14 Two-Way Analysis-of-Variance: Balanced Case

The two-way model without interaction has been illustrated in Section 12.1.2, Example 12.2.2b, and Section 12.8. In this chapter, we consider the two-way ANOVA model with interaction. In Section 14.1 we discuss the model and attendant assumptions. In Section 14.2 we consider estimable functions involving main effects and interactions. In Section 14.3 we discuss estimation of the parameters, including solutions to the normal equations using side conditions and also using a generalized inverse. In Section 14.4 we develop a hypothesis test for the interaction using a full–reduced model, and we obtain tests for main effects using the general linear hypothesis as well as the full–reduced-model approach. In Section 14.5 we derive expected mean squares from the basic definition and also using a general linear hypothesis approach. Throughout this chapter we consider only the balanced two-way model. The unbalanced case is covered in Chapter 15.

14.1 THE TWO-WAY MODEL

The two-way balanced model can be specified as follows:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk} \tag{14.1}$$

$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad k = 1, 2, \ldots, n.$$

The effect of factor A at the ith level is α_i, and the term β_j is due to the jth level of factor B. The term γ_{ij} represents the interaction AB between the ith level of A and the jth level of B. If an interaction is present, the difference $\alpha_1 - \alpha_2$, for example, is not estimable and the hypothesis H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_a$ cannot be tested. In Section 14.4, we discuss modifications of this hypothesis that are testable.

There are two experimental situations in which the model in (14.1) may arise. In the first setup, factors A and B represent two types of treatment, for example, various amounts of nitrogen and potassium applied in an agricultural experiment. We apply

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each of the ab combinations of the levels of A and B to n randomly selected experimental units. In the second situation, the populations exist naturally, for example, gender (males and females) and political preference (Democrats, Republicans, and Independents). A random sample of n observations is obtained from each of the ab populations.

Additional assumptions that form part of the model are the following:

1. $E(\varepsilon_{ijk}) = 0$ for all i, j, k.
2. $\text{var}(\varepsilon_{ijk}) = \sigma^2$ for all i, j, k.
3. $\text{cov}(\varepsilon_{ijk}, \varepsilon_{rst}) = 0$ for $(i, j, k) \neq (r, s, t)$.
4. Another assumption that we sometimes add to the model is that ε_{ijk} is $N(0, \sigma^2)$ for all i, j, k.

From assumption 1, we have $E(y_{ijk}) = \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$, and we can rewrite the model in the form

$$\begin{gathered} y_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \\ i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad k = 1, 2, \ldots, n, \end{gathered} \tag{14.2}$$

where $\mu_{ij} = E(y_{ijk})$ is the mean of a random observation in the (ij)th cell.

In the next section, we consider estimable functions of the parameters α_i, β_j, and γ_{ij}.

14.2 ESTIMABLE FUNCTIONS

In the first part of this section, we use $a = 3$, $b = 2$, and $n = 2$ for expositional purposes. For this special case, the model in (14.1) becomes

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \quad i = 1, 2, 3, \quad j = 1, 2, \quad k = 1, 2. \tag{14.3}$$

The 12 observations in (14.3) can be expressed in matrix form as

$$\begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{311} \\ y_{312} \\ y_{321} \\ y_{322} \end{pmatrix} = \left(\begin{array}{c|ccc|cc|cccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{31} \\ \gamma_{32} \end{pmatrix} + \begin{pmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{122} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \\ \varepsilon_{311} \\ \gamma_{312} \\ \varepsilon_{321} \\ \varepsilon_{322} \end{pmatrix} \tag{14.4}$$

or

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $\mathbf{y}$ is 12×1, $\mathbf{X}$ is 12×12, and $\boldsymbol{\beta}$ is 12×1. (If we added another replication, so that $n = 3$, then $\mathbf{y}$ would be 18×1, $\mathbf{X}$ would be 18×12, but $\boldsymbol{\beta}$ would remain 12×1.) The matrix $\mathbf{X}'\mathbf{X}$ is given by

$$\mathbf{X}'\mathbf{X} = \left(\begin{array}{c|ccc|cc|cccccc}
12 & 4 & 4 & 4 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
4 & 4 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
4 & 0 & 4 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
4 & 0 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
6 & 2 & 2 & 2 & 6 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
6 & 2 & 2 & 2 & 0 & 6 & 0 & 2 & 0 & 2 & 0 & 2 \\
\hline
2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right). \tag{14.5}$$

The partitioning in $\mathbf{X}'\mathbf{X}$ corresponds to that in $\mathbf{X}$ in (14.4), where there is a column for μ, three columns for the three α's, two columns for the two β's, and six columns for the six γ's.

In both $\mathbf{X}$ and $\mathbf{X}'\mathbf{X}$, the first six columns can be obtained as linear combinations of the last six columns, which are clearly linearly independent. Hence rank($\mathbf{X}$)= rank($\mathbf{X}'\mathbf{X}$) = 6 [in general, rank($\mathbf{X}$) = ab].

Since rank($\mathbf{X}$) = 6, we can find six linearly independent estimable functions of the parameters (see Theorem 12.2c). By Theorem 12.2b, we can obtain these estimable functions from $\mathbf{X}\boldsymbol{\beta}$. Using rows 1, 3, 5, 7, 9, and 11 of $E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$, we obtain $E(y_{ijk}) = \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2$:

$$\begin{aligned}
\mu_{11} &= \mu + \alpha_1 + \beta_1 + \gamma_{11}, & \mu_{12} &= \mu + \alpha_1 + \beta_2 + \gamma_{12} \\
\mu_{21} &= \mu + \alpha_2 + \beta_1 + \gamma_{21}, & \mu_{22} &= \mu + \alpha_2 + \beta_2 + \gamma_{22} \\
\mu_{31} &= \mu + \alpha_3 + \beta_1 + \gamma_{31}, & \mu_{32} &= \mu + \alpha_3 + \beta_2 + \gamma_{32}.
\end{aligned} \tag{14.6}$$

These can also be obtained from the last six rows of $\mathbf{X}'\mathbf{X}\boldsymbol{\beta}$ (see Theorem 12.2b).

By taking linear combinations of the six functions in (14.6), we obtain the following estimable functions (e.g., $\theta_1 = \mu_{11} - \mu_{21}$ and $\theta_1' = \mu_{12} - \mu_{22}$):

$$\begin{aligned}
\mu_{11} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} \\
\theta_1 &= \alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21} \quad \text{or} \quad \theta_1' = \alpha_1 - \alpha_2 + \gamma_{12} - \gamma_{22} \\
\theta_2 &= \alpha_1 - \alpha_3 + \gamma_{11} - \gamma_{31} \quad \text{or} \quad \theta_2' = \alpha_1 - \alpha_3 + \gamma_{12} - \gamma_{32} \\
\theta_3 &= \beta_1 - \beta_2 + \gamma_{11} - \gamma_{12} \quad \text{or} \quad \theta_3' = \beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}
\end{aligned} \tag{14.7}$$

$$\text{or} \quad \theta_3'' = \beta_1 - \beta_2 + \gamma_{31} - \gamma_{32}$$
$$\theta_4 = \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22}$$
$$\theta_5 = \gamma_{11} - \gamma_{12} - \gamma_{31} + \gamma_{32}.$$

The alternative expressions for θ_4 and θ_5 are of the form

$$\gamma_{ij} - \gamma_{ij'} - \gamma_{i'j} + \gamma_{i'j'}, \quad i, i' = 1, 2, 3, \quad j, j' = 1, 2, \quad i \neq i', \quad j \neq j'. \tag{14.8}$$

[For general a and b, we likewise obtain estimable functions of the form of (14.7) and (14.8).]

In θ_4 and θ_5 in (14.7), we see that there are estimable contrasts in the γ_{ij}'s, but in θ_1, θ_2, and θ_3 (and in the alternative expressions $\theta_1{}'$, $\theta_2{}'$, $\theta_3{}'$, and $\theta_3{}''$) there are no estimable contrasts in the α's alone or β's alone. (This is also true for the case of general a and b.)

To obtain a single expression involving $\alpha_1 - \alpha_2$ for later use in comparing the α values in a hypothesis test (see Section 14.4.2b), we average θ_1 and $\theta_1{}'$:

$$\begin{aligned} \tfrac{1}{2}(\theta_1 + \theta_1') &= \alpha_1 - \alpha_2 + \tfrac{1}{2}(\gamma_{11} + \gamma_{12}) - \tfrac{1}{2}(\gamma_{21} + \gamma_{22}) \\ &= \alpha_1 - \alpha_2 + \bar{\gamma}_{1.} - \bar{\gamma}_{2.}. \end{aligned} \tag{14.9}$$

For $\alpha_1 - \alpha_3$, we have

$$\begin{aligned} \tfrac{1}{2}(\theta_2 + \theta_2') &= \alpha_1 - \alpha_3 + \tfrac{1}{2}(\gamma_{11} + \gamma_{12}) - \tfrac{1}{2}(\gamma_{31} + \gamma_{32}) \\ &= \alpha_1 - \alpha_3 + \bar{\gamma}_{1.} - \bar{\gamma}_{3.}. \end{aligned} \tag{14.10}$$

Similarly, the average of θ_3, θ_3', and θ_3'' yields

$$\begin{aligned} \tfrac{1}{3}(\theta_3 + \theta_3' + \theta_3'') &= \beta_1 - \beta_2 + \tfrac{1}{3}(\gamma_{11} + \gamma_{21} + \gamma_{31}) - \tfrac{1}{3}(\gamma_{12} + \gamma_{22} + \gamma_{32}) \\ &= \beta_1 - \beta_2 + \bar{\gamma}_{.1} - \bar{\gamma}_{.2}. \end{aligned} \tag{14.11}$$

