices $s, s' \in V$, referred to as the entrance and the exit, in the Toroidal Sunflower Graph. The definition of this graph is provided in the following chapter.

Chapter 3

Graph Definition

Our aim in this chapter is provide a discussion of the sunflower graph as well as our extension to the toroidal analogue that we consider in this work. We will first begin with a description of the sunflower graph from the work of Lin and Tong [LT24].

Definition 6 (The Regular Sunflower Graph)**.** *Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ be rooted trees of height $m \geq 2$. In each tree $\mathcal{T}_i$, the root vertex has degree $d-2$, and all other internal vertices have degree $d \in \mathbb{Z}^+$. The roots $s_1, s_2, \ldots, s_n$ of trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ are connected in a cycle by the edges $\{s_1, s_2\}$, $\{s_2, s_3\}$, $\ldots$, $\{s_{n-1}, s_n\}$, $\{s_n, s_1\}$. The leaf vertices of each tree $\mathcal{T}_i$ are connected to the leaves of $\mathcal{T}_{i+1}$ for $i \in \{1, \ldots, n-1\}$, and similarly between $\mathcal{T}_n$ and $\mathcal{T}_1$. For each pair of adjacent trees, $(d-1)/2$ random perfect matchings are selected between their leaf vertices, and the corresponding edges are added to the graph. The resulting multigraph is defined to be a d-regular sunflower graph $G = (V, E)$ (see Figure 3.1).*

Next we will consider extending this notion by noting that the central cycle used in the sunflower graph is essentially just the 1-torus. Here we consider a generalization of this central cycle to that of the q-torus.

Definition 7 (Regular Toroidal Sunflower Graph)**.** *Let $\mathcal{T}_{i_1,\ldots,i_q}$ be rooted trees of height $m \geq 2$, where $i_k \in \{1, 2, \ldots, n_k\}$ for all $k \in \{1, \ldots, q\}$, $q \in \mathbb{Z}^+$. The root of each tree has degree $d-2q$, and all other internal vertices have degree $d \in \mathbb{Z}^+$, where $d \geq 2q+1$ and $d-1$ is divisible by $2q$. Denote the root vertex of $\mathcal{T}_{i_1,\ldots,i_q}$ by $s_{i_1,\ldots,i_q}$. These roots are connected by $2q$ edges forming a q-dimensional torus with an adjacency matrix*

$$A_{(i_1,\ldots,i_q),(t_1,\ldots,t_q)} = \begin{cases} 1 & \text{if } \exists!\ k_0 \in [q] \text{ s.t. } i_k = t_k \text{ for all } k \neq k_0, \\ & \text{and } i_{k_0} \equiv t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The leaf vertices of each tree $\mathcal{T}_{i_1,\ldots,i_{k_0},\ldots,i_q}$ are connected to the corresponding leaves of the neighbouring trees $\mathcal{T}_{i_1,\ldots,i_{k_0}\pm 1 \pmod{n_{k_0}},\ldots,i_q}$ (i.e., the trees connected via the toroidal structure at the root level). Specifically, for each such pair of neighbouring trees, $\frac{d-1}{2q}$ random perfect matchings are selected between their leaf vertices, and the corresponding edges are added to the graph. This construction yields a d-regular graph.

Notation 8. *For simplicity, $i = (i_1, \ldots, i_q)$ denotes the number list identifying a tree in the graph with $i_k \in [n_k], \forall k \in [q]$.*

Definition 9 (Variables Defining a Valid Toroidal Sunflower Graph)**.** *Constraints on the variables defining the graph are imposed to ensure the correct functioning of the pathfinding algorithm. The reasons for these constraints are explained throughout the thesis.*

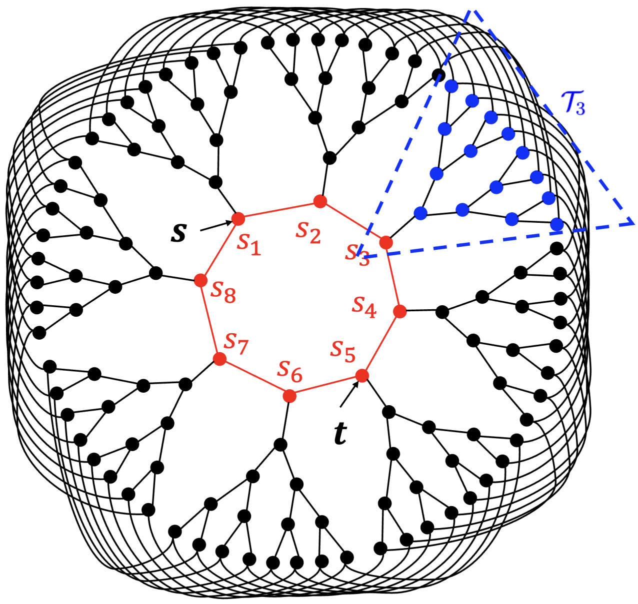

Figure 3.1: An example of the regular sunflower graph G from Definition 6 with $d = 3$, $m = 5$, $n = 8$. The s and t vertices are marked out. The tree within the dashed rectangle is the subtree Ti (in this instance $i = 3$). The leaves of the trees T_i are connected via $(d-1)/2$ random perfect matchings. We note that we only prove the expansion property for $d \geq 7$, and $d = 3$ is chosen here for visual clarity. Figure from [LT24].

i) $n_k = 2p_k$ *for all* $k \in [q]$*: Each* $n_k \in \mathbb{Z}^+$ *is the number of vertices along dimension* k *of the torus and must be even. The integers* $p_1, \dots, p_q$ *are pairwise coprime. Here,* q *is the number of torus dimensions.*

ii) $m \in \mathbb{Z}^+$ *is an odd number that defines the number of layers in the trees* $\mathcal{T}_i$ *(so that the tree height is* $m-1$*).*

iii) A valid toroidal sunflower graph requires that $d \geq 2q + 1$*, where* d *is the total degree of each vertex. This ensures that each root vertex is not only connected to the other root vertices in the toroidal structure, but also supports a tree growing from it.*

iv) Additionally, $d-1$ *must be a multiple of* $2q$ *to ensure that the leaf vertices connect evenly to the neighbouring trees.*

Next, we provide the definition of the pathfinding problem on the specific graph introduced above.

Definition 10 (Pathfinding Problem in the Toroidal Sunflower Graph)**.** *Let* $G = (V, E)$ *be a valid Toroidal Sunflower Graph as defined in Definition 7 and satisfying the additional assumptions of Definition 9. Let* $s := s_{1,\dots,1} \in V$ *denote the root vertex of the tree* $\mathcal{T}_{1,\dots,1}$*, and* $s' := s_{\frac{n_1}{2}+1,\dots,\frac{n_q}{2}+1} \in V$ *denote the root vertex of the tree* $\mathcal{T}_{\frac{n_1}{2}+1,\dots,\frac{n_q}{2}+1}$*.*

Given as input the quantum oracles from Definition 2, the entrance vertex $s \in V$*, and a classical oracle* $f_{s'}$ *with* $s' \in V$ *the exit vertex, the pathfinding problem consists of computing a path from* s *to* s' *in* G*.*

We now introduce the concepts of supervertices and superedges, adapted from [LT24], in the context of the toroidal sunflower graph. These differ from the standard definitions and form an equitable partition of the graph. This framework allows us to describe the states of the walk as linear combinations of these supervertices, as will be illustrated throughout this chapter. We also establish several key properties of these structures.

Concerning the supervertices:

Definition 11 (Sunflower Supervertex)**.** *In a toroidal sunflower graph (as defined in Definition 7), a* supervertex *is defined as the set of all vertices located at the same level of a given tree. More precisely, the supervertex S_{ji} is the set of vertices at level j in the tree $\mathcal{T}_i$, where $i = (i_1, \ldots, i_q) \in [n_1] \times \cdots \times [n_q]$.*

The collection of all such S_{ji} forms a disjoint partition of the vertex set of the graph. The cardinality of a supervertex, denoted $s_{ji} = |S_{ji}|$, represents the number of vertices it contains.

The cardinalities of the supervertices will be used to construct a reduced adjacency matrix in the supervertex basis. This representation plays a key role in enabling the quantum advantage.

Lemma 12 (Cardinality of a Supervertex)**.** *Given Definition 11 of a sunflower supervertex, the number of vertices in the supervertex S_{ji}, with $j \in [m]$ and $i = (i_1, \ldots, i_q) \in [n_1] \times \cdots \times [n_q]$, is*

$$s_{ji} = \begin{cases} 1 & \text{if } j = 1 \\ (d-2q)(d-1)^{j-2} & \text{if } j \geq 2 \end{cases} \tag{3.2}$$

Proof. Our proof of this claim is inductive. There is an only vertex at the first level ($j = 1$) by definition, the root of the tree. Since each root vertex is connected to $2q$ other root vertices, there are $d - 2q$ vertices at level $j = 2$ connected to the root. Therefore,

$$s_{j=2,i} = d - 2q.$$

Each vertex at level $j = 3$ is connected to exactly one vertex in the adjacent lower level. Since each vertex has degree d, there are $d - 1$ outgoing edges per vertex at this level of the tree. Hence,

$$s_{j=3,i} = (d-2q)(d-1).$$

Let us assume that for $3 \leq j < m$ the following holds:

$$s_{j,i} = (d-2q)(d-1)^{j-2},$$

which is satisfied for $j = 3$.

Again, each vertex at level j is connected to exactly one vertex in the adjacent lower level. Since they have degree d, each vertex branches into $d - 1$ vertices at level $j + 1$ in the tree. Therefore,

$$s_{j+1,i} = s_{j,i}(d-1) = (d-2q)(d-1)^{(j+1)-2},$$

completing the proof by induction. □

From the previous lemma, the cardinality of the toroidal sunflower graph can be readily computed. Its exponential dependence on mmm implies that, once the random walk exits the internal torus, any classical algorithm will face an exponentially small probability of returning to it (see Chapter 8).

Lemma 13 (Cardinality of the Toroidal Sunflower Graph)**.** *The total number of vertices in a valid sunflower graph of degree d and the graph is*

$$N_G = n_1 \cdots n_q \cdot |\mathcal{T}_i|, \tag{3.3}$$

where

$$|\mathcal{T}_i| = 1 + \frac{d-2q}{d-2}((d-1)^{m-1} - 1),\ \forall i = (i_1, \dots, i_q) \in [n_1] \times \cdots \times [n_q]. \tag{3.4}$$

Proof. It is clear that $N_G = n_1 \cdots n_q \cdot |\mathcal{T}_i|$. According to the Lemma 12

$$|\mathcal{T}_i| = \sum_{j=1}^{m} s_{ji} = 1 + \sum_{j=0}^{m} (d-2q)(d-1)^{j-2}. \tag{3.5}$$

Since $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$, one obtains

$$|\mathcal{T}_i| = 1 + \frac{d-2q}{d-2}\left[(d-1)^{m-1} - 1\right]. \tag{3.6}$$

□

With respect to the superedges, we define and derive their cardinality hereunder.

Definition 14 (Sunflower Superedge)**.** *In a toroidal sunflower graph (as defined in Definition 7), a* superedge $\mathcal{E}_{ji,rt}$ *is defined as the set of edges connecting the supervertices* S_{ji} *and* S_{rt}*, where* $i = (i_1, \dots, i_q)$, $t = (t_1, \dots, t_q) \in [n_1] \times \cdots \times [n_q]$*, and* $j, r \in [m]$*.*

The set of all such $\mathcal{E}_{ji,rt}$ *forms a disjoint partition of the edge set of the graph. The number of edges in the superedge* $\mathcal{E}_{ji,rt}$ *is denoted by*

$$e_{ji,rt} = |\mathcal{E}_{ji,rt}|.$$

Lemma 15. *Let* $\mathcal{E}_{ji,rt}$ *be a sunflower superedge as defined in Definition 14, with* $j, r \in [m]$ *and* $i, t \in [n_1] \times \cdots \times [n_q]$*. Then, the number of edges in* $\mathcal{E}_{ji,rt}$ *is given by*

$$e_{ji,rt} = \begin{cases} 1 & \text{if } j = r = 1, \text{ and } \exists!\ k_0 \in [q] \text{ s.t.} \\ & \quad i_k = t_k\ \forall k \neq k_0, \text{ and } i_{k_0} \equiv t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ \max\{s_{ji}, s_{rt}\} & \text{if } |j-r| = 1 \text{ and } i = t, \\ \frac{d-1}{2q}(d-2q)(d-1)^{m-2} & \text{if } j = r = m, \text{ and } \exists!\ k_0 \in [q] \text{ s.t.} \\ & \quad i_k = t_k\ \forall k \neq k_0, \text{ and } i_{k_0} \equiv t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Proof. We break our proof down into three cases

i) The first case involves the edges between two root vertices ($j = r = 1$). There is a single edge between each pair of adjacent root vertices, where adjacency is defined by $i_{k_0} = t_{k_0} \pm 1 \pmod{n_{k_0}}$, describing a circle. Therefore, $e_{ji,rt} = 1$.

ii) In the second case, $j = r \pm 1$ and $i_k = t_k$ for all $k \in [q]$. Hence, the superedge corresponds to edges between two adjacent levels within the same tree. Each vertex in the higher-level supervertex is connected by a single edge to a vertex in the lower-level one. Thus, $e_{ji,rt}$ equals the number of vertices in the higher supervertex:

$$e_{ji,rt} = \max\{s_{ji}, s_{rt}\}.$$

iii) In the third case, $j = r = m$ and $i_k = t_k$ for all $k \in [q] \setminus \{k_0\}$ with $i_{k_0} = t_{k_0} \pm 1 \pmod{n_{k_0}}$.

The cardinality of the superedge is the number of edges between the highest level m of two adjacent trees. The number of vertices at level m is $(d-2q)(d-1)^{m-2}$, and each of these vertices is connected to $2q$ adjacent trees via $d-1$ edges, distributed equally. Therefore, each vertex contributes with $\frac{d-1}{2q}$ edges to a specific adjacent tree. The total number of edges is, then,

$$e_{ji,rt} = (d-2q)(d-1)^{m-2}\,\frac{d-1}{2q}.$$

□

Chapter 4

Adjacency Matrix in the Supervertex Space

In this chapter, we calculate the adjacency matrix of the toroidal sunflower graph in the supervertex basis, making use of the properties introduced earlier. First, we describe how the graph is represented in computational terms and define the notion of a supervertex state.

Definition 16 (Computational Representation of a Graph). *Each vertex in the graph is identified with a bit string of length $\lceil \log_2(|V|) \rceil$. The edge structure is specified by the adjacency matrix $\{A_{uv}\}_{u,v \in V}$, where $A_{uv} = 1$ if there exists an edge between vertices u and v, and $A_{uv} = 0$ otherwise (including the case where u or v are not valid vertices).*

Definition 17 (Supervertex state). *Given a supervertex S_{ji} in the toroidal sunflower graph (see Definition 11), the corresponding supervertex state is defined as the uniform quantum superposition over all vertices contained in S_{ji}:*

$$|S_{ji}\rangle = \frac{1}{\sqrt{s_{ji}}} \sum_{v \in S_{ji}} |v\rangle, \tag{4.1}$$

where $s_{ji} = |S_{ji}|$ denotes the number of vertices in the supervertex.

Lemma 18 (Properties of the Adjacency Matrix). *The adjacency matrix $\{A_{u,v}\}_{u,v \in V}$ of the toroidal sunflower graph satisfies the following properties:*

i) $e_{ji,tr} = \sum_{u \in S_{ji}} \sum_{v \in S_{rt}} A_{uv}$.

ii) $\forall u \in S_{ji}$, $\sum_{v \in S_{rt}} A_{uv} = \frac{e_{ji,rt}}{s_{ji}}$.

iii) $A|S_{rt}\rangle = \sum_{j=1}^{m} \sum_i \frac{e_{ji,rt}}{\sqrt{s_{ji} s_{rt}}} |S_{ji}\rangle$.

Proof. i) By definition of $e_{ji,rt}$.

ii) Every vertex in S_{ji} has the same number of edges connecting it with a vertex of S_{rt}. Then,

$$e_{ji,tr} \overset{i)}{=} \sum_{u \in S_{ji}} \sum_{v \in S_{rt}} A_{uv} = s_{ij} \sum_{v \in S_{rt}} A_{uv}. \tag{4.2}$$

iii) Using the previous result and the definition of the sunflower supervertex state,

$$A|S_{rt}\rangle = \frac{1}{\sqrt{s_{rt}}} \sum_{v \in S_{rt}} A|v\rangle = \frac{1}{\sqrt{s_{rt}}} \sum_{v \in S_{rt}} \sum_{u \in V} A_{uv}|u\rangle = \tag{4.3}$$

$$= \sum_{ji} \frac{1}{\sqrt{s_{rt}}} \sum_{u \in S_{ji}} \sum_{v \in S_{rt}} A_{uv}|u\rangle \overset{ii)}{=} \sum_{ji} \frac{1}{\sqrt{s_{rt}}} \sum_{u \in S_{ji}} \frac{e_{ji,rt}}{s_{ji}} |u\rangle = \sum_{ji} \frac{e_{ji,rt}}{\sqrt{s_{ji} s_{rt}}} |S_{ji}\rangle \,. \tag{4.4}$$

□

Defining the adjacency matrix of the graph in the supervertex state decreases its degrees of freedom, making the quantum walk computationally efficient.

Definition 19 (Supervertex space or symmetric space)**.** *The supervertex space is given by the orthonormal basis of all the supervertex states*

$$\mathrm{span}(\{|S_{ji}\rangle : j \in [m],\ i \in [n_1] \times \cdots \times [n_q]\}). \tag{4.5}$$

It is also called the symmetric space, since it is invariant under the adjacency matrix, and we denote it by $\mathcal{S}$.

