My Notes About Γ

By David Dobor

A random variable X is said to have a gamma distribution with parameters (α, β), denoted by $\Gamma(\alpha, \beta)$, if its pdf is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} \qquad \text{where } \; x \geq 0 \;\; \alpha, \beta > 0$$

Here $\Gamma(\alpha)$ is the gamma function computed as $\int_0^\infty \mathrm{e}^{-y} y^{\alpha-1} dy$. (This reduces to the factorial function for integer α: $\Gamma(\alpha) = \alpha!$)

The Moment Generating Function:

$$\begin{aligned} M_X(t) &= \mathrm{E}\, \mathrm{e}^{tX} \\ &= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int_0^\infty \mathrm{e}^{tx} x^{\alpha-1} \mathrm{e}^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int_0^\infty x^{\alpha-1} \mathrm{e}^{-x/\frac{\beta}{1-\beta t}} dx \end{aligned}$$

To see what this integrates to, use this common trick: leave inside the integral a function that's easily recognizeable to be a `pdf`, which would therefore integrate to 1, and pull remaining constants out of the integral sign:

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha} \mathrm{e}^{tx} x^{\alpha-1} \mathrm{e}^{-x/\beta} dx \\ &= \frac{1}{\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha} \mathrm{e}^{tx} x^{\alpha-1} \mathrm{e}^{-x/\beta} dx \\ &= \left(\frac{1}{1-\beta t}\right)^\alpha \end{aligned}$$

Where the last equality follows because the thing inside the integral is a `Gamma pdf`.

First moment, the expectation, by direct integration:

$$
\begin{aligned}
\mathrm{E}\,X &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+1)\cdot\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+1)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+1)\cdot\beta}{\Gamma(\alpha)} = \alpha\beta
\end{aligned}
$$

where the last integrand is 1 because it itself is a gamma pdf with parameters $(\alpha+1, \beta)$.

Second moment, by direct integration:

$$
\begin{aligned}
\mathrm{E}\,X^2 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x^2 \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+2)\cdot\beta^2}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+2)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha+1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+2)\cdot\beta^2}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2
\end{aligned}
$$

The Variance is then:

$$
Var(X) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2
$$

Moments from the Moment Generating Function:

Differentiate the Moment Generating Function (mgf) appropriate number of times and evaluate the derivative at $t = 0$ to get the moments: differentiate once to get the expectation, twice to get the second moment, etc.. E.g.:

$$
(M_X(t))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}
$$

$$
(M_X(0))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}\Big|_{t=0} = \alpha\beta
$$

$$
(M_X(0))'' = \frac{\alpha(\alpha+1)\beta^2}{(1-\beta t)^{\alpha+2}}\Big|_{t=0} = \alpha(\alpha+1)\beta^2
$$

Distribution of Log of Gamma:

We are interested in the distribution of the natural logarithm of Gamma.
Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of Y.

We'll use the following well known theorem about computing the density of a transform, g, of some random variable X. Loosely put, if $g(X) = Y$ and g has an *inverse*, call it $h(Y)$, then the `pdf` of Y is:

$$f_Y(y) = f_X(h(y))|h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = \mathrm{e}^Y$ and $h'(y) = \mathrm{e}^y$. Therefore:

$$\begin{aligned}
f_Y(y) &= f_X(h(y))|h'(y)| \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha(y-1)} - \mathrm{e}^{-\mathrm{e}^{y/\beta}} \right) |\mathrm{e}^y| \;\; 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha y} - \mathrm{e}^{-\mathrm{e}^{y/\beta}} \right) \mathrm{e}^{-y} \mathrm{e}^{y} \;\; 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \;\; \mathrm{e}^{\left(\alpha y - e^{y/\beta} \right)} \, 1_{(-\infty,\infty)}(y)
\end{aligned}$$

This expression is a valid `pdf` – it integrates to 1 - and is said to have a `Log-Of-Gamma` `pdf` with parameters (α, β).

