# My Notes About *log* of Gamma

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A random variable $X$ is said to have a gamma distribution with parameters $(\alpha, \beta)$, denoted by $\Gamma(\alpha, \beta)$, if its `pdf` is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \cdot \mathrm{e}^{-\frac{x}{\beta}} \cdot 1_{(0,\infty)}(x) \text{ where } \alpha, \beta > 0$$

Here $\Gamma(\alpha)$ is the gamma function computed as $\int_0^\infty \mathrm{e}^{-y} y^{\alpha-1} dy$ and $1_{(0,\infty)}(x)$ stands for the indicator function (i.e. it's value is $1$ for non-negative $x$-es and $0$ elsewhere).

## *Distribution of Log of Gamma*

First, we are interested in the distribution of the natural logarithm of Gamma.
Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of $Y$.

We'll use the following well known theorem about computing the density of a transform, $g$, of some random variable $X$. Loosely put, if $g(X) = Y$ and $g$ has an *inverse*, call it $h(Y)$, then the `pdf` of $Y$ is:

$$f_Y(y) = f_X(h(y))|h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = \mathrm{e}^Y$ and $h'(y) = \mathrm{e}^y$. Therefore:

$$\begin{aligned}
f_Y(y) &= f_X(h(y))|h'(y)| \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha(y-1)} - \mathrm{e}^{-\mathrm{e}^{y/\beta}}\right) |\mathrm{e}^y| \;\; 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha y} - \mathrm{e}^{-\mathrm{e}^{y/\beta}}\right) \mathrm{e}^{-y} \mathrm{e}^y \;\; 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \;\; \mathrm{e}^{\left(\alpha y - e^{y/\beta}\right)} 1_{(-\infty,\infty)}(y)
\end{aligned}$$

This expression is a valid `pdf` – it integrates to 1 - and is said to have a `Log-Of-Gamma` `pdf` with parameters $(\alpha, \beta)$.

## *Moments of Log Of Gamma*

We now find the Moment Generating Function (`MGF`) of `Log-Of-Gamma`$(\alpha, \beta)$. By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} \ \mathrm{e}^{\left(\alpha y - e^{y/\beta}\right)} dy$$

To integrate this we use an old trick: we try to leave inside the integral sign the part of the function that integrates to 1 (this is often an easily recognizeable `pdf`); we then pull the constants that get in the way of making this integral equal to 1 out of the integral sign. Luckily, this can be done here:

$$\begin{aligned}
M_Y(t) &= \frac{1}{\beta^\alpha \ \Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{ty} \ \ \mathrm{e}^{\left(\alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha \ \Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{\left(ty + \alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha \ \Gamma(\alpha)} \cdot \frac{\beta^{\alpha+t} \ \Gamma(\alpha+t)}{\beta^{\alpha+t} \ \Gamma(\alpha+t)} \cdot \int_{-\infty}^{+\infty} \mathrm{e}^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^{\alpha+t} \ \Gamma(\alpha+t)}{\beta^\alpha \ \Gamma(\alpha)} \cdot \int_{-\infty}^{+\infty} \frac{1}{\beta^{t+\alpha} \ \Gamma(\alpha+t)} \mathrm{e}^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^t \ \Gamma(\alpha+t)}{\Gamma(\alpha)}
\end{aligned}$$

Where the last equality follows because the thing inside the integral is the `Log-Of-Gamma` `pdf` derived above and integrates to 1.

*In Short:*

$$\boxed{M_Y(t) = \frac{\beta^t \ \Gamma(\alpha+t)}{\Gamma(\alpha)} \quad \text{when } Y \text{ is } \texttt{Log-Of-Gamma}\,(\alpha, \beta)}$$

## *Cumulant Generating Function*

The Cumulant Generating Function (`CGF`) of a random variable $Y$ is defined as:

$$S_Y(t) = \log(M_Y(t))$$

It can be verified that:

$$\frac{d}{dt} S_Y(t)\Big|_{t=0} = \mathrm{E}Y \quad \text{and} \quad \frac{d^2}{dt^2} S_Y(t)\Big|_{t=0} = Var(Y)$$

## *Expectation and Variance of Log Of Gamma*

To find the expectation of `Log-Of-Gamma`, we differentiate $S_Y(t) = \log(M_Y(t))$ and evaluate the result at $t = 0$.

$$
\begin{aligned}
(S_Y(t))' = (\log(M_Y(t)))' &= \left(\log\left(\frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)}\right)\right)' \\
&= \frac{1}{\frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)}} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t\,\Gamma(\alpha+t)\right)' \\
&= \frac{\Gamma(\alpha)}{\beta^t\,\Gamma(\alpha+t)} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t \log(\beta)\,\Gamma(\alpha+t) + \beta^t\,\Gamma'(\alpha+t)\right) \\
&= \log(\beta) + \frac{\Gamma'(\alpha+t)}{\Gamma(\alpha+t)}
\end{aligned}
$$

The last term is the log-derivative of the Gamma function evaluated at $\alpha + t$. This log-derivative is called the `digamma` function and is denoted by $\psi(\cdot)$. Thus:

$$\boxed{(S_Y(t))' = \log(\beta) + \psi(\alpha+t)}$$

We *conclude* that if $Y$ is `Log-Of-Gamma`$(\alpha, \beta)$ distributed, then its expectation is given by:

$$\boxed{\mathrm{E}\,Y = \log(\beta) + \psi(\alpha)}$$

To find the variance of $Y$, we we look at the second derivative, $(S_Y(t))''$, and evaluate it at $t = 0$.

$$
\begin{aligned}
(S_Y(t))'' &= (\log(\beta) + \psi(\alpha+t))' \\
&= \psi'(\alpha+t)
\end{aligned}
$$

The derivative of `digamma` is the so-called `trigamma` function:

$$\boxed{Var(Y) = \psi'(\alpha) = \texttt{trigamma}(\alpha)}$$