Remark 20. *From Lemma 18, it is concluded that the supervertex space is invariant under the adjacency matrix.*

Lemma 21 (Restriction of the Adjacency Matrix to the Supervertex Space)**.** *The restriction of the adjacency matrix to the supervertex space yields the matrix elements*

$$\langle S_{ji}|A|S_{rt}\rangle = \begin{cases} 1 & \text{if } j,r = 1 \text{ and } \exists!\ k_0 \in [q] \text{ s.t. } i_k = t_k\, \forall k \neq k_0, \\ & \quad \text{and } i_{k_0} = t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ \frac{d-1}{2q} & \text{if } j,r = m \text{ and } \exists!\ k_0 \in [q] \text{ s.t. } i_k = t_k\, \forall k \neq k_0, \\ & \quad \text{and } i_{k_0} = t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ \sqrt{d-1} & \text{if } j = r \pm 1,\ \{j,r\} \neq \{1,2\} \text{ and } i_k = t_k\, \forall k \in [q], \\ \sqrt{d-2q} & \text{if } \{j,r\} = \{1,2\} \text{ and } i_k = t_k\, \forall k \in [q], \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. From Lemma 18 *ii)*,

$$\langle S_{ji}|A|S_{rt}\rangle = \frac{e_{ji,rt}}{\sqrt{s_{ji} s_{rt}}}. \tag{4.7}$$

We now verify each case in the expression above:

i) For the matrix elements between two adjacent supervertices at the root level, the supervertex and superdege cardinalities are $s_{ji} = s_{rt} = e_{ji,rt} = 1$. Hence, $\langle S_{ji}|A|S_{rt}\rangle = 1$.

ii) For two supervertices at the highest level that are in adjacent sunflowers, $s_{ij} = s_{rt} = (d-2q)(d-1)^{m-2}$. Moreover, $e_{ji,rt} = \frac{d-1}{2q}(d-2q)(d-1)^{m-2}$, since there are $(d-1)/2q$ perfect matchings between the leaves of distinct trees. Substituting into the formula yields $\langle S_{ji}|A|S_{rt}\rangle = \frac{d-1}{2q}$.

iii) When the two supervertices are in the same tree and at adjacent levels different from the first two, $e_{ji,rt} = \max\{s_{ij}, s_{rt}\}$. Assuming $j = r+1$, it follows that $e_{ji,rt} = s_{ji} = (d-2q)(d-1)^{j-2}$ and

$$\langle S_{ji}|A|S_{rt}\rangle = \sqrt{\frac{s_{ji}}{s_{rt}}}. \tag{4.8}$$

Substituting $s_{(r+1)i}$ for the numerator and s_{rt} for the denominator yields the desired result.

iv) The remaining case corresponds to adjacent supervertices at levels 1 and 2 of the same tree. Since $j = r+1$, $\langle S_{ji}|A|S_{rt}\rangle = \sqrt{\frac{s_{ji}}{s_{rt}}}$, but $s_{1t} = 1$ and $s_{2i} = d-2q$, and therefore, $\langle S_{ji}|A|S_{rt}\rangle = \sqrt{d-2q}$.

□

The goal of this work is to study the nullspace of the adjacency matrix. Projecting the starting state $|s\rangle$ onto the nullspace can yield information about a path that connects the entry and exit state through the internal q-dimensional torus. To achieve this, it is convenient to express the adjacency matrix in the supervertex basis, and define a Hamiltonian as the adjacency matrix restricted to this space. Following Lemma 18, one can define:

Definition 22 (Hamiltonian of the Regular Toroidal Sunflower Graph)**.** *The Hamiltonian of the Toroidal Sunflower Graph is defined to be its adjacency matrix restricted to the supervertex space:*

$$H_{ji,rt} = \begin{cases} 1, & \text{if } j = r = 1 \text{ and } \exists!\ k_0 \in [q] \text{ s.t.} \\ & \quad i_k = t_k\ \forall k \neq k_0,\ \text{and } i_{k_0} \equiv t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ \frac{d-1}{2q}, & \text{if } j = r = m \text{ and } \exists!\ k_0 \in [q] \text{ s.t.} \\ & \quad i_k = t_k\ \forall k \neq k_0,\ \text{and } i_{k_0} \equiv t_{k_0} \pm 1 \pmod{n_{k_0}}, \\ \sqrt{d-1}, & \text{if } |j-r| = 1,\ \{j,r\} \neq \{1,2\} \text{ and } i_k = t_k\ \forall k \in [q], \\ \sqrt{d-2q}, & \text{if } \{j,r\} = \{1,2\} \text{ and } i_k = t_k\ \forall k \in [q], \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

Using the Hamiltonian of the graph offers the advantage that it operates on a Hilbert space of dimension $m\prod_{k=1}^{q} n_k$, whereas the adjacency matrix A acts on a space of dimension

$$|N_G| = \prod_{k=1}^{q} n_k \left[1 + \frac{d-2q}{d-2}\left((d-1)^{m-1} - 1\right)\right], \tag{4.10}$$

as established in Lemma 13. Consequently, the Hamiltonian can be represented by a matrix of size $(m\prod_{k=1}^{q} n_k) \times (m\prod_{k=1}^{q} n_k)$. It will be shown that, in this representation, the Hamiltonian admits a decomposition as the tensor product of an $m \times m$ sparse matrix with other elementary matrices. To this end, we introduce an isometry that eases the notation.

Definition 23 ($m\prod_{k=1}^{q} n_k$-dimensional Matrix Representation Isometry of the Effective Hamiltonian)**.** *It is the isometry $V_{\mathcal{S}} : \mathbb{C}^m \otimes \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_q} \to \mathcal{S}$ defined as*

$$V_{\mathcal{S}}\left(\sum_{j=1}^{m}\sum_{i} \alpha_{ji}\,|S_{ji}\rangle\right) = \sum_{j=1}^{m}\sum_{i} \alpha_{ji}\,|b_j\rangle|a_{i_1}\rangle\cdots|a_{i_q}\rangle, \tag{4.11}$$

where $\{|b_j\rangle|a_{i_1}\rangle\cdots|a_{i_q}\rangle\}$ is an orthonormal basis and $\mathcal{S}$ denotes the symmetric subspace. The Hamiltonian expressed in this basis is referred to as the effective Hamiltonian. *It is denoted by $\tilde{H} = V_{\mathcal{S}}^{-1} H V_{\mathcal{S}}$, where H is the Hamiltonian from Definition 22.*

Finally, the effective Hamiltonian of the graph is defined as the Hamiltonian in the new basis $\{|b_j\rangle\,|a_i\rangle \mid j \in [m],\, i \in [n_1] \times \ldots \times [n_q]\}$.

Definition 24 (Effective Hamiltonian of the Toroidal Sunflower Graph in Terms of Cycle and Path Matrices)**.** *The Hamiltonian in the matrix representation can also be expressed in terms of the adjacency matrices of n_k-cycle graphs $D_0^{(n_k)}$, $\forall k \in [q]$, and the weighted path graph of m vertices D_1.*

$$\tilde{H} = (|b_1\rangle\langle b_1| + \gamma|b_m\rangle\langle b_m|) \otimes \Big((D_0^{(n_1)} \otimes I_{n_2} \otimes \cdots \otimes I_{n_q}) + \ldots + (I_{n_1} \otimes \cdots \otimes I_{n_{q-1}} \otimes D_0^{(n_q)})\Big) + \\ + D_1 \otimes I_{n_1} \otimes \cdots \otimes I_{n_q}, \tag{4.12}$$

where

$$D_0^{(n_k)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{n_k \times n_k}, \tag{4.13}$$

$$D_1 = \begin{bmatrix} 0 & w_1 & 0 & 0 & \ldots & 0 \\ w_1 & 0 & w_2 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & w_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & w_{m-2} & 0 & w_{m-1} \\ 0 & 0 & 0 & 0 & w_{m-1} & 0 \end{bmatrix}_{m \times m}, \tag{4.14}$$

and

- $\gamma = \frac{d-1}{2q}$,
- $w_1 = \sqrt{d-2q}$,
- $w_2 = \ldots = w_{m-1} = \sqrt{d-1}$,
- $n = \prod_k n_k$.

Remark 25. *The effective Hamiltonian can also be expressed as*

$$\tilde{H} = \begin{bmatrix} a & w_1 & 0 & 0 & \ldots & 0 \\ w_1 & 0 & w_2 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & w_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & w_{m-2} & 0 & w_{m-1} \\ 0 & 0 & 0 & 0 & w_{m-1} & b \end{bmatrix}, \tag{4.15}$$

where

- $a \equiv (D_0^{(n_1)} \otimes I_{n_2} \otimes \cdots \otimes I_{n_q}) + \ldots + (I_{n_1} \otimes \cdots \otimes I_{n_{q-1}} \otimes D_0^{(n_q)})$,
- $b \equiv \gamma a$,
- $\gamma = \frac{d-1}{2q}$,

- $w_1 = \sqrt{d-2q}\ I_n$,
- $w_2 = \ldots = w_{m-1} = \sqrt{d-1}\ I_n$,
- $n = \prod_k n_k$.

Chapter 5

Spectral Properties of the Effective Hamiltonian

In this chapter, we study the spectral properties of the graph, and establish its nullspace and spectral gap.

5.1 Eigenvalues and Eigenvectors

The spectral properties of the n-cycle matrix $D_0^{(n)}$ are showed below.

Property 26 (Spectral Properties of $D_0^{(n)}$). *The eigenvalues of the n-cyclic matrix $D_0^{(n)}$ are*

$$\mu_l = 2\cos\left(\frac{2\pi l}{n}\right), \quad l \in [n]. \tag{5.1}$$

The respective eigenvectors of $D_0^{(n)}$ are $|\phi_l\rangle$

$$v_l = \frac{1}{\sqrt{n}}(1, \omega^l, \ldots, \omega^{l\cdot(n-1)}), \quad l \in [n]. \tag{5.2}$$

Proof. $D_0^{(n)}$ is a circulant matrix, that is, the first column is given by a vector $(c_0, c_1, \ldots, c_{n-1})$ and the j^{th} column, by applying $j-1$ times a cyclic permutation to this vector.

$$D_0^{(n)} = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ c_2 & c_1 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix} \tag{5.3}$$

Their eigenvalues λ_j and eigenvectors $\vec{v_j}$, $\forall j \in [m]$ are well known [Gra06]:

$$\lambda_j = c_0 + c_1\omega^{-j} + c_2\omega^{-2j} + \ldots + c_{n-1}\omega^{-(n-1)j}, \tag{5.4}$$

$$\vec{v}_j = \frac{1}{\sqrt{n}}(1, \omega^j, \ldots, \omega^{(n-1)j}), \tag{5.5}$$

where $\omega = e^{\frac{2\pi i}{n}}$. Setting $c_2 = c_n = 1$, $c_j = 0 \; \forall j \neq 2, n$ yields the result shown.

□

Remark 27. *Some relevant eigenvectors and eigenvalues of $D_0^{(n)}$ are:*

i) If $l = 0 \Rightarrow \mu_0 = 2$ and $|\phi_0\rangle = 1/\sqrt{n}(1, 1, \ldots, 1) := |\phi^+\rangle$

ii) If n is even $\Rightarrow \mu_{n/2} = -2$ and $|\phi_{n/2}\rangle = 1/\sqrt{2}(1, -1, 1, -1, \ldots) := |\phi^-\rangle$

iii) If n is a multiple of 4 $\Rightarrow \mu_{n/4} = 0$, $\mu_{3n/4} = 0$ and

- $|\phi_{n/4}\rangle = 1/\sqrt{n}(1, i, -1, -i, \ldots)$
- $|\phi_{3n/4}\rangle = 1/\sqrt{n}(-i, -1, i, 1, \ldots)$

This nullspace can also be expressed by the orthogonal basis $\{|\phi^{odd}\rangle, |\phi^{even}\rangle\}$, where

- $|\phi^{odd}\rangle = 1/\sqrt{n}(1, 0, -1, 0, \ldots)$
- $|\phi^{even}\rangle = 1/\sqrt{n}(0, 1, 0, -1, \ldots)$

The previous results regarding the eigenvectors and eigenvalues are collected in the following definition.

Definition 28 (Eigenvectors and Eigenvalues of $D_0^{(n)}$)**.** *The operator $D_0^{(n)}$ admits the following eigenvectors and eigenvalues, depending on the value of n:*

- *For any $n \in \mathbb{Z}^+$, the vector $|\phi^+\rangle$ is defined:*

$$|\phi^+\rangle := \sqrt{\frac{1}{n}}(1, 1, 1, 1, \ldots), \quad \mu^+ = \mu_0 = 2. \tag{5.6}$$

- *If n is even, the vector $|\phi^-\rangle$ is defined:*

$$|\phi^-\rangle := \sqrt{\frac{1}{n}}(1, -1, 1, -1, \ldots), \quad \mu^- = \mu_{\frac{n_k}{2}} = -2. \tag{5.7}$$

- *If n is a multiple of 4, the vectors $|\phi^{odd}\rangle$ and $|\phi^{even}\rangle$ are defined:*

$$|\phi^{odd}\rangle := \sqrt{\frac{2}{n}}(1, 0, -1, 0, \ldots), \quad \mu^{odd} = \mu_{\frac{n_k}{4}, \frac{3n_k}{4}} = 0, \tag{5.8}$$

$$|\phi^{even}\rangle := \sqrt{\frac{2}{n}}(0, 1, 0, -1, \ldots), \quad \mu^{even} = \mu_{\frac{n_k}{4}, \frac{3n_k}{4}} = 0. \tag{5.9}$$

Now, we find the 0-eigenvector of the unweighted-path matrix D_1.

Property 29 (0-eigenvector of D_1)**.** *Let m be an odd integer. The weighted-path matrix D_1 defined in 24 has a unique 0-eigenvector, that is*

$$x_l = \begin{cases} 0 & \text{for even } l = 2, 4, \ldots, m-1; \\ \prod_{k=1}^{(l-1)/2} \left(-\frac{t_{2l-1}}{t_{2k}}\right) x_1 & \text{for odd } l = 1, 3, \ldots, m. \end{cases} \tag{5.10}$$

Hence, in the case that $l \geq 2$ and $d \geq 2q+1$,

$$|x_l| = \sqrt{\frac{d-2q}{d-1}}. \tag{5.11}$$

Additionally, 0 is separated from the rest of the spectrum by a gap of at least $2\sqrt{d-2q}/(m-1)$.

Proof. The proof of expression 5.10 can be found in Example 2.16 of [LT24]. Substituting d with $d-2q+2$ in that result, we obtain the weights $w_1 = \sqrt{d-2q}$ and $w_2 = \cdots = w_{n-1} = \sqrt{d-1}$, which yield the unique zero-eigenvector shown in 5.11, along with the stated spectral gap. □

Finally, we define the matrix $H_1(a,b)$ and establish a relation between its spectral gap and that of the effective Hamiltonian.

Definition 30 (Matrix $H_1(a,b)$)**.** *The matrix $H_1(a,b)$ defined here shares spectral properties with the effective Hamiltonian $\tilde{H}$ that is being studied. Let a, $b \in \mathbb{R}$, and $w_j \in \mathbb{R}$ $\forall j \in [m]$. Then,*

$$H_1(a,b) = \begin{bmatrix} a & w_1 & 0 & 0 & \dots & 0 \\ w_1 & 0 & w_2 & 0 & \dots & 0 \\ 0 & w_2 & 0 & w_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & w_{m-2} & 0 & w_{m-1} \\ 0 & 0 & 0 & 0 & w_{m-1} & b \end{bmatrix} \tag{5.12}$$

Lemma 31 (Eigenvalues and Eigenvectors of the Regular Toroidal Sunflower Graph)**.** *Consider the matrix $D_0^{(n_k)}$, its eigenvalues $\mu_{l_k} = 2\cos\left(\frac{2\pi l_k}{n_k}\right)$, and its eigenvectors $|\phi_{l_k}\rangle$, where $l_k \in \{0, \dots, n_k - 1\}$ $\forall k \in [q]$.*

Let $\lambda_j(a,b)$ be the j^{th} smallest eigenvalue of $H_1(a,b)$, where $j \in [m]$, and

$$H_1(a,b) = \begin{bmatrix} a & w_1 & 0 & 0 & \dots & 0 \\ w_1 & 0 & w_2 & 0 & \dots & 0 \\ 0 & w_2 & 0 & w_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & w_{m-2} & 0 & w_{m-1} \\ 0 & 0 & 0 & 0 & w_{m-1} & b \end{bmatrix}. \tag{5.13}$$

Let $\gamma = \frac{d-1}{2q}$, $\mu_{l_1 \dots l_q} = \mu_{l_1} + \dots + \mu_{l_q}$.

Let $\lambda_j(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$ and $|\psi_j^{l_1\dots l_q}\rangle$ be the eigenvalues and eigenvectors of $H_1(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$, where $j \in [m]$, and $l_k \in [n_k]$ $\forall k \in [q]$.

Hence:

i) $\lambda_j(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$ s.t. $j \in [m]$, $l_k \in [n_k]$, $\forall k \in [q]$ are the eigenvalues of the effective Hamiltonian $\tilde{H}$ in 22.

ii) $|\psi_j^{l_1\dots l_q}\rangle|\phi_{l_1}\rangle \dots |\phi_{l_q}\rangle$ are the associated eigenvectors of the effective Hamiltonian $\tilde{H}$ in 22.

Proof. We observe that $\{|\psi_j^{l_1\dots l_q}\rangle|\phi_{l_1}\rangle\dots|\phi_{l_q}\rangle : j\in[m], l_k\in[n_k]\ \forall k\in[q]\}$ is an orthogonal basis of the symmetric space since it is composed by the tensor product of orthogonal bases and

$$\begin{aligned}
&\tilde{H}\,|\psi_j^{l_1\dots l_q}\rangle|\phi_{l_1}\rangle\dots|\phi_{l_q}\rangle = \\
&= \left[(\mu_{l_1\dots l_q}|b_1\rangle\langle b_1| + \gamma\mu_{l_1\dots l_q}|b_m\rangle\langle b_m| + D_1)|\psi_j^{l_1\dots l_q}\rangle\right]\otimes|\phi_{l_1}\rangle\dots|\phi_{l_q}\rangle = \\
&= \lambda_j(\mu_{l_1\dots l_q},\gamma\mu_{l_1\dots l_q})|\psi_j^{l_1\dots l_q}\rangle|\phi_{l_1}\rangle\dots|\phi_{l_q}\rangle.
\end{aligned}\tag{5.14}$$

□

5.2 The Nullspace

In this section, we compute the nullspace of the adjacency matrix of the graph expressed in the supervertex basis. We begin by determining the values of $\mu_{l_1,\dots,l_q}$ for which $\lambda_j(\mu_{l_1,\dots,l_q},\gamma\mu_{l_1,\dots,l_q})$ can be zero. Due to the connection between the eigenvalues of the effective Hamiltonian $\tilde{H}$ and those of $H_1(\mu_{l_1,\dots,l_q},\,\gamma\mu_{l_1,\dots,l_q})$, we will study the conditions under which the latter matrix has zero determinant.

