Discrete Mathematics - CSCE 531 Fall 2018
In-Class Work, Day 01 (1 October 2018)

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: A and B.

Notes:

- Each of the following exercises is a different scenario.
- The solutions will be given in terms of the propositions A representing the statement "Inhabitant A is a knight," and B representing the statement "Inhabitant B is a knight."

1. A tells you that B is a knave. B says, "Neither A nor I is a knave." Determine who is a knight and who is a knave.

Most student solutions to this problem are probably far shorter than this one. In fact, many students could solve this problem easily in their heads and simply write down the solution. There are good reasons to think carefully about the steps involved, though. First, the same methods are applicable to qualitatively similar problems that are too complicated to solve in your head. Second, if one seeks to automate reasoning processes (e.g. for artificial intelligence or theorem proving), one must understand the steps well enough to describe them algorithmically. With that said …

Original Statement	Expansion Based on the Rules of the Island	Propositional Logic Representation
"A tells you that B is a knave."	There are two cases to consider: either A is a knight or A is not a knight. If A is a knight, then A tells the truth, so A's statement that B is a knave is true, so B is not a knight. On the other hand, if A is not a knight, then A is a knave, and therefore A always lies. In this case, A's statement is false, so it is not the case that B is a knave, i.e. B is a knight.	$(A \to \neg B) \wedge (\neg A \to B)$
"B says 'Neither A nor I is a knave.'"	[Similar reasoning, but slightly more succinct presentation.] If B is a knight, then neither A nor B is a knave, i.e. both A and B are knights. If B is a knave, then it is not the case that neither A nor B is a knave, i.e. either A or B is a knave (or both).	$(B \to A \wedge B) \wedge (\neg B \to \neg A \vee \neg B)$

We must determine "who is a knight and who is a knave" so that the "original statements" in the table above are (simultaneously) possible according to the rules of the island. Equivalently, we must find truth assignments for the propositions A and B so that the "propositional logic representations" above are both true. In other words, we need truth assignments such that (1) both $A \to \neg B$ and $\neg A \to B$ are true, and simultaneously (2) both $B \to A \wedge B$ and $\neg B \to \neg A \vee \neg B$ are true. Some simplification is possible, but in the interest of expediency, we can find our solution directly through the use of the following truth table:

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg A \vee \neg B$	$A \to \neg B$	$\neg A \to B$	$B \to A \wedge B$	$\neg B \to \neg A \vee \neg B$
T	T	F	F	T	F	F	T	T	T
$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	T	F	F	T	T	T	F	T
F	F	T	T	F	T	T	F	T	T

The only case for which the required propositions all hold is the one in which A is true and B is false. Therefore, A is a knight and B is a knave.

2. A tells you that "of B and I, exactly one is a knight." B tells you that only a knave would say that A is a knave. ~~Can you determine who is a knight and who is a knave?~~ Determine who is a knight and who is a knave.

We are given that "A tells you that 'of B and I, exactly one is a knight.'" A's statement is equivalent to "Either (1) A is a knight and B is not a knight, or (2) A is not a knight and B is a knight." Note that the two possibilities are mutually exclusive, so the exclusive "either-or" may be safely represented as the disjunction $(A \wedge \neg B) \vee (\neg A \wedge B)$. Now, if A is a knight, then A's statement must be true; if A is not a knight, then A's statement must be false. Thus, it must be the case that

$$\big(A \to (B \oplus A)\big) \wedge \big(\neg A \to \neg(B \oplus A)\big)$$

which is the same as

$$[\mathrm{A} \to (\mathrm{A} \wedge \neg \mathrm{B}) \vee (\neg \mathrm{A} \wedge \mathrm{B})] \wedge [\neg \mathrm{A} \to (\mathrm{A} \wedge \mathrm{B}) \vee (\neg \mathrm{A} \wedge \neg \mathrm{B})]$$

This can be simplified through the application of logical equivalences. It is typical to show fewer intermediate steps than I do here.

We begin by using the rule that $P \to Q$ is logically equivalent to $\neg P \vee Q$ for all propositions P and Q:

$$\{\neg A \vee [(A \wedge \neg B) \vee (\neg A \wedge B)]\} \wedge \{\neg\neg A \vee [(A \wedge B) \vee (\neg A \wedge \neg B)]\}.$$

Using the associativity of disjunction to remove the "extra" grouping symbols as well as the double negation yields

$$[\neg A \vee (A \wedge \neg B) \vee (\neg A \wedge B)] \wedge [A \vee (A \wedge B) \vee (\neg A \wedge \neg B)].$$

Using the associativity of disjunction and the absorption of disjunction over conjunction, we obtain

$$[\neg A \vee (A \wedge \neg B)] \wedge [A \vee (\neg A \wedge \neg B)].$$

Next, we distribute the disjunctions over the conjunctions:

$$[(\neg A \vee A) \wedge (\neg A \vee \neg B)] \wedge [(A \vee \neg A) \wedge (A \vee \neg B)].$$

