Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 7 (24 October 2018)

From Section 5.2

1. (Problem 3) Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.

   a. Show statements $P(8)$, $P(9)$, and $P(10)$, are true, completing the basis step of the proof.

      $P(8)$ is true, because postage of 8 cents can be formed using one 3-cent stamp and one 5-cent stamp. $P(9)$ is true because postage of 9 cents can be formed using three 3-cent stamps. $P(10)$ is true because postage of 10 cents can be formed using two 5-cent stamps.

   b. What is the inductive hypothesis of the proof?

      The inductive hypothesis is that $P(j)$ is true for integers with $8 \leq j \leq k$, that is, the assumption that postage of j cents can be formed using a combination of 3-cent and 5-cent stamps whenever j is an integer of at least 8 and not exceeding k.

   c. What do you need to prove in the inductive step?

      To complete the inductive step, it must be shown that $P(k+1)$ is true under the assumption of the inductive hypothesis, that is, that it is the case that postage of $k+1$ cents can be formed using a combination of 3-cent and 5-cent stamps under the assumption that postage of j cents can be formed using a combination of 3-cent and 5-cent stamps whenever j is an integer of at least 8 and not exceeding k.

   d. Complete the inductive proof for $k \geq 10$.

      For $k \geq 10$, we have $8 \leq k - 2$. Also, obviously, $k - 2 \leq k$. Thus, by the inductive hypothesis, $P(k-2)$ holds, i.e. postage of $k - 2$ cents can be formed using 3-cent and 5-cent stamps. Therefore, postage of $k + 1$ cents can be formed by adding one more 3-cent stamp to the postage used to form $k - 2$ cents.

   e. Explain why these steps show that this statement is true whenever $n \geq 8$.

      We have shown that $P(8)$, $P(9)$, and $P(10)$ are true. We have also shown that $\bigwedge_{8 \leq j \leq k} P(j) \rightarrow P(k+1)$. It follows from the principle of strong mathematical induction that $P(n)$ is true whenever $n \geq 8$. ■

2. (Problem 7) Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

   Answer: Any nonnegative integer number of dollars except 1 and 3 can be formed using just two-dollar and five-dollar bills.

   Proof: Let $P(n)$ be the proposition "n dollars can be formed using just two-dollar and five-dollar bills."

   Base case: $P(4)$ is true because a four dollar amount can be formed with a pair of two-dollar bills and no five-dollar bills. $P(5)$ is true because a five dollar amount can be formed with a single five-dollar bill and no two-dollar bills.

   Inductive hypothesis: The inductive hypothesis is that $P(j)$ is true for all integers with $4 \leq j \leq k$, that is, the assumption that amounts of j dollars can be formed using just two-dollar and five-dollar bills whenever j is an integer of at least 4 and not exceeding k.

Inductive step: To complete the inductive step, it must be shown that $P(k+1)$ is true under the assumption of the inductive hypothesis, that is, that it is the case that amounts of $k+1$ dollars can be formed using just two-dollar and five-dollar bills.

For $k \geq 5$, we have $k-1 \geq 4$. Thus, by the inductive hypothesis, $P(k-1)$ holds, i.e. amounts of $k-1$ dollars can be formed using just two-dollar and five-dollar bills. Therefore, an amount of $k+1$ dollars can be formed by adding a single two-dollar bill to the bills used to form $k-1$ dollars.

We have shown that $P(4)$ and $P(5)$ are true. We have also shown that $\bigwedge_{4\leq j\leq k} P(j) \rightarrow P(k+1)$. It follows from the principle of strong mathematical induction that $P(n)$ is true for $n \geq 4$. To complete the proof, we observe that $P(0)$ is true because a zero dollar amount can be formed with no two-dollar bills and no five-dollar bills and $P(2)$ is true because a two dollar amount can be formed with a single two-dollar bill and no five-dollar bills. ■

3. (Problem 11) Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j+2$, or $4j+3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j+1$ for some nonnegative integer j. *Hint: You may need to think in terms of current player and next player since "first" and "second" are absolute and could be confusing depending on whose turn it is.*

Proof: We are given that both players use the best strategy possible. We must prove that under this condition, if the game begins with n matches then the first player loses if $n \bmod 4 = 1$ and wins otherwise. We will have done this if we prove the slightly more general statement

$$\forall n \in \mathbb{Z}^+[P(n)]$$

where $P(n)$ is the proposition "if n matches are left at the beginning of a player's turn, the player loses if $n \bmod 4 = 1$ and wins otherwise."

