Thursday, September 11, 2014 2:27 PM

$X \sim N(\mu, \sigma^2)$ if $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

~~X~~ Log-Normal

$X \sim N(\mu, \sigma^2)$

$Y = \exp(X) \sim \text{Log-Normal}(\mu, \sigma^2)$

$$f(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln y - \mu)^2}{2\sigma^2}\right\}$$

Features of Distributions

Expectation is the weighted mean of a r.v.

Discrete, $\mu = EX = \sum_i x_i P(X = x_i)$

Continuous, $\mu = EX = \int x f(x)\,dx$

If X is continuous r.v. $g(X)$ is any transformation

$$E g(X) = \int g(x) f(x)\, dx$$

Example: $X \sim \text{Bin}(n, p)$, $EX = np$.

$X \sim \text{Poisson}(\lambda)$ $EX = \lambda$

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2, \dots$$

$$EX = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \lambda \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} e^{-\lambda} = \lambda$$

3. $X \sim \text{Exp}(\lambda)$, then $EX = \frac{1}{\lambda}$

4. $X \sim N(\mu, \sigma^2)$, then $EX = \mu$

5. $X \sim \text{LogNormal}(\mu, \sigma^2)$, $EX = e^{\mu + \frac{\sigma^2}{2}}$

Property: ① $E(aX + bY) = E(aX) + E(bY)$

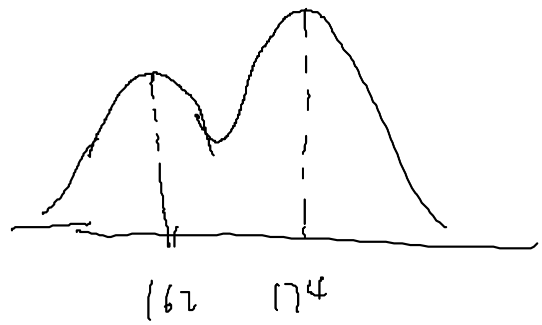
$$= aEX + bEY$$

② $E(XY) = (EX)(EY)$ if X and Y are independent

Let Z be the height

$$E(Z \mid \text{male}) = 174$$

$$E(Z \mid \text{female}) = 162$$



$$Z = 0.52X + 0.48Y$$

$$EZ = E(0.52X + 0.48Y) = 0.52EX + 0.48EY$$

$$= 0.52 \times 174 + 0.48 \times 162$$

<u>Median</u>: For a continuous r.v. X the median ν is defined as

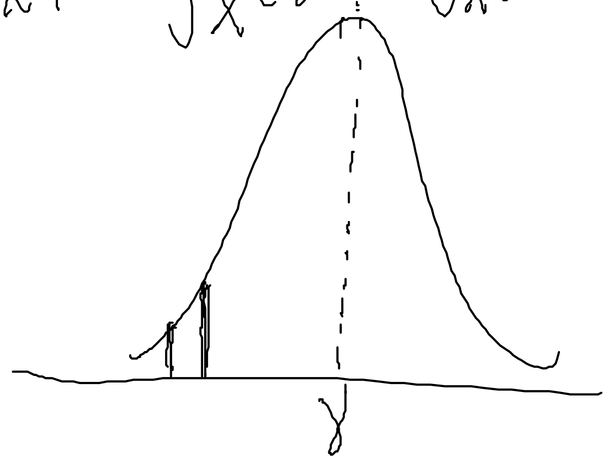
$$P(X \geq \nu) = P(X \leq \nu) = \frac{1}{2}$$

Median: is widely used in Robust statistics.

