Homework 1

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Question 1.1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices defined as:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}$$

Calculate:

- $\boldsymbol{AB}$
- $\boldsymbol{B}^T\boldsymbol{A}$

Use R to check your calculation.

Answer:
'By hand' calculation yields:

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix} -2 & 9 \\ -4 & 18 \\ -6 & 27 \end{pmatrix}$$

For example, the entry in the third row and second column of $\boldsymbol{C}$, $c_{3,2}$, is calculated as:

$$c_{3,2} = a_{3,1} \times b_{1,2} + a_{3,2} \times b_{2,2} + a_{3,3} \times b_{3,2} = 3 \times 0 + 6 \times 3 + 9 \times 1 = 27$$

Similarly,

$$\boldsymbol{B}^T\boldsymbol{A} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -6 \\ 9 & 18 & 27 \end{pmatrix}$$

whose entry in the first row second column, for example, is calculated as:

$$2 \times 2 + 1 \times 4 + (-2) \times 6 = -4$$

The following `R` code can be used to verify these results:

*Authors listed in random order. The random order was generated using `R`'s `sample()` function

```
# Solution to Question1, Homework 1

#given matrix A:
col1 <- c(1,2,3)
A <- cbind(col1, 2*col1, 3*col1)
#and matrix B:
B <- matrix((c(2, 1, -2, 0, 3, 1)), ncol = 2, nrow = 3)

#compute the following prodcts:
A %*% B
t(B) %*% A
```

Question 1.2. If $\boldsymbol{A}$ is invertible, prove that $det(\boldsymbol{A}^{-1}) = (det(\boldsymbol{A}))^{-1}$

Proof. $\boldsymbol{A}$ being invertible, consider the product $\boldsymbol{A}\boldsymbol{A}^{-1} = \mathbf{I}$. By property (b) on page 17 of the lecture notes, we have:

$$1 = det(\mathbf{I}) = det(\boldsymbol{A}\boldsymbol{A}^{-1}) = det(\boldsymbol{A})det(\boldsymbol{A}^{-1})$$

from which

$$det(\boldsymbol{A}^{-1}) = \frac{1}{det(\boldsymbol{A})}$$

□

Question 1. 3. a. If matrix $\boldsymbol{P}$is idempotent, then $\boldsymbol{Q} = \mathbf{I} - \boldsymbol{P}$ is also idempotent.

Proof. Since

$$\boldsymbol{Q}^2 = (\mathbf{I} - \boldsymbol{P})^2 = \mathbf{I}^2 - \mathbf{I}\boldsymbol{P} - \boldsymbol{P}\mathbf{I} + \boldsymbol{P}^2 = \mathbf{I} - \boldsymbol{P} - \boldsymbol{P} + \boldsymbol{P} = \mathbf{I} - \boldsymbol{P}$$

where the next-to-last equality follows because both $\mathbf{I}$ and $\boldsymbol{P}$ are idempotent. Thus

$$\boldsymbol{Q}^2 = \boldsymbol{Q}, \text{ as claimed}$$

□

Question 1. 3. b. If $\boldsymbol{X}$ is an $n \times m$ matrix with rank m, show that the following matrix $\boldsymbol{P}$ is idempotent:

$$\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$$

Proof. We have:

$$\begin{aligned}
\boldsymbol{P}^2 &= (\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T)^2 \\
&= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \\
&= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}(\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T
\end{aligned}$$

Noting that in the middle term reduces to:

$$(\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \mathbf{I}$$

We end up with:

$$\boldsymbol{P}^2 = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}$$

□

Question 1. 4. Given matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Is $\boldsymbol{A}$ positive-definite? Prove it or disprove it.

Answer:
Yes, it is positive-definite:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} =$$

$$(2x_1^2 - x_1x_2) + (-x_2x_1 + 2x_2^2 - x_2x_3) + (-x_3x_2 + 2x_3^2) =$$

$$2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 =$$

$$x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2 =$$

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 \geq 0$$

□

Question 1. 5. The Gamma $\Gamma(\alpha)$ function is defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x$$

1. Prove that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

2. Calculate $\Gamma(n)$ where n is a positive integer

3. Calculate $\int_0^\infty x^{-\alpha-1} \mathrm{e}^{-\frac{\beta}{x}} \, \mathrm{d}x$, express your result using Gamma function

Answer:

1. *Proof.* Using the definition of Gamma and integrating by parts:

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty x^\alpha \mathrm{e}^{-x} \, \mathrm{d}x = \int_0^\infty x^\alpha \, \mathrm{d}(-\mathrm{e}^{-x}) \\ &= -x^\alpha \mathrm{e}^{-x}\Big|_0^\infty + \int_0^\infty \mathrm{e}^{-x} \, \mathrm{d}(x^\alpha) \\ &= 0 + \alpha \int_0^\infty x^{\alpha-1} \mathrm{e}^{-x} \, \mathrm{d}x \\ &= \alpha\Gamma(\alpha)\end{aligned}$$

□

2. We have

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-x} \, \mathrm{d}x = 1$$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1$$

In particular, from part 1, when α is some integer n we have:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n \cdot (n-1) \cdots 2 \cdot 1 = n!$$

3. We'll do this step-by-step. First, using the substitution $y = \frac{1}{x}$ and appropriately substituting the limits of integration (and because $\mathrm{d}(\frac{1}{x}) = -\frac{1}{x^2}\mathrm{d}x)$, we have

$$\begin{aligned}\int_0^\infty x^{-\alpha-1} \mathrm{e}^{-\frac{\beta}{x}} \, \mathrm{d}x &= \int_0^\infty (1/x)^{(\alpha+1)} \mathrm{e}^{-\frac{\beta}{x}} (1/x)^{-2} \mathrm{d}(1/x) \\ &= \int_0^\infty (1/x)^{(\alpha-1)} \mathrm{e}^{-\frac{\beta}{x}} \mathrm{d}(1/x) \\ &= \int_\infty^0 y^{(\alpha-1)} \mathrm{e}^{-\beta y} \mathrm{d}(y) = -\int_0^\infty y^{(\alpha-1)} \mathrm{e}^{-\beta y} \mathrm{d}y\end{aligned}$$

Now we use the substitution $z = \beta y$. We have $\mathrm{d}y = (1/\beta)\mathrm{d}z$. So

$$\begin{aligned}-\int_0^\infty y^{(\alpha-1)} \mathrm{e}^{-\beta y} \mathrm{d}y &= -\int_0^\infty (z/\beta)^{(\alpha-1)} \mathrm{e}^{(-z)} (1/\beta) \mathrm{d}z \\ &= -(1/\beta)^{\alpha-1} (1/\beta) \int_0^\infty z^{\alpha-1} \mathrm{e}^{-z} \, \mathrm{d}z \\ &= -\beta^{-\alpha}\Gamma(\alpha)\end{aligned}$$