Note that $P(n)$ is equivalent to the compound proposition $M(n) \leftrightarrow L(n)$ where $M(n)$ is the proposition "$n \bmod 4 = 1$" and $L(n)$ is the proposition "a player who starts with n matches loses."

Base case: If $n = 1$ match is left at the beginning of a player's turn (and each player plays the best strategy possible), it is the case that $n \bmod 4 = 1$. It is also the case that the player must take the last match, thereby losing.

If $n \in \{2,3,4\}$ matches are left at the beginning of a player's turn then it is not the case that $n \bmod 4 = 1$. Also, the player can take $n-1$ matches (either 1, 2, or 3), leaving a single match at the beginning of the other player's turn. We have already shown that $P(1)$ is true, so the other player loses, i.e. the current player wins.

Inductive hypothesis: The inductive hypothesis is that $P(j)$ is true for all integers with $1 \leq j \leq k$, that is, the assumption that if j matches are left at the beginning of a player's turn and each player plays the best strategy possible, the player loses if $j \bmod 4 = 1$ and wins otherwise.

Induction step: To complete the inductive step, it must be shown that $P(k+1)$ is true under the assumption of the inductive hypothesis, that is, that if $k+1$ matches are left at the beginning of a player's turn and each player plays the best strategy possible, the player loses if $(k+1) \bmod 4 = 1$ and wins otherwise.

Observe that because $k-3 \leq k$, the inductive hypothesis applies, i.e. if $k-3$ matches are left at the beginning of a player's turn and each player plays the best strategy possible, the player loses if $(k-3) \bmod 4 = 1$ and wins otherwise.

First, suppose $(k+1) \bmod 4 = 1$. The player must take m matches, where $m \in \{1,2,3\}$. The other player can then take $4-m$ matches, since $4-m \in \{3,2,1\}$, leaving $k+1-m-(4-m) = k-3$ matches at the beginning of the current player's next turn. Since $(k+1) \bmod 4 = 1$, it is also the case that $(k-3) \bmod 4 = 1$, and by the inductive hypothesis the current player loses.

Now suppose $(k+1) \bmod 4 \neq 1$. Then $k \bmod 4 \neq 0$, so $k = 4s + r$ where $s \in \mathbb{N}$ and $r \in \{1,2,3\}$. The current player can choose to take r matches, leaving $(k+1) - r = (4s + r + 1) - r = 4s + 1 \equiv 1 (\bmod 4)$ matches at the beginning of the next player's turn. ■

4. (Inspired by Problem 25) Suppose that $P(n)$ is a propositional function. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer if

   a. $P(1)$ is true; for all nonnegative integers n, if and only if $P(n)$ is true, then $P(n+2)$ is true.

Answer: $P(n)$ is true for all positive odd integers.

Proof: Consider the propositional function $Q(i)$ which is defined to be true exactly when $P(2i+1)$ is true. Then, because $P(1)$ is true, so is $Q(0)$.

Furthermore, if $Q(k)$ is true, so is $P(2k+1)$. By letting $j = 2k+1$ so that $P(j)$ is true, we see that $P(j+2) = P(2k+3)$ is true, which finally means that $Q(k+1)$ is true.

By the principle of mathematical induction, $Q(i)$ is true for all nonnegative integers, which means that $P(2i+1)$ is true for all nonnegative integers, i.e. $P(n)$ is true for all positive odd integers.

   b. $P(1)$ and $P(2)$ are true; for all positive integers n, if $P(n)$ and $P(n+1)$ is true, then $P(n+2)$ is true.

   c. $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

Define $Q(n)$ to true exactly when is true. Then, because is true, so is . Furthermore, if $Q(n)$ is true, so is , which means that is true, which finally means that is true. By the principle of mathematical induction, $Q(n)$ is true for , which means that $P(n)$ is true whenever .

   d. $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers.