

Physical mechanics

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1 Cinematics

Having:

- \mathcal{R} as rotation matrix of S' seen from S
- \mathbf{R} the relative position of S' seen from S

the space coordinates of a point are:

$$\begin{aligned} \mathbf{r} &= \mathcal{R}^T \mathbf{r}' + \mathbf{R} \\ \mathbf{r}' &= \mathcal{R}(\mathbf{r} - \mathbf{R}) \end{aligned}$$

We can then define the antisymmetric **angular momentum matrices** as:

$$\begin{aligned} \hat{w} &= \dot{\mathcal{R}}^T \mathcal{R} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix} \\ \hat{w}' &= \mathcal{R} \dot{\mathcal{R}}^T = \begin{pmatrix} 0 & -w'_z & w'_y \\ w'_z & 0 & -w'_x \\ -w'_y & w'_x & 0 \end{pmatrix} \end{aligned}$$

And the \mathbf{a} can then define the dual vectors too:

$$\begin{aligned} \mathbf{w} &= (w_x, w_y, w_z) \\ \mathbf{w}' &= (w'_x, w'_y, w'_z) \end{aligned}$$

We get the relationships:

$$\begin{aligned} \hat{w} \mathbf{q} &= \mathbf{w} \times \mathbf{q} \\ \hat{w}' \mathbf{q} &= \mathbf{w}' \times \mathbf{q} \\ \hat{w} &= \mathcal{R}^T \hat{w}' \mathcal{R} \\ \hat{w}' &= \mathcal{R} \hat{w} \end{aligned}$$

Defining \mathbf{V} as the speed of S' as seen from S , we have the velocities (deriving the position one):

$$\begin{aligned} \mathbf{v} &= \mathcal{R}^T (\mathbf{v}' + \mathbf{w}' \times \mathbf{r}') + \mathbf{V} \\ \mathbf{v}' &= \mathcal{R}[\mathbf{v} - \mathbf{V} - \mathbf{w} \times (\mathbf{r} - \mathbf{R})] \end{aligned}$$

Then with α as the angular acceleration, A as the acceleration of S' as seen by S , and deriving this once more we get the acceleration:

$$\begin{aligned} \mathbf{a}' &= -\alpha' \times \mathbf{r}'; \text{ acimutal} \\ &\quad -2\mathbf{w}' \times \mathbf{v}'; \text{ coriolis} \\ &\quad -\mathbf{w}' \times (\mathbf{w}' \times \mathbf{r}'); \text{ centrifugal} \\ &\quad + \mathcal{R} \mathbf{a} \\ &\quad - \mathcal{R} \mathbf{A}; \text{ drag} \end{aligned}$$

$$\begin{aligned} \mathbf{a} &= \mathbf{A} + \mathcal{R}^T [\mathbf{a}' \\ &\quad + \alpha' \times \mathbf{r}' \\ &\quad + 2\mathbf{w}' \times \mathbf{v}' \\ &\quad + \mathbf{w}' \times (\mathbf{w}' \times \mathbf{r}')] \end{aligned}$$

2 Dynamics

2.1 Newton dynamics

The three newton laws:

- In absence of forces, a body remains it's constant lineal movement $\mathbf{p} = \mathbf{mv}$:

$$\mathbf{0} = \frac{d}{dt} \mathbf{p} \longrightarrow \mathbf{p} = \text{ct.}$$

- The change of lineal momentum is given by the force that acts on a body:

$$\mathbf{F} = \frac{d}{dt} \mathbf{p}; \quad \mathbf{m} = \text{ct.} \rightarrow \mathbf{F} = \mathbf{ma}$$

- Given two particles that interact, the force on particle 1 is the same strength and opposite sign than the force on particle 2.

$$\mathbf{F}_1 = -\mathbf{F}_2$$

2.2 Common differential equations and solutions

Common types of problems and their differential equations and solutions:

- Uniform rectilinear movement:

$$\ddot{q} = 0 \longrightarrow q(t) = q_0 + \dot{q}_0 t$$

- Uniformly accelerated movement:

$$\ddot{q} = a \longrightarrow q(t) = q_0 + \dot{q}_0 t + \frac{1}{2} a t^2$$

- Uniformly accelerated with friction:

$$\ddot{q} = a - b\dot{q} \longrightarrow \dot{q}(t) = \frac{a}{b} + \left(\dot{q}_0 - \frac{a}{b} \right) e^{-bt}$$

Where a/b is the limit speed.

