Discrete Mathematics - CSCE 531 Fall 2018
In Class Work, Day 15 (28 November 2018)

From Section 9.4

1. (Problem 1) Let R be the relation on the set {0,1,2,3} containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0). Find the
   a. Reflexive closure of R.

   $$\{(0,0),(1,1),(2,2),(3,3),(0,1),(1,2),(2,0),(3,0)\}$$
   $$=\{(0,0),(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(3,3)\}$$

   b. Symmetric closure of R.

   $$\{(0,1),(1,0),(1,1),(1,2),(2,1),(2,0),(0,2),(2,2),(3,0),(0,3)\}$$
   $$=\{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,1),(2,0),(2,2),(3,0)\}$$

2. (Problem 3) Let R be the relation $\{(a,b)|a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

   $$\underline{R \cup \{(b,a)|a \text{ divides } b\} = \{(a,b)|a \text{ divides } b \text{ or } b \text{ divides } a\}}$$

3. (Problem 27b) Use Warshall's algorithm to find the transitive closure of the relation on the set {1,2,3,4} containing the ordered pairs (2,1), (2,3), (3,1), (3,4), (4,1), and (4,3).

$$W_0 = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} w_{11} \vee (w_{11} \wedge w_{11}) & w_{12} \vee (w_{11} \wedge w_{12}) & w_{13} \vee (w_{11} \wedge w_{13}) & w_{14} \vee (w_{11} \wedge w_{14}) \\ w_{21} \vee (w_{21} \wedge w_{11}) & w_{22} \vee (w_{21} \wedge w_{12}) & w_{23} \vee (w_{21} \wedge w_{13}) & w_{24} \vee (w_{21} \wedge w_{14}) \\ w_{31} \vee (w_{31} \wedge w_{11}) & w_{32} \vee (w_{31} \wedge w_{12}) & w_{33} \vee (w_{31} \wedge w_{13}) & w_{34} \vee (w_{31} \wedge w_{14}) \\ w_{41} \vee (w_{41} \wedge w_{11}) & w_{42} \vee (w_{41} \wedge w_{12}) & w_{43} \vee (w_{41} \wedge w_{13}) & w_{44} \vee (w_{41} \wedge w_{14}) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} w_{11} \vee (w_{12} \wedge w_{21}) & w_{12} \vee (w_{12} \wedge w_{22}) & w_{13} \vee (w_{12} \wedge w_{23}) & w_{14} \vee (w_{12} \wedge w_{24}) \\ w_{21} \vee (w_{22} \wedge w_{21}) & w_{22} \vee (w_{22} \wedge w_{22}) & w_{23} \vee (w_{22} \wedge w_{23}) & w_{24} \vee (w_{22} \wedge w_{24}) \\ w_{31} \vee (w_{32} \wedge w_{21}) & w_{32} \vee (w_{32} \wedge w_{22}) & w_{33} \vee (w_{32} \wedge w_{23}) & w_{34} \vee (w_{32} \wedge w_{24}) \\ w_{41} \vee (w_{42} \wedge w_{21}) & w_{42} \vee (w_{42} \wedge w_{22}) & w_{43} \vee (w_{42} \wedge w_{23}) & w_{44} \vee (w_{42} \wedge w_{24}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} w_{11} \vee (w_{13} \wedge w_{31}) & w_{12} \vee (w_{13} \wedge w_{32}) & w_{13} \vee (w_{13} \wedge w_{33}) & w_{14} \vee (w_{13} \wedge w_{34}) \\ w_{21} \vee (w_{23} \wedge w_{31}) & w_{22} \vee (w_{23} \wedge w_{32}) & w_{23} \vee (w_{23} \wedge w_{33}) & w_{24} \vee (w_{23} \wedge w_{34}) \\ w_{31} \vee (w_{33} \wedge w_{31}) & w_{32} \vee (w_{33} \wedge w_{32}) & w_{33} \vee (w_{33} \wedge w_{33}) & w_{34} \vee (w_{33} \wedge w_{34}) \\ w_{41} \vee (w_{43} \wedge w_{31}) & w_{42} \vee (w_{43} \wedge w_{32}) & w_{43} \vee (w_{43} \wedge w_{33}) & w_{44} \vee (w_{43} \wedge w_{34}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} w_{11} \vee (w_{14} \wedge w_{41}) & w_{12} \vee (w_{14} \wedge w_{42}) & w_{13} \vee (w_{14} \wedge w_{43}) & w_{14} \vee (w_{14} \wedge w_{44}) \\ w_{21} \vee (w_{24} \wedge w_{41}) & w_{22} \vee (w_{24} \wedge w_{42}) & w_{23} \vee (w_{24} \wedge w_{43}) & w_{24} \vee (w_{24} \wedge w_{44}) \\ w_{31} \vee (w_{34} \wedge w_{41}) & w_{32} \vee (w_{34} \wedge w_{42}) & w_{33} \vee (w_{34} \wedge w_{43}) & w_{34} \vee (w_{34} \wedge w_{44}) \\ w_{41} \vee (w_{44} \wedge w_{41}) & w_{42} \vee (w_{44} \wedge w_{42}) & w_{43} \vee (w_{44} \wedge w_{43}) & w_{44} \vee (w_{44} \wedge w_{44}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

