Stat 8003, Homework 6

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October 8, 2014

Question 6.1. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is 1/2 versus the alternative that the probability is not 1/2. The test rejects if either 0 or 10 heads are observed.

(a) What is the significance level of the test?

(b) If in fact the probability of heads is .1, what is the power of the test?

Answer:

(a) Let X be the number of heads showing up in 10 tosses of the coin, i.e. $X \sim Binomial(10, p)$ where p is the probability of landing heads. The hypotheses are

$$H_0 : \; p = \frac{1}{2}, \quad vs \quad H_a : \; p \neq \frac{1}{2}.$$

By definition,

$$\begin{aligned}
\alpha &= P(X = 0 \mid H_0) + P(X = 10 \mid H_0) \\
&= \binom{10}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{10} + \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0 \\
&= \frac{2}{2^{10}} = \frac{1}{512} \\
&= 0.002
\end{aligned}$$

(b) If $p = 0.1$, then the probability of rejecting the null hypothesis is

$$\begin{aligned}
1 - \beta &= \binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{10}(0.1)^{10}(0.9)^0 \\
&= 0.3486784
\end{aligned}$$

Thus the significance level of this test is 0.002 and its power is 0.3487.

Question 6.2. Let $X_1, , X_n$ be a random sample from an exponential distribution with the density function

$$f(x \mid \theta) = \theta e^{-\theta x}.$$

We want to test the hypothesis

$$H_0: \theta = 1, \quad vs \quad H_a: \theta \neq 1.$$

Set the desired level of significance as $\alpha = 5\%$.

(a) Derive a generalized likelihood ratio test and show that the rejection region is of the form $\mathcal{R} = \{\bar{X} e^{-\bar{X}} \leq c\}$;

(b) Suppose n = 10. Show that the rejection region in (a) is of the form $\mathcal{R} = \{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$, where x_0 and x_1 are determined by c;

(c) When $\theta = 1$, it is known that $\sum_i X_i$ follows $Gamma(n, \frac{1}{\theta})$. How could this knowledge be used to choose c?

Answer:

(a) Since the MLE of the parameter θ for the exponential distribution is given by:

$$\hat{\theta} = 1/\bar{X},$$

We have that the GLRT is given by:

$$\begin{aligned}
\Lambda = \frac{max_{\theta \in \Theta_0 \cup \Theta_1} L(\theta; x)}{max_{\theta \in \Theta_0} L(\theta; x)} &= \frac{max_{\theta \in \Theta_0 \cup \Theta_1} \theta^n e^{-\theta \sum_i x_i}}{max_{\theta \in \Theta_0} \theta^n e^{-\theta \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n e^{-1/\bar{X} \sum_i x_i}}{\theta_0^n e^{-\theta_0 \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n e^{(-1/\bar{X})\ n\ (\sum_i x_i/n)}}{1^n e^{-1\ \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n e^{-n}}{e^{-n\ (\sum_i x_i)/n}} \\
&= (1/\bar{X})^n e^{-n} e^{n\bar{X}}
\end{aligned}$$

Our criterion for rejecting the Null Hypothesis is that this ratio be greater than some constant:

$$\Lambda \geq k^*$$

We can simplify the above expression for Λ further by taking the n-th root of both sides of this expression:

$$\begin{aligned}
\Lambda^{\frac{1}{n}} &= \left((1/\bar{X})^n e^{-n} e^{n\bar{X}}\right)^{\frac{1}{n}} \\
&\geq (k^*)^{\frac{1}{n}}
\end{aligned}$$

$\Rightarrow$

$$(1/\bar{X})\mathrm{e}^{-1}\mathrm{e}^{\bar{X}} \geq (k^*)^{\frac{1}{n}}$$

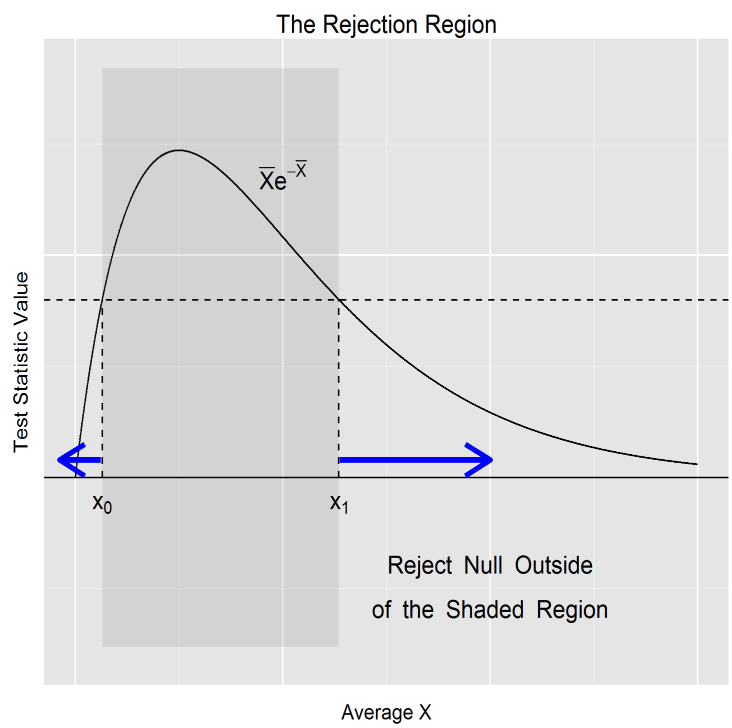
$\Rightarrow$

$$\frac{1}{(1/\bar{X})\mathrm{e}^{\bar{X}}} = \bar{X}\mathrm{e}^{-\bar{X}} \leq \frac{1}{(k^*)^{\frac{1}{n}}\mathrm{e}} = c$$

