① Bayes Theorem

| | $P(A)$ | $P(D \mid \text{Factory})$ |
|---|---|---|
| A | 0.15 | 0.02 |
| B | 0.45 | 0.005 |
| C | 0.40 | 0.01 |

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{\overset{0.02 \times 0.15}{P(D|A)P(A)}}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)}$$

$$= 0.3243$$

$$P(B|D) = 0.2432$$

$$P(C|D) = 0.4324$$

②

$$P(X=1) = \theta^2$$

$$P(X=2) = 1-\theta^2$$

(a)

$$\mu = EX = 1 \cdot \theta^2 + 2(1-\theta^2) = 2-\theta^2$$

$$m_1 = \frac{1+1+2}{3} = \frac{4}{3}$$

$$m_1 = \mu \Rightarrow 2-\theta^2 = \frac{4}{3} \qquad \theta^2 = \frac{2}{3} \qquad \theta = \pm\sqrt{\frac{2}{3}}$$

(b)

$$f(x_i|\theta) = \begin{cases} \theta^2 & x=1 \\ 1-\theta^2 & x=2 \end{cases}$$

$$f(x_i|\theta) = (\theta^2)^{2-x_i}(1-\theta^2)^{x_i-1}$$

$$L(\theta) = (\theta^2)^{\Sigma(2-x_i)}(1-\theta^2)^{\Sigma(x_i-1)} = (\theta^2)^2(1-\theta^2)$$

$$= \theta^4 - \theta^6$$

(c)

$$\frac{dL}{d\theta} = 4\theta^3 - 6\theta^5 = 0 \Rightarrow \theta^2 = \frac{2}{3} \qquad \hat{\theta} = \pm\sqrt{\frac{2}{3}}$$

(c) $\frac{dl}{d\theta} = 4\theta^3 - 6\theta^5 = 0 \Rightarrow \theta^2 = \frac{2}{3} \quad \theta = \pm\sqrt{\frac{2}{3}}$

$$\frac{d^2 l}{d\theta^2} = \left(12\theta^2 - 30\theta^4\right)\Big|_{\theta^2 = \frac{2}{3}} = 12 \times \frac{2}{3} - 30\,\frac{4}{9} < 0.$$

$$\Rightarrow \hat{\theta}_{MLE} = \pm\sqrt{\frac{2}{3}}$$

3(a) Two independent Sample Equal variance

$$T = \frac{\bar{x}_1 - \bar{x}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = 0.0353$$

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}$$

$$d.f. = n_1 + n_2 - 2 = 8$$

Fail-to-reject.

3(b) Matched Pair $H_0: \mu_{03} \geq \mu_{04}$

$H_a: \mu_{03} < \mu_{04}$

$$T = \frac{\bar{X}_{03} - \bar{X}_{04}}{S_d / \sqrt{n}} = \frac{5.12 - 5.26}{\sqrt{1.28}/\sqrt{10}} = -0.3913$$

$$d.f. = n - 1 = 9.$$

Fail-to-reject $H_0$

4 (a): $\hat{\lambda}_{MOM} = \bar{Y} = 41.7$

(b) $Y_i \sim \text{Poisson}(m_i\lambda)$

$$P(y_i|\lambda) = \frac{(m_i\lambda)^{y_i}}{y_i!}e^{-\lambda m_i} = \frac{m_i^{y_i}}{y_i!}\lambda^{y_i}e^{-\lambda m_i}$$

$$L(\lambda) = \prod P(y_i|\lambda) = \left(\prod \frac{m_i^{y_i}}{y_i!}\right)\lambda^{\Sigma y_i}e^{-\lambda\Sigma m_i}$$

$$L(\lambda) = \prod P(y_i \mid \lambda) = \left(\prod \frac{m_i^{y_i}}{y_i!}\right) \lambda^{\Sigma y_i} e^{-\lambda \Sigma m_i}$$

$$\ell(\lambda) = \log(c) + \Sigma y_i \ln \lambda - \lambda \Sigma m_i$$

$$\frac{d\ell}{d\lambda} = \frac{\Sigma y_i}{\lambda} - \Sigma m_i = 0 \Rightarrow \lambda = \frac{\Sigma y_i}{\Sigma m_i} = \frac{\bar{y}}{\bar{m}} = 6.2054$$

