Chapter 3

Review of Linear Algebra

3.1 Definition

Definition 3.1.1. *Matrix: is an array of numbers arranged as m rows, n columns, denoted by captial letters, A, B, X, Y,* $\cdots$

Example 3.1.1.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} n = 2, m = 3$$

The element of a matrix A is denoted as a_{ij} (ith row, jth column)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \tag{3.1}$$

Special Matrix

1. if m = 1, A is called a n-dimensional vector.
2. if n = m, sqaured matrix.
3. diagonal matrix: sqaured matrix, all the off diagonal-matrix elements are zero.
4. identity matrix: All diagonal elements are 1.

Example 3.1.2.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5. symmetric matrix $a_{ij} = a_{ji}$

   Example 3.1.3.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

6. lower triagonal matrix: lower elements are non-zero, i.e, a_{ij}=0, if i < j.

   Example 3.1.4.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & 3 \end{pmatrix}$$

7. upper triagonal matrix, $a_{ij} = 0$, if i > j.

   Example 3.1.5.

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

3.2 Operation of Matrix

1. Addition/Subtraction(A + B, A - B), elementwise addition, A and B have the same number of rows n and columns m.

2. Scalar-matrix product: B = λA, $b_{ij} = \lambda a_{ij}$

3. Product of two matrices, $A_{nm}B_{ms}$
   A B = C, $C_{ij} = \sum_{i=1}^{m} a_{ik}b_{kj}$, the ith row of A and the jth column of B

$$\left(a_{i1}, a_{i2}, a_{i3}, \cdots, a_{im}\right) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} \tag{3.2}$$

Example 3.2.1.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 0 \end{pmatrix}, C = AB = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Properties

1. (AB)C=A(BC)
2. AB $\neq$ BA
3. A(B+C)=AB+AC, distributive law
4. AI=IA=A
5. Transpose B=A^T if $b_{ij} = a_{ji}$, $(AB)^T \neq A^T B^T$
6. Matrix inverse: if A is a square matrix, and there exists a square matrix B, such that AB=BA=I, then we call A invertible, B=A^{-1} is the inverse of A. Not all matrices are
   invertible.

   Example 3.2.2. *An uninvertible matrix*

   $$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

   Example 3.2.3.

   $$A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}, AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

   In general,

   $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \tag{3.3}$$

if A and B are invertible

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

3.2.1 Kronecker product

Let A_{rs} and B_{uv} be two matrices, then the Kronecker product of A and B is a matrix $C_{ru,sv}$ defined as

$$C = A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1s}B \\ a_{21}B & \cdots & a_{2s}B \\ \vdots & \ddots & \vdots \\ a_{r1}B & \cdots & a_{rs}B \end{pmatrix}$$

Example 3.2.4.

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

3.3 Important Quantities

1. Trace: A is a sqaure matrix, then tr(A) = $\sum_{i=1}^{n} = a_{ii}$

   Example 3.3.1.

   $$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, tr(A) = 2 + 2 = 4$$

   Properties: tr(A+B)=tr(A)+tr(B); tr(AB)=tr(BA)

2. Determinant: A is a square matrix, recursive defition of the determinant
   n=1, A=(a_{11}), det(A)=a_{11}
   in general,

   $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, det(A) = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} det(M_{1,k})$$

   $M_{1,k}$ is the $(n-1)*(n-1)$ matrix made by the rows and columns of A except the first row and kth column.

   Example 3.3.2. *Let*

   $$A = \begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix},$$

Then

$$det(A) = a_{11}det(M_{1,1}) - a_{12}det(M_{1,2}) = 4*2 - 3*det(-3) = 8+9 = 17.$$

In general, let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$det(A) = a_{11}a_{22} - a_{12}a_{22}$$

Example 3.3.3. *Let*

$$B = \begin{pmatrix} 4 & 3 & 2 \\ -3 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} det(B) &= a_{11}det(M_{1,1}) - a_{12}det(M_{1,2}) + a_{13}det(M_{1,3}) \\ &= 4det\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} - 3det\begin{pmatrix} -3 & 0 \\ 1 & 1 \end{pmatrix} + 2det\begin{pmatrix} -3 & 2 \\ 1 & 2 \end{pmatrix} \\ &= 8 + 9 - 16 = 1 \end{aligned}$$

Properties

(a) if A is diagonal or low triangular,upper triangular matirx, det(A)=$\prod_{i=1}^{n} a_{ii}$, det(I)=1

(b) det(AB)=det(A)det(B), det(A^T)=det(A)

(c) det(A^{-1})=$(det(A))^{-}1$

(d) A matrix A is invertible, iff det(A)$\neq$0. if det(A)=0, we call A singular.

3. Rank: A is a matrix, rank(A) is the size of the largets submatrix of that has a non-zero determinant.

Example 3.3.4.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$
$$det(A) = 0, rank \neq 3$$
$$det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = 3, \textit{ rank of A is 2.}$$

Properties

(a) *if det(A)≠0, rank(A) = n.*

(b) *rank(AB) ≤ min(rank(A),rank(B))*

3.4 Special Matrices (Squared Matrices)

1. Orthogonal: A is orthogonal, iff $A^T A = AA^T = I$, (inverse of A is transpose of A).

   Example 3.4.1.

   $$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, A^T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

   A is orthogonal.

   Geometrical meaning: x $\longrightarrow$ Ax
   only rotation of the x, length are the same.

2. Idempotent: A is idempotent if A^2 = A.

   Example 3.4.2.

   $$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, AA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

   Geometrical meaning x$\longrightarrow$ Ax, projection of x to the signature line of the matrix.
   Properties: If P is idempotent, then Q=I-P is also idempotent.

3. Positive definite: Let A be a n*n matrix, x be a n-dimenstional vector, define the quadratic form:

$$q(x) = x^T Ax = \sum_{i,j=1}^{n} a_{ij} x_i x_j \tag{3.4}$$

A matrix is called positive definite iff:

(a) A is symmetric;

(b) $x^T Ax > 0, for any x \neq 0$

A matrix is called semi-positive definite if $x^T Ax \geq 0$, for any x.

Example 3.4.3. *Let*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then

$$\begin{aligned} q(x) &= x^T Ax = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2x_1 \\ &= 2x_1^2 - x_1x_2 - x_1x_2 + x_2^2 = (x_1 - x_2)^2 + x_1^2 \leq 0. \end{aligned}$$

Therefore, A is a positive definite.

Example 3.4.4.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, let X = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$q(x) = X^T BX$$
$$= (1, -1) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2$$
$$\Longrightarrow$$

B is not positive definite.

4. Eigen values/Eigen vectors: if A is a squred matrix, then consider $B = A - \lambda I, det(B) = det(A - \lambda I) = 0$
The roots of this equation is called the eigen value, the set of λ_i is called the spectrum of A.
For each λ_i, then there is a vector V_i such that $AV_i = \lambda_i V_i$, V_i is called the eigenvector. **Geometrical meaning** $\cdots$

Example 3.4.5. *Consider*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Let

$$B = A - \lambda I = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}.$$

Consequently,

$$det(B) = (2-\lambda)^2 - 1 = 0.$$

Set the determinant be zero, we can solve λ as $\lambda = 1 or 3$. When $\lambda_1 = 1$,

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

because

$$AV = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1V_1$$

Simiarly, for $\lambda_3 = 3$,

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

because

$$AV = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3V_2$$