Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 9 (31 Oct 18)

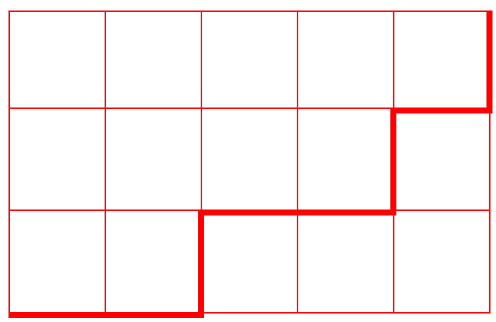
Section 6.4

1. (Problem 7). What is the coefficient of x^9 in $(2-x)^{19}$?

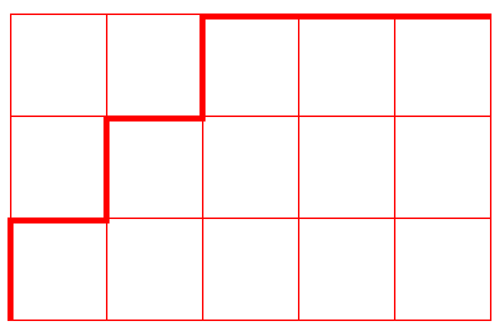
By the Binomial Theorem, the x^9 term is $\binom{19}{9}2^{10}(-x)^9$. Thus, the coefficient is $\binom{19}{9}2^{10}(-1)^9$.

2. (Inspired by Problem 33). Consider the vector (m, n) in the x, y plane where m is an integer describing the non-negative units from the origin in the x direction and n is an integer describing the non-negative units from the origin in the y direction. Now imagine that we are looking at paths along the grid between integer vertices in the x, y plane, from the origin $(0,0)$ to (m, n). Note that only movement to the right, or upward is allowed along this path (no backtracking). Your goal is to figure out how many different unique paths there could be from the origin to (m, n).

   a. For the example destination $(5,3)$, pick 2 unique paths and show how each path can be uniquely represented by a bit string consisting of m 0s and n 1s where a 0 represents a moving one unit to the right and a 1 represents a move one unit upward.



This can be represented by the bit string 00100101.



This can be represented by the bit string 10101000.

   b. Show there are $\binom{m+n}{n}$ paths from $(0,0)$ to (m, n), and compute how many unique paths there are from $(0,0)$ to $(5,3)$.

There is a one-to-one correspondence between paths and bit strings of length $m+n$ containing exactly n 1s, of which there are $\binom{m+n}{n}$. Thus, this is also the number of unique paths. For the example at hand, this is $\binom{8}{3}$.

Section 6.5

3. (Problem 5) How many ways are there to assign three distinguishable jobs (A, B, C) to five distinguishable employees if each employee can be given more than one job?

There are five independent ways to assign each of the jobs, so by the product rule there are 5^3 total ways to assign the jobs.

4. (Inspired by Problem 11) How many different ways are there to choose 8 coins from a piggy bank containing 100 identical pennies and 80 identical nickels? *Note: it's reasonable to ask whether or not order matters. However, by convention, when the objects being selected are stated to be identical, we assume that order does not matter. This is because after we're finished selecting we can't tell the difference between two different orders for selecting the same set of coins.*

The piggy bank contains $100 \geq 8$ pennies and $80 \geq 8$ nickels, so we can choose between 0 and 8 pennies, after which the remainder of the 8 coins must be nickels. Thus, there are 9 ways.

If we could tell the difference between different orders for selecting the same set of coins, there would be 2^8 ways.

5. (Problem 31) How many different strings can be made from the letters in ABRACADABRA, using all 11 letters?

This is a problem of permutations with indistinguishable objects. There are 5 A's, 2 B's, 1 C, 1 D, and 2 R's. We can imagine beginning by choosing 5 of the 11 positions in which to put the A's, then 2 of the 6 remaining positions in which to put the B's, and so forth. Thus, the total number of distinct strings that can be made is

$$\binom{11}{5}\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{2}.$$

6. (Inspired by Problem 45a) How many ways can n indistinguishable copies of the same book be placed on k distinguishable shelves?

This is problem of indistinguishable objects and distinguishable boxes. We can imagine a program consisting of the instructions "place a book on the current shelf" and "move to the next shelf." There must be n instructions of the former type, and $k - 1$ instructions of the latter type (since when we begin we are on the first shelf and we never move past the last shelf). Each such program has a total of $n + k - 1$ instructions, of which n must be chosen to be the first type. Thus, there are $\binom{n+k-1}{n}$ possible programs, and since there is a one-to-one correspondence between such programs and ways to place books on shelves, this is also the number of ways to place the books.

Section 6.4 (cont.)

7. (Problem 25). Let $n \in \mathbb{Z}^+$. Show that: $\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2}\binom{2n+2}{n+1}$.
Hint: Use the following equalities and other algebraic manipulations to write $\binom{2n}{n+1}$ *in the form* $f(n) \cdot \binom{2n}{n}$.

$$1 = \frac{n}{n}$$

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

$$(n+1)! = (n+1) \cdot n!$$

First, observe that since $2n - (n+1) = n - 1$,

$$\begin{aligned}\binom{2n}{n+1} &= \frac{(2n)!}{(n-1)!\,(n+1)!} \\ &= \frac{n}{n}\cdot\frac{1}{(n-1)!}\cdot\frac{(2n)!}{(n+1)!} \\ &= \frac{n}{n\cdot(n-1)!}\cdot\frac{(2n)!}{(n+1)\cdot n!} \\ &= \frac{n}{n+1}\cdot\frac{1}{n\cdot(n-1)!}\cdot\frac{(2n)!}{n!} \\ &= \frac{n}{n+1}\cdot\frac{(2n)!}{(n!)^2}.\end{aligned}$$

Thus

$$\begin{aligned}\binom{2n}{n+1}+\binom{2n}{n} &= \left(\frac{n}{n+1}+1\right)\cdot\frac{(2n)!}{(n!)^2} \\ \binom{2n}{n+1}+\binom{2n}{n} &= \frac{2n+1}{n+1}\cdot\frac{(2n)!}{(n!)^2} \\ \binom{2n}{n+1}+\binom{2n}{n} &= \frac{2n+2}{2n+2}\cdot\frac{2n+1}{n+1}\cdot\frac{(2n)!}{(n!)^2} \\ \binom{2n}{n+1}+\binom{2n}{n} &= \frac{1}{2}\cdot\frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\frac{(2n)!}{(n!)^2} \\ \binom{2n}{n+1}+\binom{2n}{n} &= \frac{1}{2}\frac{(2n+2)!}{[(n+1)!]^2}\end{aligned}$$