Discrete Mathematics - CSCE 531 Fall 2018
In Class Work, Day 12 (14 Nov 2018)

From Section 8.1

1. (Problem 3) A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills. Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

The ways to deposit n dollars in the vending machine fall into three categories, based on the last currency deposited: deposit $n-1$ dollars followed by a one-dollar coin, deposit $n-1$ dollars followed by a \$1 bill, or deposit $n-5$ dollars followed by a \$5 bill. Thus, the number of ways to deposit $n \geq 5$ dollars is

$$a_n = 2a_{n-1} + a_{n-5}$$

a. What are the initial conditions?

There is exactly 1 way to deposit 0 dollars: do nothing. Each way to deposit 0 dollars leads to 2 ways to deposit 1 dollar: deposit 0 dollars and then deposit a one-dollar coin or deposit 0 dollars and then deposit a \$1 bill. Thus, there are twice as many ways to deposit 1 dollar as there are ways to deposit 0 dollars. Similar statements may be made for depositing 2, 3, and 4 dollars.

Thus, the initial conditions for the recurrence relation are:

$$a_0 = 1$$
$$a_1 = 2$$
$$a_2 = 4$$
$$a_3 = 8$$
$$a_4 = 16$$

b. How many ways are there to deposit \$10 for a book of stamps?

We can “work our way up” to the answer:

$$a_5 = 2a_4 + a_0 = 2(16) + 1 = 33$$
$$a_6 = 2a_5 + a_1 = 2(33) + 2 = 68$$
$$a_7 = 2a_6 + a_2 = 2(68) + 4 = 140$$
$$a_8 = 2a_7 + a_3 = 2(140) + 8 = 288$$
$$a_9 = 2a_8 + a_4 = 2(288) + 16 = 592$$
$$a_{10} = 2a_9 + a_5 = 2(592) + 33 = 1217$$

Alternatively, we can “work our way down” to the initial conditions:

$$
\begin{aligned}
a_{10} &= 2a_9 + a_5 \\
a_{10} &= 2(2a_8 + a_4) + a_5 \\
a_{10} &= 4(2a_7 + a_3) + a_5 + 2a_4 \\
a_{10} &= 8(2a_6 + a_2) + a_5 + 2a_4 + 4a_3 \\
a_{10} &= 16(2a_5 + a_1) + a_5 + 2a_4 + 4a_3 + 8a_2 \\
a_{10} &= 33(2a_4 + a_0) + 2a_4 + 4a_3 + 8a_2 + 16a_1 \\
a_{10} &= 68a_4 + 4a_3 + 8a_2 + 16a_1 + 33a_0 \\
a_{10} &= 68 \cdot 16 + 4 \cdot 8 + 8 \cdot 4 + 16 \cdot 2 + 33 \cdot 1 \\
a_{10} &= 1217
\end{aligned}
$$

2. (Problem 11) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time

The ways to climb n stairs fall into two categories: climb $n-1$ stairs and then take one more stair or climb $n-2$ stairs and then take two more stairs at a time. Thus, the number of ways to climb n stairs satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

   a. What are the initial conditions?

There is exactly 1 way to climb 0 stairs: do nothing. There is also exactly 1 way to climb 1 stair: (do nothing and then) take one stair. Thus, the initial conditions are:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1
\end{aligned}
$$

   b. In how many ways can this person climb a flight of eight stairs?

Again, we can either "work our way up" to the term of interest or "work our way down" to the initial conditions. Doing the former will facilitate a more convenient solution to the next problem:

$$
\begin{aligned}
a_2 &= a_1 + a_0 = 2 \\
a_3 &= a_2 + a_1 = 3 \\
a_4 &= a_3 + a_2 = 5 \\
a_5 &= a_4 + a_3 = 8 \\
a_6 &= a_5 + a_4 = 13 \\
a_7 &= a_6 + a_5 = 21 \\
a_8 &= a_7 + a_6 = 34
\end{aligned}
$$

3. (Problem 19) Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds. Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds

The messages that can be sent in n microseconds fall into two categories: messages for which the final signal requires 1 microsecond and those for which the final signal requires 2 microseconds. The former are transmitted by transmitting an $n-1$ microsecond message followed by the final signal while the latter are transmitted by transmitting an $n-2$ microsecond message followed by the final signal. Thus, the number of messages that can be sent in n microseconds satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

a. What are the initial conditions?

There is one message that can be sent in 0 microseconds (the null message) and one message that can be sent in 1 microsecond (the message consisting of just the one microsecond signal). Thus,

$$a_0 = 1$$
$$a_1 = 1$$

b. How many different messages can be sent in 10 microseconds using these two signals?

