16 Analysis-of-Covariance

16.1 INTRODUCTION

In addition to the dependent variable y, there may be one or more quantitative variables that can also be measured on each experimental unit (or subject) in an ANOVA situation. If it appears that these extra variables may affect the outcome of the experiment, they can be included in the model as independent variables (x's) and are then known as *covariates* or *concomitant variables*. *Analysis of covariance* is sometimes described as a blend of ANOVA and regression.

The primary motivation for the use of covariates in an experiment is to gain precision by reducing the error variance. In some situations, analysis of covariance can be used to lessen the effect of factors that the experimenter cannot effectively control, because an attempt to include various levels of a quantitative variable as a full factor may cause the design to become unwieldy. In such cases, the variable can be included as a covariate, with a resulting adjustment to the dependent variable before comparing means of groups. Variables of this type may also occur in experimental situations in which the subjects cannot be randomly assigned to treatments. In such cases, we forfeit the causality implication of a designed experiment, and analysis of covariance is closer in spirit to descriptive model building.

In terms of a one-way model with one covariate, analysis of covariance will be successful if the following three assumptions hold.

1. *The dependent variable is linearly related to the covariate*. If this assumption holds, part of the error in the model is predictable and can be removed to reduce the error variance. This assumption can be checked by testing H_0: $\beta = 0$, where β is the slope from the regression of the dependent variable on the covariate. Since the estimated slope $\hat{\beta}$ will never be exactly 0, analysis of covariance will always give a smaller sum of squares for error than the corresponding ANOVA. However, if $\hat{\beta}$ is close to 0, the small reduction in error sum of squares may not offset the loss of a degree of freedom [see (16.27) and a comment following]. This problem is more likely to arise with multiple covariates, especially if they are highly correlated.

Linear Models in Statistics, *Second Edition*, by Alvin C. Rencher and G. Bruce Schaalje

2. *The groups (treatments) have the same slope*. In assumption 1 above, a common slope β for all k groups is implied (assuming a one-way model with k groups). We can check this assumption by testing $H_0: \beta_1 = \beta_2 = \cdots = \beta_k$, where β_i is the slope in the ith group.
3. *The covariate does not affect the differences among the means of the groups (treatments)*. If differences among the group means were reduced when the dependent variable is adjusted for the covariate, the test for equality of group means would be less powerful. Assumption 3 can be checked by performing an ANOVA on the covariate.

Covariates can be either fixed constants (values chosen by the researcher) or random variables. The models we consider in this chapter involve fixed covariates, but in practice, the majority are random. However, the estimation and testing procedures are the same in both cases, although the properties of estimators and tests are somewhat different for fixed and random covariates. For example, in the fixed-covariate case, the power of the test depends on the actual values chosen for the covariates, whereas in the random-covariate case, the power of the test depends on the population covariance matrix of the covariates.

As an illustration of the use of analysis of covariance, suppose that we wish to compare three methods of teaching language. Three classes are available, and we assign a class to each of the teaching methods. The students are free to sign up for any one of the three classes and are therefore not randomly assigned. One of the classes may end up with a disproportionate share of the best students, in which case we cannot claim that teaching methods have produced a significant difference in final grades. However, we can use previous grades or other measures of performance as covariates and then compare the students' adjusted scores for the three methods.

We give a general approach to estimation and testing in Section 16.2 and then cover specific balanced models in Sections 16.3–16.5. Unbalanced models are discussed briefly in Section 16.6. We use overparameterized models for the balanced case in Sections 16.2–16.5. and use the cell means model in Section 16.6.

16.2 ESTIMATION AND TESTING

We introduce and illustrate the analysis of covariance model in Section 16.2.1 and discuss estimation and testing for this model in Sections 16.2.2 and 16.2.3.

16.2.1 The Analysis-of-Covariance Model

In general, an analysis of covariance model can be written as

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{16.1}$$

where $\mathbf{Z}$ contains 0s and 1s, $\boldsymbol{\alpha}$ contains μ and parameters such as α_i, β_i, and γ_{ij} representing factors and interactions (or other effects); $\mathbf{X}$ contains the covariate values; and $\boldsymbol{\beta}$ contains coefficients of the covariates. Thus the covariates appear on the right side of (16.1) as independent variables. Note that $\mathbf{Z}\boldsymbol{\alpha}$ is the same as $\mathbf{X}\boldsymbol{\beta}$ in the ANOVA models in Chapters 12–14, whereas in this chapter, we use $\mathbf{X}\boldsymbol{\beta}$ to represent the covariates in the model.

We now illustrate (16.1) for some of the models that will be considered in this chapter. A one-way (balanced) model with one covariate can be expressed as

$$y_{ij} = \mu + \alpha_i + \beta x_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n, \tag{16.2}$$

where α_i is the treatment effect, x_{ij} is a covariate observed on the same sampling unit as y_{ij}, and β is a slope relating x_{ij} to y_{ij}. [If (16.2) is viewed as a regression model, then the parameters $\mu + \alpha_i \quad i = 1, 2, \ldots, k$, serve as regression intercepts for the k groups.] The kn observations for (16.2) can be written in the form $\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ as in (16.1), where

$$\mathbf{Z} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}, \quad \mathbf{X} = \mathbf{x} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{1n} \\ x_{2n} \\ \vdots \\ x_{kn} \end{pmatrix}, \tag{16.3}$$

and $\boldsymbol{\beta} = \beta$. In this case, $\mathbf{Z}$ is the same as $\mathbf{X}$ in (13.6).

For a one-way (balanced) model with q covariates, the model is

$$y_{ij} = \mu + \alpha_i + \beta_1 x_{ij1} + \cdots + \beta_q x_{ijq} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n. \tag{16.4}$$

In this case, $\mathbf{Z}$ and $\boldsymbol{\alpha}$ are as given in (16.3), and $\mathbf{X}\boldsymbol{\beta}$ has the form

$$\mathbf{X}\boldsymbol{\beta} = \begin{pmatrix} x_{111} & x_{112} & \cdots & x_{11q} \\ x_{121} & x_{122} & \cdots & x_{12q} \\ \vdots & \vdots & & \vdots \\ x_{kn1} & x_{kn2} & \cdots & x_{knq} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{pmatrix}. \tag{16.5}$$

For a two-way model with one covariate

$$y_{ijk} = \mu + \alpha_i + \delta_j + \gamma_{ij} + \beta x_{ijk} + \varepsilon_{ijk}, \tag{16.6}$$

$\mathbf{Z}\boldsymbol{\alpha}$ has the form given in (14.4), and $\mathbf{X}\boldsymbol{\beta}$ is

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{x}\beta = \begin{pmatrix} x_{111} \\ x_{112} \\ \vdots \\ x_{abn} \end{pmatrix} \beta.$$

The two-way model in (16.6) could be extended to include several covariates.

16.2.2 Estimation

We now develop estimators of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for the general case in (16.1), $\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. We assume that $\mathbf{Z}$ is less than full rank as in overparameterized ANOVA models and that $\mathbf{X}$ is full-rank as in regression models. We also assume that

$$E(\boldsymbol{\varepsilon}) = \mathbf{0} \quad \text{and} \quad \text{cov}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}.$$

The model can be expressed as

$$\begin{aligned} \mathbf{y} &= \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ &= (\mathbf{Z}, \mathbf{X})\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} + \boldsymbol{\varepsilon} \\ &= \mathbf{U}\boldsymbol{\theta} + \boldsymbol{\varepsilon}, \end{aligned} \tag{16.7}$$

where $\mathbf{U} = (\mathbf{Z}, \mathbf{X})$ and $\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$. The normal equations for (16.7) are

$$\mathbf{U}'\mathbf{U}\hat{\boldsymbol{\theta}} = \mathbf{U}'\mathbf{y},$$

which can be written in partitioned form as

$$\begin{pmatrix} \mathbf{Z}' \\ \mathbf{X}' \end{pmatrix}(\mathbf{Z}, \mathbf{X})\begin{pmatrix} \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}' \\ \mathbf{X}' \end{pmatrix}\mathbf{y},$$

$$\begin{pmatrix} \mathbf{Z}'\mathbf{Z} & \mathbf{Z}'\mathbf{X} \\ \mathbf{X}'\mathbf{Z} & \mathbf{X}'\mathbf{X} \end{pmatrix}\begin{pmatrix} \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}'\mathbf{y} \\ \mathbf{X}'\mathbf{y} \end{pmatrix}. \tag{16.8}$$

We can express (16.8) as two sets of equations in $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$:

$$\mathbf{Z}'\mathbf{Z}\hat{\boldsymbol{\alpha}} + \mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{Z}'\mathbf{y}, \tag{16.9}$$

$$\mathbf{X}'\mathbf{Z}\hat{\boldsymbol{\alpha}} + \mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}. \tag{16.10}$$

Using a generalized inverse of $\mathbf{Z}'\mathbf{Z}$, we can solve for $\hat{\boldsymbol{\alpha}}$ in (16.9):

$$\begin{aligned} \hat{\boldsymbol{\alpha}} &= (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{y} - (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}} \\ &= \hat{\boldsymbol{\alpha}}_0 - (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}}, \end{aligned} \tag{16.11}$$

where $\hat{\boldsymbol{\alpha}}_0 = (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{y}$ is a solution for the normal equations for the model $\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$ without the covariates [see (12.13)].

To solve for $\hat{\boldsymbol{\beta}}$, we substitute (16.11) into (16.10) to obtain

$$\mathbf{X}'\mathbf{Z}[(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{y} - (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}}] + \mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$$

or

$$\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{y} + \mathbf{X}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}']\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}. \tag{16.12}$$

Defining

$$\mathbf{P} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}', \tag{16.13}$$

we see that (16.12) becomes

$$\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y} - \mathbf{X}'\mathbf{P}\mathbf{y} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y}.$$

Since the elements of $\mathbf{X}$ typically exhibit a pattern unrelated to the 0s and 1s in $\mathbf{Z}$, we can assume that the columns of $\mathbf{X}$ are linearly independent of the columns of $\mathbf{Z}$. Then $\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}$ is nonsingular (see Problem 16.1), and a solution for $\hat{\boldsymbol{\beta}}$ is given by

$$\hat{\boldsymbol{\beta}} = [\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y} \tag{16.14}$$

$$= \mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}, \tag{16.15}$$

where

$$\mathbf{E}_{xx} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X} \quad \text{and} \quad \mathbf{e}_{xy} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y}. \tag{16.16}$$

For the analysis-of-covariance model (16.1) or (16.7), we denote SSE as $\text{SSE}_{y\cdot x}$. By (12.20), $\text{SSE}_{y\cdot x}$ can be expressed as

$$\begin{aligned}
\text{SSE}_{y\cdot x} &= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\theta}}'\mathbf{U}'\mathbf{y} = \mathbf{y}'\mathbf{y} - (\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}')\begin{pmatrix} \mathbf{Z}'\mathbf{y} \\ \mathbf{X}'\mathbf{y} \end{pmatrix} \\
&= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\alpha}}'\mathbf{Z}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \\
&= \mathbf{y}'\mathbf{y} - [\hat{\boldsymbol{\alpha}}_0' - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}]\mathbf{Z}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \quad \text{[by (16.11)]} \\
&= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\alpha}}_0'\mathbf{Z}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}']\mathbf{y} \\
&= \text{SSE}_y - \hat{\boldsymbol{\beta}}'\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y},
\end{aligned} \tag{16.17}$$

where $\hat{\boldsymbol{\alpha}}_0$ is as defined in (16.11), $\mathbf{P}$ is defined as in (16.13), and $\text{SSE}_y = \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\alpha}}_0'\mathbf{Z}'\mathbf{y}$ is the same as the SSE for the ANOVA model $\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$ without the covariates. Using (16.16), we can write (16.17) in the form

$$\text{SSE}_{y\cdot x} = e_{yy} - \mathbf{e}_{xy}'\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}, \tag{16.18}$$

where

$$e_{yy} = \text{SSE}_y = \mathbf{y}'(\mathbf{I} - \mathbf{P})\mathbf{y}. \tag{16.19}$$

In (16.18), we see the reduction in SSE that was noted in the second paragraph of Section 16.1. The proof that $\mathbf{E}_{xx} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}$ is nonsingular (see Problem 16.1) can be extended to show that $\mathbf{E}_{xx}$ is positive definite. Therefore, $\mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} > 0$, and $\text{SSE}_{y\cdot x} < \text{SSE}_y$.