Lemma 32 (Determinant of the Tridiagonal Matrix $H_1(a,b)$ [EMK06])**.** *Let $\beta=(\beta_0,\dots,\beta_m)$ be a $m+1$ dimensional vector defined as follows,*

$$\beta_i = \begin{cases} 1 & \text{if } i=0,\\ a & \text{if } i=1,\\ -w_{i-1}^2\beta_{i-2} & \text{if } i=2,\dots,m-1,\\ b-w_{m-1}^2\beta_{m-2} & \text{if } i=m.\end{cases}\tag{5.15}$$

The determinant of $H_1(a,b)$ is equal to β_m.

Proof. See the Theorem 2.1 in [EMK06]. □

Lemma 33. *Let m be an odd integer. Then, the determinant of the $m\times m$ matrix $H_1(a,b)$ is given by*

$$\det H_1(a,b) = b + (-1)^{\frac{m-1}{2}}\prod_{k=1}^{\frac{m-1}{2}} w_{2k}^2 a.\tag{5.16}$$

Proof. Applying the Lemma 32, $\beta_m = b - w_{m-1}^2\beta_{m-2}$, which is the value of the determinant. Then, we need to obtain the parameter β_{m-2}.

From this Lemma, $\beta_0=1$, $\beta_1=a$, $\beta_3=-w_2^2\beta_1=-w_2^2a$. Now, let us assume the induction hypothesis that for $j\in\{1,2,\dots,\frac{m-2}{2}\}$,

$$\beta_{2j+1} = (-1)^j\prod_{k=1}^{j} w_{2k}^2\beta_1.\tag{5.17}$$

It is satisfied for β_3, which corresponds to $j=1$. We now prove that, assuming the expression holds for $j\in\{1,\dots,\frac{m-2}{2}-1\}$, it is also satisfied for $j+1$.

Substituting $i=2j+1$ in equation (5.15) yields $\beta_{2j+1}=-w_{2j}^2\beta_{2j-1}$, since this choice of j corresponds to the third case of the equation. Then, for $j+1$,

$$\beta_{2(j+1)+1} = -w_{2(j+1)}^2 \beta_{2j+1}. \tag{5.18}$$

Inserting the induction hypothesis (5.17) in (5.18),

$$\beta_{2(j+1)+1} = -w_{2(j+1)}^2 (-1)^j \prod_{k=1}^{j} w_{2k}^2 \beta_1 = (-1)^{j+1} \prod_{k=1}^{j+1} w_{2k}^2 \beta_1 \tag{5.19}$$

which validates the hypothesis.

Next, since m is odd, there exists an integer $j_m \geq 1$ so that $m = 2j_m + 1$. Using (5.17),

$$\beta_{m-2} = \beta_{2(j_m-1)+1} = (-1)^{j_m-1} \prod_{k=1}^{j_m-1} w_{2k}^2 \beta_1. \tag{5.20}$$

Finally, substituting into (5.15) for the case β_m, and using the identity $\beta_1 = a$ together with the previously obtained expression for β_{m-2}, we get

$$\beta_m = b - w_{m-1}^2 \beta_{m-2} = b - (-1)^{j_m-1} w_{2j_m}^2 \prod_{k=1}^{j_m-1} w_{2k}^2 a = b + (-1)^{j_m} \prod_{k=1}^{j_m} w_{2k}^2 a, \tag{5.21}$$

where $j_m = \frac{m-1}{2}$, which completes the proof.

□

Applying this result to the case $a = \mu_{l_1,\ldots,l_q}$, and $b = \gamma\mu_{l_1,\ldots,l_q}$ yields the following Corollary.

Corollary 34. *Let $m, d, q \in \mathbb{Z}^+$ such that m is odd, $d-1 \geq 2q$, and $d-1$ is a multiple of $2q$. Let $w_1 = \sqrt{d-2q}$, $w_2 = \ldots = w_{m-1} = \sqrt{d-1}$, $\gamma = \frac{d-1}{2q}$, and consider the matrix $H_1(a,b)$ from Definition 30 with these weights. Let $\mu_{l_1\ldots l_q} = 2\sum_{k=1}^{q} \cos(2\pi l_k/n_k)$ and set $b = \gamma a$ with $a = \mu_{l_1\ldots l_q}$. Let $\lambda_j(a,b)$, $\forall j \in [m]$ be the eigenvalues of $H_1(a,b)$.*

Then,

$$\mu_{l_1\ldots l_q} = 0 \Leftrightarrow \exists j \in [m] : \lambda_j(\mu_{l_1\ldots l_q}, \gamma\mu_{l_1\ldots l_q}) = 0. \tag{5.22}$$

Proof. Since $b = \gamma a$, from Lemma 33

$$\det H_1(a,b) = b \pm \prod_{k=1}^{\frac{m-1}{2}} t_{2k}^2 a = \mu_{l_1\ldots l_q} \underbrace{\left[\frac{d-1}{2q} \pm (d-1)^{\frac{m-1}{2}}\right]}_{\neq 0}. \tag{5.23}$$

Therefore, $\det H_1(a,b) = 0 \Leftrightarrow \mu_{l_1\ldots l_q} = 0$. Additionally, it is known that $\det H_1(a,b) = 0 \Leftrightarrow \exists\lambda_j(a,b) = 0$, and therefore,

$$\exists\lambda_j(a,b) = 0 \Leftrightarrow \det H_1(a,b) = 0 \Leftrightarrow \mu_{l_1\ldots l_q} = 0$$

□

Lemma 31 and Corollary 5.2 imply that there is a 0-eigenvector $|\psi_j^{l_1\dots l_q}\rangle |\phi_{l_1}\rangle \cdots |\phi_{l_q}\rangle$ for some $j \in [m]$ only if $l_k \in [n_k]$ for all $k \in [q]$, and the values satisfy $\mu_{l_1\dots l_q} = 0$.

The next result shows that if $\mu_{l_1\dots l_q} = 0$, then there exists a unique 0-eigenvector in the first register of $|\Psi\rangle |\phi_{l_1}\rangle \cdots |\phi_{l_q}\rangle$ that makes the total vector a 0-eigenvector.

Lemma 35 (Unique 0-eigenvector of $H_1(0,0)$)**.** *The unique 0-eigenvector of $H_1(0,0)$ with weights $w_1 = \sqrt{d-1}$, $w_2 = \ldots = w_{n-1} = \sqrt{d-2q}$, and m odd is*

$$|\Psi\rangle = \frac{1}{\sqrt{N_\Psi}} \left(\sqrt{\frac{d-1}{d-2q}}, 0, 1, 0, 1 \ldots, 0, 1 \right), \tag{5.24}$$

where $N_\Psi = \frac{d-1}{d-2q} + \frac{m-1}{2}$. Then, denoting $|\Psi\rangle = (\psi_1, \ldots, \psi_q)$,

$$\psi_1 = \frac{1}{\sqrt{1 + \frac{d-2q}{d-1}\frac{m-1}{2}}}. \tag{5.25}$$

Proof. According to the Property 29 and considering that $H_1(0,0) = D_1$ is a $m \times m$ matrix, its unique 0-eigenvector is

$$|\Psi\rangle = \psi_1 \left(1, 0, \sqrt{\frac{d-2q}{d-1}}, 0, \sqrt{\frac{d-2q}{d-1}}, \ldots, 0, \sqrt{\frac{d-2q}{d-1}} \right) \tag{5.26}$$

where ψ_1 is a normalisation factor. It can also be written as

$$|\Psi\rangle = \psi_1 \sqrt{\frac{d-2q}{d-1}} \left(\sqrt{\frac{d-1}{d-2q}}, 0, 1, 0, 1, \ldots, 0, 1 \right), \tag{5.27}$$

Finally, the normalisation factor of $v = \left(\sqrt{\frac{d-1}{d-2q}}, 0, 1, 0, 1, \ldots, 0, 1\right)$ is just $\|v\| = \sqrt{N_\Psi}$ as written in the statement, since it has m entries. This yields the value of ψ_1. $\square$

The last Lemma specifies the first entry of the eigenvectors in the nullspace. Finally, we must obtain the combinations of $l_k \in [n_k]$, $\forall k \in [q]$ such that $\mu_{l_1,\ldots,l_q} = 0$.

Lemma 36 (Case $n_k = 2p_k$ with p_k coprime)**.** *Let $n_k = 2p_k$ with p_k coprime $\forall k \in [q]$. Let $l_k \in [n_k]$. Then,*

i) At most, only one $\cos(2\pi l_k/n_k)$ can vanish, and it is possible only if p_k is even, and $l_k = n_k/4$.

ii) The relation

$$\sum_k \cos\left(\frac{2\pi l_k}{n_k}\right) = 0 \tag{5.28}$$

is possible if and only if $\forall k \in [q]$,

$$\cos\left(\frac{2\pi l_k}{n_k}\right) = \begin{cases} 0 & \textit{if } l_k = \frac{n_k}{4}, \frac{3n_k}{4}, \\ 1 & \textit{if } l_k = 0, \\ -1 & \textit{if } l_k = \frac{n_k}{2}. \end{cases} \tag{5.29}$$

iii) The expression (5.28) is satisfied if and only if:

- *q is even and there are exactly $q/2$ values of $k \in [q]$ such that $l_k = 0$, and the same number of values such that $l_k = n_k/2$.*

- *q is odd, $\exists p_{k_e}$ even, $l_{k_e} \in \{n_{k_e}/4, 3n_{k_e}/4\}$, and there are exactly $\lfloor q/2 \rfloor$ values of $k \in [q]$ such that $l_k = 0$, and the same number of values such that $l_k = n_k/2$.*

Proof. We can prove that there does not exist a combination of $\{l_k\}_{k\in[q]}$ such that $l_k \neq \frac{n_k}{4}, \frac{3n_k}{4}$ and $2\sum_k \cos\left(\frac{2\pi l_k}{n_k}\right) = 0$. This is shown using the theory of cyclotomic fields.

Defining $\omega_{n_k} = e^{2\pi i/n_k}$, the terms of the sum are $2\cos\left(\frac{2\pi l_k}{n_k}\right) = e^{2\pi l_k/n_k} + e^{-2\pi l_k/n_k} = \omega_{n_k}^{l_k} + \omega_{n_k}^{-l_k}$. Therefore,

$$2\sum_k \cos\left(\frac{2\pi l_k}{n_k}\right) = \sum_k \left(\omega_{n_k}^{l_k} + \omega_{n_k}^{-l_k}\right), \tag{5.30}$$

so each term belongs to the real maximal subfield of $\mathbb{Q}(\omega_{n_k})$, that is

$$F_k = \mathbb{Q}\left(\omega_{n_k} + \omega_{n_k}^{-1}\right). \tag{5.31}$$

The cyclotomic fields satisfy

i) $\mathbb{Q}(\omega_n, \omega_m) = \mathbb{Q}(\omega_{\mathrm{lcm}(n,m)})$,

ii) $\mathbb{Q}(\omega_n) \cap \mathbb{Q}(\omega_m) = \mathbb{Q}(\omega_{\mathrm{gcd}(n,m)})$,

which means

i) $\mathbb{Q}(\omega_{n_1}, \dots, \omega_{n_q}) = \mathbb{Q}(\omega_{\mathrm{lcm}(n_1,\dots,n_q)})$,

ii) $\mathbb{Q}(\omega_{n_1}) \cap \dots \cap \mathbb{Q}(\omega_{n_q}) = \mathbb{Q}(\omega_{\mathrm{gcd}(n_1,\dots,n_q)})$.

In our case, this implies that $\forall k \in [q]$,

$$\mathbb{Q}(\omega_{n_1}, \dots, \omega_{n_k}) = \mathbb{Q}(\omega_{4p_1\cdots p_k}). \tag{5.32}$$

Once the main properties of the cyclotomic fields have been established, the problem is addressed in two steps.

Step 1 (Niven's Theorem)

According to Niven's theorem, the only rational values of $\theta \in [0°, 90°]$ for which the sine is also a rational number are $\sin 0° = 0$, $\sin 30° = \frac{1}{2}$, $\sin 90° = 1$. This can be translated to: the only rational values $r \in [0, 1[\, \cap \mathbb{Q}$ for which the cosine of $2\pi r$ is a rational number are

$$\begin{array}{cccc}
r = 0 & \cos(0) = 1 & \Rightarrow & l_k = 0, \\
r = \frac{1}{2} & \cos(\pi) = -1 & \Rightarrow & l_k = \frac{n_k}{2}, \\
r = \frac{1}{6}, \frac{5}{6} & \cos(\frac{\pi}{3}) = \cos(\frac{5\pi}{3}) = \frac{1}{2} & \Rightarrow & l_k = \frac{n_k}{6}, \frac{5n_k}{6}, \\
r = \frac{1}{3}, \frac{2}{3} & \cos(\frac{2\pi}{3}) = \cos(\frac{4\pi}{3}) = -\frac{1}{2} & \Rightarrow & l_k = \frac{n_k}{3}, \frac{2n_k}{3}, \\
r = \frac{1}{4} & \cos(\frac{\pi}{2}) = 0 & \Rightarrow & l_k = \frac{n_k}{4}, \\
r = \frac{3}{4} & \cos(\frac{3\pi}{2}) = 0 & \Rightarrow & l_k = \frac{3n_k}{4}.
\end{array}$$

Step 2 (Explanation by cases)

1. *Every* $\cos\left(\frac{2\pi l_k}{n_k}\right)$ *is rational:*

   Assume that $n_k = 2p_k$, where each p_k is coprime to the others. In our case, $r = l_k/p_k$. If none of the p_k are divisible by 3, then none of the corresponding cosines can equal $\pm\frac{1}{2}$, since this would require $r = \frac{1}{6}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3}$.

   On the other hand, if one of the p_k is divisible by 3, then it must be the only such p_k, due to the coprimality condition. In that case, the cosine corresponding to this specific p_k may be $\pm\frac{1}{2}$, but all other rational cosines must differ from $\pm\frac{1}{2}$. In fact, they must be equal to ± 1 or 0, because the values of l_k that would yield $\cos\left(\frac{2\pi l_k}{n_k}\right) = \pm\frac{1}{2}$—namely, $l_k = \frac{n_k}{6}, \quad \frac{5n_k}{6}, \quad \frac{n_k}{3}, \quad \frac{2n_k}{3}$—cannot be integers unless p_k is divisible by 3. Thus, the half integer cannot be cancelled out and $\sum_k \cos(2\pi l_k/n_k) \neq 0$.

   In short, all the cosines must equal $0, 1, -1$ in order to satisfy that relation.

   Analogously, it is deduced that there is at most one cosine equal to zero. If there exists $k \in [q]$ such that

   $$\cos\left(2\pi \frac{l_k}{n_k}\right) = 0,$$

   then $\frac{l_k}{n_k} = \frac{1}{4}$ or $\frac{3}{4}$, which implies that $l_k = \frac{p_k}{2}$ or $\frac{3p_k}{2}$, and therefore p_k must be even. However, since the p_k are pairwise coprime, all other p_k must be odd. Consequently, for every other index k, we have

   $$l_k \neq \frac{p_k}{2}, \frac{3p_k}{2},$$

   and thus the corresponding cosine is not zero.

   In conclusion, for the sum $\sum_{k=1}^{q} \cos(2\pi l_k/n_k) = 0$ to hold:

   (a) *If q is even:* None of the cosine terms can vanish, as this would leave an odd number of ± 1 terms, which cannot sum to zero. Therefore, all the cosine values must be either $+1$ or -1.

   (b) *If q is odd:* Exactly one of the cosine terms must vanish, in order to leave an even number of ± 1 terms that can cancel out.

2. *There exists an irrational* $\cos\left(\frac{2\pi l_{k_0}}{n_{k_0}}\right)$*:*

   In this case,

   $$\exists k_0 \in [q], l_{k_0} \in [n_{k_0}] \ : \ 2\cos\left(\frac{2\pi l_{k_0}}{n_{k_0}}\right) = \omega_{n_{k_0}}^{l_{k_0}} + \omega_{n_{k_0}}^{-l_{k_0}} \in \mathbb{Q}(\omega_{n_{k_0}} + \omega_{n_{k_0}}^{-1}) \setminus \mathbb{Q}. \tag{5.33}$$

Without loss of generality, one may assume that $k_0 = q$, since permuting the indices of the sequence $\{n_k\}_{k\in[q]}$ does not affect the value of the sum of cosines. Now, it is proved by contradiction that the relation $\sum_k \cos(2\pi l_k/n_k) \neq 0$ is satisfied.

Suppose, for the sake of contradiction, that there exists a collection $\{l_k\}_{k\in[q]}$ such that $\sum_k \cos\left(\frac{2\pi l_k}{n_k}\right) = 0$. Then,

$$\cos(2\pi l_q/n_q) = -\sum_{k\neq q} \cos(2\pi l_k/n_k). \tag{5.34}$$

That is

$$\omega_{n_q}^{l_q} + \omega_{n_q}^{-l_q} = \sum_{k\neq q} \omega_{n_k}^{l_k} + \omega_{n_k}^{-l_k}, \tag{5.35}$$

where $\omega_{n_q}^{l_q} + \omega_q^{-l_q} \in \mathbb{Q}(\omega_{n_q} + \omega_{n_q}^{-1}) \setminus \mathbb{Q}$.

