Quantum Advantage for Pathfinding in Regular Toroidal Sunflower Graphs

Master Thesis - Erasmus Mundus QUARMEN

David Jesús Árbol Guerrero
under the supervision of Nathan Wiebe

Department of Physics - University of Toronto

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Exponential Quantum Advantage for Pathfinding in Regular Sunflower Graphs

Jianqiang Li *[1] and Yu Tong†[2,3,4]

[1] *Department of Computer Science, Pennsylvania State University, State College, PA, USA*
[2] *Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA, USA*
[3] *Department of Mathematics, Duke University, Durham, NC, USA*
[4] *Department of Electrical and Computer Engineering, Duke University, Durham, NC, USA*

Figure: Jianqiang Li and Yu Tong, Exponential quantum advantage for pathfinding in regular sunflower graphs, 2024 [LT24].

Contents

Section 1

The Oracle Model

Classical formalism:

Definition (Classical Adjacency List Oracle)

1. $O_{G,1}(v,k)$: *Returns the k-th neighbour.*
2. $O_{G,2}(v,v')$: *Returns the multiplicity of the edge.*

Quantum formalism:

Definition (Quantum Oracles)

1. $O^Q_{G,1}\left|v,k,c\right\rangle = \left|v,k,c \oplus O_{G,1}(v,k)\right\rangle$
2. $O^Q_{G,2}\left|v,v',c\right\rangle = \left|v,v',c \oplus O_{G,2}(v,v')\right\rangle$

Block encoding algorithm: For building the non-unitary adjacency matrix.

$$||A - \alpha(\langle 0^m|_{\beta_1} \otimes I_{\beta_2})U_A(|0^m\rangle_{\beta_1} \otimes I_{\beta_2})|| \leq \epsilon_A,$$

Section 2

Graph Definition and Pathfinding Problem

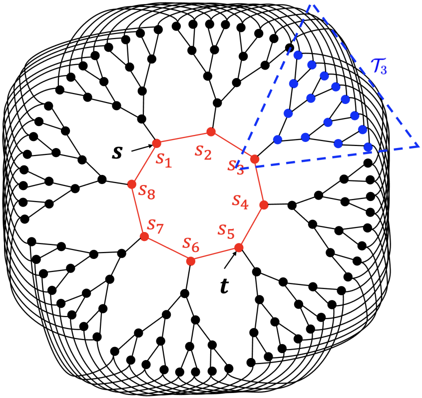


Figure: Regular Sunflower Graph [LT24].

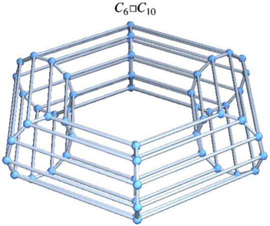
Parameters:

- d: Vertex degree
- n: Cycle length
- m: Height degree

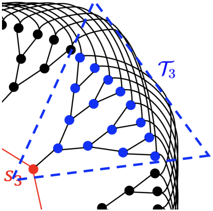
Elements:

- $\mathcal{T}_i$: $i \in [n]$: Tree
- s_i : $i \in [n]$: Roots

Entrance and exit vertices: s_1 and $s_{n/2+1}$



(a) Toroidal grid graph



(b) Regular Sunflower Graph (Zoom)

Figure: Regular toroidal sunflower graph [Wolfram Mathworld], [LT24].

Parameters:

- d: Vertex degree
- q: Number of dimensions
- n_k: Cycle length
- m: Height

Elements:

- $\mathcal{T}_i \, : \, i \in [n_1] \times \ldots \times [n_q]$: Tree
- $s_i \, : \, i \in [n_1] \times \ldots \times [n_q]$: Roots

Entrance and exit vertices: $s_{1,\ldots,1}$ and $s_{n_1/2+1,\ldots,n_q/2+1}$

Section 3

Supervertex Space and Adjacency Matrix

3.1 Supervertex Space

Definition (Supervertex)

Set of all vertices located at the same level (j) of a given tree (i): S_{ji}.

Definition (Supervertex state)

$$|S_{ji}\rangle = \frac{1}{\sqrt{s_{ji}}} \sum_{v \in S_{ji}} |v\rangle$$

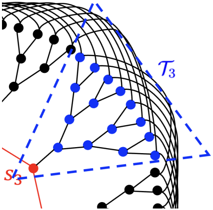


Figure: Figure from [LT24].

