Chapter 2

Review of Calculus

2.1 Limits and derivatives

As x becomes larger, the value of $1/x$ become smaller. It can not reach 0, but we can make $1/x$ as close to 0 as possible by making x large enough. This is a limit statement:

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

The (ϵ,δ)-definition of the limit of a function is as follows: Let f be a function defined on an open interval containing c and let L be a real number. Then

$$\lim_{x\to c}f(x)=L$$

means for each real $\epsilon>0$, there exists a real $\delta>0$ such that for all x with $0<|x-c|<\delta$, we have $|f(x)-L|<\epsilon$. Symbolically,

$$\forall\epsilon>0\exists\delta>0:\forall x(0<|x-c|<\delta\Rightarrow|f(x)-L|<\epsilon).$$

The first order derivative of a function $y=f(x)$ is

$$\frac{dy}{dx}=f'(x)=\lim_{\delta\to 0}\frac{\delta y}{\delta x}=\lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta}.$$

The process of calculate the derivative is called differentiation. We call dy and dx differentials. Geometrically, $f'(x_0)$ is the instant rate of change at point $(x_0,f(x_0))$, or the slope of the tangent line to the curve $y=f(x)$ at point $(x_0,f(x_0))$.

For example, $y=x^2$, then

$$\frac{dy}{dx}=\lim_{\delta\to 0}\frac{(x+\delta)^2-x^2}{\delta}=\lim_{\delta\to 0}(2x+\delta)=2x.$$

Generally, we have the power rule:

$$\frac{dx^p}{dx} = px^{p-1}.$$

We have $\frac{de^x}{dx} = e^x$, or the derivative of e^x is itself. The general exponential rule is:

$$\frac{db^x}{dx} = b^x \ln(b).$$

We have $\frac{d[\ln(x)]}{dx} = \frac{1}{x}$. The general logarithm rule is:

$$\frac{d[\log_b(x)]}{dx} = \frac{1}{x\ln(b)}$$

If $y = f(x) = c$ is a constant for any x, then $f'(x) = 0$ for all x.
Not every function has a derivative everywhere!
Properties of derivatives:

1. For any constant c and differentiable function $f(x)$, $[cf(x)]' = cf'(x)$;

   For any two differentiable functions $f(x)$ and $g(x)$

2. Sum and difference rule, $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$;
3. Product rule, $[f(x)g(x)'] = f'(x)g(x) + f(x)g'(x)$;
4. Quotient rule, $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ for $g(x) \neq 0$;
5. Chain rule, $\{f[g(x)]\}' = f'[g(x)]g'(x)$;
6. If $f(x)$ is the inverse function of $g(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

2.2 Integrals

Suppose we have a general function $y = f(t)$. For simplicity, let $f(t) > 0$ and $f(t)$ continuous. Denote

$$F(x) = \text{area under the graph of } f(t) \text{ in the interval [a,x]}.$$

Then we have, for some value z in the interval $[x, x + \delta]$

$$F(x + \delta) - F(x) = f(z)\delta,$$

or

$$\frac{F(x+\delta)-F(x)}{\delta}=f(z).$$

As δ goes to 0, z goes to x, and we have

$$F'(x)=\lim_{\delta\to 0}\frac{F(x+\delta)-F(x)}{\delta}=\lim_{z\to x}f(z)=f(x).$$

So F is antiderivative of f, and we denote

$$F(x)=\int_a^x f(t)dt.$$

This is a definite integral of the function f from a to x. f is called the integrand. We also have indefinite integral. That is, for an arbitrary constant C,

$$\int f(x)=F(x)+C.$$

Properties of integrals:

1. For any constant a, b, c and any integrable function $f(x)$,

$$\int_a^b cf(x)dx=c\int_a^b f(x)dx.$$

2. For any constant a, b, and any two integrable functions $f(x)$ and $g(x)$,

$$\int_a^b [f(x)\pm g(x)]dx=\int_a^b f(x)dx\pm\int_a^b g(x)dx.$$

2.3 Some useful results

Integration by parts. Given the existence of all integrations, we have

$$\int_a^b f(x)g'(x)dx=f(x)g(x)\mid_a^b-\int_a^b f'(x)g(x)dx.$$

L'Hospital Rule. If $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$ or $\pm\infty$, $\lim_{x\to c}f'(x)/g'(x)$ exists, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Taylor Expansion. Derivative of the derivative is called the second-order derivative

$$f^{(2)}(x) = f''(x) = [f'(x)]' = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Similarly, we can define the n-th order derivative and denote it by $f^{(n)}$.

The Taylor series of a function $f(x)$, which is infinitely differentiable in a neighborhood of a, is a power series:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

For example:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \text{ for } -1 < x \leq 1$$

Local extrema. If a function $f(x)$ has either a local maximum or a local minimum at some value x_0, and $f(x)$ is differentiable at x_0, then the tangent line must be horizontal, or $f'(x_0) = 0$. We call x_0 a local extrema if it is either a local maximum or a local minimum. On the other hand, not all solutions of $f'(x) = 0$ is a local extrema.

1. If $f'(x_0) = 0$, $f''(x_0) < 0$ and $f''(x)$ is continuous in a region around $x = x_0$, then $x = x_0$ is a local maximum.

2. If $f'(x_0) = 0$, $f''(x_0) > 0$ and $f''(x)$ is continuous in a region around $x = x_0$, then $x = x_0$ is a local minimum.