This would mean that there is an irrational number in $\mathbb{Q}(\omega_{n_q} + \omega_{n_q}^{-1}) \subseteq \mathbb{Q}(\omega_{n_q}) = \mathbb{Q}(\omega_{4p_q})$ that is also in $\mathbb{Q}(\omega_{n_1} + \omega_{n_1}^{-1}, \ldots, \omega_{n_{q-1}} + \omega_{n_{q-1}}^{-1}) \subseteq \mathbb{Q}(\omega_{n_1}, \ldots, \omega_{n_{q-1}}) = \mathbb{Q}(\omega_{4p_1\cdots p_{q-1}})$. Since p_k are coprime, as discussed in equation 5.32,

$$\mathbb{Q}(\omega_{4p_1\cdots p_{q-1}}) \cap \mathbb{Q}(\omega_{4p_q}) = \mathbb{Q}, \tag{5.36}$$

which leads to a contradiction.

$\square$

The following Lemma collects all the previous results presented in this section and identifies the nullspace of the adjacency matrix of the toroidal sunflower graph.

Lemma 37 (Nullspace of the Effective Hamiltonian)**.** *Let $w_1 = \sqrt{d-2q}$, $w_2 = \ldots = w_{m-1} = \sqrt{d-1}$, $\gamma = \frac{d-1}{2q}$, $\mu_{l_k} = \cos(\frac{2\pi l_k}{n_k})$ $\forall k \in [q]$. Let $m, d, q \in \mathbb{Z}^+$ such that m is odd, $d-1 \geq 2q$, and $d-1$ is a multiple of $2q$. Let $n_k, p_k \in \mathbb{Z}^+$ be positive integers such that $n_k = 2p_k$ with $\{p_k\}_{k\in[q]}$ pairwise coprime. Let q be even, or q odd such that $\exists\, p_{k_0}$ even.*

Then,

i) If q is even, the nullspace of the effective Hamiltonian $\tilde{H}$ is spanned by the basis

$$\mathcal{B}_0^{(2a)} = \left\{ |\Psi\rangle|\phi_{l_1}\rangle \ldots |\phi_{l_q}\rangle : l_k \in \{0, \frac{n_k}{2}\} \text{ and } |k : l_k = 0| = |k : l_k = n_k/2| = \frac{q}{2} \right\} \tag{5.37}$$

ii) If q is odd, then a nontrivial nullspace exists only if there exists an even p_{k_e}. In that case, the nullspace is spanned by

$$\mathcal{B}_0^{(2b)} = \Big\{ |\Psi\rangle|\phi_{l_1}\rangle \ldots |\phi_{l_q}\rangle : l_k \in \{0, \frac{n_k}{2}\}, \ |k : l_k = 0| = |k : l_k = n_k/2| = \frac{q-1}{2}, \text{ and } l_{k_e} \in \{\frac{n_{k_e}}{4}, \frac{3n_{k_e}}{4}\} \Big\}. \tag{5.38}$$

Here, $|\Psi\rangle = |\psi_{j_0}^{l_1\ldots l_q}\rangle$ is the unique 0-eigenvector of $H_1(0,0) = D_1$ and $|\phi_{l_k}\rangle_k$ is the eigenvector of the $D_0^{(n_k)}$ matrix associated to the eigenvalue $\mu_{l_k} = 2\cos(2\pi l_k/n_k)$ in 26.

Proof. According to Lemma 31, the eigenvectors of the effective Hamiltonian $\tilde{H}$ are of the form

$$|\psi_j^{l_1\dots l_q}\rangle \otimes |\phi_{l_1}\rangle \otimes \cdots \otimes |\phi_{l_q}\rangle, \tag{5.39}$$

where $|\psi_j^{l_1\dots l_q}\rangle$ is the j-th eigenvector of the matrix $H_1(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$, and each $|\phi_{l_k}\rangle$ is the l_k-th eigenvector of the operator $D_0^{(n_k)}$ for $k \in [q]$.

Furthermore, by Corollary 34, the matrix $H_1(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$ has a nontrivial kernel if and only if $\mu_{l_1\dots l_q} = 0$. We now determine the necessary and sufficient conditions for $\mu_{l_1\dots l_q} = \mu_{l_1} + \cdots + \mu_{l_1} = 0$.

Since $2\sum_k \cos\left(\frac{2\pi l_k}{n_k}\right) = \mu_{l_1\dots l_q}$, the Lemma 36 can be applied. Therefore, the values of μ_{l_k} and l_k can only be

$$\mu_{l_k} = \begin{cases} 0 & \text{if } l_k = \frac{n_k}{4}, \frac{3n_k}{4}, \\ 2 & \text{if } l_k = 0, \\ -2 & \text{if } l_k = \frac{n_k}{2}. \end{cases} \tag{5.40}$$

According to that Lemma, if q is even, the nullspace is given by the values of l_k in the expression 5.37. If q is odd such that $\exists p_{k_0}$ even, the nullspace is spanned by the vectors in 5.38 values of l_k.

□

Remark 38. *Broadly speaking, this means that $\mathcal{B}_0$ is composed by the vectors $|\Psi\rangle|\phi_{l_1}\rangle \dots |\phi_{l_q}\rangle$ such that the eigenvectors $|\phi_{l_k}\rangle$ have eigenvalues $\mu_{l_k} \in \{0, 2, -2\}$, and there are as many 2-eigenvectors as (-2)-eigenvectors in the entries of the total vector. When q is odd, the requirement that some p_{k_e} be even ensures $\mu_{l_{k_e}} = 0$, so that the total eigenvalue can still vanish.*

5.3 Spectral Gap of the Adjacency Matrix

The spectral gap of the adjacency matrix plays an important role in the complexity of the pathfinding algorithm and it is calculated in this section.

5.3.1 Preliminary Results

This subsection introduces two results essential to the analysis of the spectral gap of $H_1(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$. The first Lemma shows an upper bound for the norm of a Hermitian matrix in terms of its entries, while the second, presents an asymptotic upper bound of the matrix entries.

Lemma 39 (Lemma D1 from [LT24]). *Let $A = (A_{ij})_{n\times n}$ be a Hermitian matrix. Then*

$$\|A\| \leq \max_{|x\rangle:\||x\rangle\|\leq 1} \|A|x\rangle\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |A_{ij}| \tag{5.41}$$

where $\|\cdot\|_\infty$ denotes the infinity norm.

Proof. See Lemma D.1 in [LT24]. Since A is Hermitian, $\|A\|$ or $-\|A\|$ must be an eigenvalue of A. Therefore there exists $|\phi\rangle$ such that $A|\phi\rangle = \pm\|A\||\phi\rangle$ and $\||\phi\rangle\|_\infty = 1$. Consequently $\|A|\phi\rangle\|_\infty = \|A\|$, and this proves the inequality. □

Lemma 40 (Inverse of a Nonsingular Tridiagonal Matrix [dP01]). *Let $H_1(a, b)$ be the matrix in Definition 30. Let σ, δ be two m-dimensional vectors defined as follows:*

i) $\sigma_m = b, \quad \sigma_i = -\dfrac{t_i^2}{\sigma_{i+1}} \quad$ *for* $i = m-1, \dots, 2,$ *and* $\sigma_1 = a - \dfrac{t_1^2}{\sigma_2},$

ii) $\delta_1 = a, \quad \delta_i = -\dfrac{t_{i-1}^2}{\delta_{i-1}} \quad$ *for* $i = 2, \dots, m-1,$ *and* $\delta_m = b - \dfrac{t_{m-1}^2}{\sigma_{m-1}}.$

Then, the (i,j)*-th matrix element of the inverse of* $H_1(a,b)$ *is*

$$\left(H_1^{-1}(a,b)\right)_{ij} = \begin{cases} (-1)^{i+j}\, t_i \cdots t_{j-1} \frac{\sigma_{j+1} \cdots \sigma_m}{\delta_i \cdots \delta_m} & \text{if } i \le j, \\ (-1)^{i+j}\, t_j \cdots t_{i-1} \frac{\sigma_{i+1} \cdots \sigma_m}{\delta_j \cdots \delta_m} & \text{if } i > j; \end{cases} \tag{5.42}$$

with the convention that the empty product equals 1.

Proof. See the Theorem 2.1 in [dP01]. □

Corollary 41. *Let* $w_1 = \sqrt{d-2q}$, $w_2 = \cdots = w_{m-1} = \sqrt{d-1}$, $\gamma = \frac{d-1}{2q}$, *where* $d-1$ *is a multiple of* $2q$, *and let* m *be an odd integer. Assume* $\mu_{l_1 \cdots l_k} \neq 0$, *and define* $a = \mu_{l_1 \cdots l_k}$, $b = \gamma a$.

Then

$$\max_{i,j} \left(H_1^{-1}(a,b)\right)_{ij} = O(d^{1.5}/|a|^2). \tag{5.43}$$

Proof. This proof is analogous to the one in Corollary D3 of [LT24] but exchanging $d-2q$ by $d-2$ everywhere. Firstly, we compute the values of σ_i and δ_i. Considering the values of w_i for all $i \in [m-1]$, and the definition of γ, we obtain:

$$w_1 w_2 = \sqrt{(d-2q)(d-1)}, \qquad w_i w_{i+1} = d-1 \quad \forall i \in [2, m-2]. \tag{5.44}$$

Therefore, with the definitions of δ and σ in 40

$$\sigma_m = b, \qquad \sigma_i \sigma_{i+1} = -(d-1) \;\; i \in [2, m-1], \qquad \sigma_1 = a + b\,\frac{d-2q}{d-1}, \tag{5.45}$$

$$\delta_1 = a, \qquad \delta_i \delta_{i-1} = -(d-1) \;\; i \in [2, m-1], \qquad \delta_m = b + a\,\frac{d-1}{d-2q}. \tag{5.46}$$

Finally, we compute the values of

$$\sigma_m = b, \quad \sigma_i = -\frac{d-1}{b}, \quad \sigma_{i+1} = b \quad \text{for even } i \in [2, m-1], \qquad \sigma_1 = a\left(1 + \gamma \frac{d-2q}{d-1}\right), \tag{5.47}$$

$$\delta_1 = a, \quad \delta_i = -\frac{d-2q}{a}, \quad \delta_{i+1} = \frac{d-1}{d-2q}a \quad \text{for even } i \in [2, m-1], \quad \delta_m = a\left(\gamma + \frac{d-1}{d-2q}\right). \tag{5.48}$$

We now calculate the upper bounds for the matrix entries $\left(H_1^{-1}(a,b)\right)_{ij}$. Since $H_1(a,b)$ is hermitian, we only need to consider the case $i \le j$. Hence, using 40,

$$\left|\left(H_1^{-1}(a,b)\right)_{ij}\right| = \left| t_i \dots t_{j-1} \frac{\sigma_{j+1} \cdots \sigma_m}{\delta_i \dots \delta_m} \right| = \left| \frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}} \right| \cdot \left| \frac{1}{\delta_j} \right| \cdot \left| \frac{\sigma_{j+1} \cdots \sigma_m}{\delta_{j+1} \cdots \delta_m} \right|. \tag{5.49}$$

This entry is a product of three terms. Note that in the case that $i = j$, the first term is 1 and if $j = m$, the third term is 1. An upper bound for these three terms is provided hereunder.

First term:

1. *Case 1*: $j-i$ is even

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| = \left|\frac{(t_i t_{i+1}) \dots (t_{j-2} t_{j-1})}{(\delta_i \delta_{i+1}) \cdots (\delta_{j-2}\delta_{j-1})}\right| = \tag{5.50}$$

$$= \left|\frac{t_i t_{i+1}}{t_i^2} \cdots \frac{t_{j-2} t_{j-1}}{t_{j-2}^2}\right| = \tag{5.51}$$

$$= \left|\frac{t_{i+1}}{t_1} \dots \frac{t_{j-1}}{t_{j-2}}\right| = \tag{5.52}$$

$$= \begin{cases} \sqrt{\frac{d-1}{d-2q}} \text{ if } i = 1, \\ 1 \text{ if } i \geq 2. \end{cases} \tag{5.53}$$

Therefore,

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| \leq \sqrt{\frac{d-1}{d-2q}}. \tag{5.54}$$

2. *Case 1*: $j-i$ is odd

In this case, $j \geq 2$.

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| = \left|\frac{t_i \dots t_{j-2}}{\delta_i \cdots \delta_{j-2}}\right| \left|\frac{t_{j-1}}{\delta_{j-1}}\right| \leq \sqrt{\frac{d-1}{d-2q}} \left|\frac{t_{j-1}}{\delta_{j-1}}\right|. \tag{5.55}$$

Additionally, $t_{j-1} = \sqrt{d-1}$ and

$$\left|\frac{t_{j-1}}{\delta_{j-1}}\right| = \begin{cases} \frac{|a|\sqrt{d-1}}{d-2q} & \text{if } j \text{ is odd,} \\ \frac{\sqrt{d-2q}}{|a|} & \text{if } j \text{ is even,} \\ \frac{d-2q}{|a|\sqrt{d-1}} & \text{if } j \text{ is even.} \end{cases} \tag{5.56}$$

Therefore,

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| \leq \max\left\{\frac{|a|(d-1)}{(d-2q)^{3/2}}, \frac{\sqrt{d-1}}{|a|}\right\}. \tag{5.57}$$

Combining 5.54 and 5.57

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| \leq \max\left\{\frac{|a|(d-1)}{(d-2q)^{3/2}}, \frac{\sqrt{d-1}}{|a|}, \sqrt{\frac{d-1}{d-2q}}\right\}. \tag{5.58}$$

Second term: According to 5.48,

$$|\delta_j| = \begin{cases} |a| & \text{if } j = 1\ , \\ \frac{d-2q}{|a|} & \text{if } j \text{ even}\ , \\ \frac{d-1}{d-2q}|a| & \text{if } j \neq 1, m \text{ odd}\ , \\ |a|(\gamma + \frac{d-1}{d-2q}) & \text{if } j = m, \end{cases} \tag{5.59}$$

$$|\delta_j| \geq \min\left\{|a|, \frac{d-2q}{|a|}\right\}, \tag{5.60}$$

Ultimately,

$$\left|\frac{1}{\delta_j}\right| \leq \max\left\{\frac{1}{|a|}, \frac{|a|}{d-2q}\right\}. \tag{5.61}$$

Third term:

1. *Case 1*: $m - j$ is even

   That means that j is odd since m is odd. Then, using the fact that $\delta_i\delta_{i+1} = \sigma_i\sigma_{i+1}$ $i \geq 2$,

$$\left|\frac{\sigma_{j+1}\cdots\sigma_m}{\delta_{j+1}\cdots\delta_m}\right| = 1. \tag{5.62}$$

2. *Case 2:* $m - j$ is odd

$$\begin{aligned} \left|\frac{\sigma_{j+1}\cdots\sigma_m}{\delta_{j+1}\cdots\delta_m}\right| &= \left|\frac{\sigma_{j+1}}{\delta_{j+1}}\right| \cdot \left|\frac{\sigma_{j+2}\cdots\sigma_m}{\delta_{j+2}\cdots\delta_m}\right| = \\ &= \left|\frac{\sigma_{j+1}}{\delta_{j+1}}\right| = \begin{cases} \frac{b(d-2q)}{a(d-1)} = \frac{\gamma(d-2q)}{d-1} = \frac{d-2q}{2q} & \text{if } j+1 < m, \\ \left|\frac{b(d-2q)}{b(d-2q)+a(d-1)}\right| \leq 1 & \text{if } j+1 = m. \end{cases} \end{aligned} \tag{5.63}$$

   Hence,

$$\left|\frac{\sigma_{j+1}\cdots\sigma_m}{\delta_{j+1}\cdots\delta_m}\right| = \mathcal{O}\left(\frac{d-2q}{2q}\right). \tag{5.64}$$

Combining all the terms: Using the expressions 5.58, 5.61, and 5.64:

$$\left|\frac{t_i\ldots t_{j-1}}{\delta_i\cdots\delta_{j-1}}\right| \leq \max\left\{\frac{|a|(d-1)}{(d-2q)^{3/2}}, \frac{\sqrt{d-1}}{|a|}, \sqrt{\frac{d-1}{d-2q}}\right\}, \tag{5.65}$$

$$\left|\frac{1}{\delta_j}\right| \leq \max\left\{\frac{1}{|a|}, \frac{|a|}{d-2q}\right\}, \tag{5.66}$$

$$\left|\frac{\sigma_{j+1}\cdots\sigma_m}{\delta_{j+1}\cdots\delta_m}\right| = \mathcal{O}\left(\frac{d-2q}{2q}\right). \tag{5.67}$$

Assuming q constant,

$$\left|\frac{t_i \dots t_{j-1}}{\delta_i \cdots \delta_{j-1}}\right| = \mathcal{O}\left(\frac{\sqrt{d}}{|a|}\right), \quad \left|\frac{1}{\delta_j}\right| = \mathcal{O}\left(\frac{1}{|a|}\right), \quad \left|\frac{\sigma_{j+1} \cdots \sigma_m}{\delta_{j+1} \cdots \delta_m}\right| = \mathcal{O}\left(d\right). \tag{5.68}$$

$$\left(H_1^{-1}(a,b)\right)_{ij} = \mathcal{O}\left(\frac{d^{1.5}}{|a|^2}\right). \tag{5.69}$$

□

In the following two subsections, we derive an asymptotic upper bound on $\|H_1(\mu_{l_1,\dots,l_q}, \gamma\mu_{l_1,\dots,l_q})\|$ for the cases $\mu_{l_1,\dots,l_q} = 0$ and $\mu_{l_1,\dots,l_q} \neq 0$. Given the relation between the eigenvalues of $H_1(\mu_{l_1,\dots,l_q}, \gamma\mu_{l_1,\dots,l_q})$ and those of $\tilde{H}$, the asymptotic upper bound on $\|\tilde{H}\|$ is given by the smaller of the two.

5.3.2 Case 1: Non-zero Eigenvalue

For the case in which $\mu_{l_1\dots l_q} \neq 0$, the following result is obtained.