Thus we have shown that the rejection region $\mathcal{R}$ is given by:

$$\boxed{\bar{X}\mathrm{e}^{-\bar{X}} \leq \text{ some constant } c}$$

(b) Some people like to claim that a picture is worth a few words*. Here's what the rejection region from part (a) might look like:



The dashed horizontal line in this picture corresponds to the constant c from part (a). The test statistic has to be below this constant, that is $\bar{X}$ has to fall outside the shaded

*One author of this homework has heard of pictures being worth thousands of words, but he's skeptical of this claim.

region, for us to reject H_0. This picture shows that x_0 and x_1 are determined by the constant c: at these points our test statistic $\bar{X}\mathrm{e}^{-\bar{X}}$ intersects the horizontal line at level c. ($\bar{X}\mathrm{e}^{-\bar{X}}$ attains the maximum value of $1/\mathrm{e}$ when $\bar{X} = 1$. For $c \geq 1/\mathrm{e}$ we would reject H_0 because then the rejection region would cover the entire $\bar{X}$ space.)

(c) To simplify the notation, let $Y = \sum_i^{10} X_i$. We know that under the null hypothesis Y follows the $Gamma(10, 1)$ distribution. We rewrite the expression we got in part (a) as follows:

$$\bar{X}\mathrm{e}^{-\bar{X}} \leq c \quad \Rightarrow \quad \frac{Y}{10}(\mathrm{e}^{-Y})^{\frac{1}{10}} \leq c$$

$$\Rightarrow \quad Y^{10}\mathrm{e}^{-Y} \leq (10c)^{10}$$

Thus we are looking for a constant c such that

$$P\left(Y^{10}\mathrm{e}^{-Y} \leq (10c)^{10} \right) = \alpha = 0.05.$$

Or, taking the logarithms of both sides of the inequality,

$$P\left(10\log Y - Y \leq \log\, (10c)^{10} \right) = 0.05.$$

Again, for simplicity of notation, denote $\log\,(10c)^{10}$ by k. Thus, for $Y \sim Gamma(10, 1)$, we are looking for such constant k that satisfies the following inequality:

$$\boxed{P\,(\ 10\log Y - Y \leq k\ ) = 0.05}$$

One approach to computing k would be to first compute the `cdf` of the random variable $10\log Y - Y$, and then read the value of k at which this `cdf` equals 0.05.

Using this method (code attached) we obtain

$$k = 11.08295^{\dagger}.$$

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†From this k we may recover c, if need be:

$$k = \log(10c)^{10} = 11.08295$$

$$\Rightarrow$$

$$c = \frac{1}{10}\mathrm{e}^{11.08295/10}$$

$$\boxed{c = 0.3029189}$$

Question 6.3. Suppose, to be specific, that in Problem 2, the observed data are the following:

```
1.07 0.88 0.66 0.55 1.15 0.65 3.45 3.55 3.51 0.48
```

(a) Based on the result in Problem 2, will you reject H_0? What's your p-value?

(b) If we start from generalized likelihood ratio test, and use the asymptotic distribution of $2\log\Lambda$, will you reject H_0? What's you p-value?

Answer:

(a) We now use the test statistic from Problem 2 ($T = 10\log\sum_i x_i - \sum_i x_i$ where x_i are the observed data) to compute the p-value. (R code that computes these is attached). We get:

$$\begin{aligned} \mathtt{T} &= \mathtt{11.74459} \\ \mathtt{p.value} &= \mathtt{0.113} \end{aligned}$$

This p-value is too high. We decide that we can not reject H_0:

$$\boxed{\text{Accept} \quad H_0}$$

(b) We have $n = 10$ and $\bar{X} = 1.595$ from the data. Using the result that $2\log\Lambda | H_0 \sim \chi_1^2$, we obtain the test statistic

$$\begin{aligned} 2\log\Lambda &= 2\log\left(\bar{X}^{-n}\mathrm{e}^{-n}\mathrm{e}^{n\bar{X}}\right) \\ &= -2n\log\bar{X} - 2n + 2n\bar{X} \\ &= 2.562525 \end{aligned}$$

The chances of obtaining a value that is at least as extreme as 2.562525 can be calculated in R by

```
1 - pchisq(2.562525, df=1)   #ans: 0.1094237
```

This p-value is again too high. Unable to reject H_0, we again

$$\boxed{\text{Accept} \quad H_0}$$