Physical mechanics
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1 Cinematics

Having:

- $\mathcal{R}$ as rotation matrix of S' seen from S
- $\mathbf{R}$ the relative position of S' seen from S

the space coordinates of a point are:

$$r = \mathcal{R}^T r' + \mathbf{R}$$
$$r' = \mathcal{R}(r - \mathbf{R})$$

We can then define the antisymmetric **angular momentum matrices** as:

$$\hat{w} = \dot{\mathcal{R}}^T \mathcal{R} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$
$$\hat{w}' = \mathcal{R}\dot{\mathcal{R}}^T = \begin{pmatrix} 0 & -w'_z & w'_y \\ w'_z & 0 & -w'_x \\ -w'_y & w'_x & 0 \end{pmatrix}$$

And the a can then define the dual vectors too:

$$w = (w_x, w_y, w_z)$$
$$w' = (w'_x, w'_y, w'_z)$$

We get the relationships:

$$\hat{w}q = w \times q$$
$$\hat{w}'q = w' \times q$$
$$\hat{w} = \mathcal{R}^T \hat{w}' \mathcal{R}$$
$$\hat{w}' = \mathcal{R}\hat{w}$$

Defining $\mathbf{V}$ as the speed of S' as seen from S, we have the velocities (deriving the position one):

$$v = \mathcal{R}^T(v' + w' \times r') + \mathbf{V}$$
$$v' = \mathcal{R}[v - \mathbf{V} - w \times (r - \mathbf{R})]$$

Then with α as the angular acceleration, A as the acceleration of S' as seen by S, and deriving this once more we get the acceleration:

$$\begin{aligned} a' = &-\alpha' \times r'; \text{acimutal} \\ &-2w' \times v'; \text{coriolis} \\ &-w' \times (w' \times r'); \text{centrifugal} \\ &+\mathcal{R}a \\ &-\mathcal{R}\mathbf{A}; \text{drag} \end{aligned}$$

$$\begin{aligned} a = \mathbf{A} + \mathcal{R}^T[&a' \\ &+\alpha' \times r' \\ &+1w' \times v' \\ &+w' \times (w' \times r')] \end{aligned}$$

2 Dynamics

2.1 Newton dynamics

The three newton laws:

- In absence of forces, a body remains it's constant lineal movement $\mathbf{p} = \mathbf{mv}$:

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{p} \longrightarrow \mathbf{p} = \text{ct.}$$

- The change of lineal momentum is given by the force that acts on a body:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{p}; \quad \mathbf{m} = \text{ct.} \rightarrow \mathbf{F} = \mathbf{ma}$$

- Given two particles that interact, the force on particle 1 is the same strength and opposite sign than the force on particle 2.

$$\mathbf{F_1} = -\mathbf{F_2}$$

2.2 Common differential equations and solutions

Common types of problems and their differential equations and solutions:

- Uniform rectilinear movement:

$$\ddot{q} = 0 \longrightarrow q(t) = q_0 + \dot{q}_0 t$$

- Uniformly accelerated movement:

$$\ddot{q} = a \longrightarrow q(t) = q_0 + \dot{q}_0 t + \frac{1}{2}at^2$$

- Uniformly accelerated with friction:

$$\ddot{q} = a - b\dot{q} \longrightarrow \dot{q}(t) = \frac{a}{b} + \left(\dot{q}_0 - \frac{a}{b}\right)e^{-bt}$$

Where a/b is the limit speed.

- Harmonic oscillator:

$$\ddot{q} + \omega^2 q = 0 \rightarrow$$
$$q(t) = \frac{\dot{q}_0}{\omega}\sin(\omega t) + q_0\cos(\omega t)$$
$$= A\sin(\omega t + \phi)$$

Where

$$A = \sqrt{q_0^2 + \frac{\dot{q}_0^2}{\omega^2}}$$
$$\phi = \arccos\frac{\dot{q}_0}{\sqrt{q_0^2\omega^2 + \dot{q}_0^2}}$$

- Charged particle in an electric and magnetic field (Lorentz force):

$$\mathbf{F} = \mathbf{qE} + \mathbf{qv} \times \mathbf{B}$$

Ends up creating a spiral with the axes perpendicular to the magnetic field, that turns with a frequency ω_c called **cyclotronic frequency**, and a constant modulo of the speed.

2.3 Non-inertial reference frames

When applying newton dynamics from the point of view of a non-inertial reference frame, we get fictional forces:

$$\begin{aligned} m\mathbf{a}' = \mathbf{F}' \quad &; \mathcal{R}\mathbf{F} \\ -m\mathcal{R}\mathbf{A} \quad &; \mathbf{F}'_{\text{drag}} \\ -m\dot{\boldsymbol{\omega}}' \times \mathbf{r}' \quad &; \mathbf{F}'_{\text{acimutal}} \\ -2m\boldsymbol{\omega}' \times \mathbf{v}' \quad &; \mathbf{F}'_{\text{Coriolis}} \\ -m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{r}') \quad &; \mathbf{F}'_{\text{centrifugal}} \end{aligned}$$

Then, for a particle on the surface of earth we have:

$$\mathbf{a}' = \frac{\mathbf{1}}{\mathbf{m}}\mathcal{R}\mathbf{F}_{\text{ng}} + \mathbf{g}'_{\text{effective}} - \mathbf{2}\boldsymbol{\omega}' \times \mathbf{v}'$$

where $\mathbf{F}_{\text{ng}}$ is the non-gravitational force and the effective gravitational acceleration is:

$$\mathbf{g}'_{\text{effective}} = (-\omega_\mathbf{T}^\mathbf{2}\mathbf{R_T}\cos\lambda\,\text{sen}\,\lambda)\mathbf{e}'_\mathbf{y}$$
$$+(-g + \omega_T^2 R_T\cos^2\lambda)\mathbf{e}'_\mathbf{z}$$

Where λ is the latitude.