Definition (Supervertex space)

$$\mathcal{S} = \text{span}\left\{|S_{ji}\rangle : j \in [m],\ i \in [n_1] \times \cdots \times [n_q]\right\}$$

3.2 Adjacency matrix

Isometry: $|S_{ji}\rangle \longmapsto |b_j\rangle\,|a_{i_1}\rangle \dots |a_{i_q}\rangle$

Definition (Effective Hamiltonian)

$$\tilde{H} = \Big[(|b_1\rangle\langle b_1| + \gamma |b_m\rangle\langle b_m|) \otimes$$
$$\otimes \Big((D_0^{(n_1)} \otimes I_{n_2} \otimes \cdots \otimes I_{n_q}) + \ldots + (I_{n_1} \otimes \cdots \otimes I_{n_{q-1}} \otimes D_0^{(n_q)}) \Big) \Big] +$$
$$+ D_1 \otimes I_{n_1} \otimes \cdots \otimes I_{n_q}.$$

- $D_0^{(n_k)}$: *n_k-cycle adjacency matrix*
- D_1: *weighted path adjacency matrix*

Calculate: Eigenvalues and Eigenvectors, Nullspace and Spectral gap.

3.3 Nullspace

$$\mathcal{B}_0 = \{ \overbrace{|\Psi\rangle}^{D_1} \overbrace{|\phi_{l_1}\rangle \ldots |\phi_{l_q}\rangle}^{D_0} : l_k \in \{0, \frac{n_k}{2}\} \text{ and } |k : l_k = 0| = |k : l_k = n_k/2| \}$$

- $|\Psi\rangle = (\psi_1, \ldots, \psi_q)$, $|\psi_1|^2 = 1/(1 + \frac{d-2q}{d-1} \frac{m-1}{2})$.
- $|\phi_{l_1}\rangle \ldots |\phi_{l_q}\rangle$: Equally weighted trees.

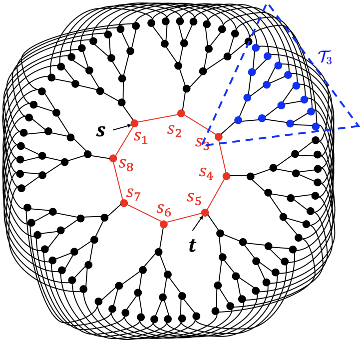


Figure: Regular Sunflower Graph [LT24].

Section 4

The Algorithm

Algorithm Pathfinding in the Toroidal Sunflower Graph [LT24].

Input: $O^{Q}_{\tilde{G},1}$, $O^{Q}_{\tilde{G},2}$, $|s\rangle$, f_t.

Output: The set of vertices of an s–t path.

```
Construct $\mathcal{V}_{\text{circ}}$ and apply FPAA: $|p\rangle = \Pi_0|s\rangle/\|\Pi_0|s\rangle\|$.
Let $\mathcal{M} = \emptyset$.
for $\chi = 1, 2, \ldots, N_s$ do
    Generate and measure $|p\rangle$. Add to $\mathcal{M}$.
end for
Find $t$ by querying $f_t$.
Generate the subgraph $G_{\text{samp}} = (\mathcal{M}, E_{\mathcal{M}})$.
Use Breadth First Search to find an $s$–$t$ path in $G_{\text{samp}}$.
```

Theorem (Robust Subspace Eigenstate Filtering [LT24])

A Hermitian. Then, $\exists \mathcal{V}_{circ}$ s.t.

$$\mathcal{V}_{circ}|0\rangle_{\alpha\beta_1}|s\rangle_{\beta_2} \approx |0\rangle_{\alpha\beta_1}\Pi_0|s\rangle_{\beta_2} + |\perp\rangle \quad s.t. \quad (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})|\perp\rangle = 0$$

with error $f(a, \Delta, \epsilon_A, \epsilon)$, using $\mathcal{O}(\alpha/\Delta \log(1/\epsilon))$ U_A.

Overhead: $1/||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)||^2 \approx m\prod_k n_k$

Theorem (Fixed-point Amplitude Amplification [GSLW19])

Reduces the overhead quadratically to $\mathcal{O}((\sqrt{m\prod_k n_k})\log(1/\epsilon'))$.