- Harmonic oscillator:

$$\begin{aligned} \ddot{q} + \omega^2 q &= 0 \rightarrow \\ q(t) &= \frac{\dot{q}_0}{\omega} \sin(\omega t) + q_0 \cos(\omega t) \\ &= A \sin(\omega t + \phi) \end{aligned}$$

Where

$$\begin{aligned} A &= \sqrt{q_0^2 + \frac{\dot{q}_0^2}{\omega^2}} \\ \phi &= \arccos \frac{\dot{q}_0}{\sqrt{q_0^2 \omega^2 + \dot{q}_0^2}} \end{aligned}$$

- Charged particle in an electric and magnetic field (Lorentz force):

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

Ends up creating a spiral with the axes perpendicular to the magnetic field, that turns with a frequency ω_c called **cyclotronic frequency**, and a constant modulo of the speed.

2.3 Non-inertial reference frames

When applying newton dynamics from the point of view of a non-inertial reference

frame, we get fictional forces:

$$\begin{aligned} m\mathbf{a}' &= \mathbf{F}' ; \mathcal{R}\mathbf{F} \\ &\quad -m\mathcal{R}\mathbf{A} ; \mathbf{F}'_{\text{drag}} \\ &\quad -m\dot{\mathbf{w}}' \times \mathbf{r}' ; \mathbf{F}'_{\text{acimutal}} \\ &\quad -2m\mathbf{w}' \times \mathbf{v}' ; \mathbf{F}'_{\text{Coriolis}} \\ &\quad -m\mathbf{w}' \times (\mathbf{w}' \times \mathbf{r}') ; \mathbf{F}'_{\text{centrifugal}} \end{aligned}$$

Then, for a particle on the surface of earth we have:

$$\mathbf{a}' = \frac{1}{m} \mathcal{R} \mathbf{F}_{\text{ng}} + \mathbf{g}'_{\text{effective}} - 2\mathbf{w}' \times \mathbf{v}'$$

where \mathbf{F}_{ng} is the non-gravitational force and the effective gravitational acceleration is:

$$\begin{aligned} \mathbf{g}'_{\text{effective}} &= (-\omega_T^2 \mathbf{R}_T \cos \lambda \sin \lambda) \mathbf{e}'_y \\ &\quad + (-g + \omega_T^2 R_T \cos^2 \lambda) \mathbf{e}'_z \end{aligned}$$

Where λ is the latitude.

3 Geometry of particle systems

3.1 Center of mass

$$\begin{aligned} \mathbf{R}_{CM} &\equiv \frac{1}{M_{\text{total}}} \sum_{i=1}^N m_i \mathbf{r}_i \\ \mathbf{R}_{CM} &\equiv \frac{1}{M_{\text{total}}} \int_{\mathcal{V}} \rho(\mathbf{r}) \mathbf{r} d\mathcal{V} \end{aligned}$$

Note that for non-euclidean coordinates we have different $d\mathcal{V}$, so the volume integrals have extra members (generic volume integrals):

- Cylindrical (for cylinder of radius R and height Z):

$$\int_0^R \int_0^Z \int_0^{2\pi} r \, d\Theta dz dr$$

- Spherical, for sphere of radius R :

$$\int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin \varphi \, d\varphi d\Theta dr$$

3.2 Inertia tensor

Symmetric tensor, defined as:

$$\mathbf{I} \equiv \sum_{i=1}^N m_i \mathbf{r}_i^2 \mathcal{I} - \sum_i m_i \mathbf{r}_i \mathbf{r}_i^T$$

In matrix form (if integrating, replace sum by volume/surface integral):

$$\sum_{i=1}^N m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix}$$

And coordinate integral form (as before, keep in mind the coordinates for the integral):

$$I_{\alpha\beta} = \int_V \rho(\mathbf{r}) [\delta_{\alpha\beta} r^2 - r^\alpha r^\beta] dV$$

I_{xx} , I_{yy} and I_{zz} are called **moments of inertia with respect to the axis**. While the non-diagonal terms are called **products of inertia**.

For **flat surfaces in the XY plane**, we have the **theorem of perpendicular axes**:

$$I_{zz} = I_{xx} + I_{yy}$$

3.3 Inertia tensor in different reference systems

We have the general formula for the inertia tensor in the reference system S' :

$$\begin{aligned} \mathbf{I}' &= \mathcal{R} [\\ &\quad \mathbf{I} - \mathbf{I}_M \\ &\quad + M(\mathbf{R} - \mathbf{R}_{CM})^2 \mathcal{I} \\ &\quad - M(\mathbf{R} - \mathbf{R}_{CM})(\mathbf{R} - \mathbf{R}_{CM})^T \\ &\quad] \mathcal{R}^T \end{aligned}$$

Where we defined the **inertia tensor of the center of mass**:

$$\mathbf{I}_M \equiv M \mathbf{R}_{CM}^2 \mathcal{I} - M \mathbf{R}_{CM} \mathbf{R}_{CM}^T$$

3.4 Particular cases

- When $S' \equiv S''_{CM}$ has the origin in the center of mass ($\mathbf{R} = \mathbf{R}_{CM}$) and **parallel**

axes to S , you get the **Steiner theorem**:

$$\mathbf{I} = \mathbf{I}''_{CM} + \mathbf{I}_M$$

The inertia tensor on S is equal to the inertia tensor from S''_{CM} plus the inertia tensor of a single particle with the same mass in the center of mass.

