My Notes About Γ

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A random variable X is said to have a gamma distribution with parameters (α, β), denoted by $\Gamma(\alpha, \beta)$, if its pdf is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}}$$

where the number $\Gamma(\alpha)$ is called the gamma fuction and is computed as $\int_0^\infty \mathrm{e}^{-y} y^{\alpha-1} dy$. (This reduces to the factorial function for integer α: $\Gamma(\alpha) = \alpha!$)

First moment, the expectation, by direct integration:

$$\begin{aligned}
E[X] &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+1) \cdot \beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+1)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+1) \cdot \beta}{\Gamma(\alpha)} = \alpha\beta
\end{aligned}$$

where the last integrand is 1 because it itself is a gamma pdf with parameters $(\alpha + 1, \beta)$.

Second moment, by direct integration:

$$\begin{aligned}
E[X^2] &= \int_0^\infty \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot x^2 \left(\frac{x}{\beta}\right)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+2) \cdot \beta^2}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+2)} \frac{1}{\beta} \left(\frac{x}{\beta}\right)^{\alpha+1} \mathrm{e}^{-\frac{x}{\beta}} dx \\
&= \frac{\Gamma(\alpha+2) \cdot \beta^2}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2
\end{aligned}$$

The Variance is then:

$$Var(X) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Moments from the Moment Generating Function:

$$\begin{aligned}
M_X(t) &= \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^\infty \mathrm{e}^{tx} x^{\alpha-1}\mathrm{e}^{-x/\beta}dx \\
&= \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x/\frac{\beta}{1-\beta t}}dx \\
&= \ldots.\ \text{same tricks as before...} \\
&= \left(\frac{1}{1-\beta t}\right)^\alpha
\end{aligned}$$

now differentiate this appropriate number of times and evaluate that derivative at $t = 0$ to get the moments: differentiate once to get the expectation, twice to get the second moment, etc.. E.g.:

$$(M_X(t))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}$$

$$(M_X(0))' = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}\Big|_{t=0} = \alpha\beta$$

$$(M_X(0))'' = \frac{\alpha(\alpha+1)\beta^2}{(1-\beta t)^{\alpha+2}}\Big|_{t=0} = \alpha(\alpha+1)\beta^2$$

Cumulant Generating Function:

This is defined as:

$$S_X(t) = \log(M_X(t)$$

So It's easily verified that:

$$\frac{d}{dt}S_X(t)\Big|_{t=0} = E[X] \quad \text{and} \quad \frac{d^2}{dt^2}S_X(t)\Big|_{t=0} = Var(X)$$

Distribution of Log of Gamma:

Let $Y = \log(X)$ where $X \sim \text{Gamma}(\alpha,\beta)$. We want the distribution of Y. In the following notation, use the indicator functions to easily track the ranges of the variables during the transformation:

$$f_X(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\ x^{\alpha-1}e^{-x/\beta}\, 1_{(0,\infty)}(x)\,.$$

Denoting the ivnverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = \mathrm{e}^Y$. Thererfore:

$$f_Y(y) = f_X(h(y))|h'(y)| = \frac{1}{\beta^\alpha\Gamma(\alpha)}\ \mathrm{e}^{(\alpha y - e^y/\beta)}\, 1_{(-\infty,\infty)}(y)\,,$$

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