More on Mathematical Induction

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We have seen several examples of proofs using mathematical induction. Here we show another example illustrating a bit more creative use of this proof technique. [1]

[1] Adapted from Carmony, "Odd Pie Fights", *Mathematics Teacher*, 1979

Odd Pie Fights

An odd number of comp-sci students stand scattered around a yard at mutually distinct distances, each armed with a freshly baked apple pie. A whistle is suddenly blown and the pie throwing contest begins. Each contestant throws a pie at the contestant nearest to her.

Use mathematical induction to show that there is at least one survivor.

Proof: We will show that the following proposition $P(n)$ is true for all positive n:

$$P(n) = \{ \text{ There is a survivor whenever } 2n+1 \text{ people standing at mutually distinct distances throw pies at each other } \}$$

Note that as n runs through all positive integers, $2n+1$ runs through all odd integers starting at 3 and greater. This is what we want - we don't want to allow for a single pie-fighter; we need at least three - since we suspect that no sane person would want to engage into a pie fight with self. These are computer science students, after all.

Base Case: $P(1)$ is true. If $n = 1$, then we have $2n + 1 = 3$ people in the pie-fight. Let the three people be A, B, and C. Let, without loss of generality, A and B be the closest pair. Since the distances between all pairs are different, we know that C is closer to either A or B. Let, again without loss of generality, A be the closest guy to C.

What happens as the contest begins? A and B through pies at each other, and C throws the pie at A. Thus we have the survivor, C. This proves the base case.

Inductive Case: If $P(k)$ is true, then $P(k+1)$ is true. Assume that $P(k)$ is true for some odd integer $k > 3$. That is, we have at least one survivor when $2k + 1$ people pie-fight as described above. We must show that $P(k + 1)$ is true. That is: we must show that there is at least one survivor whenever $2(k+1) + 1 = 2k+3$ people pie-fight as described above.

So suppose we have these $2k + 3$ people in the yard. Let A and B be the closest pair. When each person throws a pie, A and B throw pies at each other.

Now consider two cases: 1) Someone else throws a pie at either A or B and 2) No one else throws a pie at either A or B.

Case 1: Someone else throws a pie at either A or B. In this case *at least* three pies are thrown at A and B: 2 pies that they throw at each other and at least another pie that comes their way from somebody else. But this means that *at most* $(2k+3)-3=2k$ pies are thrown at the remaining $2k+1$ people. But this guarantees that there is at least one survivor: there are fewer pies than people to be hit.

Case 2: No one else throws a pie at either A or B. If nobody else throws a pie at either A or B, then the proof is simpler. Consider removing A and B with their 2 pies from the $2k+3$ people in the contest. We would be left with $2k+1$ pie-fighting each other only. By the inductive hypothesis, there is at least one survivor amongst these $2k+1$ people. This person is also the survivor amongst the initial $2k+3$ contestants (since A and B trow pies at each other).

This completes the proof. We proved the base case and we proved the inductive step using a proof by cases. From the principle of mathematical induction it follows that $P(n)$ is true for all positive integers n.

Notice that the conclusion we reached here would be false if the contest set-up were the same as before but we had an even number of people in the yard. Can you see why is it possible for everyone to be hit with a pie in this case?