Physical mechanics

David Caro
09-01-2024

1 Cinematics

Having:

- $\mathcal{R}$ as rotation matrix of S' seen from S
- $\mathbf{R}$ the relative position of S' seen from S

the space coordinates of a point are:

$$r = \mathcal{R}^T r' + \mathbf{R}$$
$$r' = \mathcal{R}(r - \mathbf{R})$$

We can then define the antisymmetric **angular momentum matrices** as:

$$\hat{w} = \dot{\mathcal{R}}^T \mathcal{R} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$
$$\hat{w}' = \mathcal{R}\dot{\mathcal{R}}^T = \begin{pmatrix} 0 & -w'_z & w'_y \\ w'_z & 0 & -w'_x \\ -w'_y & w'_x & 0 \end{pmatrix}$$

And the a can then define the dual vectors too:

$$w = (w_x, w_y, w_z)$$
$$w' = (w'_x, w'_y, w'_z)$$

We get the relationships:

$$\hat{w}q = w \times q$$
$$\hat{w}'q = w' \times q$$
$$\hat{w} = \mathcal{R}^T \hat{w}' \mathcal{R}$$
$$\hat{w}' = \mathcal{R}\hat{w}$$

Defining $\mathbf{V}$ as the speed of S' as seen from S, we have the velocities (deriving the position one):

$$v = \mathcal{R}^T(v' + w' \times r') + \mathbf{V}$$
$$v' = \mathcal{R}[v - \mathbf{V} - w \times (r - \mathbf{R})]$$

Then with α as the angular acceleration, A as the acceleration of S' as seen by S, and deriving this once more we get the acceleration:

$$\begin{aligned} a' = &-\alpha' \times r'; \text{acimutal} \\ &-2w' \times v'; \text{coriolis} \\ &-w' \times (w' \times r'); \text{centrifugal} \\ &+\mathcal{R}a \\ &-\mathcal{R}\mathbf{A}; \text{drag} \end{aligned}$$

$$\begin{aligned} a = \mathbf{A} + \mathcal{R}^T[&a' \\ &+\alpha' \times r' \\ &+1w' \times v' \\ &+w' \times (w' \times r')] \end{aligned}$$

2 Dynamics

2.1 Newton dynamics

The three newton laws:

- In absence of forces, a body remains it's constant lineal movement $\mathbf{p} = \mathbf{mv}$:

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{t}}\mathbf{p} \longrightarrow \mathbf{p} = \text{ct.}$$

- The change of lineal momentum is given by the force that acts on a body:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{t}}\mathbf{p}; \quad \mathbf{m} = \text{ct.} \rightarrow \mathbf{F} = \mathbf{ma}$$

- Given two particles that interact, the force on particle 1 is the same strength and opposite sign than the force on particle 2.

$$\mathbf{F_1} = -\mathbf{F_2}$$

2.2 Common differential equations and solutions

Common types of problems and their differential equations and solutions:

- Uniform rectilinear movement:

$$\ddot{q} = 0 \longrightarrow q(t) = q_0 + \dot{q}_0 t$$

- Uniformly accelerated movement:

$$\ddot{q} = a \longrightarrow q(t) = q_0 + \dot{q}_0 t + \frac{1}{2}at^2$$

- Uniformly accelerated with friction:

$$\ddot{q} = a - b\dot{q} \longrightarrow \dot{q}(t) = \frac{a}{b} + \left(\dot{q}_0 - \frac{a}{b}\right)e^{-bt}$$

Where a/b is the limit speed.

- Harmonic oscillator:

$$\ddot{q} + \omega^2 q = 0 \rightarrow$$
$$q(t) = \frac{\dot{q}_0}{\omega}\sin(\omega t) + q_0\cos(\omega t)$$
$$= A\sin(\omega t + \phi)$$

Where

$$A = \sqrt{q_0^2 + \frac{\dot{q}_0^2}{\omega^2}}$$
$$\phi = \arccos\frac{\dot{q}_0}{\sqrt{q_0^2\omega^2 + \dot{q}_0^2}}$$

- Charged particle in an electric and magnetic field:

$$\mathbf{F} = \mathbf{qE} + \mathbf{qv} \times \mathbf{B} \quad \text{(Lorentz force)}$$

Ends up creating a spiral with the axes perpendicular to the magnetic field, that with a frequency ω_c called **cyclotronic frequency**, and a constant modulo of the speed.

2.3 Non-inertial reference frames

When applying newton dynamics from the point of view of a non-inertial reference frame, we get fictional forces:

$$\begin{aligned} m\mathbf{a}' = \mathbf{F}' \quad &; \mathcal{R}\mathbf{F} \\ -m\mathcal{R}\mathbf{A} \quad &; \mathbf{F}'_{\text{drag}} \\ -m\dot{\boldsymbol{\omega}}' \times \mathbf{r}' \quad &; \mathbf{F}'_{\text{acimutal}} \\ -2m\boldsymbol{\omega}' \times \mathbf{v}' \quad &; \mathbf{F}'_{\text{Coriolis}} \\ -m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{r}') \quad &; \mathbf{F}'_{\text{centrifugal}} \end{aligned}$$

Then, for a particle on the surface of earth we have:

$$\mathbf{a}' = \frac{\mathbf{1}}{\mathbf{m}}\mathcal{R}\mathbf{F}_{\text{ng}} + \mathbf{g}'_{\text{effective}} - \mathbf{2}\boldsymbol{\omega}' \times \mathbf{v}'$$

where $\mathbf{F}_{\text{ng}}$ is the non-gravitational force and the effective gravitational acceleration is:

$$\mathbf{g}'_{\text{effective}} = (-\omega_{\mathbf{T}}^{\mathbf{2}}\mathbf{R_T}\cos\lambda \operatorname{sen}\lambda)\mathbf{e}'_{\mathbf{y}}$$
$$+(-g + \omega_T^2 R_T \cos^2\lambda)\mathbf{e}'_{\mathbf{z}}$$

Where λ is the latitude.