From (14.1) and assumption 1 in Section 14.1, we have

$$E(y_{ijk}) = E(\mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}),$$
$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad k = 1, 2, \ldots, n$$

or

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} \tag{14.12}$$

[see also (14.2) and (14.6)]. In Section 12.1.2, we demonstrated that for a simple additive (no-interaction) model the side conditions on the α's and β's led to redefined α^*'s and β^*'s that could be expressed as deviations from means, for example, $\alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}$. We now extend this formulation to an interaction model for μ_{ij}:

$$\begin{aligned}\mu_{ij} &= \bar{\mu}_{..} + (\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{.j} - \bar{\mu}_{..}) + (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}) \\ &= \mu^* + \alpha_i^* + \beta_j^* + \gamma_{ij}^*,\end{aligned} \tag{14.13}$$

where

$$\begin{aligned}&\mu^* = \bar{\mu}_{..}, \quad \alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}, \quad \beta_j^* = \bar{\mu}_{.j} - \bar{\mu}_{..}, \\ &\gamma_{ij}^* = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}.\end{aligned} \tag{14.14}$$

With these definitions, it follows that

$$\begin{aligned}&\sum_{i=1}^{a} \alpha_i^* = 0, \quad \sum_{j=1}^{b} \beta_j^* = 0, \\ &\sum_{i=1}^{a} \gamma_{ij}^* = 0 \quad \text{for all} \quad j = 1, 2, \ldots, b, \\ &\sum_{j=1}^{b} \gamma_{ij}^* = 0 \quad \text{for all} \quad i = 1, 2, \ldots, a.\end{aligned} \tag{14.15}$$

Using (14.12), we can write α_i^*, β_j^*, and γ_{ij}^* in (14.14) in terms of the original parameters; for example, α_i^* becomes

$$\begin{aligned}\alpha_i^* &= \bar{\mu}_{i.} - \bar{\mu}_{..} = \frac{1}{b}\sum_{j=1}^{b} \mu_{ij} - \frac{1}{ab}\sum_{ij} \mu_{ij} \\ &= \frac{1}{b}\sum_{j} (\mu + \alpha_i + \beta_j + \gamma_{ij}) - \frac{1}{ab}\sum_{ij} (\mu + \alpha_i + \beta_j + \gamma_{ij}) \\ &= \frac{1}{b}\left(b\mu + b\alpha_i + \sum_{j} \beta_j + \sum_{j} \gamma_{ij} \right) \\ &\quad - \frac{1}{ab}\left(ab\mu + b\sum_{i} \alpha_i + a\sum_{j} \beta_j + \sum_{ij} \gamma_{ij} \right) \\ &= \mu + \alpha_i + \bar{\beta}_{.} + \bar{\gamma}_{i.} - \mu - \bar{\alpha}_{.} - \bar{\beta}_{.} - \bar{\gamma}_{..} \\ &= \alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..}.\end{aligned} \tag{14.16}$$

Similarly

$$\beta_j^* = \beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..}, \tag{14.17}$$

$$\gamma_{ij}^* = \gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..}. \tag{14.18}$$

14.3 ESTIMATORS OF $\boldsymbol{\lambda}'\boldsymbol{\beta}$ AND $\boldsymbol{\sigma}^2$

We consider estimation of estimable functions $\boldsymbol{\lambda}'\boldsymbol{\beta}$ in Section 14.3.1 and estimation of σ^2 in Section 14.3.2.

14.3.1 Solving the Normal Equations and Estimating $\boldsymbol{\lambda}'\boldsymbol{\beta}$

We discuss two approaches for solving the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ and for obtaining estimates of an estimable function $\boldsymbol{\lambda}'\boldsymbol{\beta}$.

14.3.1.1 Side Conditions

From $\mathbf{X}$ and $\mathbf{y}$ in (14.4), we obtain $\mathbf{X}'\mathbf{y}$ for the special case $a = 3$, $b = 2$, and $n = 2$:

$$\mathbf{X}'\mathbf{y} = (y_{...}, y_{1..}, y_{2..}, y_{3..}, y_{1..}, y_{.2.}, y_{11.}, y_{12.}, y_{21.}, y_{22.}, y_{31.}, y_{32.})'. \tag{14.19}$$

On the basis of $\mathbf{X}'\mathbf{y}$ in (14.19) and $\mathbf{X}'\mathbf{X}$ in (14.5), we write the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ in terms of general a, b, and n:

$$\begin{aligned}
abn\hat{\mu} + bn\sum_{i=1}^{a}\hat{\alpha}_i + an\sum_{j=1}^{b}\hat{\beta}_j + n\sum_{i=1}^{a}\sum_{j=1}^{b}\hat{\gamma}_{ij} &= y_{...},\\
bn\hat{\mu} + bn\hat{\alpha}_i + n\sum_{j=1}^{b}\hat{\beta}_j + n\sum_{j=1}^{b}\hat{\gamma}_{ij} &= y_{i..}, \quad i = 1, 2, \ldots, a,\\
an\hat{\mu} + n\sum_{i=1}^{a}\hat{\alpha}_i + an\hat{\beta}_j + n\sum_{i=1}^{a}\hat{\gamma}_{ij} &= y_{.j.}, \quad j = 1, 2, \ldots, b,\\
n\hat{\mu} + n\hat{\alpha}_i + n\hat{\beta}_j + n\hat{\gamma}_{ij} &= y_{ij.}, \quad i = 1, 2, \ldots, a,\\
& \qquad\qquad j = 1, 2, \ldots, b.
\end{aligned} \tag{14.20}$$

With the side conditions $\sum_i \hat{\alpha}_i = 0$, $\sum_j \hat{\beta}_j = 0$, $\sum_i \hat{\gamma}_{ij} = 0$, and $\sum_j \hat{\gamma}_{ij} = 0$, the solution of the normal equations in (14.20) is given by

$$
\begin{aligned}
\hat{\mu} &= \frac{y_{...}}{abn} = \bar{y}_{...}, \\
\hat{\alpha}_i &= \frac{y_{i..}}{bn} - \hat{\mu} = \bar{y}_{i..} - \bar{y}_{...}, \\
\hat{\beta}_j &= \frac{y_{.j.}}{an} - \hat{\mu} = \bar{y}_{.j.} - \bar{y}_{...}, \\
\hat{\gamma}_{ij} &= \frac{y_{ij.}}{n} - \frac{y_{i..}}{bn} - \frac{y_{.j.}}{an} + \frac{y_{...}}{abn}, \\
&= \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}.
\end{aligned} \tag{14.21}
$$

These are unbiased estimators of the parameters μ^*, α_i^*, β_j^*, γ_{ij}^* in (14.14), subject to the side conditions in (14.15). If side conditions are not imposed on the parameters, then the estimators in (14.21) are not unbiased estimators of individual parameters, but these estimators can still be used in estimable functions. For example, consider the estimable function $\boldsymbol{\lambda}'\boldsymbol{\beta}$ in (14.9) (for $a = 3$, $b = 2$):

$$\boldsymbol{\lambda}'\boldsymbol{\beta} = \alpha_1 - \alpha_2 + \tfrac{1}{2}(\gamma_{11} + \gamma_{12}) - \tfrac{1}{2}(\gamma_{21} + \gamma_{22}).$$

By Theorem 12.3a and (14.21), the estimator is given by

$$
\begin{aligned}
\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}} &= \hat{\alpha}_1 - \hat{\alpha}_2 + \tfrac{1}{2}(\hat{\gamma}_{11} + \hat{\gamma}_{12}) - \tfrac{1}{2}(\hat{\gamma}_{21} + \hat{\gamma}_{22}) \\
&= \bar{y}_{1..} - \bar{y}_{...} - (\bar{y}_{2..} - \bar{y}_{...}) + \tfrac{1}{2}(\bar{y}_{11.} - \bar{y}_{1..} - \bar{y}_{.1.} + \bar{y}_{...}) \\
&\quad + \tfrac{1}{2}(\bar{y}_{12.} - \bar{y}_{1..} - \bar{y}_{.2.} + \bar{y}_{...}) - \tfrac{1}{2}(\bar{y}_{21.} - \bar{y}_{2..} - \bar{y}_{.1.} + \bar{y}_{...}) \\
&\quad - \tfrac{1}{2}(\bar{y}_{22.} - \bar{y}_{2..} - \bar{y}_{.2.} + \bar{y}_{...}).
\end{aligned}
$$

Since $\bar{y}_{11.} + \bar{y}_{12.} = 2\bar{y}_{1..}$ and $\bar{y}_{21.} + \bar{y}_{22.} = 2\bar{y}_{2..}$, the estimator $\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}} = \hat{\alpha}_1 - \hat{\alpha}_2 + \frac{1}{2}(\hat{\gamma}_{11} + \hat{\gamma}_{12}) - \frac{1}{2}(\hat{\gamma}_{21} + \hat{\gamma}_{22})$ reduces to

$$\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}} = \hat{\alpha}_1 - \hat{\alpha}_2 + \tfrac{1}{2}(\hat{\gamma}_{11} + \hat{\gamma}_{12}) - \tfrac{1}{2}(\hat{\gamma}_{21} + \hat{\gamma}_{22}) = \bar{y}_{1..} - \bar{y}_{2..}. \tag{14.22}$$

This estimator of $\alpha_1 - \alpha_2 + \frac{1}{2}(\gamma_{11} + \gamma_{12}) - \frac{1}{2}(\gamma_{21} + \gamma_{22})$ is the same as the estimator we would have for $\alpha_1^* - \alpha_2^*$, using $\hat{\alpha}_1$ and $\hat{\alpha}_2$ as estimators of α_1^* and α_2^*:

$$\widehat{\alpha_1^* - \alpha_2^*} = \hat{\alpha}_1 - \hat{\alpha}_2 = \bar{y}_{1..} - \bar{y}_{...} - (\bar{y}_{2..} - \bar{y}_{...}) = \bar{y}_{1..} - \bar{y}_{2..}.$$

By Theorem 12.3d, such estimators are BLUE. If we also assume that ε_{ijk} is $N(0, \sigma^2)$, then by Theorem 12.3h, the estimators are minimum variance unbiased estimators.