Lemma 42 (Spectral Norm of $H_1^{-1}(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q})$ when $\mu_{l_1\dots l_q} \neq 0$). *Consider the matrix $H_1(a,b)$ as in Definition 30. Let $d,q,m \in \mathbb{Z}^+$, where $d-1$ is a multiple of $2q$, and m is an odd integer. Let $w_1 = \sqrt{d-2q}$, $w_2 = \cdots = w_{m-1} = \sqrt{d-1}$, and $\gamma = \frac{d-1}{2q}$. Let $a = \mu_{l_1\dots l_q} \neq 0$, and $b = \gamma a$.*

Then, $\lambda_j(\mu_{l_1\dots l_q}, \gamma\mu_{l_1\dots l_q}) \neq 0$, $\forall j \in [m]$ and

$$\|H_1^{-1}(a,b)\| = O\left(d^{1.5} \cdot m \cdot \max_{k\in[q]} n_k^2\right). \tag{5.70}$$

Proof. This proof mimics Appendix D of [LT24]. From Lemma 34 we know that zero is not an eigenvalue since $\mu_{l_1\dots l_q} \neq 0$. From the Corollary 41

$$\max_{i,j}(H_1^{-1}(a,b))_{i,j} = O(d^{1.5}/|a|^2). \tag{5.71}$$

From Lemma 39. we know that for any hermitian matrix (actually this holds for any general matrix)

$$\|A\| \leq \max_{|x\rangle:\||x\rangle\|\leq 1} \|A|x\rangle\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |A_{ij}|. \tag{5.72}$$

Therefore,

$$\|H_1^{-1}(a,b)\| \leq \max_{1\leq i\leq n} \sum_{j=1}^{m} |H_1^{-1}(a,b)_{ij}| \leq \max_{1\leq i\leq n} \sum_{j=1}^{m} O(d^{1.5}/|a|^2) = O(d^{1.5}m/|a|^2). \tag{5.73}$$

But the minimum non-zero value that $|a|$ can take is when all $l_k = 0$, $\forall k \in [q]$ save one $l_{k_0} = 1$, with $n_{k_0} = \max_k n_k$.

$$|a| = \mu_{l_1\dots,l_q} = 2\sum_{k=1}^{q} \cos\left(\frac{2\pi l_k}{n_k}\right) \geq 2\cos\left(\frac{2\pi}{\max_{k\in[q]} n_k}\right) = \Omega\left(\frac{1}{\max_{k\in[q]} n_k}\right), \tag{5.74}$$

$$\frac{1}{|a|} = O\left(\max_{k\in[q]} n_k\right). \tag{5.75}$$

Hence,

$$\|H_1^{-1}(a,b)\| = O\left(d^{1.5}\cdot m\cdot \max_{k\in[q]} n_k^2\right). \tag{5.76}$$

□

In this case, the spectral gap is given by

$$\Delta = \Omega\left(\frac{1}{d^{1.5}\cdot m\cdot \max_k n_k^2}\right). \tag{5.77}$$

5.3.3 Case 2: Eigenvalue Equals Zero

For the case in which $\mu_{l_1\ldots l_q} \neq 0$, $H_1(a,b) = H_1(0,0) = D_1$. From Property 29, one can deduce that the spectral gap is

$$\Delta = \Omega\left(\frac{2\sqrt{d-2q}}{m-1}\right) = \Omega\left(\frac{\sqrt{d}}{m}\right). \tag{5.78}$$

5.3.4 General Result for the Spectral Gap of the Adjacency Matrix

Combining the results of the previous two subsections, we obtain the following.

Corollary 43 (Spectral Gap of the Effective Hamiltonian)**.** *Let $q = \mathcal{O}(1)$. Then, the spectral gap Δ of the effective Hamiltonian associated with the toroidal sunflower graph, as defined in Definition 24, satisfies*

$$\Delta = \Omega\left(\frac{1}{d^{1.5}\cdot m\cdot \max_k n_k^2}\right). \tag{5.79}$$

Proof. This is deduced from 5.76 and 5.78. □

Chapter 6

Overlap of the Projected Entry Vertex with the 0-eigenspace

We first compute the overlap between the projection of the initial state onto the nullspace and the subspace spanned by the vertices at level $j = 1$.

Theorem 44. *Let Π_0 be the projector onto the nullspace of the adjacency matrix of the toroidal sunflower graph. The probability of measuring a vertex at level $j = 1$ when the system is in the state $\Pi_0 |s\rangle$ is*

$$P_j(j=1) = \frac{1}{1 + \frac{d-2q}{d-1}\frac{m-1}{2}}. \tag{6.1}$$

Under the conditions d, q, $m \in \mathbb{Z}^+$ such that $d \geq 2$, $d - 2q \geq 1$, and $\frac{m}{d} \geq 1$,

$$P_j(j=1) = \Omega\left(\frac{d}{(d-2q)m}\right). \tag{6.2}$$

Proof. The probability of measuring a vertex at level $j = 1$ is the probability of measuring $|b_1\rangle$ in the first register in the representation 23. The measured state is

$$\Pi_0 |s\rangle = \sum_{\alpha \in \Lambda} \langle \Psi\phi^\alpha | b_1 a_1 \dots a_1\rangle |\Psi\phi^\alpha\rangle, \tag{6.3}$$

and therefore,

$$P_j(j=1) = \sum_i |\langle b_1 a_i | p\rangle|^2 = |\langle b_1|\Psi\rangle|^2 \sum_i |\langle a_i|\phi^\alpha\rangle|^2 \ = |\langle b_1|\Psi\rangle|^2 = |\psi_1|^2 = \left(1 + \frac{d-2q}{d-1}\frac{m-1}{2}\right)^{-1}. \tag{6.4}$$

where the value of $|\psi_1|^2$ was obtained in Lemma 35. The lower bound is given by the conditions d, q, $m \in \mathbb{Z}^+$ such that

$$d - 1 \geq \frac{d}{2}, \quad \forall d \geq 2, \tag{6.5}$$

$$d - 2q \geq 1, \tag{6.6}$$

$$\frac{m}{d} \geq 1. \tag{6.7}$$

Hence,

$$\frac{1}{1+\frac{d-2q}{d-1}\frac{m-1}{2}} \overset{6.5}{\geq} \frac{1}{1+\frac{d-2q}{d}(m-1)} \geq \frac{1}{1+(d-2q)\frac{d}{m}} \overset{6.6,6.7}{\geq} \frac{1}{2\frac{d-2q}{d}m} = \frac{d}{2(d-2q)m}. \tag{6.8}$$

□

Remark 45. *The constraints $d \geq 2$, $d - 2q \geq 1$, and $\frac{m}{d} \geq 1$ are justified as follows. First, the regular toroidal sunflower graph is only well-defined for degrees $d \geq 3$. The second condition arises from the assumption that $d - 2q$ is a positive multiple of $2q$, as defined in 9, and is necessary to guarantee that the graph's construction remains consistent. The third constraint is imposed to control the growth of the graph: the number of vertices $|N_G|$, given in Equation 4.10, scales linearly with d but exponentially with m. Therefore, requiring $m/d \geq 1$ places the problem in a regime where classical approaches become inefficient, and a quantum algorithm can provide a computational advantage.*

In the following two theorems, we compute an eigenbasis of the nullspace and its overlap with the supervertex basis. Using these results, we determine the projection of the initial state onto the nullspace in the supervertex basis, the probability of measuring a vertex in a given tree, and the norm of the projected initial state.

Theorem 46 (Nullspace Eigenbasis and Properties)**.** *Consider the effective Hamiltonian $\tilde{H}$ as defined in 24, with $n_k = 2p_k$ for all $k \in [q]$, and p_k pairwise coprime. Assume that if q is odd, there exists one even p_{k_e}. Then:*

i) Let $|\Psi\rangle$ be the unique 0-eigenvector of $H_1(0,0) = D_1$, with weights given by $w_1 = \sqrt{d-1}$ and $w_2 = \cdots = w_{n-1} = \sqrt{d-2q}$. Then, the nullspace of $\tilde{H}$ is spanned by an orthonormal basis depending on the parity of q:

- *If q is even:*

$$\mathcal{B}^{(a)} = \left\{ |\Psi\rangle \otimes |\phi^{\alpha_1}\rangle \otimes \cdots \otimes |\phi^{\alpha_q}\rangle \;\middle|\; \begin{array}{l} \alpha_k \in \{+,-\} \text{ for all } k \in [q], \\ |\{k : \alpha_k = +\}| = |\{k : \alpha_k = -\}| = \frac{q}{2} \end{array} \right\}. \tag{6.9}$$

- *If q is odd and there exists an even p_{k_e}:*

$$\mathcal{B}^{(b)} = \left\{ |\Psi\rangle \otimes |\phi^{\alpha_1}\rangle \otimes \cdots \otimes |\phi^{\alpha_q}\rangle \;\middle|\; \begin{array}{l} \alpha_k \in \{+,-\} \text{ for all } k \in [q] \setminus \{k_e\}, \\ |\{k : \alpha_k = +\}| = |\{k : \alpha_k = -\}| = \frac{q-1}{2}, \\ \alpha_{k_e} \in \{\text{even, odd}\} \end{array} \right\}. \tag{6.10}$$

Here, $|\phi^{\alpha_k}\rangle$ are eigenvectors of $D_0^{(n_k)}$, with eigenvalues μ^{α_k} as defined in Remark 27 and Definition 28.

ii) The dimension of the nullspace is:

- *If q is even: $\binom{q}{q/2}$, which asymptotically satisfies*

$$\binom{q}{q/2} \sim \frac{2^q}{\sqrt{\pi q/2}} \quad \text{as } q \to \infty.$$

- *If q is odd:* $2\binom{q-1}{(q-1)/2}$, *which satisfies*

$$2\binom{q-1}{(q-1)/2} \sim \frac{2^q}{\sqrt{\pi(q-1)/2}} \quad \textit{as } q \to \infty.$$

iii) Let $|\phi^\alpha\rangle = |\phi^{\alpha_1}\rangle \cdots |\phi^{\alpha_q}\rangle \in \mathcal{B}_0$, *with* $\alpha = (\alpha_1, \ldots, \alpha_q)$ *and* $\alpha_k \in \{+, -\}$. *Then, the overlap with the computational basis* $|a_i\rangle$ *satisfies:*

- *If q is even:*

$$|\langle \phi^\alpha | a_i \rangle| = \frac{1}{\sqrt{\prod_{k=1}^q n_k}}. \tag{6.11}$$

- *If q is odd:*

$$|\langle \phi^\alpha | a_i \rangle| = \begin{cases} 0 & \textit{if } \alpha_{k_e} = \textit{odd and } i_{k_e} \textit{ is even,} \\ & \textit{or } \alpha_{k_e} = \textit{even and } i_{k_e} \textit{ is odd,} \\ \sqrt{\frac{2}{\prod_{k=1}^q n_k}} & \textit{otherwise.} \end{cases} \tag{6.12}$$

Proof. i) From Lemma 37 and Remark 27.

ii) If q is even, then all q entries are of the form $|\phi^+\rangle$ or $|\phi^-\rangle$, with an equal number of $+$ and $-$ vectors. The number of possible configurations with exactly $q/2$ plus signs among the q positions is given by the binomial coefficient $\binom{q}{q/2}$; the remaining positions are filled with minus signs.

If q is odd and there exists an even p_{k_e}, then, according to Lemma 37, the corresponding index l_{k_e} satisfies $l_{k_e} \in \left\{\frac{n_{k_e}}{4}, \frac{3n_{k_e}}{4}\right\}$. As discussed in Remark 27, the subspace $\text{span}\{|\phi_{n_{k_e}/4}\rangle, |\phi_{3n_{k_e}/4}\rangle\}$ can be equivalently written as $\text{span}\{|\phi^{\text{odd}}\rangle, |\phi^{\text{even}}\rangle\}$. The remaining $q-1$ entries are of the form $|\phi^+\rangle$ or $|\phi^-\rangle$, again with equal numbers of $+$ and $-$ vectors, i.e., exactly $(q-1)/2$ of each. The number of such balanced configurations is given by $\binom{q-1}{(q-1)/2}$.

iii) This result is a consequence of the point *ii)* because one can deduce

$$\langle \phi^{\text{odd}} | a_{i_k} \rangle = \begin{cases} 0, & \text{if } i_k \text{ is even,} \\ \sqrt{\frac{2}{n_k}}, & \text{if } i_k \text{ is even.} \end{cases} \tag{6.13}$$

$$\langle \phi^{\text{even}} | a_{i_k} \rangle = \begin{cases} 0, & \text{if } i_k \text{ is odd,} \\ \sqrt{\frac{2}{n_k}}, & \text{if } i_k \text{ is odd.} \end{cases} \tag{6.14}$$

$$\langle \phi^+ | a_{i_k} \rangle = \sqrt{\frac{1}{n_k}}, \tag{6.15}$$

$$\langle \phi^- | a_{i_k} \rangle = (-1)^{i_k+1} \sqrt{\frac{1}{n_k}}. \tag{6.16}$$

$$\tag{6.17}$$

Therefore, if q is odd,

$$\left|\langle \phi^{\alpha_1} \ldots \phi^{\alpha_q} | a_{i_1} \ldots a_{i_q} \rangle\right| = \left|\langle \phi^{\alpha_1} | a_{i_1} \rangle \cdots \langle \phi^{\alpha_q} | a_{i_q} \rangle\right| = \frac{1}{\sqrt{\prod_k n_k}} \tag{6.18}$$

If q is odd, and either i_{k_e} is odd with $\alpha_{k_e} =$ even, or i_{k_e} is even with $\alpha_{k_e} =$ odd, then

$$\left|\langle \phi^{\alpha_1} \dots \phi^{\alpha_q} | a_{i_1} \dots a_{i_q} \rangle\right| = \left|\langle \phi^{\alpha_1} | a_{i_1} \rangle \cdots \langle \phi^{\alpha_q} | a_{i_q} \rangle\right| = 0. \tag{6.19}$$

Otherwise

$$\left|\langle \phi^{\alpha_1} \dots \phi^{\alpha_q} | a_{i_1} \dots a_{i_q} \rangle\right| = \left|\langle \phi^{\alpha_1} | a_{i_1} \rangle \cdots \langle \phi^{\alpha_q} | a_{i_q} \rangle\right| = \sqrt{\frac{2}{\prod_{k=1}^{q} n_k}}. \tag{6.20}$$

□

Theorem 47 (Projection of the Entry Vertex onto the Nullspace)**.** *Let $q \in \mathbb{Z}^+$ be the dimension of a toroidal sunflower graph. If it is odd, assume that $n_q = 2p_q$ with p_q even, which can be defined to be in any other register. Denoting*

$$\Lambda^{(a)} = \left\{(\alpha_1, \dots, \alpha_q) : \alpha_k \in \{+, -\},\ |\{k : \alpha_k = +\}| = |\{k \in [q] : \alpha_k = -\}| = \frac{q}{2}\right\}, \tag{6.21}$$

$$\Lambda^{(b)} = \left\{(\alpha_1, \dots, \alpha_{q-1}, odd) : \alpha_k \in \{+, -\},\ |\{k : \alpha_k = +\}| = |\{k \in [q] : \alpha_k = -\}| = \frac{q-1}{2}\right\}. \tag{6.22}$$

and $|\Psi\ \phi^{\alpha}\rangle = |\Psi\rangle|\phi^{\alpha_1}\rangle \dots |\phi^{\alpha_q}\rangle$*,*

i) The normalised projection of the initial state $|s\rangle = |b_1 a_1 \dots a_1\rangle$ onto the nullspace of the adjacency matrix defined in 24 is respectively

$$|p^{(a)}\rangle = \frac{1}{\sqrt{Z^{(a)}}} \sum_{\alpha \in \Lambda^{(a)}} |\Psi\phi^{\alpha}\rangle, \quad |p^{(b)}\rangle = \frac{1}{\sqrt{Z^{(b)}}} \sum_{\alpha \in \Lambda^{(b)}} |\Psi\phi^{\alpha}\rangle \tag{6.23}$$

where the normalisation factor is $Z^{(a)} = \binom{q}{q/2}$ and $Z^{(b)} = 2\binom{q-1}{(q-1)/2}$, respectively.

ii) The probability of measuring a vertex in the tree labeled by $i \equiv (i_1, \dots, i_q)$, with $i_k \in [n_k]$, is given, respectively for even and odd values of q, by:

$$P_i^{(a)}(i = (i_1, \dots, i_q)) = \frac{1}{\prod_k n_k}, \quad P_i^{(b)}(i = (i_1, \dots, i_q)) = \begin{cases} \frac{2}{\prod_k n_k} & \textit{if if } i_q \textit{ is odd,} \\ 0 & \textit{if } i_q \textit{ is even.} \end{cases} \tag{6.24}$$

iii) The norm of $\Pi_0 |s\rangle$ is

$$||\Pi_0 |s\rangle||^2 = \Omega\left(Z \frac{|\psi_1|}{\prod_k n_k}\right) = \Omega\left(\frac{1}{\sqrt{q} 2^q} \frac{d}{(d-2q)m} \frac{1}{\prod_k n_k}\right). \tag{6.25}$$

Proof. i) From Definition 28,

$$\langle \phi^{\text{odd}} | a_{i_k} \rangle = \begin{cases} 0, & \text{if } i_k \text{ is even,} \\ \sqrt{\frac{2}{n_k}}, & \text{if } i_k \text{ is odd,} \end{cases} \tag{6.26}$$

$$\langle \phi^{\text{even}} | a_{i_k} \rangle = \begin{cases} 0, & \text{if } i_k \text{ is odd,} \\ \sqrt{\frac{2}{n_k}}, & \text{if } i_k \text{ is even,} \end{cases} \tag{6.27}$$

$$\langle \phi^{+} | a_{i_k} \rangle = \sqrt{\frac{1}{n_k}}, \tag{6.28}$$

$$\langle \phi^{-} | a_{i_k} \rangle = (-1)^{i_k+1} \sqrt{\frac{1}{n_k}}. \tag{6.29}$$

$$\tag{6.30}$$

The projection of $|s\rangle = |b_1 a_1 \dots a_1\rangle$ over the nullspace is

$$\Pi_0 |s\rangle = \sum_{\alpha \in \Lambda} \langle \Psi \phi^{\alpha} | b_1 a_1 \dots a_1 \rangle |\Psi \phi^{\alpha}\rangle , \tag{6.31}$$

and the overlap given by

- If q is even:

$$\langle \Psi \, \phi^{\alpha} | b_1 a_1 \dots a_1 \rangle = \psi_1 \sqrt{\frac{1}{\prod_{k=1}^{q} n_k}}, \quad \forall \alpha \in \Lambda^{(a)}, \tag{6.32}$$