16.2.3 Testing Hypotheses

In order to test hypotheses, we assume that $\boldsymbol{\varepsilon}$ in (16.1) is distributed as $N_n(\mathbf{0}, \sigma^2\mathbf{I})$, where n is the number of rows of $\mathbf{Z}$ or $\mathbf{X}$. Using the model (16.7), we can express a hypothesis about $\boldsymbol{\alpha}$ in the form $H_0\colon \mathbf{C}\boldsymbol{\theta} = \mathbf{0}$, where $\mathbf{C} = (\mathbf{C}_1, \mathbf{O})$, so that H_0 becomes

$$H_0\colon (\mathbf{C}_1, \mathbf{O})\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \mathbf{0} \quad \text{or} \quad H_0\colon \mathbf{C}_1\boldsymbol{\alpha} = \mathbf{0}.$$

We can then use a general linear hypothesis test. Alternatively, we can incorporate the hypothesis into the model and use a full–reduced-model approach.

Hypotheses about $\boldsymbol{\beta}$ can also be expressed in the form $H_0\colon \mathbf{C}\boldsymbol{\theta} = \mathbf{0}$:

$$H_0\colon \mathbf{C}\boldsymbol{\theta} = (\mathbf{O}, \mathbf{C}_2)\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \mathbf{0} \quad \text{or} \quad H_0: \mathbf{C}_2\boldsymbol{\beta} = \mathbf{0}.$$

A basic hypothesis of interest is $H_0: \boldsymbol{\beta} = \mathbf{0}$, that is, that the covariate(s) do not belong in the model (16.1). In order to make a general linear hypothesis test of $H_0\colon \boldsymbol{\beta} = \mathbf{0}$, we need $\text{cov}(\hat{\boldsymbol{\beta}})$, where $\hat{\boldsymbol{\beta}}$ is given by (16.14) as $\hat{\boldsymbol{\beta}} = [\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y}$. Since $\mathbf{I} - \mathbf{P}$ is idempotent (see Theorems 2.13e and 2.13f), $\text{cov}(\hat{\boldsymbol{\beta}})$ can readily be found from (3.44) as

$$\begin{aligned} \text{cov}(\hat{\boldsymbol{\beta}}) &= [\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{I} - \mathbf{P})\sigma^2\mathbf{I}(\mathbf{I} - \mathbf{P})\mathbf{X}[\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1} \\ &= \sigma^2[\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1}. \end{aligned} \tag{16.20}$$

Then SSH for testing $H_0: \boldsymbol{\beta} = \mathbf{0}$ is given by Theorem 8.4a(ii) as

$$\text{SSH} = \hat{\boldsymbol{\beta}}'\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}\hat{\boldsymbol{\beta}}. \tag{16.21}$$

Using (16.16), we can express this as

$$\text{SSH} = \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}. \tag{16.22}$$

Note that SSH in (16.22) is equal to the reduction in SSE due to the covariates; see (16.17), (16.18), and (16.19).

We now discuss some specific models, beginning with the one-way model in Section 16.3.

16.3 ONE-WAY MODEL WITH ONE COVARIATE

We review the one-way model in Section 16.3.1, consider estimators of parameters in Section 16.3.2, and discuss tests of hypotheses in Section 16.3.3.

16.3.1 The Model

The one-way (balanced) model was introduced in (16.2):

$$y_{ij} = \mu + \alpha_i + \beta x_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n. \tag{16.23}$$

All kn observations can be written in the form of (16.1)

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{x}\beta + \boldsymbol{\varepsilon},$$

where $\mathbf{Z}$, $\boldsymbol{\alpha}$, and $\mathbf{x}$ are as given in (16.3).

16.3.2 Estimation

By (16.11), (13.11), and (13.12), an estimator of $\boldsymbol{\alpha}$ is obtained as

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \hat{\boldsymbol{\alpha}}_0 - (\mathbf{Z}'\mathbf{Z})^- \mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\alpha}}_0 - (\mathbf{Z}'\mathbf{Z})^- \mathbf{Z}'\mathbf{x}\hat{\beta} \\ &= \begin{pmatrix} 0 \\ \bar{y}_{1.} \\ \bar{y}_{2.} \\ \vdots \\ \bar{y}_{k.} \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{\beta}\bar{x}_{1.} \\ \hat{\beta}\bar{x}_{2.} \\ \vdots \\ \hat{\beta}\bar{x}_{k.} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{y}_{1.} - \hat{\beta}\bar{x}_{1.} \\ \bar{y}_{2.} - \hat{\beta}\bar{x}_{2.} \\ \vdots \\ \bar{y}_{k.} - \hat{\beta}\bar{x}_{2.} \end{pmatrix}\end{aligned} \tag{16.24}$$

(see Problem 16.4). In this case, with a single x, $\mathbf{E}_{xx}$ and $\mathbf{e}_{xy}$ reduce to scalars, along with e_{yy}:

$$\begin{aligned}\mathbf{E}_{xx} &= e_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2, \\ \mathbf{e}_{xy} &= e_{xy} = \sum_{ij}(x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.}), \\ e_{yy} &= \sum_{ij}(y_{ij} - \bar{y}_{i.})^2.\end{aligned} \tag{16.25}$$

Now, by (16.15), the estimator of β is

$$\hat{\beta} = \frac{e_{xy}}{e_{xx}} = \frac{\sum_{ij}(x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})}{\sum_{ij}(x_{ij} - \bar{x}_{i.})^2}. \tag{16.26}$$

By (16.18), (16.19), and the three results in (16.25), $\mathrm{SSE}_{y\cdot x}$ is given by

$$\begin{aligned} \mathrm{SSE}_{y\cdot x} &= e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = e_{yy} - \frac{e_{xy}^2}{e_{xx}} \\ &= \sum_{ij}(y_{ij} - \bar{y}_{i.})^2 - \frac{\left[\sum_{ij}(x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})\right]^2}{\sum_{ij}(x_{ij} - \bar{x}_{i.})^2}, \end{aligned} \tag{16.27}$$

which has $k(n-1) - 1$ degrees of freedom. Note that the degrees of freedom of $\mathrm{SSE}_{y\cdot x}$ are reduced by 1 for estimation of β, since $\mathrm{SSE}_y = e_{yy}$ has $k(n-1)$ degrees of freedom and e_{xy}^2/e_{xx} has 1 degree of freedom. In using analysis of covariance, the researcher expects the reduction from SSE_y to $\mathrm{SSE}_{y\cdot x}$ to at least offset the loss of a degree of freedom.

16.3.3 Testing Hypotheses

For testing hypotheses, we assume that the ε_{ij}'s in (16.23) are independently distributed as $N(0, \sigma^2)$. We begin with a test for equality of treatment effects.

16.3.3.1 Treatments

To test

$$H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_k$$

adjusted for the covariate, we use a full–reduced-model approach. The full model is (16.23), and the reduced model (with $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$) is

$$\begin{aligned} y_{ij} &= \mu + \alpha + \beta x_{ij} + \varepsilon_{ij} \\ &= \mu^* + \beta x_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n. \end{aligned} \qquad 16.28)$$

This is essentially the same as the simple linear regression model (6.1). By (6.13), SSE for this reduced model (denoted by $\mathrm{SSE}_{\mathrm{rd}}$) is given by

$$\mathrm{SSE}_{\mathrm{rd}} = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 - \frac{\left[\sum_{ij}(x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..})\right]^2}{\sum_{ij}(x_{ij} - \bar{x}_{..})^2}, \tag{16.29}$$

which has $kn - 1 - 1 = kn - 2$ degrees of freedom.

Using a notation adapted from Sections 8.2, 13.4, and 14.4, we express the sum of squares for testing H_{01} as

$$\text{SS}(\alpha|\mu, \beta) = \text{SS}(\mu, \alpha, \beta) - \text{SS}(\mu, \beta).$$

In (16.27), $\text{SSE}_{y\cdot x}$ is for the full model, and in (16.29), SSE_{rd} is for the reduced model. They can therefore be written as $\text{SSE}_{y\cdot x} = \mathbf{y}'\mathbf{y} - \text{SS}(\mu, \alpha, \beta)$ and $\text{SSE}_{\text{rd}} = \mathbf{y}'\mathbf{y} - \text{SS}(\mu, \beta)$. Hence

$$\text{SS}(\alpha|\mu, \beta) = \text{SSE}_{\text{rd}} - \text{SSE}_{y\cdot x}, \tag{16.30}$$

which has $kn - 2 - [k(n-1) - 1] = k - 1$ degrees of freedom. The test statistic for $H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ is therefore given by

$$F = \frac{\text{SS}(\alpha|\mu, \beta)/(k-1)}{\text{SSE}_{y\cdot x}/[k(n-1) - 1]}, \tag{16.31}$$

which is distributed as $F[k-1, k(n-1)-1]$ when H_{01} is true.

By (16.30), we have

$$\text{SSE}_{\text{rd}} = \text{SS}(\alpha|\mu, \beta) + \text{SSE}_{y\cdot x}.$$

Hence, SSE_{rd} functions as the "total sum of squares" for the test of treatment effects adjusted for the covariate. We can therefore denote SSE_{rd} by $\text{SST}_{y\cdot x}$, so that the expression above becomes

$$\text{SST}_{y\cdot x} = \text{SS}(\alpha|\mu, \beta) + \text{SSE}_{y\cdot x}. \tag{16.32}$$

To complete the analogy with $\text{SSE}_{y\cdot x} = e_{yy} - e_{xy}^2/e_{xx}$ in (16.27), we write (16.29) as

$$\text{SST}_{y\cdot x} = t_{yy} - \frac{t_{xy}^2}{t_{xx}}, \tag{16.33}$$

where

$$\text{SST}_{y\cdot x} = \text{SSE}_{\text{rd}}, \; t_{yy} = \sum_{ij} (y_{ij} - \bar{y}_{..})^2, \; t_{xy} = \sum_{ij} (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..}),$$
$$t_{xx} = \sum_{ij} (x_{ij} - \bar{x}_{..})^2. \tag{16.34}$$

Note that the procedure used to obtain (16.30) is fundamentally different from that used to obtain $\text{SSE}_{y\cdot x}$ and SSE_{rd} in (16.27) and (16.29). The sum of squares $\text{SS}(\alpha|\mu, \beta)$ in (16.30) is obtained as the difference between the sums of squares for full and reduced models, not as an adjustment to $\mathrm{SS}(\alpha|\mu) = n\sum_i(\bar{y}_{i.} - \bar{y}_{..})^2$ in (13.24) analogous to the adjustment used in $\mathrm{SSE}_{y\cdot x}$ and $\mathrm{SST}_{y\cdot x}$ in (16.27) and (16.33). We must use the full–reduced-model approach to compute $\mathrm{SS}(\alpha|\mu, \beta)$, because we do not have the same covariate values for each treatment and the design is therefore unbalanced (even though the n values are equal). If $\mathrm{SS}(\alpha|\mu, \beta)$ were computed in an "adjusted" manner as in (16.27) or (16.33), then $\mathrm{SS}(\alpha|\mu, \beta) + \mathrm{SSE}_{y\cdot x}$ would not equal $\mathrm{SST}_{y\cdot x}$ as in (16.32). In Section 16.4, we will follow a computational scheme similar to that of (16.30) and (16.32) for each term in the two-way (balanced) model.