- When only the origin is the same as the center of mass ($\mathbf{R} = \mathbf{R}_{CM}$), then we have the general expression:

$$\mathbf{I}'_{CM} = \mathcal{R} \mathbf{I}''_{CM} \mathcal{R}^T$$

Where S''_{CM} is the reference system from the previous point.

Given the **spectral decomposition theorem**, it's always possible to find a rotation matrix \mathcal{R} that $\mathbf{I}'_{CM} = \mathbf{I}'_D$ gets diagonalized. That new system is called **principal axes reference system**.

3.A1 Diagonalizing the inertia tensor

- Build the characteristic polynomial: $\det(\mathbf{I} - \lambda \mathcal{I})$
- Make it equal to 0 and resolve (3rd degree polynomial in worst case), that gives you the **eigenvalues** (λ).
- Get each of the **eigenvectors** (\mathbf{v}) with $\mathbf{I}\mathbf{v} = \lambda\mathbf{v}$ for each found eigenvalue λ .
- Now you get the rotation matrix \mathcal{R} that diagonalizes the inertia tensor by doing:

$$\mathcal{R} = \begin{pmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{pmatrix}$$

- And finally get the inertia tensor by applying $\mathbf{I}_D = \mathcal{R} \mathbf{I} \mathcal{R}^T$

4. Dynamics of particle systems

4.1. Movement of the center of mass

From the center of mass of a system of particles, all the internal forces all cancel out, leaving only the external forces as the ones

affecting the movement of the center of mass, so we have:

$$\mathbf{F}^{ext} = M \frac{d}{dt} \mathbf{V}_{CM}$$

$$\mathbf{V}_{CM} = \frac{d}{dt} \mathbf{R}_{CM} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{v}_i$$

4.2. Conservation of linear momentum

Given the linear momentum:

$$\mathbf{P} = \sum_{i=1}^N m_i \mathbf{v}_i \equiv M \mathbf{V}_{CM}$$

It's change depends only on the external forces (it is conserved if there's no external forces):

$$\frac{d}{dt} \mathbf{P} = \sum \mathbf{F}^{ext}$$

4.3. Conservation of angular momentum

Given the angular momentum:

$$\mathbf{L} \equiv \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i$$

It's change depends only on the torque:

$$\frac{d}{dt} \mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i \equiv \mathbf{N}$$

If all the internal forces of the system are central forces, then the angular depends only on the external torque (if there's no external torque, then it's conserved):

$$\frac{d}{dt} \mathbf{L} = \mathbf{N}^{ext}$$

4.4. Conservation of energy

Kinetic energy is:

$$T(t) \equiv \sum_{i=1}^N \frac{1}{2} m_i \mathbf{v}_i^2(t)$$

And it's variation is the **total potency**:

$$P(t) \equiv \frac{dT}{dt}(t) = \sum_{i=1}^N \mathbf{F}_i \cdot \mathbf{v}_i(t)$$

Work between times t_1 and t_2 is defined then as:

$$W(t_1 \rightarrow t_2) \equiv \int_{t_1}^{t_2} P(t) dt = \int_{t_1}^{t_2} \mathbf{F}(t) \frac{d\mathbf{r}}{dt}(t) dt$$

If all the forces that act on the particles are **conservative** forces, then we can define **potential energy** and work can also be defined as:

$$W_{t_1 \rightarrow t_2} \equiv V(t_1) - V(t_2)$$

And with this two definitions, we can define the **total energy** as:

$$E(t) = V(t) + T(t)$$

And this is conserved when we have a system where only **conservative forces** act (the ones derived from a potential).

4.5. Collisions

There's two main types of collisions:

- Elastic: When the bodies don't stick together
 - Angular momentum is conserved
 - Linear momentum is conserved
 - Energy is conserved
- Inelastic: When the bodies get stuck together
 - Angular momentum is conserved
 - Linear momentum is conserved
 - Energy is **not** conserved

4.6. Seen from the center of mass

$$\mathbf{P}'_{CM} = 0$$

$$\mathbf{L}'_{CM} = \mathcal{R} [\mathbf{L} - \mathbf{L}_M - (\mathbf{I} - \mathbf{I}_M) \boldsymbol{\omega}]$$

$$T'_{CM} = T - T_M - (\mathbf{L} - \mathbf{L}_M) \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega}^T (\mathbf{I} - \mathbf{I}_M) \boldsymbol{\omega}$$

Where $*_M$ are the entities as if there was a particle of mass M at the center of mass.

5. Rigid solid movement