- If q is odd:

$$\langle \Psi \, \phi^{\alpha} | b_1 a_1 \dots a_1 \rangle = \begin{cases} 0 & \begin{array}{l} \text{if } \alpha_{k_e} = \text{ odd} : i_{k_e} \text{ is even,} \\ \text{or } \alpha_{k_e} = \text{even} : i_{k_e} \text{ is odd,} \end{array} \\ \psi_1 \sqrt{\frac{2}{\prod_{k=1}^{q} n_k}} & \text{otherwise,} \end{cases} \quad \forall \alpha \in \Lambda^{(b)}. \tag{6.33}$$

Since the overlap is equal for all values of $\alpha \in \Lambda^{(a,b)}$, it cancels out with the normalisation constant, which yields 6.23. Finally, $Z^{(a)}$, $Z^{(b)}$ are given by the number of unit vectors in each sum, which is the dimension of the nullspace in each case. These were calculated in 46 *i)*.

ii) The probability of measuring a vertex in the tree $i = (i_1, \dots, i_q)$ is

$$P_i(i_1, \dots i_q) = \sum_j |\langle b_j a_i | p \rangle|^2 = \frac{1}{Z} \sum_{\alpha \in \Lambda} \underbrace{\left(\sum_j |\langle b_j | \Psi \rangle|^2 \right)}_{1} |\langle a_i | \phi^{\alpha} \rangle|^2. \tag{6.34}$$

Since all the non-zero values of $|\langle a_i | \phi^{\alpha} \rangle|^2$ are equal, the normalization factor Z, which accounts for the number of such terms, cancels out. Therefore, the probability $P_i(i)$ is simply given by $|\langle a_i | \phi^{\alpha} \rangle|^2$, as shown in Theorem 46 *iii)*.

iii) This is a consequence of equations 6.31, 6.32, and 6.33. Due to the orthogonality of the basis,

$$||\Pi_0 \left| s \right\rangle ||^2 = \sum_{\alpha \in \Lambda} |\langle \Psi \phi^\alpha \left| b_1 a_1 \ldots a_1 \right\rangle|^2 = \begin{cases} Z^{(a)} |\psi_1|^2 \frac{1}{\prod_{k=1}^{q} n_k} & \text{if } q \text{ is even,} \\ Z^{(b)} |\psi_1|^2 \frac{2}{\prod_{k=1}^{q} n_k} & \text{if } q \text{ is odd.} \end{cases} \tag{6.35}$$

Using the Stirling approximation

$$Z^{(a)} = \binom{q}{q/2} \approx \sqrt{\frac{2}{\pi q}} \frac{1}{2^q} = \mathcal{O}\left(\frac{1}{\sqrt{q}\, 2^q} \right), \tag{6.36}$$

$$Z^{(b)} = 2 \binom{q-1}{(q-1)/2} \approx 2 \sqrt{\frac{2}{\pi (q-1)}} \frac{1}{2^{(q-1)}} = \mathcal{O}\left(\frac{1}{\sqrt{q}\, 2^q} \right). \tag{6.37}$$

Additionally,

$$|\psi_1|^2 = P_j(j=1) = \mathcal{O}\left(\frac{d}{(d-2q)m} \right). \tag{6.38}$$

This yields the expression provided. □

Chapter 7

The Algorithm

In this chapter, we present a quantum algorithm that finds an s-s' path in the toroidal sunflower graph defined in Definition 7. As stated in Definition 10, the inputs to the problem are

i) The black-box unitaries $O^Q_{G,1}$ and $O^Q_{G,2}$ in Definition 2 that implement the adjacency list oracles 1.

ii) The entrance and exit vertices are denoted by $s := s_{1,\ldots,1}$ and $s' := s_{n_1/2+1,\ldots,n_q/2+1} \in V$, respectively. These correspond to the root vertices of the trees $\mathcal{T}_{1,\ldots,1}$ and $\mathcal{T}_{n_1/2+1,\ldots,n_q/2+1}$. The entrance vertex s is known, while the exit vertex s' is unknown and must be found by the algorithm.

iii) The indicator function $f_{s'}$ from Definition 5.

The algorithm starts by applying a polynomial function of the adjacency matrix to the entrance vertex $|s\rangle$ that projects the state onto the nullspace. Since this is a non-unitary operation, the operation succeeds with probability equal to the overlap of the kernel with $|s\rangle$. That is

$$||\Pi_0 |s\rangle||^2 = \Omega\left(\frac{d}{(d-2q)m}\frac{4^q}{q\prod_{k=1}^q n_k}\right), \tag{7.1}$$

as proved in Theorem 47. Considering $q = \mathcal{O}(1)$,

$$||\Pi_0 |s\rangle||^2 = \Omega\left(\frac{1}{m\prod_{k=1}^q n_k}\right). \tag{7.2}$$

Using fixed-point amplitude amplification, the probability can be boosted to nearly 1.

According to Corollary 43 and Theorem 46 *i)*, measuring the projected state in the computational basis yields a root vertex S_{ji} with probability

$$P_{\text{root}}(i_1,\ldots,i_q) = P_j(j=1)\cdot P_i(i_1,\ldots,i_q) = \Omega\left(\frac{1}{m\prod_k n_k}\right). \tag{7.3}$$

According to the Coupon Collector's Problem (see Appendix), a number

$$t = \mathcal{O}\left(m\prod_{k=1}^q n_k \log\left(\prod_{k=1}^q n_k\right)\right) \tag{7.4}$$

of repetitions of the projection of $|s\rangle$ onto the nullspace, followed by measurement, is sufficient to obtain a set of sampled vertices V_{samp} that contains all root vertices $S_{1,i}$ with probability greater than 2/3. In the case where q is odd and there exists $k_e \in [q]$ such that $n_{k_e} = 2p_{k_e}$ with p_{k_e} even, the index i_{k_e} is always measured to be odd. Consequently, we must apply the adjacency list oracle to determine the neighbours of all measured vertices, in order to recover a set containing all the root vertices.

Later, we identify the exit vertex s' querying $f_{s'}$, a number of times $\mathcal{O}(t) = \mathcal{O}\left(\text{poly}(m, \prod_k n_k)\right)$. Finally, using Breadth First Search on the subgraph defined by the vertex set V_s and the edges connecting them E_s yields the $s - s'$ path.

Algorithm 1 Finding an s–t Path in the Toroidal Sunflower Graph 7 [LT24].

Input: Oracles $O^Q_{\tilde{G},1}$ and $O^Q_{\tilde{G},2}$ (Definition 2), a vertex $s \in \mathcal{V}$ that is the root of $\mathcal{T}_1$ (Definition 7), and a classical oracle f_t (Definition 5).
Output: The set of vertices of an s–t path.

1: Construct circuit unitary $\mathcal{V}_{\text{circ}}$ acting on registers α, β_1, β_2 using Theorem 48 that satisfies, as in (7.7).

$$\mathcal{V}_{\text{circ}}|0\rangle_{\alpha\beta_1\beta_2} \approx |0\rangle_{\alpha\beta_1}\Pi_0|s\rangle_{\beta_2} + |\perp\rangle,$$

where $\left(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}\right)|\perp\rangle = 0$.

2: Apply fixed-point amplitude amplification to $\mathcal{V}_{\text{circ}}$ to prepare the state as in (6.23):

$$\frac{\Pi_0|s\rangle}{||\Pi_0|s\rangle||} = |p\rangle$$

by boosting the probability of getting the all-0 state upon measuring the registers α, β_1 close to 1.

3: Let $\mathcal{M} = \emptyset$.
4: **for** $\chi = 1, 2, \ldots, N_s$ **do**
5: Generate the state $|p\rangle$ through Steps 1 and 2.
6: Measure the register β_2 to obtain a vertex of $\tilde{G}$, which is added to $\mathcal{M}$.
7: **end for**
8: For each vertex in $\mathcal{M}$, add its first neighbours and the corresponding edges to $\mathcal{M}$.
9: Use all vertices in $\mathcal{M}$ to generate $G_{\text{samp}} = (V_s, E_s)$ as a subgraph of G.
10: Find t by querying f_t for each vertex in G_{samp}. If t cannot be found then abort.
11: Use Breadth First Search to find an s–t path in G_{samp}. If the path can be found then return the path. If not then abort.

Theorem 48 (Robust Subspace Eigenstate Filtering [LT24])**.** *Let A be a Hermitian matrix acting on the Hilbert space $\mathcal{H}$, with $||A|| \le \alpha$. Let $\mathcal{S}$ be an invariant subspace of A. Let H be the restriction of A to $\mathcal{S}$. We assume that 0 is an eigenvalue of H (can be degenerate), and is separated from the rest of the spectrum of H by a gap at least Δ. We also assume that A can be accessed through its $(\alpha, \mathfrak{m}, \epsilon_A)$-block encoding U_A acting on β_1, β_2 as defined in Definition 3.*

Then, there exists a unitary circuit $\mathcal{V}_{circ}$ on registers α, β_1, β_2 such that

$$||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{circ}(|0\rangle_{\alpha\beta_1} \otimes \Pi_{\mathcal{S}}) - \Pi_0\Pi_{S}|| \le \epsilon + \zeta, \tag{7.5}$$

where

$$\zeta = \frac{16\ell^2\epsilon_A}{\alpha}\left[\log\left(\frac{2\alpha}{\epsilon_A} + 1\right) + 1\right]^2 \tag{7.6}$$

and it uses $2\ell = \mathcal{O}((\alpha/\Delta)\log(1/\epsilon))$ queries to (control-) U_A and its inverse, as well as $\mathcal{O}(\ell\mathfrak{m})$ other single- or two-qubit gates. In the above $\Pi_0 : \mathcal{S} \to \mathcal{S}$ is the projection operator into the 0-eigenspace of

H, and $\Pi_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$ is the projection operator into the invariant subspace $\mathcal{S}$.

Proof. See [LT24]. □

Now, one can apply this theorem with the available information of the graph. The adjacency matrix is real and hermitian, with a symmetric subspace given by Definition 19. The spectral gap is $\Delta = \Omega\left(1/(d^{1.5} m \max_k)\right)$ for $q = \mathcal{O}(1)$, and according to Lemma 4 the matrix can be accessed via a $(d^2, \mathfrak{n}+3, \epsilon_A)$-block encoding using $\mathcal{O}(1)$ queries to the adjacency list oracle 2, $\mathcal{O}(\mathfrak{n} + \log^{2.5}(d/\epsilon_A))$ additional elementary gates, and $\mathcal{O}(\log^{2.5}(d/\epsilon_A))$ ancilla qubits.

Applying the Theorem 48, we can implement $\mathcal{V}_{\text{circ}}$ with $2\ell = \mathcal{O}((d^2/\Delta)\log(1/\epsilon))$ queries to U_A, since $\alpha = d^2$. It yields

$$\begin{aligned} &\mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle) = \mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \Pi_{\mathcal{S}}\left|s\right\rangle) = \\ &= \left|0\right\rangle_{\alpha,\beta_1} \otimes \Pi_0 \Pi_{\mathcal{S}} \left|s\right\rangle + \left|\perp\right\rangle + \left|\epsilon\right\rangle = \\ &= \left|0\right\rangle_{\alpha,\beta_1} \otimes \Pi_0 \left|s\right\rangle + \left|\perp\right\rangle + \left|\epsilon\right\rangle , \end{aligned} \tag{7.7}$$

where $\left|\perp\right\rangle$ satisfies $(\left|0\right\rangle_{\alpha\beta_1 \otimes I_{\beta_2}})\left|\perp\right\rangle = 0$, and $||\left|\epsilon\right\rangle|| \le \epsilon + \zeta$.

$\Pi_0 \left|s\right\rangle$ can be obtained by post-selecting the ancilla qubits $\left|0\right\rangle_{\alpha\beta_1}$ with a probability approximately $||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle)||^2$, assuming $||\left|\epsilon\right\rangle|| \ll ||\Pi_0 \left|s\right\rangle||$. This means an expected value of the overhead of

$$\frac{1}{||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle)||^2} \approx \frac{1}{||\Pi_0 \left|s\right\rangle||^2} \tag{7.8}$$

until we measure $\left|0\right\rangle_{\alpha\beta_1}$ because the measurement follows a geometric distribution.

However, we can use fixed-point amplitude amplification, to improve the overhead almost quadratically.

Theorem 49 (Fixed-point Amplitude Amplification [GSLW19])**.** *Let U be a unitary and Π be an orthogonal projector such that $a\left|\psi_G\right\rangle = \Pi U \left|\psi_0\right\rangle$, and $a > \delta > 0$. There is a unitary circuit $\tilde{U}$ such that $||\left|\psi_G\right\rangle - \tilde{U}\left|\psi_0\right\rangle|| \le \epsilon'$, which uses a single ancilla qubit and consists of $\mathcal{O}\left(\frac{\log(1/\epsilon')}{\delta}\right)$ U, U', $C_\Pi NOT$, $C_{\left|\psi_0\right\rangle\left\langle\psi_0\right|}NOT$ and $e^{i\phi\sigma_z}$ gates.*

Proof. See [GSLW19]. □

The overhead is improved almost quadratically to

$$\mathcal{O}\left(\frac{\log(1/\epsilon')}{||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle)||}\right) \tag{7.9}$$

where ϵ' is the error incurred in the amplitude amplification process. This is because in our case

$$U = \mathcal{V}_{\text{circ}}, \tag{7.10}$$

$$\Pi = \langle 0|_{\alpha\beta_1} \otimes I, \tag{7.11}$$

$$\left|\psi_0\right\rangle = \left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle, \tag{7.12}$$

$$a = ||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle)||, \tag{7.13}$$

$$|\psi_G\rangle = (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)/a. \tag{7.14}$$

Hence, using $\tilde{U}$ instead of $\mathcal{V}_{\text{circ}}$ one obtains, up to an error ϵ':

$$\left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - \frac{(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)}{||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)||} \right\| \leq \epsilon'. \tag{7.15}$$

Additionally, we can write a more precise overhead from (7.9),

$$\mathcal{O}\left(\frac{\log(1/\epsilon')}{||\Pi_0 |s\rangle ||}\right). \tag{7.16}$$

This can be deduced combining (7.9) with the following.

$$\begin{aligned} ||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)|| &\overset{7.7}{=} ||\Pi_0 |s\rangle + (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}) |\epsilon\rangle || \geq \\ \geq ||\Pi_0 |s\rangle || - ||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}) |\epsilon\rangle || &\geq ||\Pi_0 |s\rangle || - || |\epsilon\rangle || = \Omega(||\Pi_0 |s\rangle ||) \end{aligned} \tag{7.17}$$

where ϵ, ϵ_A were chosen so that

$$|| |\epsilon\rangle || \leq \epsilon + \zeta \leq ||\Pi_0 |s\rangle ||/2. \tag{7.18}$$

Here, we used that

$$||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}) |\epsilon\rangle || \leq ||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})|| \cdot || |\epsilon\rangle || = || |\epsilon\rangle || \tag{7.19}$$

because $(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})$ is an orthogonal projector and its norm is 1.

Thus, using Theorems 48 and 49 one can build a circuit such that

$$\left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| \leq \epsilon' + \mathcal{O}\left(\frac{\epsilon + \zeta}{||\Pi_0 |s\rangle ||}\right), \tag{7.20}$$

because

$$\begin{aligned} \left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| &= \left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - |\psi_G\rangle + |\psi_G\rangle - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| \leq \\ \leq \left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - |\psi_G\rangle \right\| + \left\| |\psi_G\rangle - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| &= \epsilon' + \left\| |\psi_G\rangle - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\|. \end{aligned} \tag{7.21}$$

Using the same notation as in 7.7,

$$(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) = \Pi_0 |s\rangle + (\langle 0| \otimes I) |\epsilon\rangle \tag{7.22}$$

$$a = \left\| (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) \right\| = \|\Pi_0 |s\rangle + (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}) |\epsilon\rangle\|, \tag{7.23}$$

and considering that $|| |\epsilon\rangle || \leq \epsilon + \zeta$:

$$\left\| |\psi_G\rangle - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| \tag{7.24}$$
$$= \left\| \frac{\Pi_0 |s\rangle + (\langle 0| \otimes I) |\epsilon\rangle}{a} - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| \tag{7.25}$$
$$= \left\| \Pi_0 |s\rangle \frac{||\Pi_0 |s\rangle || - a}{||\Pi_0 |s\rangle || \cdot a} + \frac{(\langle 0| \otimes I) |\epsilon\rangle}{a} \right\| \tag{7.26}$$
$$\leq \frac{||\Pi_0 |s\rangle || - a}{a} + \frac{\epsilon' + \zeta}{a} = \tag{7.27}$$
$$= -1 + (||\Pi_0 |s\rangle || + \epsilon' + \zeta)/a. \tag{7.28}$$

Finally, since $a = \Omega(||\Pi_0 |s\rangle ||)$, as deduced in (7.17),

$$\left\| |\psi_G\rangle - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| = \mathcal{O}\left(\frac{\epsilon + \zeta}{||\Pi_0 |s\rangle ||} \right). \tag{7.29}$$

All these results can be summarised in the following theorem.

Theorem 50 (Overhead of Projecting the Entrance Vertex onto the Nullspace)**.** *Let $|s\rangle = |b_1 a_1 \dots a_1\rangle$ be the entrance vertex of the sunflower graph $G = (V, E)$ given by Definition 7 and consider $q = \mathcal{O}(1)$.*

Let U_A be a $(d^2, \mathfrak{n}, \epsilon_A)$-block encoding of the graph matrix, which according to Lemma 4 can be obtained using $\mathcal{O}(1)$ queries to the adjacency list oracle 2, $\mathcal{O}(\mathfrak{n} + \log^{2.5}(d/\epsilon_A))$ additional elementary gates, and $\mathcal{O}(log^{2.5}(d/\epsilon_A))$ ancilla qubits, with $\mathfrak{n} = \lceil \log_2(|V|) \rceil$.