14.3.1.2 Generalized Inverse

By Corollary 1 to Theorem 2.8b, a generalized inverse of $\mathbf{X}'\mathbf{X}$ in (14.5) is given by

$$(\mathbf{X}'\mathbf{X})^- = \tfrac{1}{2}\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_6 \end{pmatrix}, \tag{14.23}$$

where the $\mathbf{O}$s are 6×6. Then by (12.13) and (14.19), a solution to the normal equations for $a = 3$ and $b = 2$ is given by

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^- \mathbf{X}'\mathbf{y} \\ &= (0, 0, 0, 0, 0, 0, \bar{y}_{11.}, \bar{y}_{12.}, \bar{y}_{21.}, \bar{y}_{22.}, \bar{y}_{31.}, \bar{y}_{32.})'. \end{aligned} \tag{14.24}$$

The estimators in (14.24) are different from those in (14.21), but they give the same estimators of estimable functions. For example, for $\boldsymbol{\lambda}'\boldsymbol{\beta} = \alpha_1 - \alpha_2 + \frac{1}{2}(\gamma_{11} + \gamma_{12}) - \frac{1}{2}(\gamma_{21} + \gamma_{22})$ in (14.9), we have

$$\begin{aligned}\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}} &= \hat{\alpha}_1 - \hat{\alpha}_2 + \tfrac{1}{2}[\hat{\gamma}_{11} + \hat{\gamma}_{12} - (\hat{\gamma}_{21} + \hat{\gamma}_{22})] \\ &= 0 - 0 + \tfrac{1}{2}[\bar{y}_{11.} + \bar{y}_{12.} - (\bar{y}_{21.} + \bar{y}_{22.})].\end{aligned}$$

It was noted preceding (14.22) that $\bar{y}_{11.} + \bar{y}_{12.} = 2\bar{y}_{1.}$ and $\bar{y}_{21.} + \bar{y}_{22.} = 2\bar{y}_{2.}$. Thus $\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}}$ becomes

$$\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}} = \tfrac{1}{2}(2\bar{y}_{1.} - 2\bar{y}_{2.}) = \bar{y}_{1..} - \bar{y}_{2..},$$

which is the same estimator as that obtained in (14.22) using $\hat{\boldsymbol{\beta}}$ in (14.21).

14.3.2 An Estimator for σ^2

For the two-way model in (14.1), assumption 2 states that $\text{var}(\varepsilon_{ijk}) = \sigma^2$ for all i, j, k. To estimate σ^2, we use (12.22), $s^2 = \text{SSE}/ab(n-1)$, where abn is the number of rows of $\mathbf{X}$ and ab is the rank of $\mathbf{X}$. By (12.20) and (12.21), we have

$$\text{SSE} = \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} = \mathbf{y}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}']\mathbf{y}.$$

With $\hat{\boldsymbol{\beta}}$ from (14.24) and $\mathbf{X}'\mathbf{y}$ from (14.19), SSE can be written as

$$\begin{aligned}\text{SSE} &= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \\ &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b} \bar{y}_{ij.}y_{ij.} \\ &= \sum_{ijk} y_{ijk}^2 - n\sum_{ij} \bar{y}_{ij.}^2. \end{aligned} \tag{14.25}$$

It can also be shown (see Problem 14.10) that this is equal to

$$\text{SSE} = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2. \tag{14.26}$$

Thus, s^2 is given by either of the two forms

$$s^2 = \frac{\sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2}{ab(n-1)} \tag{14.27}$$

$$= \frac{\sum_{ijk} y_{ijk}^2 - n\sum_{ij} \bar{y}_{ij.}^2}{ab(n-1)}. \tag{14.28}$$

By Theorem 12.3e, $E(s^2) = \sigma^2$.

14.4 TESTING HYPOTHESES

In this section, we consider tests of hypotheses for the main effects A and B and for the interaction AB. Throughout this section, we assume that $\mathbf{y}$ is $N_{abn}(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$. For expositional convenience, we sometimes illustrate with $a = 3$ and $b = 2$.

14.4.1 Test for Interaction

In Section 14.4.1.1, we express the interaction hypothesis in terms of estimable parameters, and in Sections 14.4.1.2 and 14.4.1.3, we discuss two approaches to the full–reduced-model test.

14.4.1.1 The Interaction Hypothesis

By (14.8), estimable contrasts in the γ_{ij}'s have the form

$$\gamma_{ij} - \gamma_{ij'} - \gamma_{i'j} + \gamma_{i'j'}, \quad i \neq i', \quad j \neq j'. \tag{14.29}$$

We now show that the interaction hypothesis can be expressed in terms of these estimable functions.

For the illustrative model in (14.3) with $a = 3$ and $b = 2$, the cell means in (14.12) are given in Figure 14.1. The B effect at the first level of A is $\mu_{11} - \mu_{12}$, the B effect at the second level of A is $\mu_{21} - \mu_{22}$, and the B effect at the third level of A is $\mu_{31} - \mu_{32}$.

		B	
		1	2
	1	μ_{11}	μ_{12}
A	2	μ_{21}	μ_{22}
	3	μ_{31}	μ_{32}

Figure 14.1 Cell means for the model in (14.2) and (14.12).

If these three B effects are equal, we have no interaction. If at least one effect differs from the other two, we have an interaction. The hypothesis of no interaction can therefore be expressed as

$$H_0\colon \mu_{11} - \mu_{12} = \mu_{21} - \mu_{22} = \mu_{31} - \mu_{32}. \tag{14.30}$$

To show that this hypothesis is testable, we first write the three differences in terms of the γ_{ij}'s by using (14.12). For the first two differences in (14.30), we obtain

$$\begin{aligned} \mu_{11} - \mu_{12} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} - (\mu + \alpha_1 + \beta_2 + \gamma_{12}) \\ &= \beta_1 - \beta_2 + \gamma_{11} - \gamma_{12}, \\ \mu_{21} - \mu_{22} &= \mu + \alpha_2 + \beta_1 + \gamma_{21} - (\mu + \alpha_2 + \beta_2 + \gamma_{22}) \\ &= \beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}. \end{aligned}$$

Then the equality $\mu_{11} - \mu_{12} = \mu_{21} - \mu_{22}$ in (14.30) becomes

$$\beta_1 - \beta_2 + \gamma_{11} - \gamma_{12} = \beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}$$

or

$$\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0. \tag{14.31}$$

The function $\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22}$ on the left side of (14.31) is an estimable contrast [see (14.29)]. Similarly, the third difference in (14.30) becomes

$$\mu_{31} - \mu_{32} = \beta_1 - \beta_2 + \gamma_{31} - \gamma_{32},$$

and when this is set equal to $\mu_{21} - \mu_{22} = \beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}$, we obtain

$$\gamma_{21} - \gamma_{22} - \gamma_{31} + \gamma_{32} = 0. \tag{14.32}$$

By (14.29), the function $\gamma_{21} - \gamma_{22} - \gamma_{31} + \gamma_{32}$ on the left side of (14.32) is estimable. Thus the two expressions in (14.31) and (14.32) are equivalent to the interaction hypothesis in (14.30), and the hypothesis is therefore testable.

Since the interaction hypothesis can be expressed in terms of estimable functions of γ_{ij}'s that do not involve α_i's or β_j's, we can proceed with a full–reduced-model approach. On the other hand, by (14.7), the α's and β's are not estimable without the γ's. We therefore have to redefine the main effects in order to get a test in the presence of interaction; see Section 14.4.2.

To get a reduced model from (14.1) or (14.3), we work with $\gamma_{ij}^* = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$ in (14.14), which is estimable [it can be estimated unbiasedly by $\hat{\gamma}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$ in (14.21)]. Using (14.13), the model can be expressed in terms of parameters subject to the side conditions in (14.15):

$$y_{ijk} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_{ij}^* + \varepsilon_{ijk}, \tag{14.33}$$

We can get a reduced model from (14.33) by setting $\gamma_{ij}^* = 0$.

In the following theorem, we show that H_0: $\gamma_{ij}^* = 0$ for all i, j is equivalent to the interaction hypothesis expressed as (14.30) or as (14.31) and (14.32). Since all three of these expressions involve $a = 3$ and $b = 2$, we continue with this illustrative special case.

Theorem 14.4a. Consider the model (14.33) for $a = 3$ and $b = 2$. The hypothesis H_0: $\gamma_{ij}^* = 0$, $i = 1, 2, 3$, $j = 1, 2$, is equivalent to (14.30)

$$H_0\colon \mu_{11} - \mu_{12} = \mu_{21} - \mu_{22} = \mu_{31} - \mu_{32}, \tag{14.34}$$

and to the equivalent form

$$H_0\colon \begin{pmatrix} \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} \\ \gamma_{21} - \gamma_{22} - \gamma_{31} + \gamma_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14.35}$$

obtained from (14.31) and (14.32).

