Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 03 (10 October 2018)

Notation

- $\mathcal{N} = \{0,1,2,\ldots\}$ is the set of natural numbers.
- $\mathcal{R} =$ is the set of real numbers.
- $\mathcal{R}^+$ is the set of positive real numbers.
- $\mathcal{Z} = \{\ldots,-1,0,1,\ldots\}$ is the set of integers.
- $\mathcal{Z}^+ = \{1,2,3,\ldots\}$ is the set of positive integers.

From Section 1.7

1. (Problem 1) Use a direct proof to show that the sum of two odd integers is even. *Hint: use the fact that odd integers can be represented by the formula $2k+1$ where $k \in \mathcal{Z}$, while even integers can be represented as $2k$ where $k \in \mathcal{Z}$.*

   Let p represent the proposition "$m \in \mathcal{Z}$ is odd," q represent the proposition "$n \in \mathcal{Z}$ is odd," and r represent the proposition "$m + n \in \mathcal{Z}$ is even." We need to show that $p \wedge q \to r$.

   Proof: Suppose m and n are odd. Then $m = 2k+1$ and $n = 2l+1$ for some $k, l \in \mathcal{Z}$. Therefore, $m + n = (2k+1) + (2l+1) = 2k + 2l + 2 = 2(k+l+1)$, which is even because $k + l + 1 \in \mathcal{Z}$. Therefore, the sum of two odd integers is an even integer.■

2. (Problem 7) Use a direct proof to show that every odd integer is the difference of two squares. *Hint: represent an odd integer, and then use algebraic factorization rules on polynomials.*

   Let p represent the proposition "$n \in \mathcal{Z}$ is odd" and q represent the proposition "n is the difference of two squares." We must show $p \to q$.

   Proof: Suppose $n \in \mathcal{Z}$ is odd. Then $n = 2k+1$ for some $k \in \mathcal{Z}$. Furthermore,

   $$n = 2k + 1 + k^2 - k^2 = (k^2 + 2k + 1) - (k^2) = (k+1)^2 - (k)^2.$$

   Because $k \in \mathcal{Z}$ and $k + 1 \in \mathcal{Z}$, it follows that n is the difference of two squares. ■

3. (Inspired by Problem 15) Use proof by contraposition to show that if $x + y \geq 2$, where $x, y \in \mathcal{R}$, then (either) $x \geq 1$ or $y \geq 1$ (or both).

   Let p be the proposition "$x + y \geq 2$" and q the proposition "$x \geq 1 \vee y \geq 1$." We must prove $p \to q$ by contraposition, so we must show $\neg q \to \neg p$.

   By DeMorgan's Law, $\neg q \equiv x < 1 \wedge y < 1$. Also, $\neg p \equiv x + y < 2$.

   Proof: Assume $x < 1 \wedge y < 1$. Then $x + y < 1 + 1 = 2$. ■

4. (Inspired by Problem 17b) Prove by contradiction that if $n \in \mathcal{Z}$ and $n^3 + 5$ is odd then n is even.

   Let p represent the proposition "$n \in \mathcal{Z}$," q the proposition "$n^3 + 5$ is odd," and r the proposition "n is even." Then we must prove $p \wedge q \to r$.

To prove $p \wedge q \rightarrow r$ by contradiction, we must show $\neg(p \wedge q \rightarrow r\,) \rightarrow F$. This is logically equivalent to $p \wedge q \wedge \neg r \rightarrow F$.

We begin by assuming that $p \wedge q \wedge \neg r$ is true, i.e. that $n \in \mathcal{Z}$, $n^3 + 5$ is odd, and n is not even. Because n is not even, it is odd, so $n = 2k + 1$ for some $k \in \mathcal{Z}$. It follows that $n^3 + 5 = (2k+1)^3 + 5 = (8k^3 + 12k^2 + 6k + 1) + 5 = 2(4k^3 + 6k^2 + 3k + 3)$, which is even.

We have shown that $n^3 + 5$ is not odd, i.e. that $\neg q$ is true. However, applying the simplification rule of inference to our initial assumption allows us to conclude that q is true, so we have shown that $p \wedge q \wedge \neg r \rightarrow \neg q \wedge q$. Applying the negation law to the conclusion of this conditional statement yields $p \wedge q \wedge \neg r \rightarrow F$, which completes the proof by contradiction of $p \wedge q \rightarrow r$. ■

Shorter form:

Proof: Assume that $n \in \mathcal{Z}$, $n^3 + 5$ is odd, and n is not even. Then $n = 2k + 1$ for some $k \in \mathcal{Z}$. It follows that $n^3 + 5 = (2k+1)^3 + 5 = (8k^3 + 12k^2 + 6k + 1) + 5 = 2(4k^3 + 6k^2 + 3k + 3)$, which is even. But this contradicts the assumption that $n^3 + 5$ is odd, which completes the proof by contradiction. ■