Let Π_0 be the projector onto the nullspace of the graph and let $\ell = \mathcal{O}(d^2/\Delta \log(1/\epsilon))$ be as defined in Theorem 48. There exists a unitary circuit $\tilde{U}$ such that

$$\left\| \tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) - \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||} \right\| \leq \epsilon' + \mathcal{O}\left(\frac{\epsilon + \zeta}{||\Pi_0 |s\rangle ||} \right), \tag{7.30}$$

where

$$\zeta = \frac{16\ell^2 \epsilon_A}{d^2} \left[\log\left(\frac{2d^2}{\epsilon_A} + 1 \right) + 1 \right]^2 \tag{7.31}$$

that uses

i) $\mathcal{O}\left(d^{3.5} \max_k n_k^2 \sqrt{\prod_k n_k}\, m^{1.5} \operatorname{polylog}(1/\epsilon\epsilon')\right)$ *queries to the quantum oracles defined in 2.*

ii) $\mathcal{O}\left(d^{3.5} \max_k n_k^2 \sqrt{\prod_k n_k}\, m^{2.5} \operatorname{polylog}(\prod_k n_k/\epsilon\epsilon'\epsilon_A)\right)$ *single- and two-qubit gates.*

iii) $\mathcal{O}\left(m \operatorname{polylog}(\prod_k n_k/\epsilon\epsilon')\right)$ *qubits.*

Proof. Theorem 46 can be applied under these hypothesis, identifying

- $U_A \equiv A$: The $(d, \mathfrak{n}, \epsilon_A)$-block encoding of the adjacency matrix.
- $\Delta = \Omega\left(1/(m \max_k n_k^2)\right)$.

This yields that there exists a unitary circuit $\mathcal{V}_{circ}$ on registers α, β_1, β_2 such that

$$||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes \Pi_{\mathcal{S}}) - \Pi_0 \Pi_S|| \leq \epsilon + \zeta, \tag{7.32}$$

where

$$\zeta = \frac{16\ell^2\epsilon_A}{d^2}\left[\log\left(\frac{2d^2}{\epsilon_A}+1\right)+1\right]^2 \tag{7.33}$$

using $2\ell = \mathcal{O}((d^2/\Delta)\log(1/\epsilon))$ queries to (control-) U_A and its inverse, as well as $\mathcal{O}(\ell\mathfrak{m})$ other single- or two-qubit gates. Hence,

$$\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle) = |0\rangle_{\alpha,\beta_1}\otimes\Pi_0|s\rangle + |\perp\rangle + |\epsilon\rangle\,,$$

where $|\perp\rangle$ satisfies $(|0\rangle_{\alpha\beta_1\otimes I_{\beta_2}})|\perp\rangle = 0$, and $||\,|\epsilon\rangle\,|| \leq \epsilon+\zeta$. As discussed, after post-selecting the measurements with ancilla qubit measurement$|0\rangle_{\alpha\beta_1}$ and assuming that $||\,|\epsilon\rangle\,|| \ll ||\Pi_0\,|s\rangle\,||$, the overhead is given by (7.8). However, it can be improved using Theorem 49.

Identifying,

- $U = \mathcal{V}_{\text{circ}}$,
- $\Pi = \langle 0|_{\alpha\beta_1}\otimes I$,
- $|\psi_0\rangle = |0\rangle_{\alpha\beta_1}\otimes|s\rangle$,
- $a = ||(\langle 0|_{\alpha\beta_1}\otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle)||$,
- $|\psi_G\rangle = (\langle 0|_{\alpha\beta_1}\otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle)/a$,

and choosing

$$||\,|\epsilon\rangle\,|| \leq \epsilon+\zeta \leq ||\Pi_0\,|s\rangle\,||/2, \tag{7.34}$$

we deduce that there is a unitary circuit $\tilde{U}$ such that

$$\left\|\tilde{U}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle) - \frac{(\langle 0|_{\alpha\beta_1}\otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle)}{||(\langle 0|_{\alpha\beta_1}\otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle)||}\right\| \leq \epsilon', \tag{7.35}$$

which uses a single ancilla qubit and consists of $\mathcal{O}\left(\frac{\log(1/\epsilon')}{\delta}\right)$ U, U', $C_\Pi NOT$, $C_{|\psi_0\rangle\langle\psi_0|}NOT$ and $e^{i\phi\sigma_z}$ gates.

Following the reasoning from equation (7.9) to (7.29), yields

$$\left\|\tilde{U}(|0\rangle_{\alpha\beta_1}\otimes|s\rangle) - \frac{\Pi_0\,|s\rangle}{||\Pi_0\,|s\rangle\,||}\right\| \leq \epsilon' + \mathcal{O}\left(\frac{\epsilon+\zeta}{||\Pi_0\,|s\rangle\,||}\right), \tag{7.36}$$

using

$$\mathcal{O}\left(\frac{\log(1/\epsilon')}{||\Pi_0\,|s\rangle\,||}\right) = \mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right) \tag{7.37}$$

times the circuit $\mathcal{V}_{\text{circ}}$. Here, the result

$$||\Pi_0\,|s\rangle\,|| = \Omega\left(\frac{1}{\sqrt{m\prod_k n_k}}\right) \tag{7.38}$$

from Theorem 47 was substituted.

Finally, we must determine the number of oracle queries, ancilla qubits, and additional elementary gates required to build the circuit $\tilde{U}$. To this end, recall that obtaining $\mathcal{V}_{\text{circ}}$ requires applying U_A $\mathcal{O}((d^2/\Delta)\log(1/\epsilon)) = \mathcal{O}\left((d^{3.5}m \max_k n_k^2)\log(1/\epsilon)\right)$ times.

1. Number of oracle queries:

$$\underbrace{\mathcal{O}(1)}_{\text{implementing } U_A} \times \underbrace{\mathcal{O}\left(\overbrace{(d^{3.5}m \max_k n_k^2)\log(1/\epsilon)}^{2\ell}\right)}_{\#U_A \text{ to implement } \mathcal{V}_{\text{circ}}} \times \underbrace{\mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right)}_{\text{amplitude amplification overhead}} = \tag{7.39}$$

$$= \mathcal{O}\left(d^{3.5}\max_k n_k^2 \sqrt{\prod_k n_k}\, m^{1.5}\,\text{polylog}(1/\epsilon\epsilon')\right) \tag{7.40}$$

2. Number of additional elementary gates (E.G.):

$$\underbrace{\mathcal{O}(\mathfrak{n} + \log^{2.5}(1/\epsilon_A))}_{\text{E.G. to implement } U_A} \times \underbrace{\mathcal{O}\left(\ell\right)}_{\#U_A \text{ to implement } \mathcal{V}_{\text{circ}}} \times \underbrace{\mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right)}_{\#\mathcal{V}_{\text{circ}} \text{ for amplitude amplification}} +$$

$$+ \underbrace{\mathcal{O}\left(\ell\right)}_{\text{E.G. to implement } \mathcal{V}_{\text{circ}}} \times \underbrace{\mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right)}_{\#\mathcal{V}_{\text{circ}} \text{ for amplitude amplification}} + \underbrace{\mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right)}_{\text{E.G. for amplitude amplification}} =$$

$$= \mathcal{O}(\mathfrak{n} + \log^{2.5}(1/\epsilon_A)) \times \mathcal{O}\left(\ell\right) \times \mathcal{O}\left(\sqrt{m\prod_k n_k}\log(1/\epsilon')\right) =$$

$$= \mathcal{O}(\mathfrak{n} + \log^{2.5}(1/\epsilon_A)) \times \mathcal{O}\left(d^{3.5}\max_k n_k^2 \sqrt{\prod_k n_k}\, m^{1.5}\,\text{polylog}(1/\epsilon\epsilon')\right) \tag{7.41}$$

Since $\mathfrak{n} = \lceil \log_2(|V|)\rceil$ with $|V| = \{1 + \frac{d-2q}{d-2}\left[(d-1)^{m-1} - 1\right]\}\prod_k n_k$ (see Lemma 13),

$$\mathcal{O}(\mathfrak{n}) = \mathcal{O}(m + \log(\prod_k n_k)) \subsetneq \mathcal{O}(m\log(\prod_k n_k)). \tag{7.42}$$

Therefore, the number of aditional elementary gates is

$$\mathcal{O}\left(\left(m + \log(\prod_k n_k/\epsilon_A)\right)\left(d^{3.5}\max_k n_k^2 \sqrt{\prod_k n_k}\, m^{1.5}\,\text{polylog}(1/\epsilon\epsilon')\right)\right) \subsetneq$$

$$\subsetneq \mathcal{O}\left(d^{3.5}\max_k n_k^2 \sqrt{\prod_k n_k}\, m^{2.5}\,\text{polylog}(\prod_k n_k/\epsilon\epsilon'\epsilon_A)\right) \tag{7.43}$$

3. Number of ancilla qubits (A.Q.):

$$\underbrace{\mathcal{O}(\mathfrak{n}+\log^{2.5}(1/\epsilon_A))}_{\text{A.Q. to implement } U_A} + \underbrace{\mathcal{O}\left(d+\mathfrak{n}+3\right)}_{\text{A.Q. to implement } \mathcal{V}_{\text{circ}}} + \underbrace{\mathcal{O}\left(1\right)}_{\text{A.Q. for amplitude amplification}} =$$

$$=\mathcal{O}(\mathfrak{n}+\log^{2.5}(1/\epsilon_A)) = \mathcal{O}(m+\text{polylog}(\prod_k n_k/\epsilon_A)) \subsetneq \mathcal{O}(m\,\text{polylog}(\prod_k n_k/\epsilon_A)). \quad (7.44)$$

□

Finally, the following Theorem summarises the resources needed to compute the algorithm.

Theorem 51. *Let G be the regular sunflower graph as defined in 7 with $s = S_{j=1,i=1\dots 1}$ known, $s' = S_{j=1,i=n_1/2+1\dots n_q/2+1}$, and m odd. Let $n_k = 2p_k$ such that p_k are pairwise coprime $\forall k \in [q]$. If q is odd, it is assumed that there exists an even p_{k_e}.*

Then, with probability at least $2/3$, Algorithm 1 finds an s-s' path with

$$\mathcal{O}\left(d^{3.5}\max_k n_k^2 \prod_k n_k^{1.5}\, m^{2.5}\,\text{polylog}(m\prod_k n_k)\right) \quad (7.45)$$

queries to the quantum adjacency list oracle in 2 and

$$\mathcal{O}\left(m\prod_k n_k \log\left(\prod_k n_k\right)\right) \quad (7.46)$$

queries to the indicator function oracle in 5, using in total $\text{poly}(d, m, \max_k n_k, \prod_k n_k)$ *qubits,* $\text{poly}(d, m$ $\max_k n_k, \prod_k n_k)$ *other primitive gates, and* $\mathcal{O}\left(m\prod_k n_k \log\left(\prod_k n_k\right)\right)$ *runtime for classical post-processing.*

Proof. The order of queries to the quantum adjacency list oracle was proved in Theorem 50. However, ϵ and ϵ' must be adjusted according to the value of $\|\Pi_0 \left|s\right\rangle\| = 1/\sqrt{m\prod_k n_k}$ and the requirements of the algorithm. This is why we will tackle the number of queries to the indication function first.

The order of queries to the indicator function is the same as the number of repetitions of the loop in Algorithm 1. Firstly, let us assume that the algorithm works with no errors $\epsilon = \epsilon' = \epsilon_A = 0$; that is, the resulting state is

$$\tilde{U}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle) = \frac{\Pi_0 \left|s\right\rangle}{||\Pi_0 \left|s\right\rangle||} \quad (7.47)$$

according to (7.30).

The probability of measuring a vertex in a specific tree $\mathcal{T}_i$ is

$$P_i = \Theta\left(\frac{1}{\prod_k n_k}\right) \quad \text{(Theorem 47)},$$

while the probability of measuring a vertex at level $j = 1$ is

$$P_j = \Theta\left(\frac{1}{m}\right) \quad \text{(Theorem 44)}.$$

Since these are mutually exclusive events, the probability of measuring a vertex that lies both in level

$j = 1$ and in tree $\mathcal{T}_i$ is given by

$$P_{j,i} = \Theta\left(\frac{1}{m\prod_k n_k}\right). \tag{7.48}$$

We now estimate the number of trials required to collect all root vertices s_i, using a variant of the Coupon Collector's Problem (see Appendix). Specifically, our goal is to obtain all s_i when q is even. In the case where q is odd and there exists some $n_{k_e} = 2p_{k_e}$ with p_{k_e} even, our aim is to obtain all vertices with multi-index $i = (i_1, \ldots, i_q)$ such that i_{k_e} is odd.

The formulation of the problem is analogous to that of the classical Coupon Collector's Problem. In this case we consider a random variable $T_{n'} = \sum_{k=0}^{n'-1} X_k$, where $n' = \prod_k n_k$, and each $X_k \sim \mathrm{Geo}(p_k)$ is a geometric random variable such that

$$p_k = \frac{1}{m}\frac{n'-k}{n'}. \tag{7.49}$$

Hence,

$$\mathbb{E}[T_{n'}] = mn'H_n, \tag{7.50}$$

and using the to the Markov Inequality, $P(X \geq c\mathbb{E}[X]) \leq 1/c$, for $a > 0$. Then,

$$P(T_{n'} \geq cn'mH_{n'}) \leq \frac{1}{c}, \tag{7.51}$$

which means

$$P(T_{n'} < cn'H_{n'}) < 1 - \frac{1}{c}. \tag{7.52}$$

Taking $1 - 1/c = 2/3$ as in the Appendix, yields $t = 3n'mH_{n'}$. Using the property,

$$H_n = \log(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{7.53}$$

we deduce that

$$t = \mathcal{O}\left(m\prod_k n_k \log(\prod_k n_k)\right) \tag{7.54}$$

trials are needed to obtain all the n' outcomes.

Secondly, in presence of errors, according to (7.30),

$$\left|\psi_{\text{target}}\right\rangle = \left|\psi_{\text{ideal}}\right\rangle + \left|\epsilon\right\rangle, \tag{7.55}$$

where $\left|\psi_{\text{target}}\right\rangle = \tilde{U}(\left|0\right\rangle_{\alpha\beta_1} \otimes \left|s\right\rangle)$, $\left|\psi_{\text{ideal}}\right\rangle = \Pi_0 \left|s\right\rangle / ||\Pi_0 \left|s\right\rangle||$, and $||\ \left|\epsilon\right\rangle\ || \leq \epsilon' + \mathcal{O}\left((\epsilon + \zeta)/||\Pi_0 \left|s\right\rangle||\right)$. Hence,

$$\left\langle S_{1,i} \middle| \psi_{\text{target}}\right\rangle = \left\langle S_{1,i} \middle| \psi_{\text{ideal}}\right\rangle + \left\langle S_{1,i} \middle| \epsilon\right\rangle, \tag{7.56}$$

$$
\begin{aligned}
\left|\langle S_{1,i} \mid \psi_{\text{target}}\rangle\right|^2 \geq &= \left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle - \langle S_{1,i} \mid e\rangle\right|^2 \geq \\
&\geq \left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right|^2 + \left|\langle S_{1,i} \mid e\rangle\right|^2 - 2\left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right| \cdot \left|\langle S_{1,i} \mid e\rangle\right| \geq \\
&\geq \left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right|^2 - 2\left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right| \cdot \left|\langle S_{1,i} \mid e\rangle\right| \geq \\
&\geq \left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right|^2 - 2\left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right| \cdot \|\left|\epsilon\right\rangle\| = \\
&= \left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right| \left(\left|\langle S_{1,i} \mid \psi_{\text{ideal}}\rangle\right| - 2\,\|\left|\epsilon\right\rangle\|\right)
\end{aligned} \tag{7.57}
$$

From Theorem 47,

$$
\left|\langle S_{1,i}\left|\psi_{\text{ideal}}\right\rangle\right|^2 = P(j=1,i) = P_i(i)\cdot P_j(j=1) = \Omega\left(1/(m\prod_k n_k)\right), \tag{7.58}
$$

Then, choosing ϵ, ϵ', and ϵ_A so that $2\,\|\left|\epsilon\right\rangle\| = 2\epsilon' + \mathcal{O}\left((\epsilon+\zeta)/\|\Pi_0\left|s\right\rangle\|\right)$ is at most $\left|\langle S_{1,i}\left|\psi_{\text{ideal}}\right\rangle\right|/4$, then

$$
\left|\langle S_{1,i} \mid \psi_{\text{target}}\rangle\right|^2 \geq \left|\langle S_{1,i}\left|\psi_{\text{ideal}}\right\rangle\right|^2/2, \tag{7.59}
$$

$$
\left|\langle S_{1,i} \mid \psi_{\text{target}}\rangle\right|^2 = \Omega\left(\frac{1}{m\prod_k n_k}\right). \tag{7.60}
$$

Since $\|\Pi_0\left|s\right\rangle\| = \Omega\left(1/m\prod_k n_k\right)$,

$$
\epsilon' = \mathcal{O}(1), \quad \epsilon = \zeta = \mathcal{O}\left(1/\sqrt{m\prod_k n_k}\right). \tag{7.61}
$$

This can be achieved considering the expressions of η and ℓ in Theorem 48, and choosing ϵ_A such that

$$
\epsilon_A \log^2\left(\frac{d^2}{\epsilon_A}\right) = \mathcal{O}(1/\kappa). \tag{7.62}
$$

where

$$
\kappa = d^2(\max_k n_k^4)\, m^{3.5}(\prod_k n_k^{1.5})\, \log^2(m\prod_k n_k). \tag{7.63}
$$

A suitable choice is

$$
\epsilon_A = \Theta\left(\frac{1}{\kappa\log^2(\kappa)}\right) \subsetneq \Theta\left(\frac{1}{\kappa^{1+\tau}}\right), \quad \forall\tau > 0, \tag{7.64}
$$

which satisfies the required condition.

The total number of qubits and other primitive gates needed for building the unitary $\tilde{U}$ such that

$$
\tilde{U}(\left|0\right\rangle_{\alpha\beta_!} \otimes \left|s\right\rangle) \approx \frac{\Pi_0\left|s\right\rangle}{\|\Pi_0\left|s\right\rangle\|}, \tag{7.65}
$$

are obtained from Theorem 50 and equations (3.3), (3.4); substituting this errors.

