Homework 1

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Question 1.1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices defined as:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}$$

Calculate:

- $\boldsymbol{AB}$
- $\boldsymbol{B}^T\boldsymbol{A}$

Use R to check your calculation.

Answer:
'By hand' calculation yields:

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix} -2 & 9 \\ -4 & 18 \\ -6 & 27 \end{pmatrix}$$

For example, the entry in the third row and second column of $\boldsymbol{C}$, $c_{3,2}$, is calculated as:

$$c_{3,2} = a_{3,1} \times b_{1,2} + a_{3,2} \times b_{2,2} + a_{3,3} \times b_{3,2} = 3 \times 0 + 6 \times 3 + 9 \times 1 = 27$$

Similarly,

$$\boldsymbol{B}^T\boldsymbol{A} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -6 \\ 9 & 18 & 27 \end{pmatrix}$$

whose entry in the first row second column is calculated as:

$$2 \times 2 + 1 \times 4 + (-2) \times 6 = -4$$

The following `R` code can be used to verify these results:

```
# Solution to Question1, Homework 1

#given matrix A:
col1 <- c(1,2,3)
A <- cbind(col1, 2*col1, 3*col1)
#and matrix B:
B <- matrix((c(2, 1, -2, 0, 3, 1)), ncol = 2, nrow = 3)

#compute the following prodcts:
A %*% B
t(B) %*% A
```

Question 1.2. If $\boldsymbol{A}$ is invertible, prove that $det(\boldsymbol{A}^{-1}) = (det(\boldsymbol{A}))^{-1}$

Proof. $\boldsymbol{A}$ being invertible, consider the product $\boldsymbol{A}\boldsymbol{A}^{-1} = \mathbf{I}$. By property (b) on page 17 of the lecture notes, we have:

$$1 = det(\mathbf{I}) = det(\boldsymbol{A}\boldsymbol{A}^{-1}) = det(\boldsymbol{A})det(\boldsymbol{A}^{-1})$$

from which

$$det(\boldsymbol{A}^{-1}) = \frac{1}{det(\boldsymbol{A})}$$

□

Question 1. 3. a. If matrix $\boldsymbol{P}$is idempotent, then $\boldsymbol{Q} = \mathbf{I} - \boldsymbol{P}$ is also idempotent.

Proof. Since

$$\boldsymbol{Q}^2 = (\mathbf{I} - \boldsymbol{P})^2 = \mathbf{I}^2 - \mathbf{I}\boldsymbol{P} - \boldsymbol{P}\mathbf{I} + \boldsymbol{P}^2 = \mathbf{I} - \boldsymbol{P} - \boldsymbol{P} + \boldsymbol{P} = \mathbf{I} - \boldsymbol{P}$$

where the next-to-last equality follows because both $\mathbf{I}$and $\boldsymbol{P}$are idempotent. Thus

$$\boldsymbol{Q}^2 = \boldsymbol{Q}, \text{ as claimed}$$

□

Question 1. 3. b. If $\boldsymbol{X}$is an $n \times m$ matrix with rank m, show that the following matrix $\boldsymbol{P}$is idempotent

$$\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$$

Proof. We have:

$$\begin{aligned} \boldsymbol{P}^2 &= (\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T)^2 \\ &= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \\ &= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}(\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \end{aligned}$$

Noting that in the middle term of this we've got:

$$(\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \mathbf{I}$$

We end up with:

$$\boldsymbol{P}^2 = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}$$

□

Question 1. 4. Given matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Is $\boldsymbol{A}$ positive-definite? Prove it or disprove it.

Answer:
Yes, it is positive-definite:

Proof. Blah, blah, blah. □

Question 1. 5. The Gamma $\Gamma(\alpha)$ function is defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x$$

1. Prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
2. Calculate $\Gamma(n)$ where n is a positive integer
3. Calculate $\int_0^\infty x^{-\alpha-1}\mathrm{e}^{-\frac{\beta}{x}}\,\mathrm{d}x$, express your result using Gamma function

Proof. Blah, blah, blah. □