Sensing the end is near, we apply the negation law for disjunctions, leaving

$$[T \wedge (\neg A \vee \neg B)] \wedge [T \wedge (A \wedge \neg B)].$$

Finally, the identity law for conjunction gives us

$$(\neg A \vee \neg B) \wedge (A \vee \neg B).$$

We are also given that "B tells you that only a knave would say that A is a knave." This is equivalent to "B tells you that if and only if C is a knave, C would say that A is a knave. The translation of the last phrase ("C would say that A is a knave") takes the now familiar form $(C \rightarrow \neg A) \wedge (\neg C \rightarrow A)$, which we will temporarily abbreviate with the proposition P. Translating the penultimate phrase ("if and only if C is a knave") takes a close variation of that form: $(\neg C \rightarrow P) \wedge (C \rightarrow \neg P)$, and we will abbreviate this compound proposition with Q. Finally, translating the first phrase is now routine: $(B \rightarrow Q) \wedge (\neg B \rightarrow \neg Q)$.

One way to proceed (which is different that the one I presented in class) is to construct a truth table involving the propositional variables A, B, and C, in which case we seek truth assignments satisfying $(\neg A \vee \neg B)$, $(A \vee \neg B)$, and $(B \rightarrow Q) \wedge (\neg B \rightarrow \neg Q)$. It is convenient to do so in stages. We begin with the truth table for P, which depends on A and C.

A	C	$\neg A$	$\neg C$	$C \rightarrow \neg A$	$\neg C \rightarrow A$	$P \equiv (C \rightarrow \neg A) \wedge (\neg C \rightarrow A)$
T	T	F	F	F	T	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	F

We can now construct the truth table for Q, which depends on C and P.

C	P	$\neg C$	$\neg P$	$\neg C \rightarrow P$	$C \rightarrow \neg P$	$Q \equiv (\neg C \rightarrow P) \wedge (C \rightarrow \neg P)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

Next, we construct the truth table for $R \equiv (B \rightarrow Q) \land (\neg B \rightarrow \neg Q)$.

B	Q	$\neg B$	$\neg Q$	$B \rightarrow Q$	$\neg B \rightarrow \neg Q$	$R \equiv (B \rightarrow Q) \land (\neg B \rightarrow \neg Q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Finally, we construct a combined truth table for $\neg A \lor \neg B$, $A \lor \neg B$, and R.

A	B	C	$\neg A$	$\neg B$	P	Q	$\neg A \lor \neg B$	$A \lor \neg B$	R
T	T	T	F	F	F	T	F	T	T
T	T	F	F	F	T	T	F	T	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	T	T	T	T	F
F	T	T	T	F	T	F	T	F	F
F	T	F	T	F	F	F	T	F	F
$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

There are two sets of truth assignments for which $\neg A \lor \neg B$, $A \lor \neg B$, and R are simultaneously true. They have in common that A and B are both false, meaning that A and B are both knaves, while the assignment to C has no effect. In fact, examination of the final truth table shows that the assignment to C never has an effect on the truth values of $\neg A \lor \neg B$, $A \lor \neg B$, and R. This makes intuitive sense, because Inhabitant C is hypothetical, so the statements by Inhabitants A and B must follow the rules of the island regardless of whether C is a hypothetical knight or a knave.

3. A says that B is a knave. B says, "A and I are knights." ~~Can you determine who is a knight and who is a knave?~~ Determine who is a knight and who is a knave.

"A says that B is a knave.": $(A \rightarrow \neg B) \land (\neg A \rightarrow B) \equiv (\neg A \lor \neg B) \land (A \lor B)$

"A and I are knights.":

$$\begin{aligned}
(B \rightarrow A \land B) \land [\neg B \rightarrow \neg(A \land B)] &\equiv (B \rightarrow A \land B) \land [\neg B \rightarrow (\neg A \lor \neg B)] \\
&\equiv [\neg B \lor (A \land B)] \land [B \lor (\neg A \lor \neg B)] \\
&\equiv [(\neg B \lor A) \land (\neg B \lor B)] \land [B \lor (\neg B \lor \neg A)] \\
&\equiv [(\neg B \lor A) \land (\neg B \lor B)] \land [(B \lor \neg B) \lor \neg A] \\
&\equiv [(\neg B \lor A) \land T] \land [T \lor \neg A] \\
&\equiv [\neg B \lor A] \land [T] \\
&\equiv (\neg B \lor A)
\end{aligned}$$

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A \vee B$	$\neg B \vee A$
T	T	F	F	F	T	T
$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	T	F	T	T	F
F	F	T	T	T	F	T

There is exactly one assignment of truth values to A and B that satisfies the required compound proposition: A must be true and B must be false. Thus, Inhabitant A must be a knight and B must be a knave.

4. A claims, "B and I are not the same." B says, "Of A and I, exactly one is a knight." ~~Can you determine who is a knight and who is a knave?~~ Determine who is a knight and who is a knave.

