Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 10 (5 November 2018)

Unless otherwise stated, assume that all dice, coins, etc. are fair.

From Section 7.1

1. (Problem 5) What is the probability that the sum of the numbers on two dice is even when they are rolled?

$$S = \{(1,1), (1,2), \ldots, (6,6)\}$$
$$E = \{(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), \ldots, (6,6)\}$$
$$|S| = 6 \cdot 6 = 36$$
$$|E| = 6 \cdot 3 = 18$$
$$p(E) = \frac{|E|}{|S|} = \frac{18}{36} = \frac{1}{2}$$

2. (Problem 7) What is the probability that when a coin is flipped 6 times in a row it lands heads up every time?

$$S = \{HHHHHH, HHHHHT, \ldots, TTTTTT\}$$
$$E = \{HHHHHH\}$$
$$|S| = 2^6$$
$$|E| = 1$$
$$p(E) = \frac{|E|}{|S|} = \frac{1}{2^6} = 2^{-6}$$

3. (Problem 11) What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?

$$S = \{\text{all 5 card subsets of a 52 card deck}\}$$
$$E = \{\{2\diamondsuit, 3\spadesuit, 6\heartsuit, 10\clubsuit, \mathrm{K}\heartsuit\}\}$$
$$|S| = C(52,5)$$
$$|E| = 1$$
$$p(E) = \frac{|E|}{|S|} = \frac{1}{C(52,5)}$$

4. (Inspired by Problem 29) In a super lottery, players win a fortune if they choose the eight numbers selected (without replacement) by a computer from the positive integers not exceeding 100. What is the probability that a player wins this super lottery?

The probability that a player wins this super lottery (or, more accurately, the probability of a ticket being a winning ticket, since a player can have multiple tickets) is the probability that the eight numbers selected by the computer match the eight numbers on the ticket.

$$
\begin{aligned}
&S = \{\text{all subsets of 8 integers from } \{1,2,\ldots,100\}\}\\
&E = \{\text{the set of 8 integers on the ticket}\}\\
&|S| = C(100,8)\\
&|E| = 1\\
&p(E) = \frac{|E|}{|S|} = \frac{1}{C(100,8)}
\end{aligned}
$$

5. (Problem 37) What is more likely: rolling a total of 9 when two dice are rolled, or rolling a total of 9 when three dice are rolled?

We first calculate the probability of rolling a total of nine on two dice.

$$
\begin{aligned}
&S = \{(1,1),\ldots,\}\\
&E = \{(3,6),(4,5),(5,4),(6,3)\}\\
&|S| = 36\\
&|E| = 4\\
&p(E) = \frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}
\end{aligned}
$$

We next calculate the probability of rolling a total of nine on three dice. To determine the cardinality of the event space, we imagine first rolling two distinct dice and then counting the ways that we can roll the remaining die to obtain the desired total. There are 36 outcomes of rolling two distinct dice. For the outcomes in

$$\{(1,1),(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\},$$

of which there are 11, there is no way to roll the third die to obtain a total of nine. For each of the remaining outcomes of rolling two distinct dice, there is exactly one way to roll the third die to obtain that total.

$$
\begin{aligned}
&S = \{(1,1,1),\ldots,\}\\
&E = \{(1,5,3),(1,2,6),(1,4,4),\ldots\}\\
&|S| = 6^3 = 216\\
&|E| = 36 - 11 = 25\\
&p(E) = \frac{|E|}{|S|} = \frac{25}{216}
\end{aligned}
$$

Because $\frac{25}{216} > \frac{24}{216} = \frac{1}{9}$, it is more likely to roll a total of nine on three dice than on two dice.

6. (Problem 39) Explain what is wrong with the statement that in the Monty Hall Three-Door Puzzle, the probability that the prize is behind the first door you select and the probability that the prize is behind the other of the two doors that Monty does not open are both $\frac{1}{2}$ because there are two doors left.