We display the various sums of squares for testing $H_0\colon \alpha_1 = \alpha_2 = \cdots = \alpha_k$ in Table 16.1.

16.3.3.2 Slope

We now consider a test for

$$H_{02} : \beta = 0.$$

By (16.22), the general linear hypothesis approach leads to $\mathrm{SSH} = \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}$ for testing $H_0\colon \boldsymbol{\beta} = \mathbf{0}$. For the case of a single covariate, this reduces to

$$\mathrm{SSH} = \frac{e_{xy}^2}{e_{xx}}, \tag{16.35}$$

where e_{xy} and e_{xx} are as found in (16.25). The F statistic is therefore given by

$$F = \frac{e_{xy}^2/e_{xx}}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-1]}, \tag{16.36}$$

which is distributed as $F[1, k(n-1)-1]$ when H_{02} is true.

16.3.3.3 Homogeneity of Slopes

The tests of $H_{01}\colon \alpha_1 = \alpha_2 = \cdots = \alpha_k$ and $H_{02}\colon \beta = 0$ assume a common slope for all k groups. To check this assumption, we can test the hypothesis of equal slopes in the groups

$$H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_k, \tag{16.37}$$

where β_i is the slope in the ith group. In effect, H_{03} states that the k regression lines are parallel.

TABLE 16.1 Analysis of Covariance for Testing $H_0\colon \alpha_1 = \alpha_2 = \cdots = \alpha_k$ in the One-Way Model with One Covariate

Source	SS Adjusted for Covariate	Adjusted df
Treatments	$\mathrm{SS}(\alpha\|\mu, \beta) = \mathrm{SST}_{y\cdot x} - \mathrm{SSE}_{y\cdot x}$	$k-1$
Error	$\mathrm{SSE}_{y\cdot x} = e_{yy} - e_{xy}^2/e_{xx}$	$k(n-1)-1$
Total	$\mathrm{SST}_{y\cdot x} = t_{yy} - t_{xy}^2/t_{xx}$	$kn-2$

The full model allowing for different slopes becomes

$$y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n. \tag{16.38}$$

The reduced model with a single slope is (16.23). In matrix form, the nk observations in (16.38) can be expressed as $\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{Z}$ and $\boldsymbol{\alpha}$ are as given in (16.3) and

$$\mathbf{X}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x}_k \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \tag{16.39}$$

with $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})'$. By (16.14) and (16.15), we obtain

$$\hat{\boldsymbol{\beta}} = \mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = [\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y}.$$

To evaluate $\mathbf{E}_{xx}$ and $\mathbf{e}_{xy}$, we first note that by (13.11), (13.25), and (13.26)

$$\begin{aligned} \mathbf{I} - \mathbf{P} &= \mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}' \\ &= \begin{pmatrix} \mathbf{I} - \frac{1}{n}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} - \frac{1}{n}\mathbf{J} & \cdots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I} - \frac{1}{n}\mathbf{J} \end{pmatrix}, \end{aligned} \tag{16.40}$$

where $\mathbf{I}$ in $\mathbf{I} - \mathbf{P}$ is $kn \times kn$ and $\mathbf{I}$ in $\mathbf{I} - (1/n)\mathbf{J}$ is $n \times n$. Thus

$$\mathbf{E}_{xx} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X} = \begin{pmatrix} \mathbf{x}_1'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{x}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{x}_2'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{x}_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathbf{x}_k'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{x}_k \end{pmatrix}$$

$$= \begin{pmatrix} \sum_j (x_{1j} - \bar{x}_{1.})^2 & 0 & \cdots & 0 \\ 0 & \sum_j (x_{2j} - \bar{x}_{2.})^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sum_j (x_{kj} - \bar{x}_{k.})^2 \end{pmatrix} \tag{16.41}$$

$$= \begin{pmatrix} e_{xx,1} & 0 & \cdots & 0 \\ 0 & e_{xx,2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e_{xx,k} \end{pmatrix}, \tag{16.42}$$

where $e_{xx,i} = \sum_j (x_{ij} - \bar{x}_{i.})^2$. To find $\mathbf{e}_{xy}$, we partition $\mathbf{y}$ as $\mathbf{y} = (\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_k')'$, where $\mathbf{y}_i' = (y_{i1}, y_{i2}, \ldots, y_{in})$. Then

$$\begin{aligned}
\mathbf{e}_{xy} &= \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y} \\
&= \begin{pmatrix} \mathbf{x}_1' & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{0}' & \mathbf{x}_2' & \cdots & \mathbf{0}' \\ \vdots & \vdots & & \vdots \\ \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{x}_k' \end{pmatrix} \begin{pmatrix} \mathbf{I} - \frac{1}{n}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} - \frac{1}{n}\mathbf{J} & \cdots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I} - \frac{1}{n}\mathbf{J} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{x}_1'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{y}_1 \\ \mathbf{x}_2'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{y}_2 \\ \vdots \\ \mathbf{x}_k'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{y}_k \end{pmatrix} \\
&= \begin{pmatrix} \sum_j (x_{1j} - \bar{x}_{1.})(y_{1j} - \bar{y}_{1.}) \\ \sum_j (x_{2j} - \bar{x}_{2.})(y_{2j} - \bar{y}_{2.}) \\ \vdots \\ \sum_j (x_{kj} - \bar{x}_{k.})(y_{kj} - \bar{y}_{k.}) \end{pmatrix} && (16.43) \\
&= \begin{pmatrix} e_{xy,1} \\ e_{xy,2} \\ \vdots \\ e_{xy,k} \end{pmatrix}, && (16.44)
\end{aligned}$$

where $e_{xy,i} = \sum_j (x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})$. Then, by (16.15), we obtain

$$\hat{\boldsymbol{\beta}} = \mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = \begin{pmatrix} e_{xy,1}/e_{xx,1} \\ e_{xy,2}/e_{xx,2} \\ \vdots \\ e_{xy,k}/e_{xx,k} \end{pmatrix}. \tag{16.45}$$

By analogy with (16.30), we obtain the sum of squares for the test of H_{03} in (16.37) by subtracting $\text{SSE}_{y\cdot x}$ for the full model from $\text{SSE}_{y\cdot x}$ for the reduced model, that is, $\text{SSE}(R)_{y\cdot x} - \text{SSE}(F)_{y\cdot x}$. For the full model in (16.38), $\text{SSE}(F)_{y\cdot x}$ is given by (16.18), (16.44), and (16.45) as

$$\begin{aligned} \text{SSE}(F)_{y\cdot x} &= e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = e_{yy} - \mathbf{e}'_{xy}\hat{\boldsymbol{\beta}} \\ &= e_{yy} - (e_{xy,1}, e_{xy,2}, \ldots, e_{xy,k}) \begin{pmatrix} e_{xy,1}/e_{xx,1} \\ e_{xy,2}/e_{xx,2} \\ \vdots \\ e_{xy,k}/e_{xx,k} \end{pmatrix} \\ &= e_{yy} - \sum_{i=1}^{k} \frac{e_{xy,i}^2}{e_{xx,i}}, \end{aligned} \tag{16.46}$$

which has $k(n-1) - k = k(n-2)$ degrees of freedom. The reduced model in which $H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_k = \beta$ is true is given by (16.23), for which $\text{SSE}(R)_{y\cdot x}$ is found in (16.27) as

$$\text{SSE}(R)_{y\cdot x} = e_{yy} - \frac{e_{xy}^2}{e_{xx}}, \tag{16.47}$$

which has $k(n-1) - 1$ degrees of freedom. Thus, the sum of squares for testing H_{03} is

$$\text{SSE}(R)_{y\cdot x} - \text{SSE}(F)_{y\cdot x} = \sum_{i=1}^{k} \frac{e_{xy,i}^2}{e_{xx,i}} - \frac{e_{xy}^2}{e_{xx}}, \tag{16.48}$$

which has $k(n-1) - 1 - k(n-2) = k - 1$ degrees of freedom. The test statistic is

$$F = \frac{\left[\sum_{i=1}^{k} e_{xy,i}^2/e_{xx,i} - e_{xy}^2/e_{xx}\right]/(k-1)}{\text{SSE}(F)_{y\cdot x}/k(n-2)}, \tag{16.49}$$

which is distributed as $F[k-1, k(n-2)]$ when $H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_k$ is true.

TABLE 16.2 Maturation Weight and Initial Weight (mg) of Guppy Fish

Feeding Group					
1		2		3	
y	x	y	x	y	x
49	35	68	33	59	33
61	26	70	35	53	36
55	29	60	28	54	26
69	32	53	29	48	30
51	23	59	32	54	33
38	26	48	23	53	25
64	31	46	26	37	23

If the hypothesis of equal slopes is rejected, the hypothesis of equal treatment effects can still be tested, but interpretation is more difficult. The problem is somewhat analogous to that of interpretation of a main effect in a two-way ANOVA in the presence of interaction. In a sense, the term $\beta_i x_{ij}$ in (16.38) is an interaction. For further discussion of analysis of covariance with heterogeneity of slopes, see Reader (1973) and Hendrix et al. (1982).

Example 16.3. To investigate the effect of diet on maturation weight of guppy fish (*Poecilia reticulata*), three groups of fish were fed different diets. The resulting weights y are given in Table 16.2 (Morrison 1983, p. 475) along with the initial weights x.

We first estimate β, using x as a covariate. By the three results in (16.25), we have

$$e_{xx} = 350.2857, \quad e_{xy} = 412.71429, \quad e_{yy} = 1465.7143.$$

Then by (16.26), we obtain

$$\hat{\beta} = \frac{e_{xy}}{e_{xx}} = \frac{412.7143}{350.2857} = 1.1782.$$

We now test for equality of treatment means adjusted for the covariate, $H_0\colon \alpha_1 = \alpha_2 = \alpha_3$. By (16.27), we have

$$\begin{aligned} \text{SSE}_{y\cdot x} &= e_{yy} - \frac{e_{xy}^2}{e_{xx}} = 1465.7143 - \frac{(412.7143)^2}{350.2857} \\ &= 979.4453 \end{aligned}$$

with 17 degrees of freedom. By (16.29) and (16.33), we have

$$\text{SST}_{y\cdot x} = 1141.4709$$

with 19 degrees of freedom. Thus by (16.30), we have

$$\begin{aligned}\mathrm{SS}(\alpha|\mu,\beta) &= \mathrm{SST}_{y\cdot x} - \mathrm{SSE}_{y\cdot x} = 1141.4709 - 979.4453 \\ &= 162.0256\end{aligned}$$

with 2 degrees of freedom. The F statistic is given in (16.31) as

$$F = \frac{\mathrm{SS}(\alpha|\mu,\beta)/(k-1)}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-1]} = \frac{162.0256/2}{979.4453/17} = 1.4061.$$

The p value is .272, and we do not reject $H_0\colon \alpha_1 = \alpha_2 = \alpha_3$.