Combining both Theorems:

$$\tilde{U}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle) \approx \frac{\Pi_0 |s\rangle}{||\Pi_0 |s\rangle ||}$$

using

$$\mathcal{O}\left(d^{3.5} \max_k n_k^2 \sqrt{\prod_k n_k}\, m^{1.5} \operatorname{polylog}(1/\epsilon\epsilon')\right)$$

queries to $O^Q_{G,1}$, $O^Q_{G,2}$.

Section 5

Quantum Advantage

- **Classical:**

$$\mathcal{O}((d-1)^{-(1/2-2c)\max_k n_k}) \quad \text{Exponential?}$$

$$\Omega\left(d^4 \prod_k n_k\right) \quad \text{Not yet!}$$

- **Quantum:**

$$\mathcal{O}\left(d^{3.5} \max_k n_k^2 \prod_k n_k^{1.5}\, m^{2.5}\, \mathrm{polylog}(1/\epsilon\epsilon')\right)$$

- **Quantum Advantage:**

$$\Omega\left(\frac{\sqrt{d}}{n_{\max}^{4.5}\sqrt{\prod_k n_k}\,\mathrm{polylog}(\prod_k n_k)}\right).$$

Section 6

Conclusions

Conclusions

- ✓ Extend a type of graph with a supervertex structure.
- ✓ Apply a pathfinding algorithm harnessing this structure.
- ✓ Evaluate the quantum advantage.
- ✓ Further research?

Thank you

Acknowledgements: Nathan Wiebe

References

András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe, *Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics*, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC '19, ACM, June 2019, p. 193–204.

Jianqiang Li and Yu Tong, *Exponential quantum advantage for pathfinding in regular sunflower graphs*, 2024.

Section 7

Appendix

Theorem (Robust Subspace Eigenstate Filtering [LT24])

Let A Hermitian with $\|A\| \leq a$, gap Δ, $\mathcal{S}$ invariant subspace of A, and U_A an $(a, \mathfrak{m}, \epsilon_A)$-block encoding. Then $\exists\, \mathcal{V}_{circ}$ s.t.

$$\|(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\, \mathcal{V}_{circ}(|0\rangle_{\alpha\beta_1} \otimes \Pi_{\mathcal{S}}) - \Pi_0 \Pi_{\mathcal{S}}\| \leq \epsilon + \zeta,$$

with $\zeta = f(a, \Delta, \epsilon_A, \epsilon)$, using $\mathcal{O}(\alpha/\Delta \log(1/\epsilon))$ U_A.

$$\mathcal{V}_{\text{circ}}|0\rangle_{\alpha\beta_1}|s\rangle_{\beta_2} \approx |0\rangle_{\alpha\beta_1}\Pi_0|s\rangle_{\beta_2} + |\perp\rangle \quad \text{s.t.} \quad (\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\,|\perp\rangle = 0 \tag{1}$$

Overhead:

$$\frac{1}{||(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2})\mathcal{V}_{\text{circ}}(|0\rangle_{\alpha\beta_1} \otimes |s\rangle)||^2} \approx \frac{1}{||\Pi_0\,|s\rangle\,||^2} = m\prod_k n_k \tag{2}$$

Reminder:

$$\underbrace{\left(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}\right)}_{\Pi} \underbrace{\mathcal{V}_{\text{circ}}}_{U} \underbrace{\left(|0\rangle_{\alpha\beta_1} \otimes \Pi_{\mathcal{S}}\right)}_{|\psi_0\rangle} \approx \Pi_0 \Pi_{\mathcal{S}}$$

Theorem (Fixed-point Amplitude Amplification [GSLW19])

If $\Pi U |\psi_0\rangle = \alpha |\psi_G\rangle$ *with* $\alpha > \delta > 0$. *Then,* $\exists\, \tilde{U}$ *s.t.*

$$\| \, |\psi_G\rangle - \tilde{U} |\psi_0\rangle \, \| \leq \epsilon',$$

using $\mathcal{O}\left(\frac{\log(1/\epsilon')}{\delta}\right)$ *queries to* U.

- $\alpha = \left\| \left(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}\right) \mathcal{V}_{\text{circ}} \left(|0\rangle_{\alpha\beta_1} \otimes |s\rangle\right) \right\|$
- $|\psi_G\rangle = \left(\langle 0|_{\alpha\beta_1} \otimes I_{\beta_2}\right) \mathcal{V}_{\text{circ}} \left(|0\rangle_{\alpha\beta_1} \otimes |s\rangle\right) / \alpha \approx \Pi_0 |s\rangle / \| \Pi_0 |s\rangle \|$