Discrete Mathematics - CSCE 531 Fall 2018
In-Class Exercises, Day 4 (15 Oct 18)
Functions, Sequences, and Cardinality

From Section 2.3

1. (Inspired by Problem 3) Determine whether or not each of the following defines a function from the set A of finite bit strings to $\mathcal{Z}$. Explain your reasoning.

   a. $f(S)$ is the position of a 0 bit in S.

Given a string that contains no 0 bits, the rule "the position of a 0 bit" does not assign an element of $\mathcal{Z}$. Also, given a string that contains multiple 0 bits, the rule assigns multiple elements of $\mathcal{Z}$. Thus, f is not an assignment of exactly one element of $\mathcal{Z}$ to each element of A, i.e. not a function $f: A \rightarrow \mathcal{Z}$.

   b. $f(S)$ is the number of 1 bits in S.

Every finite bit string contains a well-defined integer number of 1 bits. Thus, f is an assignment of exactly one element of $\mathcal{Z}$ to each element of A, i.e. $f: A \rightarrow \mathcal{Z}$ is a function.

   c. $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string (the string with no bits).

The mapping f is not a function, because there exist non-empty strings consisting entirely of 0 bits.

2. (Inspired by Problems 10 and 11) Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one. Also determine which ones are onto:

   a. $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d$

If $x, y \in \{a, b, c, d\}$, then $f(x) = f(y) \rightarrow x = y$. Therefore, f is one-to-one. Also, $\forall x \exists y f(y) = x$, where the domain for each quantification is $\{a, b, c, d\}$. Therefore f is onto.

   b. $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$

$f(a) = b = f(b)$, but $a \neq b$, so f is not one-to-one. Also, $\nexists x f(x) = a$, where the domain is $\{a, b, c, d\}$, so f is not onto.

   c. $f(a) = d,\ f(b) = b,\ f(c) = c,\ f(d) = d$

$f(a) = d = f(d)$, but $a \neq d$, so f is not one-to-one. Also, $\nexists x f(x) = a$, where the domain is $\{a, b, c, d\}$, so f is not onto.

3. (Inspired by Problem 15) For each of the following, determine whether or not $f: \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ is onto.
   a. $f(m, n) = m + n$

Given $x \in \mathcal{Z}$, take $m = x$ and $n = 0$. Then $f(m, n) = m + n = x + 0 = x$. Therefore, f is onto.

   b. $f(m, n) = m^2 + n^2$

The function f is not onto. Counterexamples: 3 cannot be expressed as the sum of perfect squares, so there do not exist m and n such that $f(m, n) = m^2 + n^2 = 3$. Similarly for any negative integer and many other positive integers.

   c. $f(m, n) = m$

Given $x \in \mathcal{Z}$, take $m = x$ and $n = 0$. Then $f(m, n) = m = x$. Therefore, f is onto.

   d. $f(m, n) = |n|$

The function f is not onto. Proof: Let $x < 0$ and $n \in \mathcal{Z}$. Then $|n| \geq 0$, so $x \neq |n|$. Thus, there do not exist m and n such that $f(m, n) = |n| = x$. ■

   e. $f(m, n) = m - n$

Given $x \in \mathcal{Z}$, take $m = x$ and $n = 0$. Then $f(m, n) = m - n = x - 0 = x$. Therefore, f is onto.

4. (Inspired by Problem 33) Suppose that $g: A \to B$ and $f: B \to C$ are functions.
   a. Show that if f and g are both one-to-one then $f \circ g$ is also one-to-one.

In order to prove that $f \circ g$ is one-to-one, we must show that
$f \circ g(a) = f \circ g(b)$ implies $a = b$.

Proof: By the definition of function composition,

$$f\big(g(a)\big) = f \circ g(a) \text{ and } f \circ g(b) = f\big(g(b)\big).$$

Therefore,

$$f \circ g(a) = f \circ g(b) \to f\big(g(a)\big) = f\big(g(b)\big).$$

However, f is one-to-one, so

$$f\big(g(a)\big) = f\big(g(b)\big) \to g(a) = g(b).$$

Furthermore, g is one-to-one, so

$$g(a) = g(b) \to a = b.$$

Thus,

$$f \circ g(a) = f \circ g(b) \to a = b,$$

i.e. $f \circ g$ is one-to-one. ■

   b. Show that if f and g are both onto then $f \circ g$ is also onto.

Proof: Let $c \in C$. Then, because f is onto, there exists $b \in B$ such that $f(b) = c$. Also, because g is onto, there exists $a \in A$ such that $g(a) = b$. Thus $f \circ g(a) = f(g(a)) = f(b) = c$. Since c is arbitrary and we have shown there is an element $a \in A$ such that $f \circ g(a) = c$, it follows that $f \circ g$ is onto.■

From Section 2.4

5. (Problem 13) Yes or No: Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if:

   a. $\forall n: a_n = 0$?

$$\begin{aligned} a_n &= 8 \cdot 0 - 16 \cdot 0 \\ &= 0 \end{aligned}$$

So, “Yes”

   b. $a_n = 1$?

$$\begin{aligned} a_n &= 8 \cdot 1 - 16 \cdot 1 \\ &= -8 \\ &\neq 1 \end{aligned}$$

So, “No”

   c. $a_n = 2^n$?

$$\begin{aligned} a_n &= 8 \cdot 2^{n-1} - 16 \cdot 2^{n-2} \\ &= 4 \cdot 2^n - 4 \cdot 2^n \\ &= 0 \\ &\neq 2^n \end{aligned}$$

So, “No”

   d. $a_n = 4^n$?

$$\begin{aligned} a_n &= 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2} \\ &= 2 \cdot 4^n - 4^n \\ &= 4^n \end{aligned}$$

So, “Yes”

   e. $a_n = n4^n$?

$$\begin{aligned} a_n &= 8 \cdot (n-1) \cdot 4^{n-1} - 16 \cdot (n-2) \cdot 4^{n-2} \\ &= 2 \cdot (n-1) \cdot 4^n - 1 \cdot (n-2) \cdot 4^n \\ &= [(2n-2) - (n-2)] \cdot 4^n \\ &= n4^n \end{aligned}$$

So, “Yes”

From Section 2.5

6. (Problem 3) Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between $\mathcal{N}$ and that set.
   a. All bit strings not containing the bit 0

This set is countable. The function $f(n) = 1^n$ is a one-to-one correspondence (where 1^n denotes the concatenation of n instances of the bit 1).

   b. All positive rational numbers that cannot be written with denominators less than 4

This set is countable. Follow the procedure described in Example 4 of Section 2.5, but omit the terms with denominators less than 4 (as well as any terms equal to them, which would have been omitted anyway).

   c. The real numbers not containing 0 in their decimal representation

This set is uncountable. We can directly apply Cantor's diagonalization argument.

   d. The real numbers containing a finite number of 1s in their decimal representation.

This set is uncountable. It includes as a subset the real numbers for which the number of 1s is 0. As with the previous problem, we can directly apply Cantor's diagonalization argument to this subset to show that it is uncountable, and therefore the set in question is also uncountable.

7. (Inspired by problem 11) For each of the descriptors below, give an example of two uncountable sets A and B such that the descriptor applies to $A \cap B$.
   a. Finite

$A = [0,1), B = (-1,0], A \cap B = \{0\}$.

   b. Countably infinite

$A = (0,1) \cup \mathcal{Z}, B = (-1,0) \cup \mathcal{Z}, A \cap B = \mathcal{Z}$.

   c. Uncountable

$A = (0,2), B = (1,3), A \cap B = (1,2)$.