

Stat 8003, Homework 1

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September 2, 2014

Question 1.1. Let \mathbf{A} and \mathbf{B} be two matrices defined as:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}$$

Calculate:

- \mathbf{AB}
- $\mathbf{B}^T \mathbf{A}$

Use R to check your calculation.

Answer:

‘By hand’ calculation yields:

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} -2 & 9 \\ -4 & 18 \\ -6 & 27 \end{pmatrix}$$

For example, the entry in the third row and second column of \mathbf{C} , $c_{3,2}$, is calculated as:

$$c_{3,2} = a_{3,1} \times b_{1,2} + a_{3,2} \times b_{2,2} + a_{3,3} \times b_{3,2} = 3 \times 0 + 6 \times 3 + 9 \times 1 = 27$$

Similarly,

$$\mathbf{B}^T \mathbf{A} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -6 \\ 9 & 18 & 27 \end{pmatrix}$$

whose entry in the first row second column, for example, is calculated as:

$$2 \times 2 + 1 \times 4 + (-2) \times 6 = -4$$

The following R code can be used to verify these results:

*Authors are listed in random order. The random order was generated using R's `sample()` function

```
# Solution to Question1, Homework 1

#given matrix A:
col1 <- c(1,2,3)
A <- cbind(col1, 2*col1, 3*col1)
#and matrix B:
B <- matrix((c(2, 1, -2, 0, 3, 1))), ncol = 2, nrow = 3)

#compute the following prodcets:
A %% B
t(B) %% A
```

Question 1.2. If \mathbf{A} is invertible, prove that $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$

Proof. \mathbf{A} being invertible, consider the product $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. By property (b) on page 17 of the lecture notes, we have:

$$1 = \det(\mathbf{I}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})$$

from which

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

□

Question 1. 3. a. If matrix \mathbf{P} is idempotent, then $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ is also idempotent.

Proof. Since

$$\mathbf{Q}^2 = (\mathbf{I} - \mathbf{P})^2 = \mathbf{I}^2 - \mathbf{I}\mathbf{P} - \mathbf{P}\mathbf{I} + \mathbf{P}^2 = \mathbf{I} - \mathbf{P} - \mathbf{P} + \mathbf{P} = \mathbf{I} - \mathbf{P}$$

where the next-to-last equality follows because both \mathbf{I} and \mathbf{P} are idempotent. Thus

$$\mathbf{Q}^2 = \mathbf{Q}, \text{ as claimed}$$

□

Question 1. 3. b. If \mathbf{X} is an $n \times m$ matrix with rank m , show that the following matrix \mathbf{P} is idempotent:

$$\mathbf{P} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$$

Proof. We have:

$$\begin{aligned} \mathbf{P}^2 &= (\mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T)^2 \\ &= \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \\ &= \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} (\mathbf{X}^T \mathbf{X}) (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \end{aligned}$$

Noting that in the middle term reduces to¹:

$$(\mathbf{X}^T \mathbf{X})(\mathbf{X}^T \mathbf{X})^{-1} = \mathbf{I}$$

We end up with:

$$\mathbf{P}^2 = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T = \mathbf{P}$$

□

Question 1. 4. Given matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Is \mathbf{A} positive-definite? Prove it or disprove it.

Answer:

Yes, it is positive-definite:

$$\begin{aligned} (x_1 \ x_2 \ x_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= (x_1 \ x_2 \ x_3) \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} = \\ &= (2x_1^2 - x_1x_2) + (-x_2x_1 + 2x_2^2 - x_2x_3) + (-x_3x_2 + 2x_3^2) = \\ &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 = \\ &= x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2 = \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 \geq 0 \end{aligned}$$

□

Question 1. 5. The Gamma $\Gamma(\alpha)$ function is defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

¹The inverse will exist because \mathbf{X} has rank m ($\mathbf{X}^T \mathbf{X}$ is $m \times m$ of full rank).

1. Prove that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
2. Calculate $\Gamma(n)$ where n is a positive integer
3. Calculate $\int_0^\infty x^{-\alpha-1}e^{-\frac{\beta}{x}} dx$, express your result using Gamma function

Answer:

1. *Proof.* Using the definition of Gamma and integrating by parts:

$$\begin{aligned}
 \Gamma(\alpha + 1) &= \int_0^\infty x^\alpha e^{-x} dx = \int_0^\infty x^\alpha d(-e^{-x}) \\
 &= -x^\alpha e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} d(x^\alpha) \\
 &= 0 + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx \\
 &= \alpha\Gamma(\alpha)
 \end{aligned}$$

□

2. We have

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1$$

In particular, from part 1, when α is some positive integer n , we have²:

$$\Gamma(n + 1) = n\Gamma(n) = n(n - 1)\Gamma(n - 1) = n \cdot (n - 1) \cdots 2 \cdot 1 = n!$$

3. We'll do this step-by-step. First, using the substitution $y = \frac{1}{x}$, we have that $dy = d(\frac{1}{x}) = -\frac{1}{x^2}dx$ (also, since y changes from $\infty \rightarrow 0$ as x changes from $0 \rightarrow \infty$ we appropriately flip the sign in what follows). Thus:

$$\begin{aligned}
 \int_0^\infty x^{-\alpha-1} e^{-\frac{\beta}{x}} dx &= - \int_0^\infty (1/x)^{(\alpha+1)} e^{-\frac{\beta}{x}} (1/x)^{-2} d(1/x) \\
 &= - \int_0^\infty (1/x)^{(\alpha-1)} e^{-\frac{\beta}{x}} d(1/x) \\
 &= - \int_\infty^0 y^{(\alpha-1)} e^{-\beta y} d(y) = \int_0^\infty y^{(\alpha-1)} e^{-\beta y} dy
 \end{aligned}$$

Now we use the substitution $z = \beta y$. We have $dy = (1/\beta)dz$. So

$$\begin{aligned}
 \int_0^\infty y^{(\alpha-1)} e^{-\beta y} dy &= \int_0^\infty (z/\beta)^{(\alpha-1)} e^{(-z)} (1/\beta) dz \\
 &= (1/\beta)^{\alpha-1} (1/\beta) \int_0^\infty z^{\alpha-1} e^{-z} dz \\
 &= \beta^{-\alpha} \Gamma(\alpha)
 \end{aligned}$$

²For the mathematical hair-splitters, mathematical induction can be used to formally prove this result. For the rest of the world, however, this argument should be convincing enough, we hope.