Finally, in the case with errors, there is the same probability of success as in the case with no errors (7.60), (7.48). Therefore, we also need to apply $\tilde{U}$ a number of times given by (7.54),

$$\mathcal{O}\left(m\prod_k n_k \log(\prod_k n_k)\right). \tag{7.66}$$

The total number of queries to the quantum adjacency list oracle is given by 50 *i)* times (7.66)

$$\mathcal{O}\left(d^{3.5}\max_k n_k^2 \prod_k n_k^{1.5}\, m^{2.5}\, \mathrm{polylog}(m\prod_k n_k)\right). \tag{7.67}$$

$\square$

Chapter 8

The Classical Lower Bound

In this chapter, we derive a lower bound on the probability that a classical randomized algorithm finds either a path to the exit vertex s' or a cycle that does not involve the internal torus. This probability is shown to be exponentially smaller with the graph size than than the success probability of the quantum algorithm, which is 2/3. Later, we prove that the presented quantum algorithm achieves a polynomial advantage over a classical algorithm that locates the root vertices by detecting 4-cycles in the internal torus.

Firstly, we define the enlarged toroidal sunflower graph as the toroidal sunflower graph augmented with additional disconnected vertices. While the quantum algorithm still finds the path in this enlarged graph, its inclusion restricts any classical algorithm to explore the graph contiguously. This restriction simplifies the derivation of a lower bound for the classical algorithm.

Definition 52 (The Enlarged Toroidal Sunflower Graph [LT24]). *Let the toroidal sunflower graph $G = (V, E)$ be the graph as defined in Definition 7. Let V_{aux} be a set of N_{aux} vertices. Then we call the graph $\tilde{G} = (V \cup V_{aux}, E)$ the enlarged regular sunflower graph.*

Considering $|V| = N$ and $|V_{\text{aux}}| = N_{\text{aux}}$, we can choose $N_{\text{aux}} = \Omega(N_G^2) = \Omega\left((d-1)^{2m-2}\prod_{k=1}^{q} n_k^2\right)$. Hence, if the classical algorithm applies the oracle on a vertex label that has not been explored yet, it will be exponentially unlikely that the vertex is in the graph G. The probability that this happens is

$$N_G/(N_G + N_{\text{aux}}) = O(1/N_G) = O\left(\prod_{k=1}^{q} n_k^{-2}(d-1)^{-2m+2}\right). \tag{8.1}$$

To establish the classical lower bound, we draw on a definition and a result from [LT24], concerning the pathfinding problem in a regular sunflower graph.

Definition 53 (Random Embedding [LT24]). *Let $\mathcal{T}$ be a $(d-1)$-ary tree. Let G be the regular sunflower graph 6 or the toroidal sunflower graph 7. A mapping π from $\mathcal{T}$ to G, is a random embedding of $\mathcal{T}$ into G if it satisfies the following:*

1. *The root of $\mathcal{T}$ is mapped to s.*
2. *Let $v_1, \ldots, v_{d-1}$ be the children of the root, and $a_1, \ldots, a_{d-1}$ be the neighbours of s, except one that has been randomly removed. Then $(\pi(v_1), \ldots, \pi(v_{d-1}))$ is an unbiased random permutation of $(a_1, \ldots, a_{d-1})$.*
3. *Let v be an internal vertex in $\mathcal{T}$ other than the root, let $v_1, \ldots, v_{d-1}$ be its children and u its parent, and let $a_1, \ldots, a_{d-1}$ be the neighbours of $\pi(v)$ except for $\pi(u)$. Then $(\pi(v_1), \ldots, \pi(v_{d-1}))$ is an unbiased random permutation of $(a_1, \ldots, a_{d-1})$.*

We also say that the embedding π is proper *if it is injective, and say it* exits *if $t \in \pi(\mathcal{T})$.*

Lemma 54 (Lower Classical Bound for a Regular Sunflower Graph)**.** *Let $\mathcal{T}$ be a rooted $(d-1)$-ary tree with v vertices, where $v = (d-1)^{cn}$ and $c < 1/4$. Assuming $m = n + 1$, then a random embedding π of the tree $\mathcal{T}$ into G, a regular sunflower graph, as defined in Definition 53 is improper or exits with probability at most $\mathcal{O}((d-1)^{-(1/2-2c)n})$.*

Proof. See [LT24]. □

It is easy to see that $|\mathcal{T}| = \mathcal{O}((d-1)^{h-1})$ where h is the height of the tree, so the parameter c controls the depth of the classical pathfinding algorithm, which can be considered an unbalanced tree. In this graph, finding a random embedding that exits or is improper is considered a success in both cases. This is because an improper embedding implies that the algorithm found a cycle different from the one in the bottom, and therefore traversed different trees. In conclusion, the classical algorithm extracted some information from the graph that could potentially be leveraged by a new type of randomized classical algorithm.

In order to build the lemma for the toroidal sunflower graph, we need to understand which local properties might be useful for a randomised classical algorithm to efficiently solve the pathfinding problem 10. The probability of finding the path using an embedding tree of $(d-1)^{c\sum_k n_k}$, where $c < 1/4$, is

$$\mathcal{O}\left((2q-1)^{\sum_k n_k/2}(d-1)^{-(1/2-2c)\sum_k n_k}\right), \tag{8.2}$$

which decreases exponentially with $\sum_k n_k$. this us proved in the next Lemma.

Additionally, the probability of finding a cycle out of the internal torus is

$$\mathcal{O}\left[\frac{d-1}{2q(d-2q)} \cdot (d-1)^{-(1/2-2c)\sum_k n_k}\right]. \tag{8.3}$$

However, this is not the only source of cycles that would make the random embedding improper. There can also be cycles involving only roots. Assuming that the random walk does not go back to the previous vertex, there can only be cycles with length even and greater than four.

The probability of finding a 4-cycle in the internal torus is

$$\frac{2q}{d} \times \frac{2q-1}{d-1} \times \frac{1}{d-1} \times \frac{1}{d-1} = \mathcal{O}\left(\frac{(2q)^2}{d^4}\right). \tag{8.4}$$

The problem with the toroidal sunflower is that the abundance of short cycles in the internal torus provides a local signature distinguishing its vertices. As a result, detecting a cycle shorter than the tree height m reveals that the vertex lies in the torus, which rules out exponential quantum advantage.

Lemma 55 (Lower Classical Bound for the Toroidal Sunflower Graph)**.** *Let $\mathcal{T}$ be a rooted $(d-1)$-ary tree with v vertices, where $v = (d-1)^{c\max_k n_k}$ and $c < 1/4$. Assuming $m = \max_k n_k + 1$, then a random embedding π of this tree into G, a toroidal sunflower graph as defined in Definition 53, exits or trace a cycle out of the internal torus with probability at most $\mathcal{O}((d-1)^{-(1/2-2c)\max_k n_k})$.*

Proof. Firstly, the probability of exiting is calculated. Many paths that exit are possible. However, the probability of exiting is of the order of the event with the highest probability. That is, the fastest path whose probability is

$$\mathcal{O}\left(\left(\frac{2q-1}{d-1}\right)^{\sum_k n_k/2}\right). \tag{8.5}$$

Inside the tree there are $\binom{v}{2} = \mathcal{O}(v^2)$ pairs of vertices that define a path. Hence, the probability of finding a path that exit in a random embedding is

$$\mathcal{O}(v^2) \times \mathcal{O}\left(\left(\frac{2q-1}{d-1}\right)^{\sum_k n_k/2}\right) = \mathcal{O}\left((2q-1)^{\sum_k n_k/2} \cdot (d-1)^{-(\sum_k n_k/2 - 2cn_{\max})}\right). \tag{8.6}$$

where $n_{\max} = \max_k n_k$.

Secondly, we calculate the probability of tracing a cycle outside the internal torus. This can only happen if such a cycle has vertices of the highest level and that travels through different trees. It is easy to see that it is unlikely for the lower levels in the graph $(1, \ldots, n_{\max}/2)$ to belong to the cycle. If there is a cycle that has a vertex in this levels, then there exists two paths that go up on the tree reaching the leaves. Given one of them, the probability that the other closes the cycle connecting with c is $\mathcal{O}((d-1)^{-n_{\max}/2})$. Taking into account that there are $\mathcal{O}(v^2)$ possible paths in the random embedding, the probability of finding a cycle that involves the lower levels $1, \ldots, n_{\max}/2$ is

$$\mathcal{O}(v^2)\mathcal{O}((d-1)^{-n_{\max}/2}) = \mathcal{O}((d-1)^{-(1/2-2c)n_{\max}}). \tag{8.7}$$

It is exponentially unlikely with $n_{\max}$.

We will assume that the cycle involves only the levels $n_{\max}/2+1, \ldots, n_{\max}$. Then, we consider the supervertices $S_{i,n_{\max}/2+1}, \ldots, S_{i,n_{\max}}$ in the tree $\mathcal{T}_i$. Since $s_{i,n_{\max}/2+1} = (d-2q)(d-1)^{n_{\max}/2-1}$ (see Lemma (12)), there are $(d-1)^{n_{\max}/2}$ subtrees in $\mathcal{T}_i$ with height $n_{\max}/2 - 1$. Given a random embedding from a tree $\mathcal{T}$, a cycle is detected when two points of the tree a, b have the same image under π; $\pi(a) = \pi(b) \in G$. Now, let c be their closest common ancestor vertex, and let P_1, P_2 be the paths that connect a-c and b-c in $\mathcal{T}$.

There are two possible scenarios: $c \in \{a, b\}$, or $c \notin \{a, b\}$.

i) Case $c \notin \{a, b\}$: The paths $\pi(P_1)$ and $\pi(P_2)$ must visit a sequence of subtrees after the leaves are reached. Assume that $\pi(P_1)$ finishes in the tree $\mathcal{T}_i$. Given the last subtree visited by $\pi(P_1)$, we must calculate the probability that $\pi(P_2)$ ends in the same subtree. Assuming that the second-last subtree visited by $\pi(P_2)$ was in one of the $2q$ trees adjacent to $\mathcal{T}_i$. Then, the probability of forming a cycle is at most

$$\underbrace{\frac{1}{2q}}_{\pi(P_2)\text{ ending in tree }\mathcal{T}_i} \times \underbrace{\frac{1}{(d-2q)(d-1)^{n_{\max}/2-1}}}_{\pi(P_2)\text{ ending in the correct subtree of }\mathcal{T}_i} = \mathcal{O}\left(\frac{1}{2q(d-1)^{n_{\max}/2}}\right). \tag{8.8}$$

i) Case $c \in \{a, b\}$: Without loss of generality, we may assume that $c = a$. Therefore, the probability that the last subtree visited by $\pi(P_2)$ is the one containing a is calculated in the same way:

$$\mathcal{O}\left(\frac{1}{2q(d-1)^{n_{\max}/2}}\right). \tag{8.9}$$

Finally, since there are $\binom{v}{2} = \mathcal{O}(v^2)$ possible paths in the embedded tree $\mathcal{T}$, the probability of encountering a cycle outside the internal torus is given by

$$\mathcal{O}(v^2) \times \mathcal{O}\left(\frac{1}{2q(d-1)^{n_{\max}/2}}\right) = \mathcal{O}\left((d-1)^{-(1/2-2c)\,n_{\max}}\right). \tag{8.10}$$

This represents the highest probability among the two events: either exiting or forming a cycle outside of the internal torus.

□

According to the previous result, the only viable approach for a classical algorithm to solve the pathfinding problem is to first identify the internal vertices by detecting 4-cycles, and then apply a breadth-first search algorithm to find the path within the subgraph corresponding to the internal torus.

Using an argument analogous to the Coupon Collector's Problem,

$$\Omega\left(d^4 \prod_k n_k\right) \tag{8.11}$$

attempts to detect 4-cycles are required in order to identify all internal vertices with probability greater than 2/3. Since each 4-cycle detection involves four queries to the adjacency list oracle, the overall query complexity of the classical algorithm is given by (8.11).

Finally, assuming $m = \max_k n_k$, as stated in Lemma 55, and using the bounds from (8.11) and Theorem 50, the query complexities of the classical and quantum algorithms are given by

- **Classical:**

$$\Omega\left(d^4 \prod_k n_k\right). \tag{8.12}$$

- **Quantum:** Denoting $n_{\max} = \max_k n_k$,

$$\mathcal{O}\left(d^{3.5} n_{\max}^{4.5} \prod_k n_k^{1.5} \text{ polylog}(\prod_k n_k)\right). \tag{8.13}$$

The quantum advantage is at least

$$\Omega\left(\frac{\sqrt{d}}{n_{\max}^{4.5}\sqrt{\prod_k n_k}\,\text{polylog}(\prod_k n_k)}\right). \tag{8.14}$$

In conclusion, we need a high vertex degree to obtain a significant advantage.

Chapter 9

Conclusions

In this work, we have extended the results of Li and Tong [LT24], who designed a quantum algorithm that finds a path between two input vertices, entrance and exit, with exponential advantage over any classical algorithm. Building on their construction, we introduced a new type of graph, which we call the Toroidal Sunflower Graph (Definition 7), derived from the original Sunflower Graph (Definition 6). Our aim was to achieve exponential quantum advantage using a similar algorithmic approach. However, we only obtained a polynomial advantage. This limitation arises from a local property of the toroidal sunflower graph that makes it easier for a classical algorithm to navigate it.

The toroidal sunflower graph is defined to preserve an equitable partition structure. This allows a quantum random walk starting at the entrance vertex to be represented as a sequence of states in a reduced basis, even though the original basis is defined by associating each basis state with a single vertex. The reduced basis consists of states that are superpositions of vertex states belonging to the same set of the equitable partition. These are referred to as supervertex states (Definition 17), and the space spanned by this basis is called the supervertex space (Definition 19).

The matrix elements of the adjacency matrix of this graph can be expressed as tensor products and sums of elementary weighted path and cycle matrices, whose eigenvalues and eigenvectors are known (Definition 24). Using certain analytical results, we ultimately determine the spectral gap of the adjacency matrix (Corollary 43) and an eigenbasis for its nullspace (Theorem 46). These results allow us to compute the projection of the entrance state onto the nullspace $\Pi_0 |s\rangle$, its norm, and the probability of measuring a specific vertex or a set of vertices within the same supervertex (Theorem 47). We find that the probability of measuring a vertex in the internal toroidal structure of the graph — which contains both the entrance and the exit vertices — is $\mathcal{O}(1/m)$, where m is an odd, positive integer that controls the exponential growth of the graph (Theorem 44).

Then, we present the algorithm that computes the nullspace of the graph, similar to the one introduced in [LT24]. By applying Singular Value Transformation (Theorem 48), we can compute $\Pi_0 |s\rangle$ with an overhead that scales as $\mathcal{O}(1/\|\Pi_0 |s\rangle \|^2)$. Using Fixed-Point Amplitude Amplification (Theorem 49), we improve this scaling almost quadratically. By adjusting the precision of both algorithms, we compute the query complexity to the quantum adjacency list oracle, the number of ancilla qubits, other primitive gates, and the post-processing runtime (Theorems 50, and 51).

In the final chapter, we prove that a local structure exists within the internal torus of the graph that allows it to be classically navigated. Any other randomized classical algorithm involving vertices outside this structure is exponentially unlikely to yield information about the position of the random walker. This insight came at the end of my research, as I initially expected to obtain an exponential advantage. Assuming constant torus dimension, we obtain a polynomial quantum advantage, as shown in (8.14). In future work, we plan to modify the graph by introducing random edges in the internal torus, to block classical information extraction and recover the exponential quantum advantage.

Appendix. Coupon Collector's Problem

Statement of the problem: *Given a random experiment with replacement and n possible outcomes, each occurring with equal probability $p = 1/n$, how many trials are required to observe all n distinct outcomes at least once with probability greater than $2/3$?*

Let $X_k \sim \mathrm{Geo}\left(\frac{n-k}{n}\right)$ be a geometric random variable for each $k \in \{0, \dots, n-1\}$. That is, X_k denotes the number of Bernoulli trials with success probability $p = \frac{n-k}{n}$ required to obtain the first success. It is known that $\mathbb{E}[X_k] = 1/p_k = n/(n-k)$.

Define the random variable

$$T_n = \sum_{k=0}^{n-1} X_k, \tag{1}$$

which represents the total number of trials required to observe all n distinct outcomes when sampling from a uniform distribution over n elements. Hence,

$$\mathbb{E}[T_n] = \sum_{k=0}^{n-1} \mathbb{E}[X_k] = \sum_{k=0}^{n-1} n/(n-k) = n \sum_{k=0}^{n} 1/k = nH_n, \tag{2}$$

where $H_n = \sum_{k=0}^{n} 1/k$ is the n-th harmonic number.

According to the Markov Inequality, $P(X \geq c\mathbb{E}[X]) \leq 1/c$, for $a > 0$. Then,

$$P(T_n \geq cnH_n) \leq \frac{1}{c}, \tag{3}$$

which means

$$P(T_n < cnH_n) > 1 - \frac{1}{c}. \tag{4}$$

Setting $1 - 1/c = 2/3$ yields $t = 3nH_n$. The harmonic numbers satisfy the asymptotic expansion

$$H_n = \log(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{5}$$

where γ is the Euler–Mascheroni constant. Therefore, $t = \mathcal{O}(n \log n)$ trials are sufficient to obtain all n outcomes with probability at least $2/3$.

Bibliography

[CCD$^+$03] Andrew M. Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A. Spielman. Exponential algorithmic speedup by a quantum walk. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, STOC03, page 59–68. ACM, June 2003.

[dP01] C.M. da Fonseca and J. Petronilho. Explicit inverses of some tridiagonal matrices. *Linear Algebra and its Applications*, 325(1):7–21, 2001.

[EMK06] Moawwad El-Mikkawy and Abdelrahman Karawia. Inversion of general tridiagonal matrices. *Applied Mathematics Letters*, 19(8):712–720, 2006.

[Gra06] Robert M. Gray. *Toeplitz and Circulant Matrices: A Review*, volume 2 of *Foundations and Trends in Communications and Information Theory*. Now Publishers, 2006.

[GSLW19] András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC '19, page 193–204. ACM, June 2019.

[LT24] Jianqiang Li and Yu Tong. Exponential quantum advantage for pathfinding in regular sunflower graphs, 2024.