My Notes About *log* of Gamma

By David Dobor

A random variable X is said to have a gamma distribution with parameters (α, β) , denoted by $\Gamma(\alpha, \beta)$, if its pdf is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha - 1} \cdot e^{-\frac{x}{\beta}} \cdot 1_{(0, \infty)}(x) \text{ where } \alpha, \beta > 0$$

Here $\Gamma(\alpha)$ is the gamma function computed as $\int_0^\infty \mathrm{e}^{-y} y^{\alpha-1} dy$ and $1_{(0,\infty)}(x)$ stands for the indicator function (i.e. it's value is 1 for non-negative x-es and 0 elsewhere).

Distribution of Log of Gamma

First, we are interested in the distribution of the natural logarithm of Gamma. Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of Y.

We'll use the following well known theorem about computing the density of a transform, g, of some random variable X. Loosely put, if g(X) = Y and g has an *inverse*, call it h(Y), then the pdf of Y is:

$$f_Y(y) = f_X(h(y))|h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by h(Y), we have $X = h(Y) = e^Y$ and $h'(y) = e^y$. Therefore:

$$f_Y(y) = f_X(h(y))|h'(y)|$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \left(e^{\alpha(y-1)} - e^{-e^{y/\beta}} \right) |e^y| \ 1_{(-\infty,\infty)}(y)$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \left(e^{\alpha y} - e^{-e^{y/\beta}} \right) e^{-y} e^y \ 1_{(-\infty,\infty)}(y)$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \ e^{(\alpha y - e^{y/\beta})} 1_{(-\infty,\infty)}(y)$$

This expression is a valid pdf – it integrates to 1 - and is said to have a Log-Of-Gamma pdf with parameters (α, β) .

Moments of Log Of Gamma

We now find the Moment Generating Function (MGF) of Log-Of-Gamma(α, β). By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} e^{ty} \frac{1}{\beta^{\alpha} \Gamma(\alpha)} e^{(\alpha y - e^{y\beta})} dy$$

To integrate this we use an old trick: we try to leave inside the integral sign the part of the function that integrates to 1 (this is often an easily recognizeable pdf); we then pull the constants that get in the way of making this integral equal to 1 out of the integral sign. Luckily, this can be done here:

$$M_{Y}(t) = \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{ty} e^{(\alpha y - e^{y/\beta})} dy$$

$$= \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{(ty + \alpha y - e^{y/\beta})} dy$$

$$= \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \cdot \frac{\beta^{\alpha+t} \Gamma(\alpha+t)}{\beta^{\alpha+t} \Gamma(\alpha+t)} \cdot \int_{-\infty}^{+\infty} e^{(y(\alpha+t) - e^{y/\beta})} dy$$

$$= \frac{\beta^{\alpha+t} \Gamma(\alpha+t)}{\beta^{\alpha} \Gamma(\alpha)} \cdot \int_{-\infty}^{+\infty} \frac{1}{\beta^{t+\alpha} \Gamma(\alpha+t)} e^{(y(\alpha+t) - e^{y/\beta})} dy$$

$$= \frac{\beta^{t} \Gamma(\alpha+t)}{\Gamma(\alpha)}$$

Where the last equality follows because the thing inside the integral is the Log-Of-Gamma pdf derived above and integrates to 1.

In Short.

$$M_Y(t) = rac{eta^t \; \Gamma(lpha + t)}{\Gamma(lpha)} \quad ext{when Y is Log-Of-Gamma } (lpha, eta)$$

Cumulant Generating Function

The Cumulant Generating Function (CGF) of a random variable Y is defined as:

$$S_Y(t) = \log(M_Y(t))$$

It can be verified that:

$$\left. \frac{d}{dt} S_Y(t) \right|_{t=0} = \operatorname{E} Y \quad \text{and} \quad \left. \frac{d^2}{dt^2} S_Y(t) \right|_{t=0} = Var(Y)$$

Expectation and Variance of Log Of Gamma

To find the expectation of Log-Of-Gamma, we differentiate $S_Y(t) = \log(M_Y(t))$ and evaluate the result at t = 0.

$$(S_Y(t))' = (\log(M_Y(t)))' = \left(\log\left(\frac{\beta^t \Gamma(\alpha+t)}{\Gamma(\alpha)}\right)\right)'$$

$$= \frac{1}{\frac{\beta^t \Gamma(\alpha+t)}{\Gamma(\alpha)}} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t \Gamma(\alpha+t)\right)'$$

$$= \frac{\Gamma(\alpha)}{\beta^t \Gamma(\alpha+t)} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t \log(\beta) \Gamma(\alpha+t) + \beta^t \Gamma'(\alpha+t)\right)$$

$$= \log(\beta) + \frac{\Gamma'(\alpha+t)}{\Gamma(\alpha+t)}$$

The last term is the log-derivative of the Gamma function evaluated at $\alpha + t$. This log-derivative is called the digamma function and is denoted by $\psi(\cdot)$. Thus:

$$(S_Y(t))' = \log(\beta) + \psi(\alpha + t)$$

We conclude that if Y is $Log-Of-Gamma(\alpha, \beta)$ distributed, then its expectation is given by:

$$EY = \log(\beta) + \psi(\alpha)$$

To find the variance of Y, we we look at the second derivative, $(S_Y(t))''$, and evaluate it at t=0.

$$(S_Y(t))'' = (\log(\beta) + \psi(\alpha + t))'$$

= $\psi'(\alpha + t)$

The derivative of digamma is the so-called trigamma function:

$$Var(Y)=\psi^{''}(\alpha)={
m trigamma}\,(\alpha)$$