[2] Schlaikjer, E. M., *Mus. Comp. Zoöl. Bull.*, **76**, 83–84 (1935).
[3] Peterson, O. A., *Ann. Carnegie Mus.*, **20**, art. 14 (1931).

DIFFERENTIABLE MANIFOLDS IN EUCLIDEAN SPACE[1]

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We present here a summary of some theorems on the imbedding of abstract differentiable manifolds in Euclidean space E^n and on the approximation to such manifolds by analytic manifolds. As a corollary it is noted that any differentiable manifold may be given an analytic Riemannian metric.

I. *The Imbedding of a Differentiable Manifold in Euclidean Space.*—Let M be a topological space with neighborhoods $U_1, U_2, \ldots$. Let each U_i be homeomorphic with the interior of the unit m-sphere S^m. If U_i and U_j have common points U_{ij}, then the homeomorphisms of U_i and U_j with S^m induce a mapping of one part of S^m on another part. If all such maps are of class C^r (i.e., have continuous partial derivatives through the rth order), $r \geqq 1$, with non-vanishing Jacobian, we say M is *differentiable*, and *of class* C^r.

If M is in E^n and each point of M is in a neighborhood which may be defined by expressing $n - m$ of the coördinates in terms of the remaining m, the functions being of class C^r, then M is of class C^r in the above sense; we say *M is of class C^r in E^n*. Suppose M is of class C^r, and is mapped into E^n. The n coördinates at points of M are n functions defined over M. If these functions are of class C^s (with the obvious definition for $s \leqq r$), and are independent (so that m independent directions at any point of M go into m independent directions in E^n), we call the map of M in E^n a *regular C^r-map*. Such a map is locally one-one: a neighborhood of any point of M is mapped in a one-one manner in E^n.

THEOREM I. *Any m-manifold of class C^r ($r \geqq 1$ finite or infinite) may be imbedded by a regular C^r-map in E^{2m}, and by such a map in a one-one manner in E^{2m+1}.*

The proof runs as follows: If M is closed, a finite number of neighborhoods $U_1, \ldots, U_\nu$ cover M. Corresponding to these neighborhoods we define functions $f_1, \ldots, f_\mu$, $\mu = (m + 1)\nu$, of class C^r over M, which, used as coördinates, map M in a regular C^r-manner in E^μ. If $r = 1$, we next approximate to M by a manifold of class C^2. We now project M or the new manifold along straight lines into spaces of lower dimension, till we have it in E^{2m+1} or in E^{2m}. If M is not closed, we define such a map successively over larger and larger parts of M.

II. *Approximations to Differentiable Manifolds by Analytic Manifolds.*—A manifold of class C^r in E^n was defined above; it is *analytic* if the functions defining its neighborhoods are analytic. If M and M^* are homeomorphic manifolds in E^n and $\eta(p)$ is a positive continuous function defined on M, we say *M^* approximates to M through the rth order with an error* $< \eta(p)$ if the distance from any point p of M to the corresponding point p^* of M^* is $< \eta(p)$, and corresponding partial derivatives of order $\leqq r$ (in a suitable coördinate system) differ by $<\eta(p)$.

THEOREM II. *Let M be of class C^r in E^n ($r \geqq 1$ finite), and let $\eta(p)$ be a positive continuous function defined on M. Then there is an analytic manifold M^* in E^n which approximates to M through the rth order with an error* $<\eta(p)$.

To prove the theorem, we first define a positive function f, analytic in $E^n - M$, and approaching 0 as we approach M. The points $f = c > 0$ define a "tube" about M; the $(n - m)$-plane orthogonal to M at p cuts $f = c$ in an $(n - m - 1)$-sphere. We define in an analytic fashion a "center" to the tube; the set of center points form M^*.