To test $H_0\colon \beta = 0$, we use (16.36):

$$\begin{aligned}F &= \frac{e_{xy}^2/e_{xx}}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-1]} = \frac{(412.7143)^2/350.2857}{979.4453/17} \\ &= 8.4401.\end{aligned}$$

The p-value is .0099, and we reject $H_0 : \beta = 0$.

To test the hypothesis of equal slopes in the groups, $H_0\colon \beta_1 = \beta_2 = \beta_3$, we first estimate β_1, β_2, and β_3 using (16.45):

$$\hat{\beta}_1 = .7903, \quad \hat{\beta}_2 = 1.9851, \quad \hat{\beta}_3 = .8579.$$

Then by (16.46) and (16.47),

$$\mathrm{SSE}(F)_{y\cdot x} = 880.5896, \quad \mathrm{SSE}(R)_{y\cdot x} = 979.4453.$$

The difference $\mathrm{SSE}(R)_{y\cdot x} - \mathrm{SSE}(F)_{y\cdot x}$ is used in the numerator of the F statistic in (16.49):

$$F = \frac{(979.4453 - 880.5896)/2}{880.5896/(3)(5)} = .8420.$$

The p value is .450, and we do not reject $H_0\colon \beta_1 = \beta_2 = \beta_3$. □

16.4 TWO-WAY MODEL WITH ONE COVARIATE

In this section, we discuss the two-way (balanced) fixed-effects model with one covariate. The model was introduced in (16.6) as

$$y_{ijk} = \mu + \alpha_i + \gamma_j + \delta_{ij} + \beta x_{ijk} + \varepsilon_{ijk}, \tag{16.50}$$

$$i = 1, 2, \ldots, a, \;\; j = 1, 2, \ldots, c, \;\; k = 1, 2, \ldots, n,$$

where α_i is the effect of factor A, γ_j is the effect of factor C, δ_{ij} is the AC interaction effect, and x_{ijk} is a covariate measured on the same experimental unit as y_{ijk}.

16.4.1 Tests for Main Effects and Interactions

In order to find $\text{SSE}_{y\cdot x}$, we consider the hypothesis of no overall treatment effect, that is, no A effect, no C effect, and no interaction (see a comment preceding Theorem 14.4b). By analogy to (16.28), the reduced model is

$$y_{ijk} = \mu^* + \beta x_{ijk} + \varepsilon_{ijk}. \tag{16.51}$$

By analogy to (16.29), SSE for the reduced model is given by

$$\begin{aligned} \text{SSE}_{\text{rd}} &= \sum_{i=1}^{a}\sum_{j=1}^{c}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 - \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{...})(y_{ijk} - \bar{y}_{...})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{...})^2} \\ &= \sum_{ijk} y_{ijk}^2 - \frac{y_{...}^2}{acn} - \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{...})(y_{ijk} - \bar{y}_{...})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{...})^2}. \end{aligned} \tag{16.52}$$

By analogy to (16.27), SSE for the full model in (16.50) is

$$\begin{aligned} \text{SSE}_{y\cdot x} &= \sum_{ijk}(y_{ijk} - \bar{y}_{ij.})^2 - \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})^2} \\ &= \sum_{ijk} y_{ijk}^2 - \sum_{ij}\frac{y_{ij.}^2}{n} - \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})^2}, \end{aligned} \tag{16.53}$$

which has $ac(n-1) - 1$ degrees of freedom. Note that the degrees of freedom for $\text{SSE}_{y\cdot x}$ have been reduced by 1 for the covariate adjustment.

Now by analogy to (16.30), the overall sum of squares for treatments is

$$\begin{aligned} \text{SS}(\alpha, \gamma, \delta|\mu, \beta) &= \text{SSE}_{\text{rd}} - \text{SSE}_{y\cdot x} \\ &= \sum_{ij}\frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{acn} + \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})^2} \\ &\quad - \frac{\left[\sum_{ijk}(x_{ijk} - \bar{x}_{...})(y_{ijk} - \bar{y}_{...})\right]^2}{\sum_{ijk}(x_{ijk} - \bar{x}_{...})^2}, \end{aligned} \tag{16.54}$$

which has $ac - 1$ degrees of freedom.

Using (14.47), (14.69), and (14.70), we can partition the term $\sum_{ij} y_{ij.}^2/n - y_{...}^2/acn$ in (16.54), representing overall treatment sum of squares, as in (14.40):

$$\sum_{ij} \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{acn} = cn\sum_i (\bar{y}_{i..} - \bar{y}_{...})^2 + an\sum_j (\bar{y}_{.j.} - \bar{y}_{...})^2$$
$$+ n\sum_{ij} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$
$$= \text{SSA}_y + \text{SSC}_y + \text{SSAC}_y. \tag{16.55}$$

To conform with this notation, we define

$$\text{SSE}_y = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2.$$

We have an analogous partitioning of the overall treatment sum of squares for x:

$$\sum_{ij} \frac{x_{ij.}^2}{n} - \frac{x_{...}^2}{acn} = \text{SSA}_x + \text{SSC}_x + \text{SSAC}_x, \tag{16.56}$$

where, for example

$$\text{SSA}_x = cn\sum_{i=1}^{a} (\bar{x}_{i..} - \bar{x}_{...})^2.$$

We also define

$$\text{SSE}_x = \sum_{ijk} (x_{ijk} - \bar{x}_{ij.})^2.$$

The "overall treatment sum of products" $\sum_{ij} x_{ij.}y_{ij.}/n - x_{...}y_{...}/acn$ can be partitioned in a manner analogous to that in (16.55) and (16.56) (see Problem 16.8):

$$\sum_{ij} \frac{x_{ij.}y_{ij.}}{n} - \frac{x_{...}y_{...}}{acn} = cn\sum_i (\bar{x}_{i..} - \bar{x}_{...})(\bar{y}_{i..} - \bar{y}_{...}) + an\sum_j (\bar{x}_{.j.} - \bar{x}_{...})(\bar{y}_{.j.} - \bar{y}_{...})$$
$$+ n\sum_{ij} (\bar{x}_{ij.} - \bar{x}_{i..} - \bar{x}_{.j.} + \bar{x}_{...})(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})$$
$$= \text{SPA} + \text{SPC} + \text{SPAC}. \tag{16.57}$$

We also define

$$\text{SPE} = \sum_{ijk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.}).$$

We can now write $\text{SSE}_{y\cdot x}$ in (16.53) in the simplified form

$$\text{SSE}_{y\cdot x} = \text{SSE}_y - \frac{(\text{SPE})^2}{\text{SSE}_x}.$$

We display these sums of squares and products in Table 16.3.

We now proceed to develop hypothesis tests for factor A, factor C, and the interaction AC. The orthogonality of the balanced design is lost when adjustments are made for the covariate [see comments following (16.34); see also Bingham and Feinberg (1982)]. We therefore obtain a "total" for each term (A, C, or AC) by adding the error SS or SP to the term SS or SP for each of x, y and xy (see the entries for $A + E$, $C + E$, and $AC + E$ in Table 16.3). These totals are analogous to $\text{SST}_{y\cdot x} = \text{SS}(\alpha|\mu, \beta) + \text{SSE}_{y\cdot x}$ in (16.32) for the one-way model. The totals are used to obtain sums of squares adjusted for the covariate in a manner analogous to that employed in the one-way model [see (16.30) or the "treatments" line in Table 16.1]. For example, the adjusted sum of squares $\text{SSA}_{y\cdot x}$ for factor A is obtained as follows:

$$\text{SS}(A + E)_{y\cdot x} = \text{SSA}_y + \text{SSE}_y - \frac{(\text{SPA} + \text{SPE})^2}{\text{SSA}_x + \text{SSE}_x}, \tag{16.58}$$

$$\text{SSE}_{y\cdot x} = \text{SSE}_y - \frac{(\text{SPE})^2}{\text{SSE}_x}, \tag{16.59}$$

$$\text{SSA}_{y\cdot x} = \text{SS}(A + E)_{y\cdot x} - \text{SSE}_{y\cdot x}. \tag{16.60}$$

From inspection of (16.58), (16.59), and (16.60), we see that $\text{SSA}_{y\cdot x}$ has $a-1$ degrees of freedom. The statistic for testing $H_{01}\colon \alpha_1 = \alpha_2 = \cdots = \alpha_a$, corresponding to the

TABLE 16.3 Sums of Squares and Products for *x* and *y* in a Two-Way Model

	SS and SP Corrected for the Mean		
Source	y	x	xy
A	SSA_y	SSA_x	SPA
C	SSC_y	SSC_x	SPC
AC	SSAC_y	SSAC_x	SPAC
Error	SSE_y	SSE_x	SPE
$A + E$	$\text{SSA}_y + \text{SSE}_y$	$\text{SSA}_x + \text{SSE}_x$	SPA + SPE
$C + E$	$\text{SSC}_y + \text{SSE}_y$	$\text{SSC}_x + \text{SSE}_x$	SPC + SPE
$AC + E$	$\text{SSAC}_y + \text{SSE}_y$	$\text{SSAC}_x + \text{SSE}_x$	SPAC + SPE

TABLE 16.4 Value of Crops *y* and Size *x* of Farms in Three Iowa Counties

	County					
	1		2		3	
Landlord–Tenant	y	x	y	x	y	x
Related	6399	160	2490	90	4489	120
	8456	320	5349	154	10026	245
	8453	200	5518	160	5659	160
	4891	160	10417	234	5475	160
	3491	120	4278	120	11382	320
Not related	6944	160	4936	160	5731	160
	6971	160	7376	200	6787	173
	4053	120	6216	160	5814	134
	8767	280	10313	240	9607	239
	6765	160	5124	120	9817	320

Source: Ostle and Mensing (1975, p. 480).

main effect of A, is then given by

$$F = \frac{\text{SSA}_{y \cdot x}/(a-1)}{\text{SSE}_{y \cdot x}/[ac(n-1)-1]}, \tag{16.61}$$

which is distributed as $F[a-1, ac(n-1)-1]$ if H_{01} is true. Tests for factor C and the interaction AC are developed in an analogous fashion.

Example 16.4a. In each of three counties in Iowa, a sample of farms was taken from farms for which landlord and tenant are related and also from farms for which landlord and tenant are not related. Table 16.4 gives the data for $y =$ value of crops produced and $x =$ size of farm.