$$\frac{d^2\ell}{d\lambda^2} = -\frac{\Sigma y_i}{\lambda^2} < 0$$

(c) $H_0: \lambda = 5.9$

$H_a: \lambda \neq 5.9$

$$GLRT = \frac{\sup_{\lambda} L(\lambda)}{\sup_{\lambda \in \Theta_0} L(\lambda)} = \frac{L(\lambda = 6.2054)}{L(\lambda = 5.9)} = \frac{6.2^{\Sigma y_i} e^{-6.2 \Sigma m_i}}{5.9^{\Sigma y_i} e^{-5.9 \Sigma m_i}}$$

$$= \left(\frac{6.2}{5.9}\right)^{\Sigma y_i} e^{-0.3 \Sigma m_i}$$

(d) $2 \log \Lambda \sim \chi_1^2$

$$2 \log \Lambda = 2\left(\Sigma y_i \log \frac{6.2}{5.9} - 0.3 \Sigma m_i\right) = 7.309$$

$$\chi^2_{1, 0.05} = 3.84$$

Since $2 \log \Lambda = 7.309 > 3.84$. reject the $H_0$

FWER Family wise Error Rate

$H_{0i}: \theta_i = 0$

$H_{1i}: \theta_i \neq 0$ $\quad i = 1, 2, 3, \cdots, p.$

$H_0: \theta_1 = \theta_2 = \cdots \theta_p = 0$
$H_a$: At least one is non-zero

$$\delta_i = \begin{cases} 0 & \text{Fail-to-reject } H_{0i} \\ 1 & \text{Reject } H_{0i} \end{cases}$$

FWER = P( At least one rejection is a false rejection)

if $p=1$, FWER = Type I error.

If $p > 1$. if $\left\{\begin{array}{l}\delta_i = 1 \\ \text{Reject}\end{array}\right.$ if p-value < 0.05.

does this guarantee that $FWER \leq 0.05$ ?

Bonferroni correct: reject if p value $\leq \frac{0.05}{p}$

Confidence intervals    Estimation Methods

Point Estimate is always wrong.

100(1-α)% CI is an interval $(L(X), U(X))$ that traps the parameter of interest, in with 100(1-α)% "confidence".

Thus for all $\theta \in \Theta$

$$P(\underset{\theta}{L}(X) < \theta < \underset{\theta}{U}(X)) \geq 1 - \alpha.$$

$L(X)$ $U(X)$ are functions of data only.

$H_0: \mu = 2$

$H_a: \mu \neq 2$    $(2.1, 2.7)$

$H_0. \mu = 2$

$H_a. \mu \neq 2.$ if $\mu = 2.0001$

$(2.00003 \quad 2.00013)$

$[2.3, \quad 3.3]$

Z2. Pivot Method.    Pivotal Quantity

Pivot: function of data and the parameter, which have the distribution independent of the parameter of interest.

Example 7.2.1 (GDP)

$$Y_i = \log(\text{GDP - } i\text{-th Country})$$

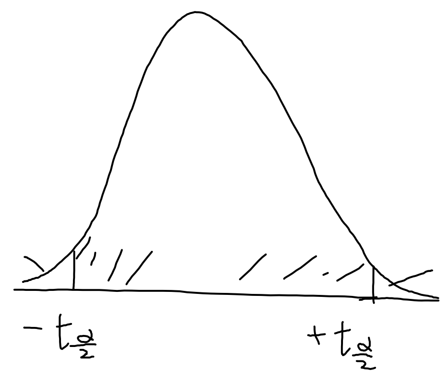
$$Y_i \sim N(\mu, \sigma^2)$$

$(1-\alpha)$ CI for $\mu$.

Sol: $\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{Y}-\mu}{S/\sqrt{n}} \sim T_{n-1}$$




$$1-\alpha = P\left(-t_{\frac{\alpha}{2}} < T_{n-1} < t_{\frac{\alpha}{2}}\right)$$

$$= P\left(-t_{\frac{\alpha}{2}} < \frac{\bar{Y}-\mu}{S/\sqrt{n}} < t_{\frac{\alpha}{2}}\right)$$

$$= P\left(-t_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} < \bar{Y}-\mu < t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right)$$

$$= P\left(\bar{Y} - t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} < \mu < \bar{Y} + t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right)$$

$(1-\alpha)$ CI for $\mu$: $\left[\bar{Y} - t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}, \bar{Y} + t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right]$

$$= \bar{Y} \pm t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}.$$

Normal mean, $(1-\alpha)$ CI: $\bar{Y} \pm t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$.