The statement neglects the fact that the probabilities of the relevant events are not equal. In particular, without loss of generality (WLOG), assume you choose the first door. Let (i, j) represent the outcome that the prize is behind door i and Monty opens door j. Then the sample space is

$$S = \{(1,2), (1,3), (2,3), (3,2)\}$$

If the prize is behind the door you choose, then Monty randomly opens one of the other two doors. Thus, the probabilities of the individual events for this case are given by the probability that the prize is behind the door you chose, which is $\frac{1}{3}$, multiplied by the probability that Monty opens a specific door, which is $\frac{1}{2}$.

$$\mathrm{p}(1,2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$
$$\mathrm{p}(1,3) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

In contrast, if the prize is not behind the door you choose, then Monty has no choice in which door to open. Thus, the probabilities for the events in this case are just

$$\mathrm{p}(2,3) = \frac{1}{3}$$
$$\mathrm{p}(3,2) = \frac{1}{3}$$

The events resulting in you winning the prize by sticking with your original choice are

$$E_{stick} = \{(1,2), (1,3)\}$$

for which the total probability is

$$p(E_{stick}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

The events resulting in you winning the prize by switching doors are

$$E_{change} = \{(2,3), (3,2)\}$$

for which the total probability is

$$p(E_{change}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

With this analysis in mind, we can see that the strategy of sticking with your original choice wins exactly when you choose the right door to begin with, whereas the strategy of switching doors wins exactly when you choose a wrong door to begin with. The latter condition is twice as likely, and therefore the latter strategy is twice as likely to result in you winning the prize.

Note that on the actual game show, Monty Hall was at liberty to vary from the formal rules of this puzzle (see https://en.wikipedia.org/wiki/Monty_Hall_problem#History).

From Section 7.2

7. (Problem 5) A pair of dice is loaded. The probability that a 4 appears on the first die is $\frac{2}{7}$, and the probability that a 3 appears on the second die is $\frac{2}{7}$. Other outcomes for each die appear with probability $\frac{1}{7}$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

$$S = \{(1,1), \dots, \}$$
$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$
$$p(1,6) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$$
$$p(2,5) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$$
$$p(3,4) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$$
$$p(4,3) = \frac{2}{7} \cdot \frac{2}{7} = \frac{4}{49}$$
$$p(5,2) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$$
$$p(6,1) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$$
$$p(7) = p(1,6) + p(2,5) + (3,4) + p(4,3) + (5,2) + p(6,1) = \frac{9}{49}$$

8. (Problem 7) What is the probability of these events when we randomly select a permutation of {1,2,3,4}?

In each case, $|S| = 24$ because the sample space is

$$\begin{aligned} S = \{&1234,1243,1324,1342,1423,1432,\\ &2134,2143,2314,2341,2413,2431,\\ &3124,3142,3214,3241,3412,3421,\\ &4123,4132,4213,4231,4312,4321\}. \end{aligned}$$

a. 1 precedes 4.

$$\begin{aligned} E &= \{1234,1243,1324,1342,1423,1432,\\ &\quad 2134,2143,2314,3124,3142,3214\} \\ |E| &= 12 \\ p(E) &= \frac{12}{24} = \frac{1}{2} \end{aligned}$$

b. 4 precedes 1.

$$\begin{aligned} E &= \{2341,2413,2431,3241,3412,3421,\\ &\quad 4123,4132,4213,4231,4312,4321\} \\ |E| &= 12 \\ p(|E|) &= \frac{12}{24} = \frac{1}{2} \end{aligned}$$

c. 4 precedes 1 AND 4 precedes 2

$$\begin{aligned} E &= \{3412,3421,4123,4132,4213,4231,4312,4321\} \\ |E| &= 8 \\ p(|E|) &= \frac{8}{24} = \frac{1}{3} \end{aligned}$$

d. 4 precedes 1, AND 4 precedes 2, AND 4 precedes 3.

$$\begin{aligned} E &= \{4123,4132,4213,4231,4312,4321\} \\ |E| &= 6 \\ p(|E|) &= \frac{6}{24} = \frac{1}{4} \end{aligned}$$

e. 4 precedes 3 AND 2 precedes 1.

$$\begin{aligned} E &= \{2143,2413,2431,4213,4231,4321\} \\ |E| &= 6 \\ p(|E|) &= \frac{6}{24} = \frac{1}{4} \end{aligned}$$