PROOF. To establish the equivalence of $\gamma_{ij}^* = 0$ and the first equality in (14.35), we find an expression for each γ_{ij} by setting $\gamma_{ij}^* = 0$. For γ_{12} and γ_{12}^*, for example, we use (14.18) to obtain

$$\gamma_{12}^* = \gamma_{12} - \bar{\gamma}_{1.} - \bar{\gamma}_{.2} + \bar{\gamma}_{..}. \tag{14.36}$$

Then $\gamma_{12}^* = 0$ gives

$$\gamma_{12} = \bar{\gamma}_{1.} + \bar{\gamma}_{.2} - \bar{\gamma}_{..}.$$

Similarly, from (14.18) and the equalities $\gamma_{11}^* = 0$, $\gamma_{21}^* = 0$, and $\gamma_{22}^* = 0$, we obtain

$$\gamma_{11} = \bar{\gamma}_{1.} + \bar{\gamma}_{.1} - \bar{\gamma}_{..}, \quad \gamma_{21} = \bar{\gamma}_{2.} + \bar{\gamma}_{.1} - \bar{\gamma}_{..}, \quad \gamma_{22} = \bar{\gamma}_{2.} + \bar{\gamma}_{.2} - \bar{\gamma}_{..}.$$

When these are substituted into $\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22}$, we obtain

$$\begin{aligned}\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} &= \bar{\gamma}_{1.} + \bar{\gamma}_{.1} - \bar{\gamma}_{..} - (\bar{\gamma}_{1.} + \bar{\gamma}_{.2} - \bar{\gamma}_{..}) \\ &\quad - (\bar{\gamma}_{2.} + \bar{\gamma}_{.1} - \bar{\gamma}_{..}) + \bar{\gamma}_{2.} + \bar{\gamma}_{.2} - \bar{\gamma}_{..} \\ &= 0,\end{aligned}$$

which is the first equality in (14.35). The second equality in (14.35) is obtained similarly.

To show that the first equality in (14.34) is equivalent to the first equality in (14.35), we substitute $\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$ into $\mu_{11} - \mu_{12} = \mu_{21} - \mu_{22}$:

$$\begin{aligned}0 &= \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} \\ &= \mu + \alpha_1 + \beta_1 + \gamma_{11} - (\mu + \alpha_1 + \beta_2 + \gamma_{12}) \\ &\quad - (\mu + \alpha_2 + \beta_1 + \gamma_{21}) + \mu + \alpha_2 + \beta_2 + \gamma_{22} \\ &= \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22}.\end{aligned}$$

Similarly, the second equality in (14.34) is equivalent to the second equality in (14.35). □

In Section 14.4.1.2, we obtain the test for interaction based on the normal equations, and in Section 14.4.1.3, we give the test based on a generalized inverse.

14.4.1.2 Full–Reduced-Model Test Based on the Normal Equations

In this section, we develop the full–reduced-model test for interaction using the normal equations. We express the full model in terms of parameters subject to side conditions, as in (14.33)

$$y_{ijk} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_{ij}^* + \varepsilon_{ijk}, \tag{14.37}$$

where $\mu^* = \bar{\mu}_{..}$, $\alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}$, $\beta_j^* = \bar{\mu}_{.j} - \bar{\mu}_{..}$, and $\gamma_{ij}^* = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$ are as given in (14.14). The reduced model under H_0: $\gamma_{ij}^* = 0$ for all i and j is

$$y_{ijk} = \mu^* + \alpha_i^* + \beta_j^* + \varepsilon_{ijk}. \tag{14.38}$$

Since we are considering a balanced model, the parameters μ^*, α_i^*, and β_j^* (subject to side conditions) in the reduced model (14.38) are the same as those in the full model (14.37) [in (14.44), the estimates in the two models are also shown to be the same].

Using the notation of Chapter 13, the sum of squares for testing H_0: $\gamma_{ij}^* = 0$ is given by

$$\mathrm{SS}(\gamma|\mu, \alpha, \beta) = \mathrm{SS}(\mu, \alpha, \beta, \gamma) - \mathrm{SS}(\mu, \alpha, \beta). \tag{14.39}$$

The estimators $\hat{\mu}$, $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_{ij}$ in (14.21) are unbiased estimators of μ^*, α_i^*, β_j^*, γ_{ij}^*. Extending $\mathbf{X}'\mathbf{y}$ in (14.19) from $a = 3$ and $b = 2$ to general a and b, we obtain

$$\begin{aligned}
\mathrm{SS}(\mu, \alpha, \beta, \gamma) &= \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \\
&= \hat{\mu}y_{...} + \sum_{i=1}^{a} \hat{\alpha}_i y_{i..} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j.} + \sum_{i=1}^{a}\sum_{j=1}^{b} \hat{\gamma}_{ij} y_{ij.} \\
&= \bar{y}_{...}y_{...} + \sum_i (\bar{y}_{i..} - \bar{y}_{...})y_{i...} + \sum_j (\bar{y}_{.j.} - \bar{y}_{...})y_{.j.} \\
&\quad + \sum_{ij} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})y_{ij.} \\
&= \frac{y_{...}^2}{abn} + \left(\sum_i \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}\right) + \left(\sum_j \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}\right) \\
&\quad + \left(\sum_{ij} \frac{y_{ij.}^2}{n} - \sum_i \frac{y_{i..}^2}{bn} - \sum_j \frac{y_{.j.}^2}{an} + \frac{y_{...}^2}{abn}\right) \qquad (14.40) \\
&= \sum_{ij} \frac{y_{ij.}^2}{n}. \qquad (14.41)
\end{aligned}$$

Note that we would obtain the same result using $\hat{\boldsymbol{\beta}}$ in (14.24) (extended to general a and b).

For the reduced model in (14.38), the $\mathbf{X}_1$ matrix and $\mathbf{X}_1'\mathbf{y}$ vector for $a = 3$ and $b = 2$ consist of the first six columns of $\mathbf{X}$ in (14.4‘) and the first six elements of $\mathbf{X}'\mathbf{y}$ in (14.19). We thus obtain

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{pmatrix} 12 & 4 & 4 & 4 & 6 & 6 \\ 4 & 4 & 0 & 0 & 2 & 2 \\ 4 & 0 & 4 & 0 & 2 & 2 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 6 & 2 & 2 & 2 & 6 & 0 \\ 6 & 2 & 2 & 2 & 0 & 6 \end{pmatrix}, \quad \mathbf{X}_1'\mathbf{y} = \begin{pmatrix} y_{...} \\ y_{1..} \\ y_{2..} \\ y_{3..} \\ y_{.1.} \\ y_{.2.} \end{pmatrix}. \tag{14.42}$$

From the pattern in (14.42), we see that for general a and b the normal equations for the reduced model become

$$
\begin{aligned}
abn\hat{\mu} + bn\sum_{i=1}^{a}\hat{\alpha}_i + an\sum_{j=1}^{b}\hat{\beta}_j &= y_{\ldots}, \\
bn\hat{\mu} + bn\hat{\alpha}_i + n\sum_{j=1}^{b}\hat{\beta}_j &= y_{i..}, \quad i = 1, 2, \ldots, a, \\
an\hat{\mu} + n\sum_{i=1}^{a}\hat{\alpha}_i + an\hat{\beta}_j &= y_{.j.}, \quad j = 1, 2, \ldots, b.
\end{aligned}
\tag{14.43}
$$

Using the side conditions $\sum_i \hat{\alpha}_i = 0$ and $\sum_j \hat{\beta}_j = 0$, we obtain the solutions

$$
\hat{\mu} = \frac{y_{\ldots}}{abn} = \bar{y}_{\ldots}, \quad \hat{\alpha}_i = \frac{y_{i..}}{bn} - \hat{\mu} = \bar{y}_{i..} - \bar{y}_{\ldots}, \quad \hat{\beta}_j = \frac{y_{.j.}}{an} - \hat{\mu} = \bar{y}_{.j.} - \bar{y}_{\ldots}. \tag{14.44}
$$

These solutions are the same as those for the full model in (14.21), as expected in the case of a balanced model.

The sum of squares for the reduced model is therefore

$$
\begin{aligned}
\mathrm{SS}(\mu, \alpha, \beta) &= \hat{\boldsymbol{\beta}}_1'\mathbf{X}_1'\mathbf{y} \\
&= \frac{y_{\ldots}^2}{abn} + \left(\sum_i \frac{y_{i..}^2}{bn} - \frac{y_{\ldots}^2}{abn}\right) + \left(\sum_j \frac{y_{.j.}^2}{an} - \frac{y_{\ldots}^2}{abn}\right),
\end{aligned}
$$

and the difference in (14.39) is

$$
\begin{aligned}
\mathrm{SS}(\gamma|\mu, \alpha, \beta) &= \mathrm{SS}(\mu, \alpha, \beta, \gamma) - \mathrm{SS}(\mu, \alpha, \beta) \\
&= \sum_{ij}\frac{y_{ij.}^2}{n} - \sum_i \frac{y_{i..}^2}{bn} - \sum_j \frac{y_{.j.}^2}{an} + \frac{y_{\ldots}^2}{abn}.
\end{aligned}
\tag{14.45}
$$

The error sum of squares is given by

$$
\begin{aligned}
\mathrm{SSE} &= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \\
&= \sum_{ijk} y_{ijk}^2 - \sum_{ij}\frac{y_{ij.}^2}{n}
\end{aligned}
\tag{14.46}
$$

(see Problem 14.13b). In terms of means rather than totals, (14.45) and (14.46) become

$$
\mathrm{SS}(\gamma|\mu, \alpha, \beta) = n\sum_{ij}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{\ldots})^2, \tag{14.47}
$$

$$
\mathrm{SSE} = \sum_{ijk}(y_{ijk} - \bar{y}_{ij.})^2. \tag{14.48}
$$

There are ab parameters involved in the hypothesis H_0: $\gamma_{ij}^* = 0$, $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b$. However, the $a + b$ side conditions $\sum_i \gamma_{ij}^* = 0$ for $j = 1, 2, \ldots, b$ and $\sum_j \gamma_{ij}^* = 0$ for $i = 1, 2, \ldots, a$ impose $a - 1 + b - 1$ restrictions. With the additional condition $\sum_{i=1}^a \sum_{j=1}^b \gamma_{ij}^* = 0$, we have a total of $a + b - 2 + 1 = a + b - 1$ restrictions. Therefore the degrees of freedom for SS($\gamma|\mu, \alpha, \beta$) are $ab - (a + b - 1) = (a - 1)(b - 1)$ (see Problem 14.14).