From Section 9.5

4. (Problem 1) Which of these relations on {0,1,2,3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.
   a. {(0,0), (1,1), (2,2), (3,3)}

Symmetric, reflexive, transitive, so equivalence

   b. {(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)}

Not reflexive (1,1), not transitive [(0,2) and (2,3) but not (0,3)], therefore not an equivalence

   c. {(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)}

Is an equivalence

   d. {(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)}

Not an equivalence because not transitive (missing (1,2), e.g)

   e. {(0,0), (0,1), (0,2), (1,0), (1,1)), (1,2), (2,0), (2,2), (3,3)}

Not symmetric (e.g. (2,1)), not transitive (e.g. (2,1)), therefore not an equivalence

5. (Problem 3) Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.
   a. $\{(f, g) \mid f(1) = g(1)\}$

This is reflexive, since $f(1) = f(1)$ for every function f.

This is symmetric, since $f(1) = g(1)$ implies $g(1) = f(1)$.

This is transitive, since $f(1) = g(1)$ and $g(1) = h(1)$ implies $f(1) = h(1)$.

Thus, this is an equivalence relation.

   b. $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

Not transitive. Counterexample: f is the zero function, g is zero for evens and one for odds, h is the one function.

   c. $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

Not reflexive, not symmetric, not transitive

   d. $\{(f, g) \mid \text{there exists } C \in \mathbb{Z}, \text{such that for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$

Is an equivalence

   e. $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$

Not reflexive. Consider $f(n) = n$. $f(0) = 0 \neq 1 = f(1)$.

Symmetric.

Not transitive. Suppose the following hold:

$$f(0) = g(1) = a$$
$$f(1) = g(0) = b \neq a$$
$$g(0) = h(1)$$
$$g(1) = h(0)$$

Then $h(1) = b \neq a = f(0)$.

6. (Inspired by Problem 25) Let R be the relation on the set of all finite length bit strings such that sRt (i.e. $(s, t) \in R$) if and only if s and t contain the same number of 1s is an equivalence relation. Show that R is an equivalence relation.

Let $n_1(s)$ be the number of 1s in s. Then sRt if and only if $n_1(s) = n_1(t)$.

Reflexive: $n_1(s) = n_1(s)$ so sRs.

Symmetric: Suppose sRt. Then $n_1(s) = n_1(t)$, so $n_1(t) = n_1(s)$, i.e. tRs.