We first obtain the sums of squares and products listed in Table 16.3, where factor A is relationship status and factor C is county. These are given in Table 16.5, where,

TABLE 16.5 Sums of Squares and Products for *x* and *y*

	SS and SP Corrected for the Mean		
Source	y	x	xy
A	2,378,956.8	132.30	17,740.8
C	8,841,441.3	7724.47	249,752.8
AC	1,497,572.6	2040.20	41,440.3
Error	138,805,865	106,870	3,427,608.6
$A+E$	141,184,822	107,002.3	3,445,349.4
$C+E$	147,647,306	114,594.5	3,677,361.4
$AC+E$	140,303,437	108,910.2	3,469,048.9

for example, $\mathrm{SSA}_y = 2378956.8$, $\mathrm{SSA}_y + \mathrm{SSE}_y = 141{,}184{,}822$, and $\mathrm{SPAC} + \mathrm{SPE} = 3{,}469{,}048.9$.

By (16.58), (16.59), and (16.60), we have

$$\mathrm{SS}(A+E)_{y\cdot x} = 30{,}248{,}585, \quad \mathrm{SSE}_{y\cdot x} = 28{,}873{,}230,$$
$$\mathrm{SSA}_{y\cdot x} = 1{,}375{,}355.1.$$

Then by (16.61), we have

$$\begin{aligned} F &= \frac{\mathrm{SSA}_{y\cdot x}/(a-1)}{\mathrm{SSE}_{y\cdot x}/[ac(n-1)-1]} \\ &= \frac{1{,}375{,}355.1/1}{28{,}873{,}230/23} = \frac{1{,}375{,}355.1}{1{,}255{,}357.8} = 1.0956. \end{aligned}$$

The p value is .306, and we do not reject $H_0\colon \alpha_1 = \alpha_2$.

Similarly, for factor C, we have

$$F = \frac{766{,}750.1/2}{1{,}255{,}357.8} = .3054$$

with $p = .740$. For the interaction AC, we obtain

$$F = \frac{932{,}749.5/2}{1{,}255{,}357.8} = .3715$$

with $p = .694$. □

16.4.2 Test for Slope

To test the hypothesis $H_{02}\colon \beta = 0$, the sum of squares due to β is $(\mathrm{SPE})^2/\mathrm{SSE}_x$, and the F statistic is given by

$$F = \frac{(\mathrm{SPE})^2/\mathrm{SSE}_x}{\mathrm{SSE}_{y\cdot x}/[ac(n-1)-1]}, \tag{16.62}$$

which (under H_{02} and also H_{03} below) is distributed as $F[1, ac(n-1)-1]$.

Eample 16.4b. To test $H_0\colon \beta = 0$ for the farms data in Table 16.4, we use SPE and SSE_x from Table 16.5 and $\mathrm{SSE}_{y\cdot x}$ in Example 16.4a. Then by (16.62), we obtain

$$\begin{aligned} F &= \frac{(\mathrm{SPE})^2/\mathrm{SSE}_x}{\mathrm{SSE}_{y\cdot x}/[ac(n-1)-1]} \\ &= \frac{(3{,}427{,}608.6)^2/106{,}870}{1{,}255{,}357.8} = 87.5708. \end{aligned}$$

The p value is 2.63×10^{-9}, and $H_0\colon \beta = 0$ is rejected. □

16.4.3 Test for Homogeneity of Slopes

The test for homogeneity of slopes can be carried out separately for factor A, factor C, and the interaction AC. We describe the test for homogeneity of slopes among the levels of A. The hypothesis is

$$H_{03}: \beta_1 = \beta_2 = \cdots = \beta_a;$$

that is, the regression lines for the a levels of A are parallel. The intercepts, of course, may be different. To obtain a slope estimator $\hat{\beta}_i$ for the ith level of A, we define SSE_x and SPE for the ith level of A:

$$\text{SSE}_{x,i} = \sum_{j=1}^{c}\sum_{k=1}^{n}(x_{ijk} - \bar{x}_{ij.})^2, \quad \text{SPE}_i = \sum_{jk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.}). \tag{16.63}$$

Then $\hat{\beta}_i$ is obtained as

$$\hat{\beta}_i = \frac{\text{SPE}_i}{\text{SSE}_{x,i}},$$

and the sum of squares due to β_i is $(\text{SPE}_i)^2/\text{SSE}_{x,i}$.

By analogy to (16.46), the sum of squares for the full model in which the β_i's are different is given by

$$\text{SS}(F) = \text{SSE}_y - \sum_{i=1}^{a}\frac{(\text{SPE}_i)^2}{\text{SSE}_{x,i}},$$

and by analogy to (16.47), the sum of squares in the reduced model with a common slope is

$$\text{SS}(R) = \text{SSE}_y - \frac{(\text{SPE})^2}{\text{SSE}_x}.$$

Our test statistic for $H_{03}: \beta_1 = \beta_2 = \cdots = \beta_a$ is then similar to (16.49):

$$\begin{aligned} F &= \frac{[\text{SS}(R) - \text{SS}(F)]/(a-1)}{\text{SS}(F)/[ac(n-1)-1]} \\ &= \frac{\left[\sum_{i=1}^{a}(\text{SPE}_i)^2/\text{SSE}_{x,i} - (\text{SPE})^2/\text{SSE}_x\right]/(a-1)}{[\text{SSE}_y - \sum_{i=1}^{a}(\text{SPE}_i)^2/\text{SSE}_{x,i}]/[ac(n-1)-a]}, \end{aligned} \tag{16.64}$$

which (under H_{03}) is distributed as $F[a-1, ac(n-1)-a]$. The tests for homogeneity of slopes for C and AC are constructed in a similar fashion.

Example 16.4c. To test homogeneity of slopes for factor A, we first find $\hat{\beta}_1$ and $\hat{\beta}_2$ for the two levels of A:

$$\hat{\beta}_1 = \frac{\text{SPE}_1}{\text{SSE}_{x,1}} = \frac{2{,}141{,}839.8}{61{,}359.2} = 34.9066,$$

$$\hat{\beta}_2 = \frac{\text{SPE}_2}{\text{SSE}_{x,2}} = \frac{1{,}285{,}768.8}{45{,}510.8} = 28.2519.$$

Then

$$\text{SS}(F) = \text{SSE}_y - \sum_{i=1}^{2} \frac{(\text{SPE}_i)^2}{\text{SSE}_{x,i}} = 27{,}716{,}088.7,$$

$$\text{SS}(R) = \text{SSE}_y - \frac{(\text{SPE})^2}{\text{SSE}_x} = 28{,}873{,}230.$$

The difference is $\text{SS}(R) - \text{SS}(F) = 1{,}157{,}140.94$. Then by (16.64), we obtain

$$F = \frac{1{,}157{,}140.94/1}{27{,}716{,}088.7/22} = .9185.$$

The p value is .348, and we do not reject $H_0\colon \beta_1 = \beta_2$.

For homogeneity of slopes for factor C, we have

$$\hat{\beta}_1 = 23.2104, \quad \hat{\beta}_2 = 50.0851, \quad \hat{\beta}_3 = 31.6693,$$

$$F = \frac{9{,}506{,}034.16/2}{19{,}367{,}195.5/21} = 5.1537$$

with $p = .0151$. □

16.5 ONE-WAY MODEL WITH MULTIPLE COVARIATES

16.5.1 The Model

In some cases, the researcher has more than one covariate available. Note, however, that each covariate decreases the error degrees of freedom by 1, and therefore the inclusion of too many covariates may lead to loss of power.

For the one-way model with q covariates, we use (16.4):

$$\begin{aligned} y_{ij} &= \mu + \alpha_i + \beta_1 x_{ij1} + \beta_2 x_{ij2} + \cdots + \beta_q x_{ijq} + \varepsilon_{ij} \\ &= \mu + \alpha_i + \boldsymbol{\beta}' \mathbf{x}_{ij} + \varepsilon_{ij}, \qquad (16.65) \\ i &= 1, 2, \ldots, k, \;\; j = 1, 2, \ldots, n, \end{aligned}$$

where $\boldsymbol{\beta}' = (\beta_1, \beta_2, \ldots, \beta_q)$ and $\mathbf{x}_{ij} = (x_{ij1}, x_{ij2}, \ldots, x_{ijq})'$. For this model, we wish to test $H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ and $H_{02}: \boldsymbol{\beta} = \mathbf{0}$. We will also extend the model to allow for a different $\boldsymbol{\beta}$ vector in each of the k groups and test equality of these $\boldsymbol{\beta}$ vectors.

The model in (16.65) can be written in matrix notation as

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $\mathbf{Z}$ and $\boldsymbol{\alpha}$ are given following (16.3) and $\mathbf{X}\boldsymbol{\beta}$ is as given by (16.5):

$$\mathbf{X}\boldsymbol{\beta} = \begin{pmatrix} x_{111} & x_{112} & \cdots & x_{11q} \\ x_{121} & x_{122} & \cdots & x_{12q} \\ \vdots & \vdots & & \vdots \\ x_{kn1} & x_{kn2} & \cdots & x_{knq} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{pmatrix}.$$

The vector $\mathbf{y}$ is $kn \times 1$ and the matrix $\mathbf{X}$ is $kn \times q$. We can write $\mathbf{y}$ and $\mathbf{X}\boldsymbol{\beta}$ in partitioned form corresponding to the k groups:

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_k \end{pmatrix}, \quad \mathbf{X}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_k \end{pmatrix} \boldsymbol{\beta}, \tag{16.66}$$

where

$$\mathbf{y}_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{in} \end{pmatrix} \quad \text{and} \quad \mathbf{X}_i = \begin{pmatrix} x_{i11} & x_{i12} & \cdots & x_{i1q} \\ x_{i21} & x_{i22} & \cdots & x_{i2q} \\ \vdots & \vdots & & \vdots \\ x_{in1} & x_{in2} & \cdots & x_{inq} \end{pmatrix}.$$

16.5.2 Estimation

We first obtain $\mathbf{E}_{xx}$, $\mathbf{e}_{xy}$, and e_{yy} for use in $\hat{\boldsymbol{\beta}}$ and $\text{SSE}_{y \cdot x}$. By (16.16), $\mathbf{E}_{xx}$ can be expressed as

$$\mathbf{E}_{xx} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{X}.$$

Using $\mathbf{X}$ partitioned as in (16.66) and $\mathbf{I} - \mathbf{P}$ in the form given in (16.40), $\mathbf{E}_{xx}$ becomes

$$\mathbf{E}_{xx} = \sum_{i=1}^{k} \mathbf{X}_i' \left(\mathbf{I} - \frac{1}{n}\mathbf{J} \right) \mathbf{X}_i \tag{16.67}$$

(see Problem 16.10). Similarly, using $\mathbf{y}$ partitioned as in (16.66), $\mathbf{e}_{xy}$ is given by (16.16) as

$$\mathbf{e}_{xy} = \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y} = \sum_{i=1}^{k} \mathbf{X}_i'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{y}_i. \tag{16.68}$$

By (16.19) and (16.40), we have

$$e_{yy} = \mathbf{y}'(\mathbf{I} - \mathbf{P})\mathbf{y} = \sum_{i=1}^{k} \mathbf{y}_i'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{y}_i. \tag{16.69}$$

The elements of $\mathbf{E}_{xx}$, $\mathbf{e}_{xy}$, and e_{yy} are extensions of the sums of squares and products found in the three expressions in (16.25).