To test H_0: $\gamma_{ij}^* = 0$ for all i, j, we therefore use the test statistic

$$F = \frac{\mathrm{SS}(\gamma|\mu, \alpha, \beta)/(a-1)(b-1)}{\mathrm{SSE}/ab(n-1)}, \tag{14.49}$$

which is distributed as $F[(a-1)(b-1), ab(n-1)]$ if H_0 is true (see Section 12.7.2).

14.4.1.3 Full–Reduced-Model Test Based on a Generalized Inverse

We now consider a matrix development of SSE and SS($\gamma|\mu, \alpha, \beta$) based on a generalized inverse. By (12.21), SSE $= \mathbf{y}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}']\mathbf{y}$. For our illustrative model with $a = 3$, $b = 2$, and $n = 2$, the matrix $\mathbf{X}'\mathbf{X}$ is given in (14.5) and a generalized inverse $(\mathbf{X}'\mathbf{X})^-$ is provided in (14.23). The 12×12 matrix $\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is then given by

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' = \tfrac{1}{2}\begin{pmatrix} \mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{J} & \cdots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{J} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \mathbf{jj}' & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{jj}' & \cdots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{jj}' \end{pmatrix}, \tag{14.50}$$

where $\mathbf{J}$ and $\mathbf{O}$ are 2×2 and $\mathbf{j}$ is 2×1 (see Problem 14.17). The vector $\mathbf{y}$ in (14.4) can be written as

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_{11} \\ \mathbf{y}_{12} \\ \mathbf{y}_{21} \\ \mathbf{y}_{22} \\ \mathbf{y}_{31} \\ \mathbf{y}_{32} \end{pmatrix}, \tag{14.51}$$

where $\mathbf{y}_{ij} = \begin{pmatrix} y_{ij1} \\ y_{ij2} \end{pmatrix}$, $i = 1, 2, 3$, $j = 1, 2$. By (12.21), (14.50), and (14.51), SSE becomes

$$\begin{aligned} \mathrm{SSE} &= \mathbf{y}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}']\mathbf{y} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{y} \\ &= \sum_{ijk} y_{ijk}^2 - \tfrac{1}{2}\textstyle\sum_{ij} \mathbf{y}_{ij}'\mathbf{jj}'\mathbf{y}_{ij} = \sum_{ijk} y_{ijk}^2 - \tfrac{1}{2}\sum_{ij} y_{ij.}^2, \end{aligned}$$

which is the same as (14.46) with $n = 2$.

For SS($\gamma|\mu, \alpha, \beta$), we obtain

$$\begin{aligned} \mathrm{SS}(\gamma|\mu, \alpha, \beta) &= \mathrm{SS}(\mu, \alpha, \beta, \gamma) - \mathrm{SS}(\mu, \alpha, \beta) \\ &= \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - \hat{\boldsymbol{\beta}}_1'\mathbf{X}_1'\mathbf{y} \\ &= \mathbf{y}'[\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}_1']\mathbf{y}, \end{aligned} \tag{14.52}$$

where $\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is as found in (14.50) and $\mathbf{X}_1$ consists of the first six columns of $\mathbf{X}$ in (14.4). The matrix $\mathbf{X}_1'\mathbf{X}_1$ is given in (14.42), and a generalized inverse of $\mathbf{X}_1'\mathbf{X}_1$ is given by

$$(\mathbf{X}_1'\mathbf{X}_1)^- = \frac{1}{12}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{14.53}$$

Then

$$\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}_1' = \frac{1}{12}\begin{pmatrix} 4\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ 2\mathbf{J} & 4\mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} \\ \mathbf{J} & -\mathbf{J} & 4\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & 4\mathbf{J} & -\mathbf{J} & \mathbf{J} \\ \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & 4\mathbf{J} & 2\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & 4\mathbf{J} \end{pmatrix}, \tag{14.54}$$

where $\mathbf{J}$ is 2×2. For the difference between (14.50) and (14.54), we obtain

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}_1' = \frac{1}{12}\begin{pmatrix} 2\mathbf{J} & -2\mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} \\ -2\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & -2\mathbf{J} & -\mathbf{J} & \mathbf{J} \\ \mathbf{J} & -\mathbf{J} & -2\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & -2\mathbf{J} \\ \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & -2\mathbf{J} & 2\mathbf{J} \end{pmatrix}, \tag{14.55}$$

where $\mathbf{J}$ is 2×2.

To show that SS($\gamma|\mu, \alpha, \beta$) $= \mathbf{y}'[\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}']\mathbf{y}$ in (14.52) is equal to the formulation of SS($\gamma|\mu, \alpha, \beta$) shown in (14.45), we first write (14.45) in matrix notation:

$$\sum_{i=1}^{3}\sum_{j=1}^{2}\frac{y_{ij.}^2}{2}-\sum_{i=1}^{3}\frac{y_{i..}^2}{4}-\sum_{j=1}^{2}\frac{y_{.j.}^2}{6}+\frac{y_{...}^2}{12}=\mathbf{y}'\left(\tfrac{1}{2}\mathbf{A}-\tfrac{1}{4}\mathbf{B}-\tfrac{1}{6}\mathbf{C}+\tfrac{1}{12}\mathbf{D}\right)\mathbf{y}. \qquad (14.56)$$

We now find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. For $\frac{1}{2}\sum_{ij} y_{ij.}^2 = \frac{1}{2}\mathbf{y}'\mathbf{A}\mathbf{y}$, we have by (14.50) and (14.51),

$$\tfrac{1}{2}\sum_{ij} y_{ij.}^2 = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{2}\mathbf{y}_{ij}'\mathbf{j}\mathbf{j}'\mathbf{y}_{ij},$$

where $\mathbf{j}$ is 2×1. This can be written as

$$\begin{aligned}\tfrac{1}{2}\sum_{ij} y_{ij.}^2 &= \tfrac{1}{2}(\mathbf{y}_{11}', \mathbf{y}_{12}', \ldots, \mathbf{y}_{32}')\begin{pmatrix}\mathbf{jj}' & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \mathbf{jj}' & \cdots & \mathbf{O}\\ \vdots & \vdots & & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{jj}'\end{pmatrix}\begin{pmatrix}\mathbf{y}_{11}\\ \mathbf{y}_{12}\\ \vdots\\ \mathbf{y}_{32}\end{pmatrix} \qquad (14.57)\\ &= \tfrac{1}{2}\mathbf{y}'\mathbf{A}\mathbf{y},\end{aligned}$$

where

$$\mathbf{A} = \begin{pmatrix}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \mathbf{J} & \cdots & \mathbf{O}\\ \vdots & \vdots & & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{J}\end{pmatrix},$$

and $\mathbf{J}$ is 2×2. Note that by (14.50), we also have $\frac{1}{2}\mathbf{A} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$.

For the second term in (14.56), $\frac{1}{4}\sum_i y_{i..}^2$, we first use (14.51) to write $y_{i..}$ and $y_{i..}^2$ as

$$y_{i..} = \sum_{jk} y_{ijk} = \sum_k y_{i1k} + \sum_k y_{i2k} = \mathbf{y}_{i1}'\mathbf{j} + \mathbf{y}_{i2}'\mathbf{j} = (\mathbf{y}_{i1}', \mathbf{y}_{i2}')\begin{pmatrix}\mathbf{j}\\ \mathbf{j}\end{pmatrix},$$

$$y_{i..}^2 = (\mathbf{y}_{i1}', \mathbf{y}_{i2}')\begin{pmatrix}\mathbf{j}\\ \mathbf{j}\end{pmatrix}(\mathbf{j}', \mathbf{j}')\begin{pmatrix}\mathbf{y}_{i1}\\ \mathbf{y}_{i2}\end{pmatrix} = (\mathbf{y}_{i1}', \mathbf{y}_{i2}')\begin{pmatrix}\mathbf{jj}' & \mathbf{jj}'\\ \mathbf{jj}' & \mathbf{jj}'\end{pmatrix}\begin{pmatrix}\mathbf{y}_{i1}\\ \mathbf{y}_{i2}\end{pmatrix}.$$

Thus $\frac{1}{4}\sum_{i=1}^{3} y_{i..}^2$ can be written as

$$\frac{1}{4}\sum_{i=1}^{3} y_{i..}^2 = \frac{1}{4}(\mathbf{y}_{11}', \mathbf{y}_{12}', \ldots, \mathbf{y}_{32}')\begin{pmatrix} \mathbf{J} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{J} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{J} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{J} \end{pmatrix}\begin{pmatrix} \mathbf{y}_{11} \\ \mathbf{y}_{12} \\ \vdots \\ \mathbf{y}_{31} \end{pmatrix} \tag{14.58}$$

$$= \frac{1}{4}\mathbf{y}'\mathbf{B}\mathbf{y}.$$

Similarly, the third term of (14.56), $\frac{1}{6}\sum_{j=1}^{2} y_{.j.}^2$, can be written as

$$\frac{1}{6}\sum_{j=1}^{2} y_{.j.}^2 = \frac{1}{6}\mathbf{y}'\begin{pmatrix} \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\ \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} \\ \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\ \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} \\ \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\ \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{J} \end{pmatrix}\mathbf{y} = \frac{1}{6}\mathbf{y}'\mathbf{C}\mathbf{y}. \tag{14.59}$$

For the fourth term of (14.56), $y_{...}^2/12$, we have

$$y_{...} = \sum_{ijk} y_{ijk} = \mathbf{y}'\mathbf{j}_{12},$$

$$\frac{1}{12}y_{...}^2 = \frac{1}{12}\mathbf{y}'\mathbf{j}_{12}\mathbf{j}_{12}'\mathbf{y} = \frac{1}{12}\mathbf{y}'\mathbf{J}_{12}\mathbf{y} = \frac{1}{12}\mathbf{y}'\mathbf{D}\mathbf{y}, \tag{14.60}$$

where $\mathbf{j}_{12}$ is 12×1 and $\mathbf{J}_{12}$ is 12×12. To conform with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in (14.57), (14.58), and (14.59), we write $\mathbf{D} = \mathbf{J}_{12}$ as

$$\mathbf{D} = \mathbf{J}_{12} = \begin{pmatrix} \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} & \mathbf{J} \end{pmatrix},$$

where $\mathbf{J}$ is 2×2.