3 Geometry of particle systems

3.1 Center of mass

$$\mathbf{R}_{CM} \equiv \frac{1}{M_{\text{total}}}\sum_{i=1}^{N} m_i\mathbf{r}_i$$
$$\mathbf{R}_{CM} \equiv \frac{1}{M_{\text{total}}}\int_{\mathcal{V}} \rho(\mathbf{r})\mathbf{r}\,d\mathcal{V}$$

Note that for non-euclidean coordinates we have different $d\mathcal{V}$, so the volume integrals have extra members (generic volume integrals):

- Cylindrical (for cylinder of radius R and height Z):

$$\int_0^R\int_0^Z\int_0^{2\pi} r \quad d\Theta dz dr$$

- Spherical, for sphere of radius R:

$$\int_0^R\int_0^{2\pi}\int_0^{\pi} r^2\sin\varphi \quad d\varphi d\Theta dr$$

3.2 Inertia tensor

Symmetric tensor, defined as:

$$\mathbf{I} \equiv \sum_{i=1}^{N} m_i \mathbf{r}_i^2 \mathcal{I} - \sum_{i}^{N} m_i \mathbf{r}_i \mathbf{r}_i^T$$

$$\mathbf{I} \equiv \int_{\mathcal{V}} \rho \mathbf{r}^2 \mathcal{I} - \sum_{i}^{N} m_i \mathbf{r}_i \mathbf{r}_i^T$$

In matrix form (if integrating, replace sum by volume/surface integral):

$$\sum_{i=1}^{N} \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix}$$

And coordinate integral form (as before, keep in mind the coordinates for the integral):

$$I_{\alpha\beta} = \int_{\mathcal{V}} \rho(\mathbf{r})[\delta_{\alpha\beta} r^2 - r^\alpha r^\beta] d\mathcal{V}$$

I_{xx}, I_{yy} and I_{zz} are called **moments of inertia with respect to the axis**. While the non-diagonal terms are called **products of inertia**.
For **flat surfaces in the XY plane**, we have the **theorem of perpendicular axes**:

$$I_{zz} = I_{xx} + I_{yy}$$

3.3 Inertia tensor in different reference systems

We have the general formula for the inertia tensor in the reference system S':

$$\begin{aligned} \mathbf{I}' = \mathcal{R}[\\ &\mathbf{I} - \mathbf{I}_M \\ &+M(\mathbf{R} - \mathbf{R}_{CM})^2 \mathcal{I} \\ &-M(\mathbf{R} - \mathbf{R}_{CM})(\mathbf{R} - \mathbf{R}_{CM})^T \\ ]\mathcal{R}^T \end{aligned}$$

Where we defined the **inertia tensor of the center of mass**:

$$\mathbf{I}_M \equiv M \mathbf{R}_{CM}^2 \mathcal{I} - M \mathbf{R}_{CM} \mathbf{R}_{CM}^T$$

3.4 Particular cases

- When $S' \equiv S''_{CM}$ has the origin in the center of mass ($\mathbf{R} = \mathbf{R}_{CM}$) and **parallel axes to** S, you get the **Steiner theorem**:

  $$\mathbf{I} = \mathbf{I}''_{CM} + \mathbf{I}_M$$

  The inertia tensor on S is equal to the inertia tensor from S''_{CM} plus the inertia tensor of a single particle with the same mass in the center of mass.

- When only the origin is the same as the center of mass ($\mathbf{R} = \mathbf{R}_{CM}$), then we have the general expression:

  $$\mathbf{I}'_{CM} = \mathcal{R} \mathbf{I}''_{CM} \mathcal{R}^T$$

  Where S''_{CM} is the reference system from the previous point.

  Given the **spectral decomposition theorem**, it's always possible to find a rotation matrix $\mathcal{R}$ that $\mathbf{I}'_{CM} = \mathbf{I}'_D$ gets diagonalized. That new system is called **principal axes reference system**.

3.A1 Diagonalizing the inertia tensor

- Build the characteristic polynomial: $\det(\mathbf{I} - \lambda \mathcal{I})$

- Make it equal to 0 and resolve (3rd degree polynomial in worst case), that gives you the **eigenvalues** (λ).

- Get each of the **eigenvectors** ($\mathbf{v}$) with $\mathbf{I}\mathbf{v} = \lambda \mathbf{v}$ for each found eigenvalue λ.

- Now you get the rotation matrix $\mathcal{R}$ that diagonalizes the inertia tensor by doing:

  $$\mathcal{R} = \begin{pmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{pmatrix}$$

- And finally get the inertia tensor by applying $\mathbf{I}_D = \mathcal{R} \mathbf{I} \mathcal{R}^T$