To examine the elements of the matrix $\mathbf{E}_{xx}$, we first note that $\mathbf{I} - (1/n)\mathbf{J}$ is symmetric and idempotent and therefore $\mathbf{X}_i'[\mathbf{I} - (1/n)\mathbf{J})]\mathbf{X}_i$ in (16.67) can be written as

$$\begin{aligned} \mathbf{X}_i'(\mathbf{I} - (1/n)\mathbf{J})\mathbf{X}_i &= \mathbf{X}_i'(\mathbf{I} - (1/n)\mathbf{J})'(\mathbf{I} - (1/n)\mathbf{J})\mathbf{X}_i \\ &= \mathbf{X}_{ci}'\mathbf{X}_{ci}, \end{aligned} \tag{16.70}$$

where $\mathbf{X}_{ci} = [\mathbf{I} - (1/n)\mathbf{J}]\mathbf{X}_i$ is the centered matrix

$$\mathbf{X}_{ci} = \begin{pmatrix} x_{i11} - \bar{x}_{i.1} & x_{i12} - \bar{x}_{i.2} & \cdots & x_{i1q} - \bar{x}_{i.q} \\ x_{i21} - \bar{x}_{i.1} & x_{i22} - \bar{x}_{i.2} & \cdots & x_{i2q} - \bar{x}_{i.q} \\ \vdots & \vdots & & \vdots \\ x_{in1} - \bar{x}_{i.1} & x_{in2} - \bar{x}_{i.2} & \cdots & x_{inq} - \bar{x}_{i.q} \end{pmatrix} \tag{16.71}$$

[see (7.33) and Problem 7.15], where $\bar{x}_{i.2}$, for example, is the mean of the second column of $\mathbf{X}_i$, that is, $\bar{x}_{i.2} = \sum_{j=1}^{n} x_{ij2}/n$. By Theorem 2.2c(i), the diagonal elements of $\mathbf{X}_{ci}'\mathbf{X}_{ci}$ are

$$\sum_{j=1}^{n} (x_{ijr} - \bar{x}_{i.r})^2, \qquad r = 1, 2, \ldots, q, \tag{16.72}$$

and the off-diagonal elements are

$$\sum_{j=1}^{n} (x_{ijr} - \bar{x}_{i.r})(x_{ijs} - \bar{x}_{i.s}), \qquad r \neq s. \tag{16.73}$$

By (16.67) and (16.72), the diagonal elements of $\mathbf{E}_{xx}$ are

$$\sum_{i=1}^{k}\sum_{j=1}^{n} (x_{ijr} - \bar{x}_{i.r})^2, \qquad r = 1, 2, \ldots, q, \tag{16.74}$$

and by (16.67) and (16.73), the off-diagonal elements are

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ijr}-\bar{x}_{i.r})(x_{ijs}-\bar{x}_{i.s}), \quad r \neq s. \tag{16.75}$$

These are analogous to $e_{xx} = \sum_{ij}(x_{ij}-\bar{x}_{i.})^2$ in (16.25).

To examine the elements of the vector $\mathbf{e}_{xy}$, we note that by an argument similar to that used to obtain (16.70), $\mathbf{X}_i'[\mathbf{I}-(1/n)\mathbf{J}]\mathbf{y}_i$ in (16.68) can be written as

$$\mathbf{X}_i'[\mathbf{I}-(1/n)\mathbf{J})\mathbf{y}_i = \mathbf{X}_i'[\mathbf{I}-(1/n)\mathbf{J}]'[\mathbf{I}-(1/n)\mathbf{J}]\mathbf{y}_i = \mathbf{X}_{ci}'\mathbf{y}_{ci},$$

where $\mathbf{X}_{ci}$ is as given in (16.71) and

$$\mathbf{y}_{ci} = \begin{pmatrix} y_{i1}-\bar{y}_{i.} \\ y_{i2}-\bar{y}_{i.} \\ \vdots \\ y_{in}-\bar{y}_{i.} \end{pmatrix}$$

with $\bar{y}_{i.} = \sum_{j=1}^{n} y_{ij}/n$. Thus the elements of $\mathbf{X}_{ci}'\mathbf{y}_{ci}$ are of the form

$$\sum_{j=1}^{n}(x_{ijr}-\bar{x}_{i.r})(y_{ij}-\bar{y}_{i.}) \quad r = 1, 2, \ldots, q,$$

and by (16.68), the elements of $\mathbf{e}_{xy}$ are

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ijr}-\bar{x}_{i.r})(y_{ij}-\bar{y}_{i.}) \quad r = 1, 2, \ldots, q.$$

Similarly, e_{yy} in (16.69) can be written as

$$\begin{aligned} e_{yy} &= \sum_{i=1}^{k}\mathbf{y}_i'\left(\mathbf{I}-\frac{1}{n}\mathbf{J}\right)'\left(\mathbf{I}-\frac{1}{n}\mathbf{J}\right)\mathbf{y}_i = \sum_{i=1}^{k}\mathbf{y}_{ci}'\mathbf{y}_{ci} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{i.})^2. \end{aligned} \tag{16.76}$$

By (16.15), we obtain

$$\hat{\boldsymbol{\beta}} = \mathbf{E}_{xx}^{-1}\mathbf{e}_{xy},$$

where $\mathbf{E}_{xx}$ is as given by (16.67) and $\mathbf{e}_{xy}$ is as given by (16.68). Likewise, by (16.18), we have

$$\mathrm{SSE}_{y\cdot x} = e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}, \tag{16.77}$$

where e_{yy} is as given in (16.69) or (16.76). The degrees of freedom of $\mathrm{SSE}_{y\cdot x}$ are $k(n-1) - q$.

By (16.11) and (13.12), we obtain

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \hat{\boldsymbol{\alpha}}_0 - (\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{X}\hat{\boldsymbol{\beta}} \\ &= \begin{pmatrix} 0 \\ \bar{y}_{1.} \\ \bar{y}_{2.} \\ \vdots \\ \bar{y}_{k.} \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{1.} \\ \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{2.} \\ \vdots \\ \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{k.} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{y}_{1.} - \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{1.} \\ \bar{y}_{2.} - \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{2.} \\ \vdots \\ \bar{y}_{k.} - \hat{\boldsymbol{\beta}}'\bar{\mathbf{x}}_{k.} \end{pmatrix}\end{aligned} \tag{16.78}$$

$$= \begin{pmatrix} \bar{y}_{1.} - (\hat{\beta}_1\bar{x}_{1.1} + \hat{\beta}_2\bar{x}_{1.2} + \cdots + \hat{\beta}_q\bar{x}_{1.q}) \\ \bar{y}_{2.} - (\hat{\beta}_1\bar{x}_{2.1} + \hat{\beta}_2\bar{x}_{2.2} + \cdots + \hat{\beta}_q\bar{x}_{2.q}) \\ \vdots \\ \bar{y}_{k.} - (\hat{\beta}_1\bar{x}_{k.1} + \hat{\beta}_2\bar{x}_{k.2} + \cdots + \hat{\beta}_q\bar{x}_{k.q}) \end{pmatrix}. \tag{16.79}$$

16.5.3 Testing Hypotheses

16.5.3.1 Treatments

To test

$$H_{01}\colon \alpha_1 = \alpha_2 = \cdots = \alpha_k$$

adjusted for the q covariates, we use the full–reduced-model approach as in Section 16.3.3.1. The full model is given by (16.65), and the reduced model (with $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$) is

$$\begin{aligned} y_{ij} &= \mu + \alpha + \boldsymbol{\beta}'\mathbf{x}_{ij} + \varepsilon_{ij} \\ &= \mu^* + \boldsymbol{\beta}'\mathbf{x}_{ij} + \varepsilon_{ij}, \end{aligned} \tag{16.80}$$

which is essentially the same as the multiple regression model (7.3). By (7.37) and (7.39) and by analogy with (16.33),

$$\mathrm{SSE}_{\mathrm{rd}} = \mathrm{SST}_{y\cdot x} = t_{yy} - t'_{xy}\mathbf{T}_{xx}^{-1}\mathbf{t}_{xy}, \tag{16.81}$$

where t_{yy} is

$$t_{yy} = \sum_{ij} (y_{ij} - \bar{y}_{..})^2,$$

the elements of $\mathbf{t}_{xy}$ are

$$\sum_{ij} (x_{ijr} - \bar{x}_{..r})(y_{ij} - \bar{y}_{..}), \quad r = 1, 2, \ldots, q,$$

and the elements of $\mathbf{T}_{xx}$ are

$$\sum_{ij} (x_{ijr} - \bar{x}_{..r})(x_{ijs} - \bar{x}_{..s}), \quad r = 1, 2, \ldots, q, \quad s = 1, 2, \ldots, q.$$

Thus, by analogy with (16.30), we use (16.81) and (16.77) to obtain

$$\begin{aligned} \mathrm{SS}(\alpha|\mu, \beta) &= \mathrm{SST}_{y\cdot x} - \mathrm{SSE}_{y\cdot x} \\ &= t_{yy} - \mathbf{t}'_{xy}\mathbf{T}_{xx}^{-1}\mathbf{t}_{xy} - e_{yy} + \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} \\ &= \sum_{ij} (y_{ij} - \bar{y}_{..})^2 - \sum_{ij} (y_{ij} - \bar{y}_{i.})^2 - \mathbf{t}'_{xy}\mathbf{T}_{xx}^{-1}\mathbf{t}_{xy} + \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} \\ &= n\sum_{i} (\bar{y}_{i.} - \bar{y}_{..})^2 - \mathbf{t}'_{xy}\mathbf{T}_{xx}^{-1}\mathbf{t}_{xy} + \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}, \end{aligned} \tag{16.82}$$

which has $k-1$ degrees of freedom (see Problem 16.13). We display these sums of squares and products in Table 16.6.

The test statistic for $H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ is

$$F = \frac{\mathrm{SS}(\alpha|\mu, \beta)/(k-1)}{\mathrm{SSE}_{y\cdot x}/[k(n-1) - q]}, \tag{16.83}$$

which (under H_{01}) is distributed as $F[k-1, k(n-1) - q]$.

TABLE 16.6 Analysis-of-Covariance Table for Testing H_{01}: $\alpha_1 = \alpha_2 = \cdots = \alpha_k$ in the One-Way Model with q Covariates

Source	SS Adjusted for the Covariate	Adjusted df
Treatments	$\mathrm{SS}(\alpha\|\mu, \beta) = \mathrm{SST}_{y\cdot x} - \mathrm{SSE}_{y\cdot x}$	$k - 1$
Error	$\mathrm{SSE}_{y\cdot x} = e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}$	$k(n-1) - q$
Total	$\mathrm{SST}_{y\cdot x} = t_{yy} - \mathbf{t}'_{xy}\mathbf{T}_{xx}^{-1}\mathbf{t}_{xy}$	$kn - q - 1$

16.5.3.2 Slope Vector

To test

$$H_{02}: \boldsymbol{\beta} = \mathbf{0},$$

the sum of squares is given by (16.22) as

$$\mathrm{SSH} = \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy},$$

where $\mathbf{E}_{xx}$ is as given by (16.67) and $\mathbf{e}_{xy}$ is the same as in (16.68). The F statistic is then

$$F = \frac{\mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}/q}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-q]}, \tag{16.84}$$

which is distributed as $F[q, k(n-1)-q]$ if $H_{02}: \boldsymbol{\beta} = \mathbf{0}$ is true.