Now, combining (14.57)–(14.60), we obtain the matrix of the quadratic form in (14.56):

$$\tfrac{1}{2}\mathbf{A} - \tfrac{1}{4}\mathbf{B} - \tfrac{1}{6}\mathbf{C} + \tfrac{1}{12}\mathbf{D} = \frac{1}{12}\begin{pmatrix} 2\mathbf{J} & -2\mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} \\ -2\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & -2\mathbf{J} & -\mathbf{J} & \mathbf{J} \\ \mathbf{J} & -\mathbf{J} & -2\mathbf{J} & 2\mathbf{J} & \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} & -\mathbf{J} & \mathbf{J} & 2\mathbf{J} & -2\mathbf{J} \\ \mathbf{J} & -\mathbf{J} & \mathbf{J} & -\mathbf{J} & -2\mathbf{J} & 2\mathbf{J} \end{pmatrix}, \qquad (14.61)$$

which is the same as (14.55). Thus the matrix version of SS($\gamma|\mu$, α, β) in (14.52) is equal to SS($\gamma|\mu$, α, β) in (14.45):

$$\mathbf{y}'[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-}\mathbf{X}_1']\mathbf{y} = \sum_{ij}\frac{y_{ij.}^2}{n} - \sum_i \frac{y_{i..}^2}{bn} - \sum_j \frac{y_{.j.}^2}{an} + \frac{y_{...}^2}{abn}.$$

14.4.2 Tests for Main Effects

In Section 14.4.2.1, we develop a test for main effects using the full–reduced–model approach. In Section 14.4.2.2, a test for main effects is obtained using the general linear hypothesis approach. Throughout much of this section, we use $a = 3$ and $b = 2$, where a is the number of levels of factor A and b is the number of levels of factor B.

14.4.2.1 Full–Reduced-Model Approach

If interaction is present in the two-way model, then by (14.9) and (14.10), we cannot test H_0: $\alpha_1 = \alpha_2 = \alpha_3$ (for $a = 3$) because $\alpha_1 - \alpha_2$ and $\alpha_1 - \alpha_3$ are not estimable. In fact, there are no estimable contrasts in the α's alone or the β's alone (see Problem 14.2). Thus, if there is interaction, the effect of factor A is different for each level of factor B and vice versa.

To examine the main effect of factor A, we consider $\alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}$, as defined in (14.14). This can be written as

$$\begin{aligned} \alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..} &= \sum_{j=1}^{b}\frac{\mu_{ij}}{b} - \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\mu_{ij}}{ab} \\ &= \frac{1}{b}\sum_j\left(\mu_{ij} - \sum_i \frac{\mu_{ij}}{a}\right) \\ &= \frac{1}{b}\sum_j (\mu_{ij} - \bar{\mu}_{.j}). \qquad (14.62) \end{aligned}$$

The expression in parentheses, $\mu_{ij} - \bar{\mu}_{.j}$, is the effect of the ith level of factor A at the jth level of factor B. Thus in (14.62), $\alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}$ is expressed as the average effect of the ith level of factor A (averaged over the levels of B). This definition leads to the side condition $\sum_i \alpha_i^* = 0$.

Since the α_i^*'s are estimable [see (14.21) and the comment following], we can use them to express the hypothesis for factor A. For $a = 3$, this becomes

$$H_0\colon \alpha_1^* = \alpha_2^* = \alpha_3^*, \tag{14.63}$$

which is equivalent to

$$H_0\colon \alpha_1^* = \alpha_2^* = \alpha_3^* = 0 \tag{14.64}$$

because $\sum_i \alpha_i^* = 0$.

The hypothesis $H_0\colon \alpha_1^* = \alpha_2^* = \alpha_3^*$ in (14.63) states that there is no effect of factor A when averaged over the levels of B. Using $\alpha_i^* = \bar{\mu}_{i.} - \bar{\mu}_{..}$, we can express $H_0\colon \alpha_1^* = \alpha_2^* = \alpha_3^*$ in terms of means:

$$H_0\colon \bar{\mu}_{1.} - \bar{\mu}_{..} = \bar{\mu}_{2.} - \bar{\mu}_{..} = \bar{\mu}_{3.} - \bar{\mu}_{..},$$

which can be written as

$$H_0\colon \bar{\mu}_{1.} = \bar{\mu}_{2.} = \bar{\mu}_{3.}.$$

The values for the cell means in Figure 14.2 illustrate a situation in which H_0 holds in the presence of interaction.

Because H_0 in (14.63) or (14.64) is based on an average effect, many texts recommend that the interaction AB be tested first, and if it is found to be significant, then the main effects should not be tested. However, with the main effect of A defined as the average effect over the levels of B and similarly for the effect of B, the tests for A and B can be carried out even if AB is significant. Admittedly, interpretation requires more care, and the effect of a factor may change if the number of levels of the other factor is altered. But in many cases useful information can be gained about the main effects in the presence of interaction.

		B 1	2	means
A	1	$\mu_{11} = 5$	$\mu_{12} = 1$	$\bar{\mu}_{1.} = 3$
	2	$\mu_{21} = 4$	$\mu_{22} = 2$	$\bar{\mu}_{2.} = 3$
	3	$\mu_{31} = 3$	$\mu_{32} = 3$	$\bar{\mu}_{3.} = 3$

Figure 14.2 Cell means illustrating $\bar{\mu}_{1.} = \bar{\mu}_{2.} = \bar{\mu}_{3.}$ in the presence of interaction.

Under H_0: $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$, the full model in (14.33) reduces to

$$y_{ijk} = \mu^* + \beta_j^* + \gamma_{ij}^* + \varepsilon_{ijk}. \tag{14.65}$$

Because of the orthogonality of the balanced model, the estimators of μ^*, β_j^*, and γ_{ij}^* in (14.65) are the same as in the full model. If we use $\hat{\mu}$, $\hat{\beta}_j$, and $\hat{\gamma}_{ij}$ in (14.21) and elements of $\mathbf{X}'\mathbf{y}$ in (14.19) extended to general a, b, and n, we obtain

$$\mathrm{SS}(\mu, \beta, \gamma) = \hat{\mu} y_{...} + \sum_{j=1}^{b} \hat{\beta}_j y_{.j.} + \sum_{i=1}^{a} \sum_{j=1}^{b} \hat{\gamma}_{ij} y_{ij.},$$

which, by (14.40), becomes

$$\begin{aligned} \mathrm{SS}(\mu, \beta, \gamma) = {} & \frac{y_{...}^2}{abn} + \left(\sum_j \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn} \right) \\ & + \left(\sum_{ij} \frac{y_{ij.}^2}{n} - \sum_i \frac{y_{i..}^2}{bn} - \sum_j \frac{y_{.j.}^2}{an} + \frac{y_{...}^2}{abn} \right). \end{aligned} \tag{14.66}$$

From (14.40) and (14.66), we have

$$\begin{aligned} \mathrm{SS}(\alpha|\mu, \beta, \gamma) &= \mathrm{SS}(\mu, \alpha, \beta, \gamma) - \mathrm{SS}(\mu, \beta, \gamma) \\ &= \sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}. \end{aligned} \tag{14.67}$$

For the special case of $a = 3$, we see by (14.7) that there are two linearly independent estimable functions involving the three α's. Therefore, $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$ has 2 degrees of freedom. In general, $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$ has $a - 1$ degrees of freedom.

In an analogous manner, for factor B we obtain

$$\begin{aligned} \mathrm{SS}(\beta|\mu, \alpha, \gamma) &= \mathrm{SS}(\mu, \alpha, \beta, \gamma) - \mathrm{SS}(\mu, \alpha, \gamma) \\ &= \sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}, \end{aligned} \tag{14.68}$$

which has $b - 1$ degrees of freedom.

In terms of means, we can express (14.67) and (14.68) as

$$\mathrm{SS}(\alpha|\mu, \beta, \gamma) = bn \sum_{i=1}^{a} (\bar{y}_{i..} - \bar{y}_{...})^2, \tag{14.69}$$

$$\mathrm{SS}(\beta|\mu, \alpha, \gamma) = an\sum_{j=1}^{b}(\bar{y}_{.j.} - \bar{y}_{...})^2. \tag{14.70}$$

It is important to note that the full–reduced-model approach leading to $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$ in (14.67) *cannot* be expressed in terms of matrices in a manner analogous to that in (14.52) for the interaction, namely, $\mathrm{SS}(\gamma|\mu, \alpha, \beta) = \mathbf{y}'[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-}\mathbf{X}_1']\mathbf{y}$. The matrix approach is appropriate for the interaction because there are estimable functions of the γ_{ij}'s that do not involve μ or the α_i or β_j terms. In the case of the A main effect, however, we cannot obtain a matrix $\mathbf{X}_1$ by deleting the three columns of $\mathbf{X}$ corresponding to α_1, α_2, and α_3 because contrasts of the form $\alpha_1 - \alpha_2$ are not estimable without involving the γ_{ij}'s [see (14.9) and (14.10)].

If we add the sums of squares for factor A, B, and the interaction in (14.67), (14.68), and (14.45), we obtain $\sum_{ij} y_{ij.}^2/n - y_{...}^2/abn$, which is the overall sum of squares for "treatments," $\mathrm{SS}(\alpha, \beta, \gamma|\mu)$. This can also be seen in (14.40). In the following theorem, the three sums of squares are shown to be independent.

