Stat 8003, Homework 3

Group G: Andrew Schneider, Abdulsalam Hdadi, David Dobor

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Question 3.1. Consider a bivariate distribution with $P(X = 1, Y = 2) = 0.4, P(X = 2, Y = 3) = 0.6$. Find the correlation coefficient between X and Y.

Answer: We fill the following table with the given joint probability density. (If X and Y attain other values, the joint density at those values must be zero, since the given probabilities already sum up to 1.) The marginal probabilities are given at the right and at the bottom margins of the table. All entries in the joint table sum up to 1; the entries in the margins also sum up to 1:

	$X = 1$	$X = 2$	$P(Y = y)$:
$Y = 2$	2/5	0	2/5
$Y = 3$	0	3/5	3/5
$P(X = x)$:	2/5	3/5	***1***

The correlation coefficient here is 1. We verify this by direct computation:

$$E[X] = 1 \cdot \frac{2}{5} + 2 \cdot \frac{3}{5} = \frac{8}{5}$$
$$E[X^2] = 1^2 \cdot \frac{2}{5} + 2^2 \cdot \frac{3}{5} = \frac{14}{5}$$
$$Var(X) = E[X^2] - (E[X])^2 = \frac{14}{5} - \left(\frac{8}{5}\right)^2 = \frac{6}{25}$$

$$E[Y] = 2 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{13}{5}$$
$$E[Y^2] = 2^2 \cdot \frac{2}{5} + 3^2 \cdot \frac{3}{5} = 7$$
$$Var(Y) = E[Y^2] - (E[X])^2 = 7 - \left(\frac{13}{5}\right)^2 = \frac{6}{25}$$

$$E[XY] = 1 \cdot 2 \cdot \frac{2}{5} + 2 \cdot 3 \cdot \frac{3}{5} = \frac{22}{5}$$
$$E[X] \cdot E[Y] = \frac{8}{5} \cdot \frac{13}{5} = \frac{104}{25}$$
$$Cov(X,Y) = E[XY] - E[X] \cdot E[Y] = \frac{22}{5} - \frac{104}{25} = \frac{6}{25}$$

Therefore, the correlation coefficient is:

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{6/25}{\sqrt{6/25 \cdot 6/25}} = 1$$

Question 3.2. Find two random variables X and Y , such that $Cov(X,Y) = 0$ but X and Y are not independent.

Answer: Let X just be Bernoulli:

$$X = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ +1 & \text{with probability } \frac{1}{2} \end{cases}$$

And Let Y be a `r.v.` that directly depends on the outcome of X as follows:

$$Y = \begin{cases} 0 & \text{whenever } X = -1 \\ -8003 & \text{with probability } \frac{1}{2} \text{ whenever } X = 1 \\ +8003 & \text{with probability } \frac{1}{2} \text{ whenever } X = 1 \end{cases}$$

Now since:

$$\begin{aligned} E[XY] &= (-1) \cdot 0 \cdot P(X = -1) + \\ &\quad + 1 \cdot 8003 \cdot P(X = 1, Y = 1) + \\ &\quad + 1 \cdot (-8003) \cdot P(X = 1, Y = -1) \\ &= 0 \end{aligned}$$

And since $E[X]$ is also 0, the the covariance $Cov(X,Y) = E[XY] - E[X] \cdot E[Y] = 0$, although the `r.v.`s are by definition dependent.

Question 3.3. In the Example of `GDP`. Assume that the data follows a gamma distribution $\Gamma(\alpha, \beta)$.

(a) Derive the estimator for α, β using the methods of moments;

(b) Compare the density of the data vs the fitted curve.

Answer:

Question 3.4. For any random variable X, let $MX(t) = Eexp(Xt)$ and $SX(t) = log(MX(t))$. $MX(t)$ is called the moment generating function and $SX(t)$ is the cumulant generating function. It is known that (Can you prove it? Not required.)

$$\frac{d}{dt}S_X(t)\Big|_{t=0} = E[X] \quad \text{and} \quad \frac{d^2}{dt^2}S_X(t)\Big|_{t=0} = Var(X)$$

Use this fact to answer the following questions.

(a) Assume that X follows a Gamma distribution with parameter α and β. Calculate the cumulant generating function of $\log(X)$;

(b) Calculate $E\log X$ and $Var(\log X)$. Write your final result by using the digamma function $\psi(x)$ and trigamma function $\psi_1(x)$, where $\psi(x) = (\log\Gamma(x))'$ and $\psi_1(x) = (\log\Gamma(x))''$.

(c) Match the first and second moment of $\log(X)$, and derive the `MOM` estimator of α and β.
(Hint: in `R`, you can use `digamma(x)`, `trigamma(x)`, and `limma::trigammaInverse(x)`.)

(d) Apply your estimator to the `GDP` dataset and estimate the parameters α and β.

Answer:
We first derive the distribution of $\log X$:

Distribution of Log of Gamma:

Let $Y = \log(X)$ where $X \sim \Gamma(\alpha, \beta)$. We want the distribution of Y.
In the following notation, use the indicator functions to easily track the ranges of the variables during the transformation:

$$f_X(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\ x^{\alpha-1}e^{-x/\beta}\,1_{(0,\infty)}(x)\,.$$

We'll use the following well known theorem about computing the density of a transform, g, of some random variable X. Very loosely put, if $g(X) = Y$ and g has an *inverse*, call it $h(Y)$, then the `pdf` of Y is:

$$f_Y(y) = f_X(h(y))|h'(y)|$$

So here, denoting the inverse of $Y = \log(X)$ by $h(Y)$, we have $X = h(Y) = \mathrm{e}^Y$ and $h'(y) = \mathrm{e}^y$. Therefore:

$$
\begin{aligned}
f_Y(y) &= f_X(h(y))|h'(y)| \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha(y-1)} - \mathrm{e}^{-\mathrm{e}^y/\beta}\right) \, |\mathrm{e}^y| \;\, 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\mathrm{e}^{\alpha y} - \mathrm{e}^{-\mathrm{e}^y/\beta}\right) \, \mathrm{e}^{-y}\mathrm{e}^y \;\, 1_{(-\infty,\infty)}(y) \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \;\, \mathrm{e}^{(\alpha y - e^y/\beta)} \, 1_{(-\infty,\infty)}(y)
\end{aligned}
$$

OK, so far so good. We found the `pdf` of `Log-Gamma`.

Now we find its moment generating function. By definition:

$$
M_Y(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} \;\, \mathrm{e}^{(\alpha y - e^y/\beta)} dy
$$

To integrate this we use the same trick of leaving inside the integral sign the part of the function that integrates to 1, getting constants appropriately out of the integral. Luckily, this can be done here:

$$
\begin{aligned}
M_Y(t) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{ty} \;\, \mathrm{e}^{(\alpha y - e^y/\beta)} dy \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} \mathrm{e}^{(ty + \alpha y - e^y/\beta)} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)\beta^\alpha \Gamma(\alpha + t)} \int_{-\infty}^{+\infty} \mathrm{e}^{y(t+\alpha) - e^y/\beta} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} \frac{1}{\beta^\alpha \Gamma(\alpha + t)} \mathrm{e}^{y(t+\alpha) - e^y/\beta} dy \\
&= \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)}
\end{aligned}
$$

Where the last equality follows because the thing inside the integral is a `Log-Gamma` `pdf` and integrates to 1.

Next, by definition, the *cumulant generating function* is just the log of this:

$$
\log M_Y(t) = \log \left(\frac{\Gamma(\alpha + t)}{\Gamma(\alpha)} \right)
$$