Transitive: Suppose sRt and tRu. Then $n_1(s) = n_1(t)$ and $n_1(t) = n_1(u)$, so $n_1(s) = n_1(u)$, i.e. sRu.

7. (Problem 41) Which of these collections of subsets are partitions of {1,2,3,4,5,6}?
    a. {1,2},{2,3,4},{4,5,6}

This is not a partition because 4 is included in two of the subsets (as well as 2).

    b. {1},{2,3,6},{4},{5}

The subsets form a partition because the subsets are disjoint and their union is {1,2,3,4,5,6}.

    c. {2,4,6},{1,3,5}

This is a partition.

    d. {1,4,5},{2,6}

This is not a partition. The subsets are disjoint, but their union lacks 3, which is an element of the original set.

8. (Problem 43) Which of these collections of subsets are partitions of the set of bit strings of length 8?

    a. The set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01.

Given that we are told the sets are subsets of the set of bit strings of length 8, we conclude that this is a partition. The union of the three subsets is the set of all bit strings of length 8. Also, the three subsets are disjoint, so they form a partition.

    b. The set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11.

This is not a partition because a string could belong to more than one of the subsets (e.g. 11001001 belongs to all four). Therefore the subsets are not disjoint.

    c. The set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11.

Under the same assumption as part (a), this is a partition because the subsets are disjoint and their union is the original set.

    d. The set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00.

This is not a partition because the union of the subsets is not the original set. In particular, strings ending with 01 and 10 are not included in any of the subsets.

e. The set of bit strings that contain $3k$ ones for some nonnegative integer k; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

This is a partition, because strings containing 0, 3, or 6 ones belong to the first subset (only); strings containing 1, 4, or 7 ones belong to the second subset (only); and strings containing 2, 5, or 8 ones belong to the third subset (only). Every bit string of length 8 falls into one of these categories and therefore belongs to one of the subsets. Thus, the subsets are disjoint and their union is the original set.

From Section 9.6

9. (Problem 1) Which of these relations on $\{0,1,2,3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.
   a. $\{(0,0), (1,1), (2,2), (3,3)\}$

This is a partial ordering because it is reflexive, antisymmetric and transitive.

   b. $\{(0,0), (1,1), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

This is not transitive (counterexample: (3,2) and (2,0) but not (3,0)) and not antisymmetric (counterexample: (2,3) and (3,2)).

   c. $\{(0,0), (1,1), (1,2), (2,2), (3,3)\}$

This is a partial ordering.

   d. $\{(0,0), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

This is a partial ordering.

   e. $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

This is not a partial ordering because it is neither transitive (counterexample: (2,0) and (0,1) but not (2,1)) nor antisymmetric (counterexample: (2,0) and (0,2)).

10. (Problem 5) Which of these are posets?
   a. $(\mathbb{Z}, =)$

This is a poset because equality on the integers is reflexive, antisymmetric, and transitive.

   b. $(\mathbb{Z}, \neq)$

This is not a poset because inequality on the integers is neither reflexive, antisymmetric, nor transitive.

c. $(\mathbb{Z}, \geq)$

This is a poset because the inclusive inequality relation on the integers is reflexive, antisymmetric, and transitive.

d. $(\mathbb{Z}, \nmid)$

This is not a poset because the “does not divide” relation on the integers is not reflexive (every integer divides itself).

11. (Problem 33) Answer these questions for the poset $(\{3,5,9,15,24,45\}, |)$.

a. Find the maximal elements.

24 and 45

b. Find the minimal elements.

3 and 5

c. Is there a greatest element?

No, because there are two maximal elements.

d. Is there a least element?

No, because there are two minimal elements.

e. Find all upper bounds of $\{3,5\}$.

15 and 45

f. Find the least upper bound of $\{3,5\}$, if it exists.

15

g. Find all lower bounds of $\{15,45\}$.

3, 5, and 15

h. Find the greatest lower bound of $\{15,45\}$, if it exists.

15