Solutions to Homework 1

Exercise 1

Adding page 5 as asked (pages 3 and 5 link to each other) gives the new link matrix:

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

As before, we are looking for an eigenvector $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T$ corresponding to the eigenvalue 1 (which exists because $\mathbf{A}$ is column-stochastic). This $\mathbf{x}$ will be our ranking. Solve $\mathbf{Ax} = \mathbf{x}$, i.e. $(\mathbf{A} - \mathbf{I})\mathbf{x} = 0$, for $\mathbf{x}$:

$$-x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0 \tag{1}$$

$$\frac{1}{3}x_1 - x_2 = 0 \tag{2}$$

$$\frac{1}{3}x_1 + \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4 + x_5 = 0 \tag{3}$$

$$\frac{1}{3}x_1 + \frac{1}{2}x_2 - x_4 = 0 \tag{4}$$

$$\frac{1}{2}x_3 - x_5 = 0 \tag{5}$$

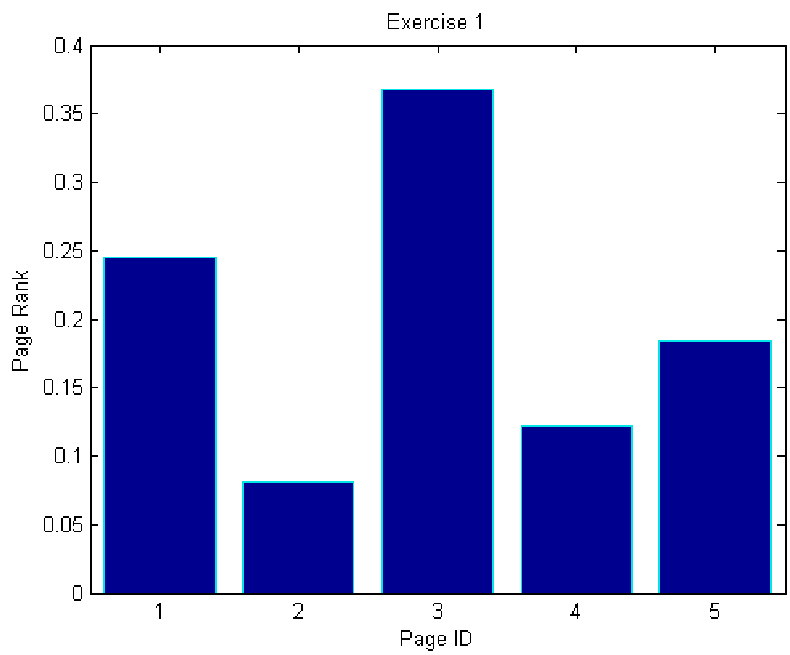
Solve 'by hand' first, then confirm solution with `MATLAB`. Set $x_3 = 2$ and $x_5 = 1$ to satisfy equation (5). Then subtract (4) from (3) to obtain $x_4 = 2/3$, plug x_3 and x_4 into equation (1) to get $x_1 = 4/3$. Finally plug x_1 into equation (2) to get $x_2 = 4/9$.

So we've got one eigenvector: $\mathbf{x} = [4/3, 4/9, 2, 2/3, 1]^T$ corresponding to $\lambda = 1$; there are many more solutions - as many as you'd like, all collinear to $\mathbf{x}$. (We can agree to pick the one that ensures the components of $\mathbf{x}$ sum to 1, i.e. scale $\mathbf{x}$ by dividing each component by the sum of $\mathbf{x}$'s components.)

Here's some MATLAB that does the same:

```
A = [ 0          0       1/2     1/2        0   ;
     1/3         0        0       0         0   ;
     1/3        1/2       0      1/2        1   ;
     1/3        1/2       0       0         0   ;
      0          0       1/2      0         0 ];

[V D] = eig(A1);
disp('the eigenvector associated with eigenvalue 1:')
V(:,1)
x = V(:,1)/sum(V(:,1))
figure(1)
bar(x)
```



Thus the new ranking is

$$x = [\ 0.2449,\ 0.0816,\ 0.3673,\ 0.1224,\ 0.1837\ ]$$

and page 3 is now ranked the highest.

Exercise 7

Let $\mathbf{A}$ and $\mathbf{S}$ be column stochastic matrices, i.e. $\sum_i A_{ij} = 1$ and $\sum_i S_{ij} = 1$ for every j-th column of $\mathbf{A}$ or $\mathbf{S}$. Let $0 \leq m \leq 1$ and let $\mathbf{M} = m\mathbf{A} + (1-m)\mathbf{S}$. Consider the sum of the elements in $\mathbf{M}$'s j-th column:

$$\begin{aligned}
\sum_i M_{ij} &= \sum_i (mA_{ij} + (1-m)S_{ij}) \\
&= \sum_i mA_{ij} + \sum_i (1-m)S_{ij} \\
&= m\sum_i A_{ij} + (1-m)\sum_i S_{ij} \\
&= m + (1-m) \\
&= 1
\end{aligned}$$

This being true for any j, $\mathbf{M}$ is column-stochastic.

Exercise 11

Here the link matrix $\mathbf{A}$ is the same as in problem 1. Set $m = 0.15$ and write down the matrix $\mathbf{M} = 0.85\mathbf{A} + 0.15\mathbf{S}$ where $\mathbf{S}$ is the 5×5 matrix each element of which is $1/5$.

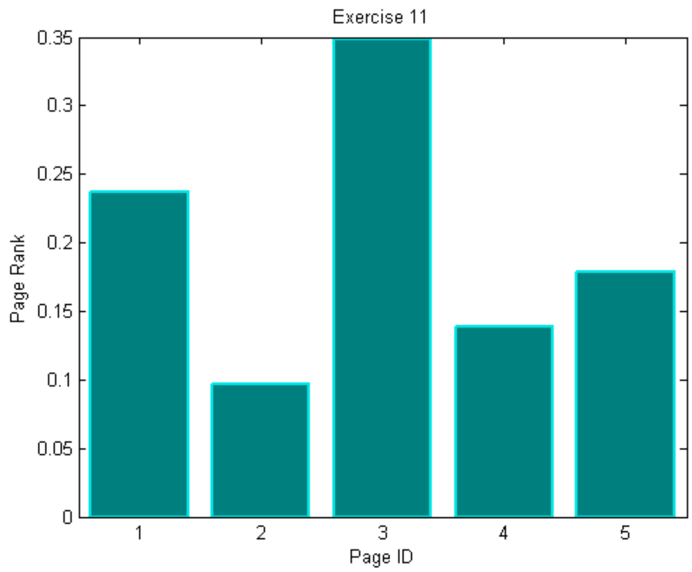
```
A = [ 0          0       1/2     1/2        0   ;
     1/3         0        0       0         0   ;
     1/3        1/2       0      1/2        1   ;
     1/3        1/2       0       0         0   ;
      0          0       1/2      0         0 ];

S = 1/5 * ones(5);
m = 0.15;
M = (1 - m)*A + m*S
[V D] = eig(M)
% diag(D)
ranks = V(:,1) / sum(V(:,1))
bar(ranks)
```

Here the output is:

$$ranks = [\ 0.2371,\ 0.0972,\ 0.3489,\ 0.1385,\ 0.1783\ ]$$

Slightly different scores but the relative ranking of the pages is the same.


Exercise 11
Page Rank
Page ID
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5