16.5.3.3 Homogeneity of Slope Vectors

The tests of $H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ and $H_{02}: \boldsymbol{\beta} = \mathbf{0}$ assume a common coefficient vector $\boldsymbol{\beta}$ for all k groups. To check this assumption, we can extend the model (16.65) to obtain a full model allowing for different slope vectors:

$$y_{ij} = \mu + \alpha_i + \boldsymbol{\beta}'_i\mathbf{x}_{ij} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n. \tag{16.85}$$

The reduced model with a single slope vector is given by (16.65). We now develop a test for the hypothesis

$$H_{03}: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_k,$$

that is, that the k regression planes (for the k treatments) are parallel.

By extension of (16.46) and (16.47), we have

$$\mathrm{SSE}(F)_{y\cdot x} = e_{yy} - \sum_{i=1}^{k} \mathbf{e}'_{xy,i}\mathbf{E}_{xx,i}^{-1}\mathbf{e}_{xy,i}, \tag{16.86}$$

$$\mathrm{SSE}(R)_{y\cdot x} = e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}, \tag{16.87}$$

where

$$\mathbf{E}_{xx,i} = \mathbf{X}'_i[\mathbf{I} - (1/n)\mathbf{J}]\mathbf{X}_i \quad \text{and} \quad \mathbf{e}_{xy,i} = \mathbf{X}'_i[\mathbf{I} - (1/n)\mathbf{J}]\mathbf{y}_i$$

are terms in the summations in (16.67) and (16.68). The degrees of freedom for $\mathrm{SSE}(F)_{y\cdot x}$ and $\mathrm{SSE}(R)_{y\cdot x}$ are $k(n-1) - kq = k(n-q-1)$ and $k(n-1) - q$, respectively. Note that $\text{SSE}(R)_{y\cdot x}$ in (16.87) is the same as $\text{SSE}_{y\cdot x}$ in (16.77). The estimator of $\boldsymbol{\beta}_i$ for the ith group is

$$\hat{\boldsymbol{\beta}}_i = \mathbf{E}_{xx,i}^{-1}\mathbf{e}_{xy,i}. \tag{16.88}$$

By analogy to (16.48), the sum of squares for testing $H_{03}\colon \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_k$ is $\text{SSE}(R)_{y\cdot x} - \text{SSE}(F)_{y\cdot x} = \sum_{i=1}^k \mathbf{e}_{xy,i}\mathbf{E}_{xx,i}^{-1}\mathbf{e}_{xy,i} - \mathbf{e}_{xy}'\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}$, which has $k(n-1) - q - [k(n-1) - kq] = q(k-1)$ degrees of freedom. The test statistic for $H_{03}\colon \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_k$ is

$$F = \frac{[\text{SSE}(R)_{y\cdot x} - \text{SSE}(F)_{y\cdot x}]/q(k-1)}{\text{SSE}(F)_{y\cdot x}/k(n-q-1)}, \tag{16.89}$$

which is distributed as $F[q(k-1), k(n-q-1)]$ if H_{03} is true. Note that if n is not large, $n-q-1$ may be small, and the test will have low power.

Example 16.5. In Table 16.7, we have instructor rating y and two course ratings x_1 and x_2 for five instructors in each of three courses (Morrison 1983, p. 470).

We first find $\hat{\boldsymbol{\beta}}$ and $\text{SSE}_{y\cdot x}$. Using (16.67), (16.68), and (16.69), we obtain

$$\mathbf{E}_{xx} = \begin{pmatrix} 1.0619 & 0.6791 \\ 0.6791 & 1.2363 \end{pmatrix}, \quad \mathbf{e}_{xy} = \begin{pmatrix} 1.0229 \\ 1.9394 \end{pmatrix}, \quad e_{xy} = 3.6036.$$

Then by (16.15), we obtain

$$\hat{\boldsymbol{\beta}} = \mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = \begin{pmatrix} -0.0617 \\ 1.6026 \end{pmatrix}.$$

By (16.77) and (16.81), we have

$$\text{SSE}_{y\cdot x} = .5585, \quad \text{SST}_{y\cdot x} = .7840.$$

TABLE 16.7 Instructor Rating y and Two Course Ratings x_1 and x_2 in Three Courses

Course								
1			2			3		
y	x_1	x_2	y	x_1	x_2	y	x_1	x_2
2.14	2.71	2.50	2.77	2.29	2.45	1.11	1.74	1.82
1.34	2.00	1.95	1.23	1.83	1.64	2.41	2.19	2.54
2.50	2.66	2.69	1.37	1.78	1.83	1.74	1.40	2.23
1.40	2.80	2.00	1.52	2.18	2.24	1.15	1.80	1.82
1.90	2.38	2.30	1.81	2.14	2.11	1.66	2.17	2.35

Then by (16.82), we see that

$$\mathrm{SS}(\alpha|\mu, \beta) = \mathrm{SST}_{y\cdot x} - \mathrm{SSE}_{y\cdot x} = .2254.$$

The F statistic for testing $H_0\colon \alpha_1 = \alpha_2 = \alpha_3$ is given by (16.83) as

$$F = \frac{\mathrm{SS}(\alpha|\mu, \beta)/(k-1)}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-q]} = \frac{.2254/2}{.5585/10} = 2.0182, \qquad p = .184.$$

To test $H_{02}\colon \boldsymbol{\beta} = \mathbf{0}$, we use (16.84) to obtain

$$F = \frac{\mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}/q}{\mathrm{SSE}_{y\cdot x}/[k(n-1)-q]} = 27.2591, \qquad p = 8.95 \times 10^{-5}.$$

Before testing homogeneity of slope vectors, $H_0\colon \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3$, we first obtain estimates of $\boldsymbol{\beta}_1,\ \boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$ using (16.88):

$$\hat{\boldsymbol{\beta}}_1 = \mathbf{E}_{xx,1}^{-1}\mathbf{e}_{xy,1} = \begin{pmatrix} .4236 & .1900 \\ .1900 & .4039 \end{pmatrix}^{-1} \begin{pmatrix} .2786 \\ .6254 \end{pmatrix} = \begin{pmatrix} -0.0467 \\ 1.5703 \end{pmatrix},$$

$$\hat{\boldsymbol{\beta}}_2 = \begin{pmatrix} .2037 & .2758 \\ .2758 & .4161 \end{pmatrix}^{-1} \begin{pmatrix} .4370 \\ .6649 \end{pmatrix} = \begin{pmatrix} -0.1781 \\ 1.7159 \end{pmatrix},$$

$$\hat{\boldsymbol{\beta}}_3 = \begin{pmatrix} .4346 & .2133 \\ .2133 & .4163 \end{pmatrix}^{-1} \begin{pmatrix} .3073 \\ .6492 \end{pmatrix} = \begin{pmatrix} -0.0779 \\ 1.5993 \end{pmatrix}.$$

Then by (16.86) and (16.87), we obtain

$$\mathrm{SSE}(F)_{y\cdot x} = e_{yy} - \sum_{i=1}^{3} \mathbf{e}'_{xy,i}\mathbf{E}_{xx,i}^{-1}\mathbf{e}_{xy,i} = .55725,$$

$$\mathrm{SSE}(R)_{y\cdot x} = e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy} = .55855.$$

The F statistic for testing $H_0\colon \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3$ is then given by (16.89) as

$$F = \frac{[\mathrm{SSE}(R)_{y\cdot x} - \mathrm{SSE}(F)_{y\cdot x}]/q(k-1)}{\mathrm{SSE}(F)_{y\cdot x}/k(n-q-1)}$$

$$= \frac{.0012993/4}{.55725/6} = .003498.$$

□

16.6 ANALYSIS OF COVARIANCE WITH UNBALANCED MODELS

The results in previous sections are for balanced ANOVA models to which covariates have been added. The case in which the ANOVA model is itself unbalanced before the addition of a covariate was treated by Hendrix et al. (1982), who also discussed heterogeneity of slopes. The following approach, based on the cell means model of Chapter 15, was suggested by Bryce (1998).

For an analysis-of-covariance model with a single covariate and a common slope β, we extend the cell means model (15.3) or (15.18) as

$$\mathbf{y} = (\mathbf{W}, \mathbf{x})\begin{pmatrix} \boldsymbol{\mu} \\ \beta \end{pmatrix} + \boldsymbol{\varepsilon} = \mathbf{W}\boldsymbol{\mu} + \beta\mathbf{x} + \boldsymbol{\varepsilon}. \tag{16.90}$$

This model allows for imbalance in the n_{ij}'s as well as the inherent imbalance in analysis of covariance models [see Bingham and Feinberg (1982) and a comment following (16.34)]. The vector $\boldsymbol{\mu}$ contains the means for a one-way model as in (15.2), a two-way model as in (15.17), or some other model. Hypotheses about main effects, interactions, the covariate, or other effects can be tested by using contrasts on $\begin{pmatrix} \boldsymbol{\mu} \\ \beta \end{pmatrix}$ as in Section 15.3.

The hypothesis $H_{02}\colon \beta = 0$ can be expressed in the form $H_{02}\colon (0, \ldots, 0, 1)\begin{pmatrix} \boldsymbol{\mu} \\ \beta \end{pmatrix} = 0$. To test H_{02}, we use a statistic analogous to (15.29) or (15.32). To test homogeneity of slopes, $H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_k$ for a one-way model (or $H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_a$ for the slopes of the a levels of factor A in a two-way model, and so on), we expand the model (16.90) to include the β_i's

$$\mathbf{y} = (\mathbf{W}, \mathbf{W}_x)\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\beta} \end{pmatrix} + \boldsymbol{\varepsilon} = \mathbf{W}\boldsymbol{\mu} + \mathbf{W}_x\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{16.91}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)'$ and $\mathbf{W}_x$ has a single value of x_{ij} in each row and all other elements are 0s. (The x_{ij} in $\mathbf{W}_x$ is in the same position as the corresponding 1 in $\mathbf{W}$.) Then $H_{03}\colon \beta_1 = \beta_2 = \cdots = \beta_k$ can be expressed as $H_{03}\colon (\mathbf{O}, \mathbf{C})\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\beta} \end{pmatrix} = \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$, where $\mathbf{C}$ is a $(k-1) \times k$ matrix of rank $k-1$ such that $\mathbf{Cj} = \mathbf{0}$. We can test $H_{03}\colon \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ using a statistic analogous to (15.33).

Constraints on the μ's and the β's can be introduced by inserting nonsingular matrices $\mathbf{A}$ and $\mathbf{A}_x$ into (16.91):

$$\mathbf{y} = \mathbf{W}\mathbf{A}^{-1}\mathbf{A}\boldsymbol{\mu} + \mathbf{W}_x\mathbf{A}_x^{-1}\mathbf{A}_x\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \tag{16.92}$$

The matrix $\mathbf{A}$ has the form illustrated in (15.37) for constraints on the μ's. The matrix $\mathbf{A}_x$ provides constraints on the β's. For example, if

$$\mathbf{A}_x = \begin{pmatrix} \mathbf{j}' \\ \mathbf{C} \end{pmatrix},$$

where $\mathbf{C}$ is a $(k-1) \times k$ matrix of rank $k-1$ such that $\mathbf{Cj} = \mathbf{0}$ as above, then the model (16.92) has a common slope. In some cases, the matrices $\mathbf{A}$ and $\mathbf{A}_x$ would be the same.

