

My Notes About Γ

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A random variable X is said to have a gamma distribution with parameters (α, β) , denoted by $\Gamma(\alpha, \beta)$, if its pdf is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \quad \text{where } x \geq 0 \quad \alpha, \beta > 0$$

Here $\Gamma(\alpha)$ is the gamma function computed as $\int_0^\infty e^{-y} y^{\alpha-1} dy$. (This reduces to the factorial function for integer α : $\Gamma(\alpha) = \alpha!$)

The Moment Generating Function:

$$\begin{aligned} M_X(t) &= E e^{tX} \\ &= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x/\frac{\beta}{1-\beta t}} dx \end{aligned}$$

To see what this integrates to, use this common trick: leave inside the integral a function that's easily recognizable to be a pdf, which would therefore integrate to 1, and pull remaining constants out of the integral sign:

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha} e^{tx} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha} e^{tx} x^{\alpha-1} e^{-x/\beta} dx \\ &= \left(\frac{1}{1-\beta t}\right)^\alpha \end{aligned}$$

Where the last equality follows because the thing inside the integral is a Gamma pdf.

First moment, the expectation, by direct integration:

$$\begin{aligned} E X &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^\alpha e^{-\frac{x}{\beta}} dx \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta} \left(\frac{x}{\beta}\right)^\alpha e^{-\frac{x}{\beta}} dx \\ &= \frac{\Gamma(\alpha+1) \cdot \beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+1)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^\alpha e^{-\frac{x}{\beta}} dx \\ &= \frac{\Gamma(\alpha+1) \cdot \beta}{\Gamma(\alpha)} = \alpha\beta \end{aligned}$$

where the last integrand is 1 because it itself is a gamma pdf with parameters $(\alpha+1, \beta)$.

Second moment, by direct integration:

$$\begin{aligned} E X^2 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x^2 \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} dx \\ &= \frac{\Gamma(\alpha+2) \cdot \beta^2}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+2)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-\frac{x}{\beta}} dx \\ &= \frac{\Gamma(\alpha+2) \cdot \beta^2}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2 \end{aligned}$$

The Variance is then:

$$Var(X) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Moments from the Moment Generating Function:

Differentiate the Moment Generating Function (mgf) appropriate number of times and evaluate the derivative at $t = 0$ to get the moments: differentiate once to get the expectation, twice to get the second moment, etc.. E.g.:

$$(M_X(t))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}$$

$$(M_X(0))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}} \Big|_{t=0} = \alpha\beta$$

$$(M_X(0))'' = \frac{\alpha(\alpha+1)\beta^2}{(1-\beta t)^{\alpha+2}} \Big|_{t=0} = \alpha(\alpha+1)\beta^2$$

Cumulant Generating Function:

This is defined as:

$$S_X(t) = \log(M_X(t))$$

So It's easily verified that:

$$\left. \frac{d}{dt} S_X(t) \right|_{t=0} = E[X] \quad \text{and} \quad \left. \frac{d^2}{dt^2} S_X(t) \right|_{t=0} = \text{Var}(X)$$

Distribution of Log of Gamma:

Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of Y .

In the following notation, use the indicator functions to easily track the ranges of the variables during the transformation:

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} 1_{(0, \infty)}(x).$$

We'll use the following well known theorem about computing the density of a transform, g , of some random variable X . Very loosely put, if $g(X) = Y$ and g has an *inverse*, call it $h(Y)$, then the pdf of Y is:

$$f_Y(y) = f_X(h(y)) |h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = e^Y$ and $h'(y) = e^y$. Therefore:

$$\begin{aligned} f_Y(y) &= f_X(h(y)) |h'(y)| \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(e^{\alpha(y-1)} - e^{-e^{y/\beta}} \right) |e^y| 1_{(-\infty, \infty)}(y) \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(e^{\alpha y} - e^{-e^{y/\beta}} \right) e^{-y} e^y 1_{(-\infty, \infty)}(y) \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{(\alpha y - e^{y/\beta})} 1_{(-\infty, \infty)}(y) \end{aligned}$$

OK, so far so good. We found the pdf of Log-Of-Gamma.

Now we find its moment generating function. By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} e^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{(\alpha y - e^{y/\beta})} dy$$

To integrate this we use the same trick of leaving inside the integral sign the part of the function that integrates to 1, getting constants appropriately out of the integral. Luckily, this can be done here:

$$\begin{aligned}
M_Y(t) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{ty} e^{(\alpha y - e^{y/\beta})} dy \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{(ty + \alpha y - e^{y/\beta})} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha) \beta^\alpha \Gamma(\alpha + t)} \int_{-\infty}^{+\infty} e^{y(t + \alpha) - e^{y/\beta}} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} \frac{1}{\beta^\alpha \Gamma(\alpha + t)} e^{y(t + \alpha) - e^{y/\beta}} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)}
\end{aligned}$$

Where the last equality follows because the thing inside the integral is a Log-Gamma pdf and integrates to 1.

Next, by definition, the *cumulant generating function* is just the log of this:

$$\log M_Y(t) = \log \left(\frac{\Gamma(\alpha + t)}{\Gamma(\alpha)} \right)$$