1 2 5 7 10000

Mode: γ

$$\gamma = \inf\{x : f_X(t) \le f_X(x) \quad \forall t\}$$



Variance $V(X) = E(X - EX)^2$

$$V(X) = E\left(X^2 - 2X\,EX + (EX)^2\right)$$
$$= EX^2 - 2E(X\,EX) + E(EX)^2$$
$$= EX^2 - 2(EX)EX + (EX)^2$$
$$= EX^2 - (EX)^2$$

Second moment $\left(\text{First moment}\right)^2$

r-th moment : EX^r

Properties: ① $V(X) = EX^2 - (EX)^2$

② $V(aX+b) = V(aX) = a^2 V(X)$

③ $V(Y_1+Y_2) = V(Y_1)+V(Y_2)+2\,Cov(Y_1,Y_2)$

$Cov(Y_1, Y_2) = E(Y_1-EY_1)(Y_2-EY_2) = EY_1Y_2 - EY_1 \cdot EY_2$

$$\rho = \frac{Cov(Y_1,Y_2)}{\sqrt{V(Y_1)V(Y_2)}} \qquad -1 \le \rho \le 1$$

ρ: how strongly Y_1 and Y_2 are linearly related?

If $\rho = 0$: Y_1 and Y_2 are not linearly related.

If Y_1 and Y_2 are independent, $\rho = 0$

$\rho = 0 \not\Rightarrow$ independence

⑤ $V(Y_1 + Y_2) = V(Y_1) + V(Y_2)$ if Y_1 and Y_2 are independant

① $X \sim Bin(n, p)$. $\quad V(X) = np(1-p)$.

② $X \sim Poisson(\lambda)$, $\quad V(X) = \lambda$

③ $X \sim N(\mu, \sigma^2)$ $\quad V(X) = \sigma^2$.

④ $X \sim LogNormal(\mu, \sigma^2)$ $\quad V(X) = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}$

$EX = \underline{e^{\mu + \sigma^2/2}}$

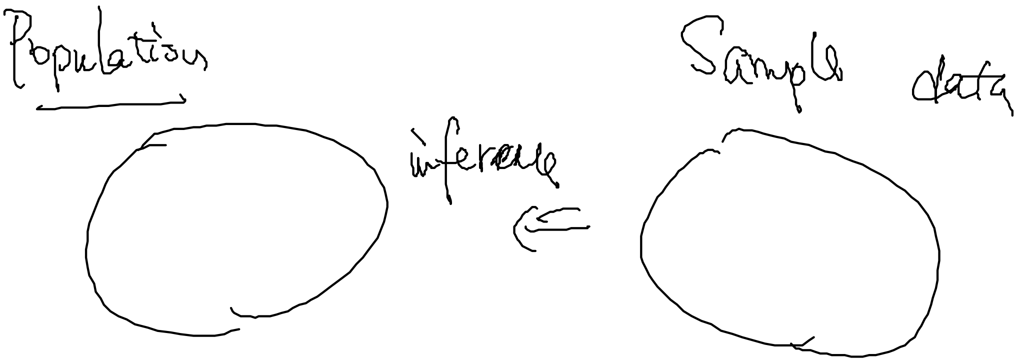
$$EX^2 = V(X) + (EX)^2 = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2} + e^{2\mu + \sigma^2}$$

$$= e^{\sigma^2} e^{2\mu + \sigma^2}$$

⑥ $\begin{cases} \hat{\theta}_1 & \underline{E}\hat{\theta}_1 \neq \theta \\ \hat{\theta}_2 & E\hat{\theta}_2 = \theta \end{cases}$ $\quad V(\hat{\theta}_1) < V(\hat{\theta}_2)$

$\hat{\theta}_1$ is better.

Estimation.



Probability Distribution:

Normal(μ, σ^2)

Log-Normal(μ, σ^2)

Gamma(α, β)

Parametric Model

Nonparametric Model

Poisson(λ)

Point Estimation

Hypothesis Testing

Confidence interval

Collect the data

Formulate the model

Diagnosis of the model

| Confidence interval

Diagonosis of the model
Calculate Statistic.
Draw conclusion

Point Estimation.