PROBLEMS

16.1 Show that if the columns of $\mathbf{X}$ are linearly independent of those of $\mathbf{Z}$, then $\mathbf{X}'(\mathbf{I}-\mathbf{P})\mathbf{X}$ is nonsingular, as noted preceding (16.14).

16.2 (**a**) Show that $\text{SSE}_{y \cdot x} = e_{yy} - \mathbf{e}'_{xy}\mathbf{E}_{xx}^{-1}\mathbf{e}_{xy}$ as in (16.18).

(**b**) Show that $e_{yy} = \mathbf{y}'(\mathbf{I}-\mathbf{P})\mathbf{y}$ as in (16.19).

16.3 Show that for $H_0\colon \boldsymbol{\beta} = \mathbf{0}$, we have $\text{SSH} = \hat{\boldsymbol{\beta}}'\mathbf{X}'(\mathbf{I}-\mathbf{P})\mathbf{X}\hat{\boldsymbol{\beta}}$ as in (16.21).

16.4 Show that $\hat{\boldsymbol{\alpha}} = (0,\ \bar{y}_{1.} - \hat{\beta}\bar{x}_{1.}, \ldots, \bar{y}_{k.} - \hat{\beta}\bar{x}_{k.})'$ as in (16.24).

16.5 Show that $e_{xx} = \sum_{ij}(x_{ij} - \bar{x}_{i.})^2$, $e_{xy} = \sum_{ij}(x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})$, and $e_{yy} = \sum_{ij}(y_{ij} - \bar{y}_{i.})^2$, as in (16.25).

16.6 (**a**) Show that $\mathbf{E}_{xx}$ has the form shown in (16.41).

(**b**) Show that $\mathbf{e}_{xy}$ has the form shown in (16.43).

16.7 Show that the sums of products in (16.52) and (16.53) can be written as $\sum_{ijk}(x_{ijk} - \bar{x}_{ij.})(y_{ijk} - \bar{y}_{ij.}) = \sum_{ijk} x_{ijk}y_{ijk} - n\sum_{ij}\bar{x}_{ij.}\bar{y}_{ij.}$ and $\sum_{ijk}(x_{ijk} - \bar{x}_{...})(y_{ijk} - \bar{y}_{...}) = \sum_{ijk} x_{ijk}y_{ijk} - acn\bar{x}_{...}\bar{y}_{...}$.

16.8 Show that the "treatment sum of products" $\sum_{ij} x_{ij.}y_{ij.}/n - x_{...}y_{...}/acn$ can be partitioned into the three sums of products in (16.57).

16.9 (**a**) Express the sums of squares and test statistic for factor C in a form analogous to those for factor A in (16.58), (16.60), and (16.61).

(**b**) Express the sums of squares and test statistic for the interaction AC in a form analogous to those for factor A in (16.58), (16.60), and (16.61).

16.10 (**a**) Show that $\mathbf{E}_{xx} = \sum_{i=1}^{k} \mathbf{X}'_i[\mathbf{I} - (1/n)\mathbf{J}]\mathbf{X}_i$ as in (16.67).

(**b**) Show that $\mathbf{e}_{xy} = \sum_{i=1}^{k} \mathbf{X}'_i[\mathbf{I} - (1/n)\mathbf{J}]\mathbf{y}_i$ as in (16.68).

(**c**) Show that $e_{yy} = \sum_{i=1}^{k} \mathbf{y}'_i[\mathbf{I} - (1/n)\mathbf{J}]\mathbf{y}_i$ as in (16.69).

16.11 Show that the elements of $\mathbf{X}'_{ic}\mathbf{X}_{ic}$ are given by (16.72) and (16.73).

16.12 Show that $\hat{\boldsymbol{\alpha}}$ has the form given in (16.78).

16.13 Show that $\sum_{ij}(y_{ij} - \bar{y}_{..})^2 - \sum_{ij}(y_{ij} - \bar{y}_{i.})^2 = n\sum_i(\bar{y}_{i.} - \bar{y}_{..})^2$ as in (16.82).

16.14 In Table 16.8 we have the weight gain y and initial weight x of pigs under four diets (treatments).

(**a**) Estimate β.

(**b**) Test $H_0\colon \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ using F in (16.31).

TABLE 16.8 Gain in Weight *y* and Initial Weight *x* of Pigs

Treatment							
1		2		3		4	
y	x	y	x	y	x	y	x
165	30	180	24	156	34	201	41
170	27	169	31	189	32	173	32
130	20	171	20	138	35	200	30
156	21	161	26	190	35	193	35
167	33	180	20	160	30	142	28
151	29	170	25	172	29	189	36

Source: Ostle and Malone (1988, p. 445).

(**c**) Test $H_0\colon \beta = 0$ using F in (16.36).

(**d**) Estimate β_1, β_2, β_3, and β_4 and test homogeneity of slopes $H_0\colon \beta_1 = \beta_2 = \beta_3 = \beta_4$ using F in (16.49).

16.15 In a study to investigate the effect of income and geographic area of residence on daily calories consumed, three people were chosen at random in each of the 18 income–zone combinations. Their daily caloric intake y and age x are recorded in Table 16.9.

(**a**) Obtain the sums of squares and products listed in Table 16.3, where zone is factor A and income group is factor C.

(**b**) Calculate $\text{SS}(A + E)_{y\cdot x}$, $\text{SSE}_{y\cdot x}$, and $\text{SSA}_{y\cdot x}$ using (16.58), (16.59), and (16.60). For factor A calculate F by (16.61) for $H_0\colon \alpha_1 = \alpha_2 = \alpha_3$. Similarly, obtain the F statistic for factor C and the interaction.

(**c**) Using SPE, SSE_x, and $\text{SSE}_{y\cdot x}$ from parts (a) and (b), calculate the F statistic to test $H_0\colon \beta = 0$.

(**d**) Calculate the separate slopes for the three levels of factor A, find SS(F) and SS(R), and test for homogeneity of slopes. Repeat for factor C.

16.16 In a study to investigate differences in ability to distinguish aurally between environmental sounds, 10 male subjects and 10 female subjects were assigned randomly to each of two levels of treatment (experimental and control). The variables were x = pretest score and y = posttest score on auditory discrimination. The data are given in Table 16.10.

We use the posttest score y as the dependent variable and the pretest score x as the covariate. This gives the same result as using the gain score (post–pre) as the dependent variable and the pretest as the covariate (Hendrix et al. 1978).

(**a**) Obtain the sums of squares and products listed in Table 16.3, where treatment is factor A and gender is factor C.

TABLE 16.9 Caloric Intake *y* and Age *x* for People Classified by Geographic Zone and Income Group

Income Group	Zone 1		Zone 2		Zone 3	
	y	*x*	*y*	*x*	*y*	*x*
1	1911	46	1318	80	1127	74
	1560	66	1541	67	1509	71
	2639	38	1350	73	1756	60
2	1034	50	1559	58	1054	83
	2096	33	1260	74	2238	47
	1356	44	1772	44	1599	71
3	2130	35	2027	32	1479	56
	1878	45	1414	51	1837	40
	1152	59	1526	34	1437	66
4	1297	68	1938	33	2136	31
	2093	43	1551	40	1765	56
	2035	59	1450	39	1056	70
5	2189	33	1183	54	1156	47
	2078	36	1967	36	2660	43
	1905	38	1452	53	1474	50
6	1156	57	2599	35	1015	63
	1809	52	2355	64	2555	34
	1997	44	1932	79	1436	54

Source: Ostle and Mensing (1975, p. 482).

TABLE 16.10 Pretest Score *x* and Posttest Score *y* on Auditory Discrimination

Male				Female			
Exp.[a]		Control		Exp.		Control	
x	*y*	*x*	*y*	*x*	*y*	*x*	*y*
58	71	35	49	64	71	68	70
57	69	31	69	39	71	52	64
63	71	54	69	69	71	53	67
66	70	65	65	56	76	43	63
45	65	54	63	67	71	54	63
51	69	37	55	39	65	35	53
62	69	64	66	32	66	62	65
58	66	69	69	62	70	67	69
52	61	70	69	64	68	51	68
59	63	39	57	66	68	42	61

[a]Experimental.

Source: Hendrix (1967, pp. 154–157).

TABLE 16.11 Initial Age x_1, Initial Weight x_2, and Rate of Gain y of 40 Pigs

Treatment 1			Treatment 2			Treatment 3			Treatment 4		
x_1	x_2	y	x_1	x_2	y	x_1	x_2	y	x_1	x_2	y
78	61	1.40	78	74	1.61	78	80	1.67	77	62	1.40
90	59	1.79	99	75	1.31	83	61	1.41	71	55	1.47
94	76	1.72	80	64	1.12	79	62	1.73	78	62	1.37
71	50	1.47	75	48	1.35	70	47	1.23	70	43	1.15
99	61	1.26	94	62	1.29	85	59	1.49	95	57	1.22
80	54	1.28	91	42	1.24	83	42	1.22	96	51	1.48
83	57	1.34	75	52	1.29	71	47	1.39	71	41	1.31
75	45	1.55	63	43	1.43	66	52	1.39	63	40	1.27
62	41	1.57	62	50	1.29	67	40	1.56	62	45	1.22
67	40	1.26	67	40	1.26	67	40	1.36	67	39	1.36

Source: Snedecor and Cochran (1967, p. 440).

(**b**) Calculate $\text{SS}(A+E)_{y\cdot x}$, $\text{SSE}_{y\cdot x}$, and $\text{SSA}_{y\cdot x}$ using (16.58), (16.59), and (16.60). For factor A calculate F by (16.61) for $H_0\colon \alpha_1 = \alpha_2$. Similarly, obtain the F statistic for factor C and the interaction.

(**c**) Using SPE, SSE_x, and $\text{SSE}_{y\cdot x}$ from parts (a) and (b), calculate the F statistic to test $H_0\colon \beta = 0$.

(**d**) Calculate the separate slopes for the two levels of factor A, find SS(F) and SS(R), and test for homogeneity of slopes. Repeat for factor C.

16.17 In an experiment comparing four diets (treatments), the weight gain y (pounds per day) of pigs was recorded along with two covariates, initial age x_1 (days) and initial weight x_2 (pounds). The data are presented in Table 16.11.

(**a**) Using (16.67), (16.68), and (16.69), find $\mathbf{E}_{xx}$, $\mathbf{e}_{xy}$, and e_{yy}. Find $\hat{\boldsymbol{\beta}}$.

(**b**) Using (16.77), (16.81), and (16.82), find $\text{SSE}_{y\cdot x}$, $\text{SST}_{y\cdot x}$, and $\text{SS}(\alpha|\mu, \beta)$. Then test $H_0\colon \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, adjusted for the covariates, using the F statistic in (16.83).

(**c**) Test $H_0\colon \boldsymbol{\beta} = \mathbf{0}$ using (16.84).

(**d**) Find $\hat{\boldsymbol{\beta}}_1$, $\hat{\boldsymbol{\beta}}_2$, $\hat{\boldsymbol{\beta}}_3$, and $\hat{\boldsymbol{\beta}}_4$ using (16.88). Find $\text{SSE}(F)_{y\cdot x}$ and $\text{SSE}(R)_{y\cdot x}$ using (16.86) and (16.87). Test $H_0\colon \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3 = \boldsymbol{\beta}_4$ using (16.89).