This recurrence relation and this set of initial conditions are the same as those for the previous problem. Thus, we can continue "working up" from where we left off:

$$a_9 = a_8 + a_7 = 55$$
$$a_{10} = a_9 + a_8 = 89$$

From Section 8.2

4. (Problem 5) How many different messages can be transmitted in n microseconds using the two signals described in Problem 3 above (Rosen's Problem 19 in Section 8.1)?

This problem is identical to Example 4 in Section 8.2 except that the initial conditions are different. In particular, the recurrence relation is

$$a_n = a_{n-1} + a_{n-2},$$

so the characteristic equation is

$$r^2 - r - 1 = 0.$$

which has roots $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$, so the solution has the form

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

However, the initial conditions are $a_0 = 1$ and $a_1 = 1$. Thus, α_1 and α_2 satisfy the simultaneous linear equations

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 \quad \text{and}$$
$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1.$$

Of course, $\left(\frac{1+\sqrt{5}}{2}\right)^0 = \left(\frac{1-\sqrt{5}}{2}\right)^0 = 1$, so $\alpha_2 = 1 - \alpha_1$. Substituting into the second equation yields

$$1 = \frac{1+\sqrt{5}}{2}\alpha_1 + \frac{1-\sqrt{5}}{2}(1-\alpha_1),$$

for which the solution is

$$\alpha_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, \text{and therefore}$$
$$\alpha_2 = 1 - \alpha_1 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

Substituting α_1 and α_2 into the solution yields

$$a_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$
$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

5. (Problem 45) Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of 1 month and six new pairs of rabbits at the age of 2 months and every month afterward. None of the rabbits ever dies or leaves the island.
   a. Construct a recurrence relation for the number of pairs of rabbits on the island n months after one newborn pair is placed on the island.

Let month 0 be the month in which the first pair of rabbits is placed on the island, a_n the total number of pairs present in month n, and b_n the number of pairs born in month n. Then $a_n = a_{n-2} + b_{n-1} + b_n$. Also, $b_n = 2b_{n-1} + 6a_{n-2}$. Furthermore, $b_{n-1} = a_{n-1} - a_{n-2}$. Thus,

$$\begin{aligned} a_n &= a_{n-2} + b_{n-1} + (2b_{n-1} + 6a_{n-2}) \\ &= 7a_{n-2} + 3b_{n-1} \\ &= 7a_{n-2} + 3(a_{n-1} - a_{n-2}) \\ &= 3a_{n-1} + 4a_{n-2} \end{aligned}$$

b. By solving the recurrence relation in (a) determine the number of pairs of rabbits on the island n months after one pair is placed on the island.

The characteristic equation of the recurrence relation is

$$r^2 - 3r - 4 = (r-4)(r+1) = 0$$

which has the roots

$$r_1 = 4$$
$$r_2 = -1.$$

Thus, the solution has the form $a_n = \alpha_1(4)^n + \alpha_2(-1)^n$ and the initial conditions

$$a_0 = 1 = \alpha_1(4)^0 + \alpha_2(-1)^0 = \alpha_1 + \alpha_2$$
$$a_1 = 3 = \alpha_1(4)^1 + \alpha_2(-1)^1 = 4\alpha_1 - \alpha_2$$

Solving these simultaneous linear equations yields

$$\alpha_1 = \frac{4}{5}$$
$$\alpha_2 = \frac{1}{5}$$

so that the solution is

$$\begin{aligned} a_n &= \frac{4}{5}(4)^n + \frac{1}{5}(-1)^n \\ &= \frac{1}{5}[4^{n+1} + (-1)^n]. \end{aligned}$$

Note: the initial conditions in the back of the book are off by a factor of 2 (perhaps due to confusion of individual rabbits vs. pairs of rabbits).

6. (Problem 23) Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a. Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

$$a_n = -2^{n+1} = -2 \cdot 2^n = (-3+1) \cdot 2^n = 3 \cdot -2^n + 2^n = 3a_{n-1} + 2^n$$

b. Use Theorem 5 to find all solutions of this recurrence relation.

The associated linear homogeneous equation is $a_n = 3a_{n-1}$, which has the characteristic equation $r - 3 = 0$ and the solution $a_n^{(h)} = \alpha 3^n$, where α is a constant determined by the initial conditions (in part (c)). In part (a) we showed that $a_n = -2^{n+1}$ is a particular solution of the nonhomogeneous equation. Thus, from Theorem 5 we can conclude that every solution is of the form $a_n = \alpha 3^n - 2^{n+1}$.

c. Find the solution with $a_0 = 1$.

$$1 = a_0 = \alpha 3^0 - 2^{0+1} = \alpha - 2 \Rightarrow \alpha = 3 \Rightarrow a_n = 3^{n+1} - 2^{n+1}$$