Stat 8003, Homework 6

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Question 6.1. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is 1/2 versus the alternative that the probability is not 1/2. The test rejects if either 0 or 10 heads are observed.

(a) What is the significance level of the test?

(b) If in fact the probability of heads is .1, what is the power of the test?

Answer:
...This part is done... Write this up later... Shouldn't take long...

Question 6.2. Let $X_1, , X_n$ be a random sample from an exponential distribution with the density function

$$f(x \mid \theta) = \theta \mathrm{e}^{-\theta x}.$$

We want to test the hypothesis

$$H_0: \ \theta = 1, \quad vs \quad H_a: \ \theta \neq 1.$$

Set the desired level of significance as $\alpha = 5\%$.

(a) Derive a generalized likelihood ratio test and show that the rejection region is of the form $\mathcal{R} = \{\bar{X}\mathrm{e}^{-\bar{X}} \leq c\}$;

(b) Suppose n = 10. Show that the rejection region in (a) is of the form $\mathcal{R} = \{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$, where x_0 and x_1 are determined by c;

(c) When $\theta = 1$, it is known that $\sum_i X_i$ follows $Gamma(n, \frac{1}{\theta})$. How could this knowledge be used to choose c?

Answer:

(a) Since the `MLE` of the parameter θ for the exponential distribuiton is given by:

$$\hat{\theta} = 1/\bar{X},$$

We have that the `GLRT` is given by:

$$\begin{aligned}
\Lambda = \frac{max_{\theta \in \Theta_0 \cup \Theta_1} L(\theta; x)}{max_{\theta \in \Theta_0} L(\theta; x)} &= \frac{max_{\theta \in \Theta_0 \cup \Theta_1} \theta^n \mathrm{e}^{-\theta \sum_i x_i}}{max_{\theta \in \Theta_0} \theta^n \mathrm{e}^{-\theta \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n \mathrm{e}^{-1/\bar{X} \sum_i x_i}}{\theta_0^n \mathrm{e}^{-\theta_0 \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n \mathrm{e}^{(-1/\bar{X})\ n\ (\sum_i x_i/n)}}{1^n \mathrm{e}^{-1\ \sum_i x_i}} \\
&= \frac{(1/\bar{X})^n \mathrm{e}^{-n}}{\mathrm{e}^{-n\ (\sum_i x_i)/n}} \\
&= (1/\bar{X})^n \mathrm{e}^{-n} \mathrm{e}^{n\bar{X}}
\end{aligned}$$

Our criterion for rejecting the Null Hypothesis is that this ratio be greater than some constant:

$$\Lambda \geq k^*$$

We can simplify the above expression for Λ further by taking the n-th root of both sides of this expression:

$$\begin{aligned}
\Lambda^{\frac{1}{n}} &= \left((1/\bar{X})^n \mathrm{e}^{-n} \mathrm{e}^{n\bar{X}} \right)^{\frac{1}{n}} \\
&\geq (k^*)^{\frac{1}{n}}
\end{aligned}$$

$\Rightarrow$

$$(1/\bar{X})\mathrm{e}^{-1}\mathrm{e}^{\bar{X}} \geq (k^*)^{\frac{1}{n}}$$

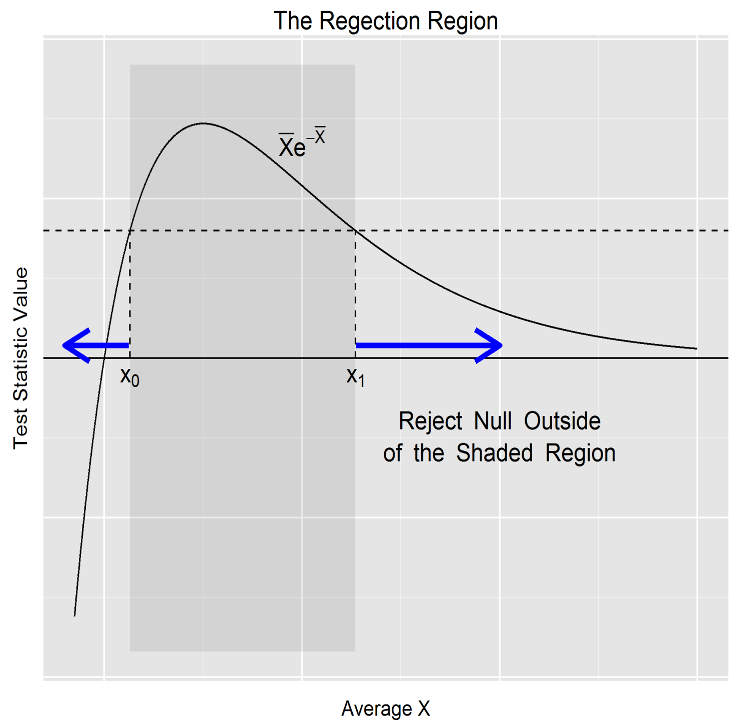
$\Rightarrow$

$$\frac{1}{(1/\bar{X})\mathrm{e}^{\bar{X}}} = \bar{X}\mathrm{e}^{-\bar{X}} \leq \frac{1}{(k^*)^{\frac{1}{n}}\mathrm{e}} = c$$

Thus we have shown that the rejection region $\mathcal{R}$ is given by:

$$\boxed{\bar{X}\mathrm{e}^{-\bar{X}} \leq \text{ some constant } c}$$

(b) Some people like to claim that a picture is worth a few words[1]. Let's see what the rejection region from part (a) might look like:



The dashed horizontal line in this picture corresponds to the constant c from part (a). The test statistic has to be below this constant, that is $\bar{X}$ has to fall outside the shaded region, for us to reject H_0. This picture shows that x_0 and x_1 are determined by the constant c: at these points our test statistic $\bar{X}e^{-\bar{X}}$ intersects the horizontal line at level c.

(c) //TODO: Figure this out later ...

Question 6.3. Suppose, to be specific, that in Problem 2, the observed data are the following:

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1.07 0.88 0.66 0.55 1.15 0.65 3.45 3.55 3.51 0.48
```

(a) Based on the result in Problem 2, will you reject H_0? What's your p-value?

(b) If we start from generalized likelihood ratio test, and use the asymptotic distribution of $2 \log \Lambda$, will you reject H_0? What's you p-value?

Answer:

[1]One author of this homework has heard of pictures being worth thousands of words, but he's skeptical of this claim.