Moments of Log Of Gamma

We now find the Moment Generating Function (`MGF`) of `Log-Of-Gamma`(α, β). By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} \;\; \mathrm{e}^{\left(\alpha y - e^{y\beta} \right)} dy$$

To integrate this we use an old trick: we try to leave inside the integral sign the part of the function that integrates to 1 (this is often an easily recognizeable `pdf`); we then pull the constants that get in the way of making this integral equal to 1 out of the integral sign.

Luckily, this can be done here:

$$\begin{aligned}
M_Y(t) &= \frac{1}{\beta^\alpha\,\Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{ty}\ \ \mathrm{e}^{\left(\alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha\,\Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{\left(ty + \alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha\,\Gamma(\alpha)} \cdot \frac{\beta^{\alpha+t}\,\Gamma(\alpha+t)}{\beta^{\alpha+t}\,\Gamma(\alpha+t)} \cdot \int_{-\infty}^{+\infty} \mathrm{e}^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^{\alpha+t}\,\Gamma(\alpha+t)}{\beta^\alpha\,\Gamma(\alpha)} \cdot \int_{-\infty}^{+\infty} \frac{1}{\beta^{t+\alpha}\,\Gamma(\alpha+t)} \mathrm{e}^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)}
\end{aligned}$$

Where the last equality follows because the thing inside the integral is the `Log-Of-Gamma` `pdf` derived above and integrates to 1.

In Short:

$$\boxed{M_Y(t) = \frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)} \quad \text{when } Y \text{ is } \texttt{Log-Of-Gamma}\,(\alpha,\beta)}$$

Cumulant Generating Function

The Cumulant Generating Function (`CGF`) of a random variable Y is defined as:

$$S_Y(t) = \log(M_Y(t))$$

It can be verified that:

$$\frac{d}{dt} S_Y(t)\Big|_{t=0} = \mathrm{E}\,Y \quad \text{and} \quad \frac{d^2}{dt^2} S_Y(t)\Big|_{t=0} = Var(Y)$$

Expectation and Variance of Log Of Gamma

To find the expectation of `Log-Of-Gamma`, we differentiate $S_Y(t) = \log(M_Y(t))$ and evaluate the result at $t = 0$.

$$
\begin{aligned}
(S_Y(t))' = (\log(M_Y(t)))' &= \left(\log\left(\frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)}\right)\right)' \\
&= \frac{1}{\frac{\beta^t\,\Gamma(\alpha+t)}{\Gamma(\alpha)}} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t\,\Gamma(\alpha+t)\right)' \\
&= \frac{\Gamma(\alpha)}{\beta^t\,\Gamma(\alpha+t)} \cdot \frac{1}{\Gamma(\alpha)} \cdot \Big(\beta^t \log(\beta)\;\Gamma(\alpha+t) + \beta^t\;\Gamma'(\alpha+t)\Big) \\
&= \log(\beta) + \frac{\Gamma'(\alpha+t)}{\Gamma(\alpha+t)}
\end{aligned}
$$

The last term is the log-derivative of the Gamma function evaluated at $\alpha + t$. This log-derivative is called the `digamma` function and is denoted by $\psi(\cdot)$. Thus:

$$\boxed{(S_Y(t))' = \log(\beta) + \psi(\alpha+t)}$$

We *conclude* that if Y is $\texttt{Log-Of-Gamma}(\alpha, \beta)$ distributed, then its expectation is given by:

$$\boxed{\mathrm{E}\,Y = \log(\beta) + \psi(\alpha)}$$

To find the variance of Y, we we look at the second derivative, $(S_Y(t))''$, and evaluate it at $t = 0$.

$$
\begin{aligned}
(S_Y(t))'' &= (\log(\beta) + \psi(\alpha+t))' \\
&= \psi'(\alpha+t)
\end{aligned}
$$

The derivative of `digamma` is the so-called `trigamma` function:

$$\boxed{Var(Y) = \psi''(\alpha) = \texttt{trigamma}(\alpha)}$$