

# My Notes About $\log$ of Gamma

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September 15, 2014

A random variable  $X$  is said to have a gamma distribution with parameters  $(\alpha, \beta)$ , denoted by  $\Gamma(\alpha, \beta)$ , if its pdf is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \cdot e^{-\frac{x}{\beta}} \cdot 1_{(0,\infty)}(x) \text{ where } \alpha, \beta > 0$$

Here  $\Gamma(\alpha)$  is the gamma function computed as  $\int_0^\infty e^{-y} y^{\alpha-1} dy$  and  $1_{(0,\infty)}(x)$  stands for the indicator function (i.e. it's value is 1 for non-negative  $x$ -es and 0 elsewhere).

## *Distribution of Log of Gamma*

First, we are interested in the distribution of the natural logarithm of Gamma. Let  $Y = \log(X)$  where  $X \sim \Gamma(\alpha, \beta)$ . We want the distribution of  $Y$ .

We'll use the following well known theorem about computing the density of a transform,  $g$ , of some random variable  $X$ . Loosely put, if  $g(X) = Y$  and  $g$  has an *inverse*, call it  $h(Y)$ , then the pdf of  $Y$  is:

$$f_Y(y) = f_X(h(y)) |h'(y)|$$

So here, denoting the inverse of  $Y = \log(X)$  by  $h(Y)$ , we have  $X = h(Y) = e^Y$  and  $h'(y) = e^y$ . Therefore:

$$\begin{aligned} f_Y(y) &= f_X(h(y)) |h'(y)| \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left( e^{\alpha(y-1)} - e^{-e^{y/\beta}} \right) |e^y| 1_{(-\infty, \infty)}(y) \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left( e^{\alpha y} - e^{-e^{y/\beta}} \right) e^{-y} e^y 1_{(-\infty, \infty)}(y) \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{(\alpha y - e^{y/\beta})} 1_{(-\infty, \infty)}(y) \end{aligned}$$

This expression is a valid pdf – it integrates to 1.

## Moments of Log Of Gamma

We now find the Moment Generating Function (MGF) of  $\Gamma(\alpha, \beta)$ . By definition:

$$M_Y(t) = \int_{-\infty}^{+\infty} e^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{(\alpha y - e^{y/\beta})} dy$$

To integrate this we use an old trick: we try to leave inside the integral sign the part of the function that integrates to 1 (this is often an easily recognizeable pdf); we then pull the constants that get in the way of making this integral equal to 1 out of the integral sign. Luckily, this can be done here:

$$\begin{aligned} M_Y(t) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{ty} e^{(\alpha y - e^{y/\beta})} dy \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} e^{(ty + \alpha y - e^{y/\beta})} dy \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \frac{\beta^{\alpha+t} \Gamma(\alpha+t)}{\beta^{\alpha+t} \Gamma(\alpha+t)} \cdot \int_{-\infty}^{+\infty} e^{(y(\alpha+t) - e^{y/\beta})} dy \\ &= \frac{\beta^{\alpha+t} \Gamma(\alpha+t)}{\beta^\alpha \Gamma(\alpha)} \cdot \int_{-\infty}^{+\infty} \frac{1}{\beta^{t+\alpha} \Gamma(\alpha+t)} e^{(y(\alpha+t) - e^{y/\beta})} dy \\ &= \frac{\beta^t \Gamma(\alpha+t)}{\Gamma(\alpha)} \end{aligned}$$

Where the last equality follows because the thing inside the integral is the Log-Of-Gamma pdf we derived above and it integrates to 1.

*In Short:*

$M_Y(t) = \frac{\beta^t \Gamma(\alpha+t)}{\Gamma(\alpha)} \quad \text{when } Y \text{ is Log-Of-Gamma } (\alpha, \beta)$
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## Cumulant Generating Function

The Cumulant Generating Function (CGF) of a random variable  $Y$  is defined as:

$$S_Y(t) = \log(M_Y(t))$$

It can be verified that:

$$\left. \frac{d}{dt} S_Y(t) \right|_{t=0} = E Y \quad \text{and} \quad \left. \frac{d^2}{dt^2} S_Y(t) \right|_{t=0} = Var(Y)$$

## Expectation and Variance of Log Of Gamma

To find the expectation of `Log-Of-Gamma`, we differentiate  $S_Y(t) = \log(M_Y(t))$  and evaluate the result at  $t = 0$ .

$$\begin{aligned}
 (S_Y(t))' &= (\log(M_Y(t)))' = \left( \log \left( \frac{\beta^t \Gamma(\alpha + t)}{\Gamma(\alpha)} \right) \right)' \\
 &= \frac{1}{\frac{\beta^t \Gamma(\alpha + t)}{\Gamma(\alpha)}} \cdot \frac{1}{\Gamma(\alpha)} \cdot (\beta^t \Gamma(\alpha + t))' \\
 &= \frac{\Gamma(\alpha)}{\beta^t \Gamma(\alpha + t)} \cdot \frac{1}{\Gamma(\alpha)} \cdot (\beta^t \log(\beta) \Gamma(\alpha + t) + \beta^t \Gamma'(\alpha + t)) \\
 &= \log(\beta) + \frac{\Gamma'(\alpha + t)}{\Gamma(\alpha + t)}
 \end{aligned}$$

The last term is the log-derivative of the Gamma function evaluated at  $\alpha + t$ . This log-derivative is called the digamma function and is denoted by  $\psi(\cdot)$ . Thus:

$$(S_Y(t))' = \log(\beta) + \psi(\alpha + t)$$

One conclusion is that if  $Y$  is `Log-Of-Gamma`, its expectation is given by:

$$E Y = \log(\beta) + \psi(\alpha)$$

To find what the variance of  $Y$  is, we look at the second derivative  $(S_Y(t))''$  and evaluate it at  $t = 0$ .

$$\begin{aligned}
 (S_Y(t))'' &= (\log(\beta) + \psi(\alpha + t))' \\
 &= \psi'(\alpha + t)
 \end{aligned}$$

The derivative of digamma is the so-called `trigamma` function. Thus

$$Var(Y) = \psi''(\alpha) = \text{trigamma}(\alpha)$$