Discrete Mathematics - CSCE 531 Fall 2018
In Class Work, Day 14 (26 November 2018)

From Section 9.1

1. (Problem 7) Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if:
   a. $x \neq y$

      Reflexive:
      - For all $a \in \mathbb{Z}$ is $(a, a) \in R$?
      - No, because it is false that $a \neq a$.

      Symmetric:
      - For all $a, b \in \mathbb{Z}$ does $(a, b) \in R \to (b, a) \in R$ hold?
      - Yes, because $a \neq b \to b \neq a$.

      Antisymmetric:
      - For all $a, b \in \mathbb{Z}$ does $(a, b) \in R \wedge (b, a) \in R \to a = b$ hold?
      - No, because it is false that $a \neq b \wedge b \neq a \to a = b$.

      Transitive:
      - For all $a, b, c \in \mathbb{Z}$ does $(a, b) \in R \wedge (b, c) \in R \to (a, c) \in R$ hold?
      - No, because it is false that $a \neq b \wedge b \neq c \to a \neq c$.

   b. $xy \geq 1$

Reflexive: no; symmetric: yes; antisymmetric: no; transitive: yes.

   c. $x = y + 1$ or $x = y - 1$

Reflexive: no; symmetric: yes; antisymmetric: no; transitive: no.

   d. $x \equiv y (\mathrm{mod}\, 7)$

Reflexive: yes; symmetric: yes; antisymmetric: no; transitive: yes.

   e. x is a multiple of y

Reflexive: yes; symmetric: no; antisymmetric: yes; transitive: yes.

Can you find an example in which x is a multiple of y, y is a multiple of x, and $x \neq y$? Can you prove that no such example exists?

   f. x and y are both negative or both nonnegative

Reflexive: yes; symmetric: yes; antisymmetric: no; transitive: yes.

g. $x = y^2$

Reflexive: no; symmetric: no; antisymmetric: yes; transitive: no.

h. $x \geq y^2$

Reflexive: no; symmetric: no; antisymmetric: yes; transitive: yes.

From Section 9.3

2. (Problem 1) Represent each of these relations on $\{1,2,3\}$ with a matrix (with the elements of this set listed in increasing order).
   a. $\{(1,1), (1,2), (1,3)\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   b. $\{(1,2), (2,1), (2,2), (3,3)\}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   c. $\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   d. $\{(1,3), (3,1)\}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3. (Problem 9) How many nonzero entries does the matrix representing the relation R on $A = \{1,2,3, \dots ,100\}$ consisting of the first 100 positive integers have if R is:
   a. $\{(a, b) | a > b\}$?

$$\sum_{i=1}^{100} (i - 1)$$

   b. $\{(a, b) | a \neq b\}$?

$$100 \cdot 99$$

c. $\{(a, b) | a = b + 1\}$?

99

d. $\{(a, b) | a = 1\}$?

100

e. $\{(a, b) | ab = 1\}$?

1

4. (Inspired by Problem 13) Let R be the relation represented by the matrix:

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the matrix representing

a. R^{-1} [Note: $(a, b) \in R \leftrightarrow (b, a) \in R^{-1}$. This is the matrix of the inverse of the relation, *not* the (multiplicative) inverse of the matrix of the relation.]

$$M_{R_1^{-1}} = M_{R_1}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b. $\overline{R}$ [Note: $(a, b) \in R \leftrightarrow (a, b) \notin \overline{R}$.]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

c. R^2 [Hint: See pages 178-183 for help with general matrix multiplication and how to use Boolean products to find $R^2 = R \odot R$]

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$