Parametric Estimation

$N(\mu, \sigma^2)$

$\pi_1 N(\mu_1, \sigma_1^2) + \pi_2 N(\mu_2, \sigma_2^2)$

$\text{Gamma}(\alpha, \beta)$

$\text{LogNormal}(\mu, \sigma^2)$

§3. Methods of Moments MOM

Principle: Assume a parametric model, match the moments of the distribution to the moments of the sample

$$Y_1, Y_2, \dots, Y_n \overset{\text{iid}}{\sim} f(y)$$

$$m_r = \frac{1}{n} \sum_i Y_i^r \rightarrow \text{r-th moment of the sample}$$

$$m_1 = \mu = EY_1$$

$$\begin{cases} m_1 = \mu = EY_1 \\ m_2 = EY_1^2 \\ \vdots \\ m_r = EY_1^r \end{cases}$$

GDP example: Y_i be the GDP of the i-th country

$$Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\begin{cases} m_1 = EY = \mu \\ m_2 = EY^2 = \sigma^2 + \mu^2 \end{cases}$$

$$\begin{cases} \hat{\mu} = m_1 \\ \hat{\sigma}^2 = m_2 - m_1^2 \end{cases}$$

Method II: $Y_i \sim$ Log Normal (μ, σ^2)

$$\begin{cases} m_1 = EY = e^{\mu + \frac{\sigma^2}{2}} \\ m_2 \quad \ldots \sigma^2 \quad 2(\mu + \frac{\sigma^2}{2}) \quad \ldots \sigma^2 \end{cases}$$

$$m_2 = EY^2 = e^{\sigma^2} \cdot e^{2\left(\mu + \frac{\sigma^2}{2}\right)} = m_1^2 e^{\sigma^2}$$

$$\begin{cases} \hat{\mu} = \log(m_1) - \frac{\hat{\sigma}^2}{2} \\ \hat{\sigma}^2 = \log \frac{m_2}{m_1^2} \end{cases}$$

Method III: $Y_i \sim \text{LogNormal}(\mu, \sigma^2)$

$X_i \sim \log(Y_i) \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{cases} m_1' = \frac{1}{n}\sum X_i = \mu \\ m_2' = \frac{1}{n}\sum X_i^2 = \mu^2 + \sigma^2 \end{cases}$$

$$\begin{cases} \hat{\mu} = m_1' \\ \hat{\sigma}^2 = m_2' - m_1'^2 \end{cases}$$

Poisson $EX = \lambda \quad = m_1 \qquad \hat{\lambda} = m_1$

$V(X) = \lambda \quad EX^2 = \lambda^2 + \lambda \Rightarrow \hat{\lambda} = \_$

Bias $\beta \to$ parameter $\hat{\beta} \to$ estimate

$$\text{bias} = E\hat{\beta} - \beta$$

$\hat{\beta}$ is unbiased if $\text{Bias} = 0$

② **Consistency**, $\hat{\beta} \xrightarrow{P} \beta$ $n \to \infty$

$$P\left(\left|\hat{\beta} - \beta\right| < \epsilon\right) \to 1 \quad \text{as } n \to \infty$$

③ **Efficiency**:

Bias - Variance Tradeoff:

$$MSE = E\left(\hat{\beta} - \beta\right)^2$$

$$= E\left(\hat{\beta} - E\hat{\beta} + E\hat{\beta} - \beta\right)^2$$

$$= E\left(\hat{\beta} - E\hat{\beta}\right)^2 + \cancel{2E(\hat{\beta} - E\hat{\beta})(E\hat{\beta} - \beta)} \quad 0$$

9

$$+ E\left(E\hat{\beta} - \beta\right)^2$$

$$= V(\hat{\beta}) + \left(\text{bias}(\hat{\beta})\right)^2$$

$X_1 \cdots X_n \sim \text{LogNormal}(\mu, \sigma^2)$

$\begin{cases} \hat{\mu}_1 \\ \hat{\sigma}^2_1 \end{cases}$ $\begin{cases} \hat{\mu}_2 \\ \hat{\sigma}^2_2 \end{cases}$

$\mu = 0 \quad \sigma = 1$

1000

1. Generate sample $X \sim \text{LogNormal}(0, 1)$
2. Calculate $\begin{pmatrix} \hat{\mu}_1 \\ \hat{\sigma}^2_1 \end{pmatrix}$ $\begin{pmatrix} \hat{\mu}_1 \\ \hat{\sigma}^2_2 \end{pmatrix}$
3. Calculate squared error $(\hat{\mu} - \mu)^2 + (\hat{\sigma}^2 - \sigma^2)^2$
4. Replicate 1 - 3 2,000 times
5. Take the average