5. (Problem 27) Prove that if $n \in \mathcal{Z}^+$, then n is odd if and only if $5n + 6$ is odd. *Note that "iff" is a biconditional/bi-implication and should be proven by proving each of the two implications separately.*

   Proof: Let p be the proposition "$n \in \mathcal{Z}^+$," q the proposition "n is odd," and r the proposition "$5n + 6$ is odd." We must prove that $p \rightarrow (q \leftrightarrow r)$, which is logically equivalent to $p \rightarrow [(q \rightarrow r) \wedge (r \rightarrow q)]$, which is in turn equivalent to $[(p \wedge q) \rightarrow r] \wedge [(p \wedge r) \rightarrow q]$ (details are left to the reader).

   First, assume that $(p \wedge q)$ holds, i.e. that $n \in \mathcal{Z}^+$ is odd. Then $n = 2k + 1$ for some $k \in \mathcal{Z}$, from which it follows that $5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1$, which is odd. We have shown $(p \wedge q) \rightarrow r$.

   Now assume that $(p \wedge r)$ holds, i.e. that $n \in \mathcal{Z}^+$ and $5n + 6$ is odd. Then $5n + 6 = 2k + 1$ for some $k \in \mathcal{Z}$. Therefore $5n = 2k + 1 - 6 = 2k - 5$. Direct proof doesn't seem to be leading anywhere, so let's try contraposition, i.e. show that $\neg q \rightarrow \neg(p \wedge r)$. This is logically equivalent to $\neg q \rightarrow (\neg p \vee \neg r)$. Assume that n is not odd, i.e. that n is even. Then $n = 2k$ for some $k \in \mathcal{Z}$, so $5n + 6 = 10k + 6 = 2(5k + 3)$, which is even because $5k + 3 \in \mathcal{Z}$. We have shown that $\neg q \rightarrow \neg r$. It follows that $\neg q \rightarrow (\neg p \vee \neg r)$. ■

   Shorter form:

   Proof: First, assume that $n \in \mathcal{Z}^+$ is odd. Then $n = 2k + 1$ for some $k \in \mathcal{Z}$, from which it follows that $5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1$, which is odd. Now assume that $n \in \mathcal{Z}^+$ is not odd. Then $n = 2k$ for some $k \in \mathcal{Z}$, so $5n + 6 = 10k + 6 = 2(5k + 3)$, which is even because $5k + 3 \in \mathcal{Z}$. ■

<u>From Section 1.8</u>

6. (Problem 1) Prove that $n^2 + 1 \geq \cancel{2}2^n$ when n is a positive integer with $1 \leq n \leq 4$. *Hint: proof by cases.*

We will use an exhaustive proof, which is a special type of proof by cases.

Proof: There are four cases to consider:

If $n = 1$ then $n^2 + 1 = 1^2 + 1 = 2 \geq 2 = 2^1 = 2^n$.
If $n = 2$ then $n^2 + 1 = 2^2 + 1 = 5 \geq 4 = 2^2 = 2^n$.
If $n = 3$ then $n^2 + 1 = 3^2 + 1 = 10 \geq 8 = 2^3 = 2^n$.
If $n = 4$ then $n^2 + 1 = 4^2 + 1 = 17 \geq 16 = 2^4 = 2^n$.

Thus, in every case for which $n \in \mathcal{Z}^+$ with $1 \leq n \leq 4$, it is the case that $n^2 + 1 \geq 2^n$. ■

7. (Problem 11) Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube. *Hint: find an example that meets the criteria.*

   There are at least three such pairs: -1 and 0, 0 and 1 (which works both ways), and 8 and 9.

   Proof: One such pair is $0^2 = 0 \in \mathcal{Z}$ and $1^3 = 1 \in \mathcal{Z}$. ■

8. (Problem 41) Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite). *Hint: use "without loss of generality" and then break the problem into sub-parts.*

   Proof: Without loss of generality (WLOG), assume the removed corners are the top left and top right squares (if this is not the case, we can reorient the board so that it is). Then the top row has six remaining squares, which we can tile with three dominoes. Each of the other rows has eight squares, so we can tile each of them with four dominoes. ■

9. (Problem 43) Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares. *Hint: create an interim step by proving a particular property of the board first, and then show how to fill in all such boards with dominoes.*

   Proof: We are given that the total number of squares is even. Also, because the board is rectangular, the total number of squares is mn, where m is the number of rows and n is the number of columns (or ranks and files for the chess aficionados). If both m and n were odd, then mn would also be odd, so either m or n is even (or both). WLOG, assume that m is even (otherwise we can reorient the board so that it is). Then $m = 2k$ for some $k \in \mathcal{Z}$, and each of the n columns can be tiled with k tiles. ■