95% CI for log GDP: $[8.59, 8.97]$

$(1-\alpha)$ Coverage Probability. $-$ 95%

The prob. that the parameter $\theta$ falls in the interval $\geq$ 95%.

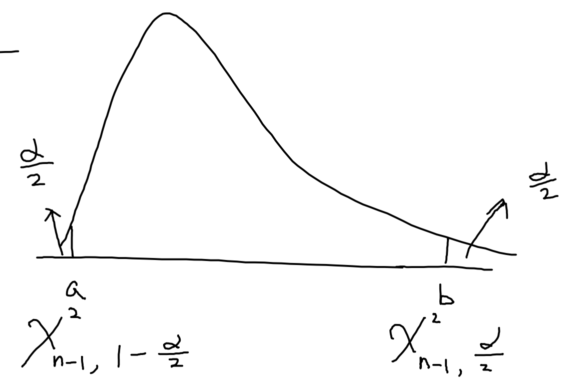
$\theta$ is fixed. Randomness.

$$P\left(\theta \in [l(X).\ u(X)]\right) \geq 1-\alpha$$

Example 7.2.2. $(1-\alpha)$ CI for $\sigma^2$.

$$\frac{S^2}{\sigma^2} \sim \frac{\chi^2_{n-1}}{n-1}$$

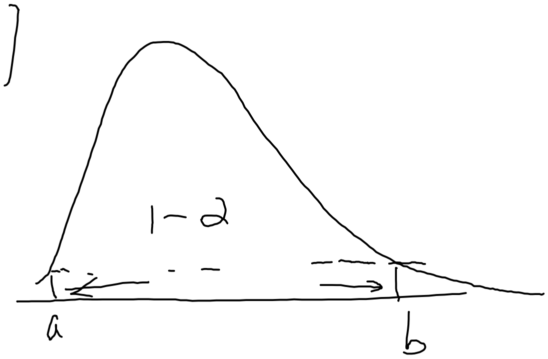
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$




$$1-\alpha = P\left(\chi^2_{n-1\ 1-\frac{\alpha}{2}} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,\frac{\alpha}{2}}\right)$$

$$= P\left(\frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1\ 1-\frac{\alpha}{2}}}\right)$$

95% CI : $[1.45\ .\ 2.18]$




$$\begin{cases} \int_a^b d\chi^2_{n-1} = 1-\alpha \\ f_{\chi^2_{n-1}}(a) = f_{\chi^2_{n-1}}(b). \end{cases}$$

let $a = [0,\ \chi^2_{n-1,1-\alpha}]$ $\quad b = \chi^2_{n-1,\ \alpha - F_{\chi^2_{n-1}}(a)}$

$a = [0, \chi^2_{n-1,1-\alpha}]$ $\quad \chi^2_{n-1}, \alpha - F_{\chi^2_{n-1}}(a)$

choose $a$ such that $|f(a) - f(b)|$ attains the minimum.

$$CI: [1.4, \ 2.2]$$

CLT Suppose $Y_1 \cdots Y_n$ i.i.d follow a distribution

with $EY_i = \mu$, $V(Y_i) = \sigma^2$

then as $n \to \infty$

$$\frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \xrightarrow{D} N(0,1)$$

CI for $\mu$: $\quad \bar{Y} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

Example 7.2.3

$p$:

$X_i$ Indicator whether the i-th individual chose for economy.

$p$: probability.

$X_i \sim \text{Bernoulli}(p)$ $\qquad X = \sum X_i$

$$\hat{p} = \frac{X}{N} = 0.41$$

$\bar{X}$

$EX_i = p \qquad V(X_i) = p(1-p)$

CLT: $\dfrac{\sqrt{n}(\hat{p} - p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1)$ $\qquad V(\hat{p}) = \dfrac{p(1-p)}{n}$

$(1-\alpha)$ CI : $-z_{\frac{\alpha}{2}} < \dfrac{\sqrt{n}(\hat{p}-p)}{\sqrt{p(1-p)}} < z_{\frac{\alpha}{2}}$

$2 \qquad\qquad v\,p(1-p)$

$$\Leftrightarrow \hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}} < p < \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}$$

$$\left[\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \quad \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

95% CI for $p$ is $(0.38, \ 0.44)$

Variance Stabilization Transformation (VST)

$$V(\hat{p}) = \frac{p(1-p)}{n}$$

$$V(g(\hat{p})) \approx \text{constant}$$