From theorems I and II follows

THEOREM III. *Any m-manifold of class C^r ($r \geqq 1$ finite) is homeomorphic with an analytic manifold in E^{2m+1}, the homeomorphism being of class C^r.*

We may define ds^2 on the manifold as the ds^2 in E^{2m+1}; hence

THEOREM IV. *Any manifold M of class C^r ($r \geqq 1$ finite) may be given an analytic Riemannian metric, the g_{ij} being of class C^r in terms of the original neighborhoods in M.*

III. *Imbedding of Manifolds in Families of Analytic Manifolds.*—We state here a generalization of Theorem II for certain classes of manifolds in E^n. We say M is in *regular position in E^n* if there exist $n - m$ independent continuous unit vector functions $v_1(p), \ldots, v_{n-m}(p)$ defined over M with the following property: Each point p_0 of M is in a neighborhood U of p_0 in M which is an m-cell, and such that any vector through two points of U makes an angle $>\rho(p_0)$ with the $(n - m)$-plane determined by the $v_i(p_0)$; $\rho(p)$ is a positive continuous function defined on M. If M is differentiable, the condition reduces to: The normal $(n - m)$-planes to points of M may be determined by $n - m$ vector functions on M. The class of such (differentiable) manifolds is the same as the class of manifolds which may be determined by the simultaneous vanishing of $n - m$ (independent) differentiable functions.[2]

THEOREM V. *Let M be an m-manifold of class C^r ($r \geqq 1$ finite) in regular position in E^n, and let $\eta(p)$ be a positive continuous function on M. Then M can be imbedded in an $(n - m)$-parameter family of manifolds $M(c_1, \ldots, c_{n-m})$, each $|c_i| < 1$, such that*

(1) $M(0, \ldots, 0) = M$.

(2) *$M(c_1, \ldots, c_{n-m})$ is analytic if $(c_1, \ldots, c_{n-m}) \neq (0, \ldots, 0)$.*

(3) *Each $M(c_1, \ldots, c_{n-m})$ approximates to M through the rth order with an error $< \eta(p)$.*

(4) *The manifolds fill out a neighborhood of M in a one-one way.*

[1] Presented to the Am. Math. Society, Sept., 1935.

[2] If M is differentiable, it is in regular position if and only if the normal sphere-space is a product space. See the following paper, especially 3 (C) and 8 (d).

SPHERE-SPACES[1]

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1. *Introduction.*—Spaces often occur in which the points themselves are spaces of some simple sort, for instance spheres of a given dimension. The set of all great circles on a sphere is such a space. Some general types of sphere-spaces are given in §3 below, and some specific examples in §8. Locally, sphere-spaces are product spaces (see §2); but in the large, this may no longer hold. In this note we define invariants which serve to distinguish different sphere-spaces when they have the same "base space." The proofs will be given in a later paper.

2. *Definitions.*—A sphere space is defined as follows: Let K be a topological space (usually a manifold or complex). To each point p of K let there correspond a point set $S(p)$. Let $U_1, U_2, \ldots$ be a set of neighborhoods covering K. To each U_i let there correspond a function $\xi_i(p, q)$, where p ranges over U_i and q ranges over the unit l-sphere S^l in Euclidean space E^{l+1}; for p fixed, $\xi_i(p, q)$ is a one-one map of S^l onto $S(p)$. Set $\xi_i'(p, q') = q$ whenever $\xi_i(p, q) = q'$. Suppose U_i and U_j have common points U_{ij}. Then $\xi_{ij}(p, q) = \xi_j'(p, \xi_i(p, q))$ for fixed p in U_{ij} is a one-one map of S^l onto itself; we assume that these maps are differentiable with non-vanishing Jacobian, and that they vary continuously with p. If these conditions are satisfied, we call the resulting system a *sphere-space* $S(K)$.

The space K we call the *base space.* The set of all pairs (p, q), where q is in $S(p)$, we call the *total space* $\mathfrak{S}$;[2] if the spheres $S(p)$ are non-intersecting (as parts of another space), we may let $\mathfrak{S}$ be simply the points q. $\mathfrak{S}$ is evidently a topological space.

If the maps $\xi_{ij}(p, q)$ (p fixed) are orthogonal transformations, we call the sphere-space *regular.* All the sphere-spaces described below are regular; in fact, we can define these maps by projecting the spheres $S(p)$ onto $S(p_0)$ in a simple manner for p near p_0.