My Notes About log of Gamma

By David Dobor

A random variable X is said to have a gamma distribution with parameters (α, β), denoted by $\Gamma(\alpha, \beta)$, if its `pdf` is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \cdot e^{-\frac{x}{\beta}} \cdot 1_{(0,\infty)}(x) \ \text{ where } \ \alpha, \beta > 0$$

Here $\Gamma(\alpha)$ is the gamma function computed as $\int_0^\infty e^{-y} y^{\alpha-1} dy$ and $1_{(0,\infty)}(x)$ stands for the indicator function (i.e. it's value is 1 for non-negative x-es and 0 elsewhere).

Distribution of Log of Gamma

First, we are interested in the distribution of the natural logarithm of Gamma. Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of Y.

We'll use the following well known theorem about computing the density of a transform, g, of some random variable X. Loosely put, if $g(X) = Y$ and g has an *inverse*, call it $h(Y)$, then the `pdf` of Y is:
$$f_Y(y) = f_X(h(y))|h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = e^Y$ and $h'(y) = e^y$. Therefore:

$$
\begin{aligned}
f_Y(y) &= f_X(h(y))|h'(y)| \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(e^{\alpha(y-1)} - e^{-e^{y/\beta}} \right) |e^y| \ 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(e^{\alpha y} - e^{-e^{y/\beta}} \right) e^{-y} e^y \ 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \ e^{\left(\alpha y - e^{y/\beta}\right)} 1_{(-\infty,\infty)}(y)
\end{aligned}
$$

This expression is a valid `pdf` – it integrates to 1 - and is said to have a `Log-Of-Gamma` `pdf` with parameters (α, β).

Moments of Log Of Gamma

We now find the Moment Generating Function (MGF) of `Log-Of-Gamma`(α, β). By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} e^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} \; e^{\left(\alpha y - e^{y/\beta}\right)} dy$$

To integrate this we use an old trick: we try to leave inside the integral sign the part of the function that integrates to 1 (this is often an easily recognizeable `pdf`); we then pull the constants that get in the way of making this integral equal to 1 out of the integral sign. Luckily, this can be done here:

$$
\begin{aligned}
M_Y(t) &= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{ty} \; e^{\left(\alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{\left(ty + \alpha y - e^{y/\beta}\right)} dy \\
&= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \cdot \frac{\beta^{\alpha+t} \; \Gamma(\alpha+t)}{\beta^{\alpha+t} \; \Gamma(\alpha+t)} \cdot \int_{-\infty}^{+\infty} e^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^{\alpha+t} \; \Gamma(\alpha+t)}{\beta^\alpha \; \Gamma(\alpha)} \cdot \int_{-\infty}^{+\infty} \frac{1}{\beta^{t+\alpha} \; \Gamma(\alpha+t)} e^{\left(y(\alpha+t) - e^{y/\beta}\right)} dy \\
&= \frac{\beta^t \; \Gamma(\alpha+t)}{\Gamma(\alpha)}
\end{aligned}
$$

Where the last equality follows because the thing inside the integral is the `Log-Of-Gamma` `pdf` derived above and integrates to 1.

In Short:

$$\boxed{M_Y(t) = \frac{\beta^t \; \Gamma(\alpha+t)}{\Gamma(\alpha)} \quad \text{when } Y \text{ is } \texttt{Log-Of-Gamma}(\alpha, \beta)}$$

Cumulant Generating Function

The Cumulant Generating Function (CGF) of a random variable Y is defined as:

$$S_Y(t) = \log(M_Y(t))$$

It can be verified that:

$$\frac{d}{dt} S_Y(t) \Big|_{t=0} = \mathrm{E}\, Y \quad \text{and} \quad \frac{d^2}{dt^2} S_Y(t) \Big|_{t=0} = Var(Y)$$

Expectation and Variance of Log Of Gamma

To find the expectation of `Log-Of-Gamma`, we differentiate $S_Y(t) = \log(M_Y(t))$ and evaluate the result at $t = 0$.

$$
\begin{aligned}
(S_Y(t))' = (\log(M_Y(t)))' &= \left(\log \left(\frac{\beta^t \, \Gamma(\alpha + t)}{\Gamma(\alpha)} \right) \right)' \\
&= \frac{1}{\frac{\beta^t \, \Gamma(\alpha+t)}{\Gamma(\alpha)}} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t \, \Gamma(\alpha + t) \right)' \\
&= \frac{\Gamma(\alpha)}{\beta^t \, \Gamma(\alpha + t)} \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\beta^t \log(\beta) \, \Gamma(\alpha + t) + \beta^t \, \Gamma'(\alpha + t) \right) \\
&= \log(\beta) + \frac{\Gamma'(\alpha + t)}{\Gamma(\alpha + t)}
\end{aligned}
$$

The last term is the log-derivative of the Gamma function evaluated at $\alpha + t$. This log-derivative is called the `digamma` function and is denoted by $\psi(\cdot)$. Thus:

$$\boxed{(S_Y(t))' = \log(\beta) + \psi(\alpha + t)}$$

We *conclude* that if Y is `Log-Of-Gamma`(α, β) distributed, then its expectation is given by:

$$\boxed{\mathrm{E}\,Y = \log(\beta) + \psi(\alpha)}$$

To find the variance of Y, we we look at the second derivative, $(S_Y(t))''$, and evaluate it at $t = 0$.

$$
\begin{aligned}
(S_Y(t))'' &= (\log(\beta) + \psi(\alpha + t))' \\
&= \psi'(\alpha + t)
\end{aligned}
$$

The derivative of `digamma` is the so-called `trigamma` function:

$$\boxed{Var(Y) = \psi''(\alpha) = \texttt{trigamma}(\alpha)}$$