"A claims, ` B and I are not the same.'": First, consider the statement "A and B are not the same."

$$(A \to \neg B) \wedge (\neg A \to B) \equiv (\neg A \vee \neg B) \wedge (A \vee B)$$

Now, consider that the previous statement is a claim made by Inhabitant A.

$$\begin{aligned}
&\{A \to [(\neg A \vee \neg B) \wedge (A \vee B)]\} \wedge \{\neg A \to \neg[(\neg A \vee \neg B) \wedge (A \vee B)]\} \\
&\equiv \{\neg A \vee [(\neg A \vee \neg B) \wedge (A \vee B)]\} \wedge \{A \vee \neg[(\neg A \vee \neg B) \wedge (A \vee B)]\} \\
&\equiv \{[\neg A \vee (\neg A \vee \neg B)] \wedge [\neg A \vee (A \vee B)]\} \wedge \{A \vee [\neg(\neg A \vee \neg B) \vee \neg(A \vee B)]\} \\
&\equiv \{[(\neg A \vee \neg B)] \wedge [T]\} \wedge \{A \vee [(A \wedge B) \vee (\neg A \wedge \neg B)]\} \\
&\equiv \{\neg A \vee \neg B\} \wedge \{[A \vee (A \wedge B)] \vee [A \vee (\neg A \wedge \neg B)]\} \\
&\equiv (\neg A \vee \neg B) \wedge \{A \vee (A \wedge B) \vee A \vee (\neg A \wedge \neg B)\} \\
&\equiv (\neg A \vee \neg B) \wedge \{A \vee (\neg A \wedge \neg B)\} \\
&\equiv (\neg A \vee \neg B) \wedge \{(A \vee \neg A) \wedge (A \vee \neg B)\} \\
&\equiv (\neg A \vee \neg B) \wedge (A \vee \neg B) \\
&\equiv (\neg A \wedge A) \vee \neg B \\
&\equiv \neg B
\end{aligned}$$

"B says, `Of A and I, exactly one is a knight.'"

First, consider the statement "Of A and B, exactly one is a knight": $(A \to \neg B) \wedge (\neg A \to B)$

This is identical to A's claim. Following the same line of reasoning as above, the fact that the statement is a claim made by Inhabitant B therefore reduces to the requirement that $\neg A$ is true. Combining these results, we see that both Inhabitant A and Inhabitant B must be knaves.

5. A tells you, “At least one of the following is true: that I am a knight or that B is a knight.” B claims, “A would say that I am a knave.” ~~Can you determine who is a knight and who is a knave?~~ Determine who is a knight and who is a knave.

“A tells you, `At least one of the following is true: that I am a knight or that B is a knight.' “:

$$\begin{aligned}&[A \to (A \vee B)] \wedge [\neg A \to \neg(A \vee B)]\\ &\equiv [\neg A \vee (A \vee B)] \wedge [A \vee (\neg A \wedge \neg B)]\\ &\equiv [T] \wedge [A \vee \neg B]\\ &\equiv A \vee \neg B\end{aligned}$$

“A would say that B is a knave.”: $(A \to \neg B) \wedge (\neg A \to B)$

This is identical to the inhabitants’ claims in the previous problem. It is a claim made by B, and therefore reduces to the requirement that $\neg A$ is true. Conjoining this with $A \vee \neg B$ yields $\neg A \wedge \neg B$, so both A and B must be knaves.

6. A says, “B and I are both knights or both knaves.” B claims, “A and I are the same.” ~~Can you determine who is a knight and who is a knave?~~ Determine who is a knight and who is a knave.

“B and A are both knights or both knaves.”: $(A \wedge B) \vee (\neg A \wedge \neg B)$

“A says, `B and I are both knights or both knaves.'”:

$$\begin{aligned}&\{A \to [(A \wedge B) \vee (\neg A \wedge \neg B)]\} \wedge \{\neg A \to \neg[(A \wedge B) \vee (\neg A \wedge \neg B)]\}\\ &\equiv \{\neg A \vee (A \wedge B) \vee (\neg A \wedge \neg B)\} \wedge \{A \vee [\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B)]\}\\ &\equiv (\neg A \vee B) \wedge \{A \vee [(\neg A \vee \neg B) \wedge (A \vee B)]\}\\ &\equiv (\neg A \vee B) \wedge \{[A \vee (\neg A \vee \neg B)] \wedge [A \vee (A \vee B)]\}\\ &\equiv (\neg A \vee B) \wedge (A \vee B)\\ &\equiv B\end{aligned}$$

Thus, B must be a knight.

“A and B are the same.”:

$$\begin{gathered}(A \to B) \wedge (\neg A \to \neg B)\\ (\neg A \vee B) \wedge (A \vee \neg B)\\ [(\neg A \vee B) \wedge A] \vee [(\neg A \vee B) \wedge \neg B]\\ [(\neg A \wedge A) \vee (B \wedge A)] \vee [(\neg A \wedge \neg B) \vee (B \wedge \neg B)]\\ (B \wedge A) \vee (\neg A \wedge \neg B)\end{gathered}$$

This is identical to the statement that B and A are both knights or both knaves. The fact that B makes this claim implies that A is a knight.