8.5. Hierarchical Model and MCMC

Example 8.5.1 Baseball example

Let Y_i be the batting average at the first 45 - at bats.

$$45 \cdot Y_i \sim \text{Bin}(45, P_i)$$

$$\begin{cases} Y_i \sim N(\theta_i, \sigma^2) \\ \theta_i \sim N(\mu, \tau^2) \quad i = 1, 2, \ldots 18 \\ \mu \sim N(0, 1000) \\ (\sigma^2, \tau^2)^{-1} \sim \text{Gamma}(0.001, 0.001) \end{cases}$$

$$\hat{\theta}_i = E(\theta_i \mid \tilde{Y})$$

MCMC. Gibbs Sampler:

$$\theta_i \mid Y, \mu, \tau^2, \sigma^2$$

$$\mu \mid Y, \theta_i, \tau^2, \sigma^2$$

$$\tau^2 \mid Y, \theta_i, \mu, \sigma^2$$

$$\sigma^2 \mid Y, \theta_i, \mu, \tau^2$$

$$\theta_i \mid \text{Rest} \sim N(My_i + (1-M)\mu, \ M\sigma^2) \qquad M = \frac{\tau^2}{\tau^2 + \sigma^2}$$

$$\mu \mid \text{Rest} \sim$$

$$\theta_i \overset{iid}{\sim} N(\mu, \tau^2) \Rightarrow \bar{\theta} \sim N\left(\mu, \frac{\tau^2}{n}\right)$$

$$\mu \sim N(0, 1000)$$

$$\mu \mid \text{Rest} \sim N\left(M_1 \bar{\theta} \cancel{+ (1-M_1) \cdot 0}, \ M_1 \frac{\tau^2}{n}\right) \quad \text{where } M_1 = \frac{1000}{1000 + \frac{\tau^2}{n}}$$

$\sigma^2 \mid \text{Rest}$ $\quad Y_i \sim N(\theta_i, \sigma^2)$

$$l(y_i) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{(y_i-\theta_i)^2}{2\sigma^2}\right\}$$

$$l(y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{\Sigma(y_i-\theta_i)^2}{2\sigma^2}\right\}$$

Since $\sigma^2 \sim \text{INGamma}(\alpha, \beta)$

$$\Rightarrow l(\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left\{-\frac{\beta}{\sigma^2}\right\}$$

$$\sigma^2 \mid \text{Rest} \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{\Sigma(y_i-\theta_i)^2}{2\sigma^2}\right\} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left\{-\frac{\beta}{\sigma^2}\right\}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{n}{2}+1} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{\Sigma(y_i-\theta_i)^2}{2}+\beta\right)\right\}$$

$$\sigma^2 \mid \text{Rest} \sim \text{INGamma}\left(\alpha+\frac{n}{2}, \frac{\Sigma(y_i-\theta_i)^2}{2}+\beta\right)$$

$$\tau^2 \mid \text{Rest} \sim \text{INGamma}\left(\alpha+\frac{n}{2}, \frac{\Sigma(\theta_i-\mu)^2}{2}+\beta\right)$$

BUGS Bayesian Inference Using Gibbs Sampling

$$\begin{cases} Y_i \mid \theta_i \overset{ind}{\sim} N(\theta_i, \sigma^2) \\ \theta_i \sim N(\mu, \tau^2) \end{cases}$$

$$E\, \theta_i \mid Y_i = MY_i + (1-M)\mu$$

① Shrinkage effect.

② Borrowing the strength.

8.6. Empirical Bayes

X_i be the batting average for the first 45 at-bats

VST.

$$Y_i = 2\sqrt{n}\arcsin\sqrt{X_i} = 2\sqrt{45}\arcsin\sqrt{X_i}$$

$$\theta_1 = 2\sqrt{45}\arcsin\sqrt{p_i}$$

$$\begin{cases} Y_i \mid \theta_i \overset{ind}{\sim} N(\theta_i, 1) \\ \theta_i \overset{ind}{\sim} N(\mu, \tau^2) \end{cases}$$

$$\theta_i \mid Y \sim N(MY_i + (1-M)\mu, M) \qquad M = \frac{\tau^2}{1+\tau^2}$$

$$\hat{\theta}_i = E\theta_i \mid Y = My_i + \underline{(1-M)}\mu$$

$$f(y_i \mid \mu, \tau^2) = \int f(y_i \mid \theta_i) f(\theta_i \mid \mu, \tau^2)\, d\theta_i$$

$$EY_i = E(\underline{EY_i \mid \theta_i}) = E\theta_i = \mu$$

$$EY_i^2 = E(EY_i^2 \mid \theta_i) = E(\theta_i^2 + 1) = \mu^2 + \tau^2 + 1$$

Set $\begin{cases} \mu = m_1 \\ \mu^2 + \tau^2 + 1 = m_2 \end{cases} \Rightarrow \begin{cases} \hat{\mu} = m_1 \\ \hat{\tau}^2 = (m_2 - 1 - m_1^2)_+ \end{cases}$

$$= \max(0, m_2 - 1 - m_1^2)$$

$$\hat{M} = \frac{\hat{\tau}^2}{1+\hat{\tau}^2}$$

$$\hat{\theta}_i = \hat{M} y_i + (1-\hat{M})\hat{\mu}$$

$$= \hat{M} y_i + (1-\hat{M})\bar{y}$$

borrowing the strength

Lindley - James - Stein Estimator.

$$Y_i \overset{ind}{\sim} N(\theta_i, \sigma^2) \qquad i = 1, 2, \cdots p$$

<u>MLE</u>: $\hat{\theta}_i = y_i$

$$\hat{\theta}_{i,JS} = \left(1 - \frac{(p-2)\sigma^2}{\underline{\sum Y_i^2}}\right) y_i + \frac{(p-2)\sigma^2}{\sum y_i^2}\, 0$$

If $p > 2$, $E\sum(\hat{\theta}_{i,JS} - \theta_i)^2 < E\sum(Y_i - \theta_i)^2$

Lindley-James-Stein Estimator

$$\hat{\theta}_{i,LJS} = \hat{M}_{LJS})\, y_i + (1-\hat{M}_{LJS})\, \bar{y}$$

$$\hat{M}_{LJS} = 1 - \frac{(p-3)\sigma^2}{\Sigma (y_i - \bar{y})^2}$$

Nonparametric empirical Bayes approach.

Two-group models

$H_0^i: \theta_i = 0. \mapsto$ i-th gene is NOT significant

$H_a^i: \theta_i = 1$ $\downarrow$ is significant

$\underline{t_i} \approx z_i \to$ test statistic

$$z_i \mid H_0^i \sim N(0,1) = f_0(x)$$

$$z_i \mid H_1^i \sim f_1(x)$$

$$z_i \overset{ind}{\sim} (1-\theta_i) f_0(z_i) + \underline{\theta_i f_1(z_i)}$$

$$\theta_i \overset{iid}{\sim} \text{Bernoulli}(\pi_1)$$

local fdr

$$P(\theta_i = 0 \mid z) = \frac{\pi_0 f_0(z_i)}{\pi_0 f_0(z_i) + \pi_1 f_1(z_i)} = \frac{\pi_0 f_0(z_i)}{f(z)}$$

$$z_i \overset{iid}{\sim} \pi_0 f_0(z_i) + \pi_1 f(z_i)$$

$\hat{f}(z) \approx$ kernel density

$$\text{local fdr} = \frac{\pi_0 f_0(z_i)}{\hat{f}(z_i)} \approx \frac{f_0(z_i)}{\hat{f}(z_i)}$$

Sun and Cai (2007, JASA)