Theorem 14.4b. If $\mathbf{y}$ is $N_{abn}(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$, then $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$, $\mathrm{SS}(\beta|\mu, \alpha, \gamma)$, and $\mathrm{SS}(\gamma|\mu, \alpha, \beta)$ are independent.

Proof. This follows from Theorem 5.6c; see Problem 14.23. □

Using (14.45), (14.46), (14.67), and (14.68), we obtain the analysis-of-variance (ANOVA) table given in Table 14.1.

TABLE 14.1 ANOVA Table for a Two-Way Model with Interaction

Source of Variation	df	Sum of Squares
Factor A	$a-1$	$\sum_i \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$
Factor B	$b-1$	$\sum_j \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$
Interaction	$(a-1)(b-1)$	$\sum_{ij} \frac{y_{ij.}^2}{n} - \sum_i \frac{y_{i..}^2}{bn} - \sum_j \frac{y_{.j.}^2}{an} + \frac{y_{...}^2}{abn}$
Error	$ab(n-1)$	$\sum_{ijk} y_{ijk}^2 - \sum_{ij} \frac{y_{ij.}^2}{n}$
Total	$abn-1$	$\sum_{ijk} y_{ijk}^2 - \frac{y_{...}^2}{abn}$

The test statistic for factor A is

$$F = \frac{\mathrm{SS}(\alpha|\mu, \beta, \gamma)/(a-1)}{\mathrm{SSE}/ab(n-1)}, \tag{14.71}$$

which is distributed as $F[a-1, ab(n-1)]$ if H_0: $\alpha_1^* = \alpha_2^* = \cdots = \alpha_a^* = 0$ is true. For factor B, we use $\mathrm{SS}(\beta|\mu, \alpha, \gamma)$ in (14.68), and the F statistic is given by

$$F = \frac{\mathrm{SS}(\beta|\mu, \alpha, \gamma)/(b-1)}{\mathrm{SSE}/ab(n-1)},$$

which is distributed as $F[b-1, ab(n-1)]$ if H_0: $\beta_1^* = \beta_2^* = \cdots = \beta_b^* = 0$ is true. In Section 14.4.2.2, these F statistics are obtained by the general linear hypothesis approach. The F distributions can thereby be justified by Theorem 12.7c.

Example 14.4. The moisture content of three types of cheese made by two methods was recorded by Marcuse (1949) (format altered). Two cheeses were measured for each type and each method. If *method* is designated as factor A and *type* is factor B, then $a = 2$, $b = 3$, and $n = 2$. The data are given in Table 14.2, and the totals are shown in Table 14.3.

The sum of squares for factor A is given by (14.67) as

$$\begin{aligned}
\mathrm{SS}(\alpha|\mu, \beta, \gamma) &= \sum_{i=1}^{2} \frac{y_{i..}^2}{(3)(2)} - \frac{y_{...}^2}{(2)(3)(2)} \\
&= \tfrac{1}{6}[(221.98)^2 + (220.81)^2] - \tfrac{1}{12}(442.79)^2 \\
&= .114075.
\end{aligned}$$

TABLE 14.2 Moisture Content of Two Cheeses from Each of Three Different Types Made by Two Methods

	Type of Cheese		
Method	1	2	3
1	39.02	35.74	37.02
	38.79	35.41	36.00
2	38.96	35.58	35.70
	39.01	35.52	36.04

TABLE 14.3 Totals for Data in Table 14.2

		B		
A	1	2	3	Totals
1	$y_{11.} = 77.81$	$y_{12.} = 71.15$	$y_{13.} = 73.02$	$y_{1..} = 221.98$
2	$y_{21.} = 77.97$	$y_{22.} = 71.10$	$y_{23.} = 71.74$	$y_{2..} = 220.81$
Totals	$y_{.1.} = 155.78$	$y_{.2.} = 142.25$	$y_{.3.} = 144.76$	$y_{...} = 442.79$

Similarly, for factor B we use (14.68):

$$\begin{aligned}\mathrm{SS}(\beta|\mu,\ \alpha,\ \gamma) &= \sum_{j=1}^{3} \frac{y_{.j.}^2}{(2)(2)} - \frac{y_{...}^2}{12} \\ &= \tfrac{1}{4}[(155.78)^2 + (142.25)^2 + (144.76)^2] - \tfrac{1}{12}(442.79)^2 \\ &= 25.900117.\end{aligned}$$

For error, we use (14.46) to obtain

$$\begin{aligned}\mathrm{SSE} &= \sum_{ijk} y_{ijk}^2 - \tfrac{1}{2}\textstyle\sum_{ij} y_{ij.}^2 \\ &= (39.02)^2 + (38.79)^2 + \cdots + (36.04)^2 - \tfrac{1}{2}[(77.81)^2 + \cdots + (71.74)^2] \\ &= 16{,}365.56070 - 16364.89875 = .661950.\end{aligned}$$

The total sum of squares is given by

$$\mathrm{SST} = \sum_{ijk} y_{ijk}^2 - \frac{y_{...}^2}{12} = 26.978692.$$

The sum of squares for interaction can be found by (14.45) or by subtracting all other terms from the total sum of squares:

$$\begin{aligned}\mathrm{SS}(\gamma|\mu,\ \alpha,\ \beta) &= 26.978692 - .114075 - 25.900117 - .661950 \\ &= .302550.\end{aligned}$$

With these sums of squares, we can compute mean squares and F statistics as shown in Table 14.4.

Only the F test for *type* is significant, since $F_{.05,1,6} = 5.99$ and $F_{.05,2,6} = 5.14$. The p value for *type* is .0000155. The p values for *method* and the *interaction* are .3485 and .3233, respectively.

TABLE 14.4 ANOVA for the Cheese Data in Table 14.2

Source of Variation	Sum of Squares	df	Mean Square	F
Method	0.114075	1	0.114075	1.034
Type	25.900117	2	12.950058	117.381
Interaction	0.302550	2	0.151275	1.371
Error	0.661950	6	0.110325	
Total	26.978692	11		

Note that in Table 14.2, the difference between the two replicates in each cell is very small except for the cell with method 1 and type 3. This suggests that the replicates may be repeat measurements rather than true replications; that is, the experimenter may have measured the same piece of cheese twice rather than measuring two different cheeses. □

14.4.2.2 General Linear Hypothesis Approach

We now obtain SS$(\alpha|\mu, \beta, \gamma)$ for $a = 3$ and $b = 2$ by an approach based on the general linear hypothesis. Using $\alpha_i^* = \alpha_i - \bar{\alpha}_. + \bar{\gamma}_{i.} - \bar{\gamma}_{..}$ in (14.16), the hypothesis $H_0: \alpha_1^* = \alpha_2^* = \alpha_3^*$ in (14.63) can be expressed as $H_0: \alpha_1 + \bar{\gamma}_{1.} = \alpha_2 + \bar{\gamma}_{2.} = \alpha_3 + \bar{\gamma}_{3.}$ or

$$H_0\colon\ \alpha_1 + \tfrac{1}{2}(\gamma_{11} + \gamma_{12}) = \alpha_2 + \tfrac{1}{2}(\gamma_{21} + \gamma_{22}) = \alpha_3 + \tfrac{1}{2}(\gamma_{31} + \gamma_{32}) \qquad (14.72)$$

[see also (14.9) and (14.10)]. The two equalities in (14.72) can be expressed in the form

$$H_0\colon \begin{pmatrix} \alpha_1 + \frac{1}{2}\gamma_{11} + \frac{1}{2}\gamma_{12} - \alpha_3 - \frac{1}{2}\gamma_{31} - \frac{1}{2}\gamma_{32} \\ \alpha_2 + \frac{1}{2}\gamma_{21} + \frac{1}{2}\gamma_{22} - \alpha_3 - \frac{1}{2}\gamma_{31} - \frac{1}{2}\gamma_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Rearranging the order of the parameters to correspond to the order in $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}, \gamma_{31}, \gamma_{32})'$ in (14.4), we have

$$H_0\colon \begin{pmatrix} \alpha_1 - \alpha_3 + \frac{1}{2}\gamma_{11} + \frac{1}{2}\gamma_{12} - \frac{1}{2}\gamma_{31} - \frac{1}{2}\gamma_{32} \\ \alpha_2 - \alpha_3 + \frac{1}{2}\gamma_{21} + \frac{1}{2}\gamma_{22} - \frac{1}{2}\gamma_{31} - \frac{1}{2}\gamma_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (14.73)$$

which can now be written in the form $H_0\colon \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ with

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{14.74}$$

By Theorem 12.7b(iii), the sum of squares corresponding to $H_0\colon \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ is

$$\text{SSH} = (\mathbf{C}\hat{\boldsymbol{\beta}})'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}\hat{\boldsymbol{\beta}}. \tag{14.75}$$

Substituting $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}$ from (12.13), SSH in (14.75) becomes

$$\text{SSH} = \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y} = \mathbf{y}'\mathbf{A}\mathbf{y}. \tag{14.76}$$

Using $\mathbf{C}$ in (14.74), $(\mathbf{X}'\mathbf{X})^{-}$ in (14.23), and $\mathbf{X}$ in (14.4), we obtain

$$\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \tfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}, \tag{14.77}$$

$$\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}' = \tfrac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad [\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1} = \tfrac{4}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{14.78}$$

Then $\mathbf{A} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ in (14.76) becomes

$$\mathbf{A} = \tfrac{1}{12}\begin{pmatrix} 2\mathbf{J} & -\mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & 2\mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & -\mathbf{J} & 2\mathbf{J} \end{pmatrix}, \tag{14.79}$$

where $\mathbf{J}$ is 4×4. This can be expressed as

$$\mathbf{A} = \tfrac{1}{12}\begin{pmatrix} 2\mathbf{J} & -\mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & 2\mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & -\mathbf{J} & 2\mathbf{J} \end{pmatrix} = \tfrac{1}{12}\begin{pmatrix} 3\mathbf{J} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & 3\mathbf{J} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & 3\mathbf{J} \end{pmatrix} - \tfrac{1}{12}\begin{pmatrix} \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} \\ \mathbf{J} & \mathbf{J} & \mathbf{J} \end{pmatrix}. \tag{14.80}$$

