Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 11 (7 November 2018)

From Section 7.3

1. (Problem 1). Suppose that E and F are events in a sample space, $p(E) = \frac{1}{3}$, $p(F) = \frac{1}{2}$, and $p(E|F) = \frac{2}{5}$. Find $p(F \mid E)$.

By Bayes' Theorem,

$$p(F|E) = \frac{p(F) \cdot p(E|F)}{p(E)} = \frac{1/2 \cdot 2/5}{1/3} = \frac{3}{5}.$$

2. (Inspired by Problem 5) Suppose that 8.0% of Russian athletes use steroids, that a Russian athlete who uses steroids tests positive 96% of the time, and that a Russian athlete who does not use steroids tests positive 9.0% of the time. What is the approximate numerical value of the probability that a randomly selected Russian athlete who tests positive actually uses steroids?

Let S be the event that a randomly selected Russian athlete uses steroids. Then $p(S) = 0.08$. Let T be the event that a randomly selected Russian athlete tests positive. Then $p(T|S) = 0.96$ and $P(T|\overline{S}) = 0.09$. By Bayes' Theorem, the probability that a randomly selected Russian athlete who tests positive actually uses steroids is

$$p(S|T) = \frac{p(T|S) \cdot p(S)}{p(T|S) \cdot p(S) + p(T|\overline{S}) \cdot p(\overline{S})}$$
$$p(S|T) = \frac{0.96 \cdot 0.08}{0.96 \cdot 0.08 + 0.09 \cdot (1 - 0.08)}$$
$$p(S|T) \approx .48$$

From Section 7.4

3. (Problem 3) What is the expected number of times a 6 appears when a fair die is rolled 10 times?

Let X be the random variable that equals the number of times a 6 appears. The rolls are independent, so each roll is a Bernoulli trial, with the appearance of a 6 being a "success." Thus, X has a binomial distribution with parameters $n = 10$ and $p = 1/6$. The expected number of successes of such a random variable is np, so the expected number of times a 6 appears in ten rolls is

$$E(X) = 10 \cdot \frac{1}{6} = \frac{10}{6}.$$

4. (Inspired by Problem 9) Refer to Examples 2 and 4 of Section 3.3. Consider a list of n distinct integers. Suppose that the probability that x occurs in the list is $2/3$, and that if it is in

the list then it is equally likely to occur anywhere in the list. Find the average number of comparisons used by the linear search algorithm to find x or to determine that it is not in the list.

Let X be the random variable that equals the number of comparisons used and let F be the event that x is found in the list. From Examples 2 and 4, the expected number of comparisons is $E(X|F) = n + 2$ if x is found in the list and $E(X|\overline{F}) = 2n + 2$ if it is not.
We are given that the probability of the former is $p(F) = 2/3$ and that of the latter is therefore $p(\overline{F}) = 1 - 2/3 = 1/3$. The expectation operator is linear, and in particular, the expectation of a (weighted) sum is the (weighted) sum of the expectations. Thus, the total expectation is

$$E(X) = p(F) \cdot E(X|F) + p(\overline{F}) \cdot E(X|\overline{F})$$
$$E(X) = \frac{2}{3} \cdot (n + 2) + \frac{1}{3} \cdot (2n + 2)$$
$$E(X) = \frac{4}{3}n + 2\,.$$

If we only counted the comparisons of x to the elements of the list (and not the comparisons of the loop index to the limit), then the total expectation would be

$$\frac{2}{3} \cdot \frac{n+1}{2} + \frac{1}{3} \cdot n = \frac{2}{3}n + \frac{1}{3}.$$

5. (Problem 27) What is the variance of the number of heads that come up when a fair coin is flipped 10 times?

Let X be the random variable that equals the number of heads that come up.
when $n = 10$ independent Bernoulli trials are performed where, on each trial, $p = 1/2$ is the probability of success and $q = 1 - 1/2 = 1/2$ is the probability of failure. By Example 18, the variance of X is

$$npq = 10 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.5$$

6. (Problem 11) Suppose that we roll a fair die until a 6 comes up or we have rolled it 10 times (whichever happens first). What is the expected number of times we roll the die?

Let X be the random variable that equals the number of die rolls. Each time we roll the die, the probability of the event that a 6 does not come up is $5/6$. These events are independent, so the probability of rolling the die $0 \leq i \leq 10$ times and not getting a 6 is $(5/6)^i$.

Furthermore, the probability of rolling the die $0 \leq i - 1 \leq 8$ times without rolling a 6 and then getting a 6 on the next roll is $p(X = i) = (5/6)^{i-1} \cdot (1/6)$.

Finally, observe that once we have rolled the die 9 times without rolling a 6, we can conclude that the total number of rolls will be 10, regardless of the outcome of the 10th roll. The probability of this event is $p(X = 10) = (5/6)^9$.

Thus, the expected number of times to roll the die is

$$E(X) = \sum_{i=1}^{10} p(X = i) \cdot i$$
$$E(X) = \left(\frac{5}{6}\right)^9 \cdot 10 + \sum_{i=1}^{9} \left[\left(\frac{5}{6}\right)^{i-1} \cdot \frac{1}{6}\right] \cdot i$$
$$E(X) \approx 5.03$$

As an aside, if we removed the upper limit on the number of rolls, then we would have a random variable with a geometric distribution. In that case, we would also have $E(X) = 1/p = 6$.

From Section 7.2

7. (Problem 17) If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events. *Hints:*

- *If A and B are sets then $A \cap B$ and $\overline{A} \cap B$ are disjoint.*
- *If S and T are disjoint sets then $S = (S \cup T) - T$.*
- *Use Theorems 7.1 and 7.2.*

Theorem: If E and F are independent events, then $\overline{E}$ and F are necessarily independent events.

Proof: We are given that E and F are independent events, i.e. that $p(E \cap F) = p(E)p(F)$. Also, by Theorem 1 of Section 7.1, $p(E) = 1 - p(\overline{E})$. Furthermore, observe that the events $E \cap F$ and $\overline{E} \cap F$ are disjoint (Rosen proves this rigorously in his solution on page S-46), which implies by Theorem 1 of Section 7.2 that

$$\begin{aligned} p(\overline{E} \cap F) &= p([E \cap F] \cup [\overline{E} \cap F]) - p(E \cap F) \\ &= p(F) - p(E \cap F) \\ &= p(F) - p(E)p(F) \\ &= [1 - p(E)]p(F) \\ &= p(\overline{E})p(F) \end{aligned}$$

which shows that $\overline{E}$ and F are independent events. ■