To evaluate $\mathbf{y}'\mathbf{A}\mathbf{y}$, we redefine $\mathbf{y}$ in (14.51) as

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_{11} \\ \mathbf{y}_{12} \\ \mathbf{y}_{21} \\ \mathbf{y}_{22} \\ \mathbf{y}_{31} \\ \mathbf{y}_{32} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} \mathbf{y}_{i1} \\ \mathbf{y}_{i2} \end{pmatrix} = \mathbf{y}_i. \tag{14.81}$$

Then (14.76) becomes

$$
\begin{aligned}
\mathrm{SSH} = \mathbf{y}'\mathbf{A}\mathbf{y} &= \tfrac{1}{12}(\mathbf{y}_1', \mathbf{y}_2', \mathbf{y}_3') \begin{pmatrix} 3\mathbf{J}_4 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & 3\mathbf{J}_4 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & 3\mathbf{J}_4 \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{pmatrix} - \tfrac{1}{12}\mathbf{y}'\mathbf{J}_{12}\mathbf{y} \\
&= \tfrac{3}{12}\sum_{i=1}^{3} \mathbf{y}_i'\mathbf{J}_4\mathbf{y}_i - \tfrac{1}{12}\mathbf{y}'\mathbf{J}_{12}\mathbf{y} \\
&= \frac{1}{4}\sum_i \mathbf{y}_i'\mathbf{j}_4\mathbf{j}_4'\mathbf{y}_i - \tfrac{1}{12}\mathbf{y}'\mathbf{j}_{12}\mathbf{j}_{12}'\mathbf{y} \\
&= \sum_i \frac{y_{i..}^2}{4} - \frac{y_{...}^2}{12},
\end{aligned}
$$

which is the same as $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$ in (14.67) with $a = 3$ and $b = n = 2$.

The sum of squares for testing the B main effect can be obtained similarly using a general linear hypothesis approach (see Problem 14.25).

14.5 EXPECTED MEAN SQUARES

We find expected mean squares by direct evaluation of the expected value of sums of squares and also by a matrix method based on the expected value of quadratic forms.

14.5.1 Sums-of-Squares Approach

The expected mean squares for the tests in Table 14.1 are given in Table 14.5. Note that these are expressed in terms of α_i^*, β_j^*, and γ_{ij}^* subject to the side

TABLE 14.5 Expected Mean Squares for a Two-Way ANOVA

Source	Sum of Squares	Mean Square	Expected Mean Square
A	$\mathrm{SS}(\alpha\|\mu, \beta, \gamma)$	$\dfrac{\mathrm{SS}(\alpha\|\mu, \beta, \gamma)}{a-1}$	$\sigma^2 + bn\sum_i \dfrac{\alpha_i^{*2}}{a-1}$
B	$\mathrm{SS}(\beta\|\mu, \alpha, \gamma)$	$\dfrac{\mathrm{SS}(\beta\|\mu, \alpha, \gamma)}{b-1}$	$\sigma^2 + an\sum_j \dfrac{\beta_j^{*2}}{b-1}$
AB	$\mathrm{SS}(\gamma\|\mu, \alpha, \beta)$	$\dfrac{\mathrm{SS}(\gamma\|\mu, \alpha, \beta)}{(a-1)(b-1)}$	$\sigma^2 + n\sum_{ij} \dfrac{\gamma_{ij}^{*2}}{(a-1)(b-1)}$
Error	SSE	$\dfrac{\mathrm{SSE}}{ab(n-1)}$	σ^2

conditions $\sum_i \alpha_i^* = 0$, $\sum_j \beta_j^* = 0$, and $\sum_i \gamma_{ij}^* = \sum_j \gamma_{ij}^* = 0$. These expected mean squares can be derived by inserting the model $y_{ijk} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_{ij}^* + \varepsilon_{ijk}$ in (14.33) into the sums of squares and then finding expected values. We illustrate this approach for the first expected mean square in Table 14.5.

To find the expected value of $\text{SS}(\alpha|\mu, \beta, \gamma) = \sum_i y_{i..}^2/bn - y_{...}^2/abn$ in (14.67), we first note that by using assumption 1 in Section 14.1, we can write assumptions 2 and 3 in the form

$$E(\varepsilon_{ijk}^2) = \sigma^2 \quad \text{for all} \quad i, j, k, \tag{14.82}$$

$$E(\varepsilon_{ijk}\varepsilon_{rst}) = 0 \quad \text{for all} \quad (i, j, k) \neq (r, s, t). \tag{14.83}$$

Using these results, along with assumption 1 and the side conditions in (14.15), we can show that $E(y_{...}^2) = a^2b^2n^2\mu^{*2} + abn\sigma^2$ as follows:

$$\begin{aligned}
E(y_{...}^2) &= E\left(\sum_{ijk} y_{ijk}\right)^2 = E\left[\sum_{ijk}(\mu^* + \alpha_i^* + \beta_j^* + \gamma_{ij}^* + \varepsilon_{ijk})\right]^2 \\
&= E\left(abn\mu^* + bn\sum_i \alpha_i^* + an\sum_j \beta_j^* + n\sum_{ij}\gamma_{ij}^* + \sum_{ijk}\varepsilon_{ijk}\right)^2 \\
&= E\left[a^2b^2n^2\mu^{*2} + 2abn\mu^*\sum_{ijk}\varepsilon_{ijk} + \left(\sum_{ijk}\varepsilon_{ijk}\right)^2\right] \\
&= a^2b^2n^2\mu^{*2} + E\left(\sum_{ijk}\varepsilon_{ijk}^2\right) + E\left(\sum_{ijk \neq rst}\varepsilon_{ijk}\varepsilon_{rst}\right) \\
&= a^2b^2n^2\mu^{*2} + abn\sigma^2.
\end{aligned}$$

It can likewise be shown that

$$E\left(\sum_{i=1}^a y_{i..}^2\right) = ab^2n^2\mu^{*2} + b^2n^2\sum_{i=1}^a \alpha_i^{*2} + abn\sigma^2 \tag{14.84}$$

(see Problem 14.27). Thus

$$\begin{aligned}
E\left[\frac{\text{SS}(\alpha|\mu, \beta, \gamma)}{a-1}\right] &= \frac{1}{a-1}E\left(\sum_i \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}\right) \\
&= \frac{1}{a-1}\left[\frac{ab^2n^2\mu^{*2}}{bn} + \frac{b^2n^2\sum_i \alpha_i^{*2}}{bn} + \frac{abn\sigma^2}{bn} - \frac{a^2b^2n^2\mu^{*2}}{abn} - \frac{abn\sigma^2}{abn}\right] \\
&= \frac{1}{a-1}\left[(a-1)\sigma^2 + bn\sum_i \alpha_i^{*2}\right].
\end{aligned}$$

The other expected mean squares in Table 14.5 can be obtained similarly (see Problem 14.28).

14.5.2 Quadratic Form Approach

We now obtain the first expected mean square in Table 14.2 using a matrix approach. We illustrate with $a = 3$, $b = 2$, and $n = 2$. By (14.75), we obtain

$$E[\mathrm{SS}(\alpha|\mu, \beta, \gamma)] = E\{(\mathbf{C}\hat{\boldsymbol{\beta}})'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}\hat{\boldsymbol{\beta}}\}. \tag{14.85}$$

The matrix $\mathbf{C}$ contains estimable functions, and therefore by (12.44) and (12.45), we have $E(\mathbf{C}\hat{\boldsymbol{\beta}}) = \mathbf{C}\boldsymbol{\beta}$ and $\mathrm{cov}(\mathbf{C}\hat{\boldsymbol{\beta}}) = \sigma^2\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}'$. If we define $\mathbf{G}$ to be the 2×2 matrix $[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}$, then by Theorem 5.2a, (14.85) becomes

$$\begin{aligned} E[\mathrm{SS}(\alpha|\mu, \beta, \gamma)] &= E[(\mathbf{C}\hat{\boldsymbol{\beta}})'\mathbf{G}(\mathbf{C}\hat{\boldsymbol{\beta}})] \\ &= \mathrm{tr}[\mathbf{G}\ \mathrm{cov}(\mathbf{C}\hat{\boldsymbol{\beta}})] + [E(\mathbf{C}\hat{\boldsymbol{\beta}})]'\mathbf{G}[E(\mathbf{C}\hat{\boldsymbol{\beta}})] \\ &= \mathrm{tr}(\mathbf{G}\sigma^2\mathbf{G}^{-1}) + (\mathbf{C}\boldsymbol{\beta})'\mathbf{G}(\mathbf{C}\boldsymbol{\beta}) \\ &= 2\sigma^2 + \boldsymbol{\beta}'\mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}\boldsymbol{\beta} \qquad (14.86) \\ &= 2\sigma^2 + \boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta}, \qquad (14.87) \end{aligned}$$

where $\mathbf{L} = \mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}\mathbf{C}$. Using $\mathbf{C}$ in (14.74) and $[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}']^{-1}$ in (14.78), $\mathbf{L}$ becomes

$$\mathbf{L} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -4 & -4 & 0 & 0 & 4 & 4 & -2 & -2 & -2 & -2 \\ 0 & -4 & 8 & -4 & 0 & 0 & -2 & -2 & 4 & 4 & -2 & -2 \\ 0 & -4 & -4 & 8 & 0 & 0 & -2 & -2 & -2 & -2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 & 0 & 0 & 2 & 2 & -1 & -1 & -1 & -1 \\ 0 & 4 & -2 & -2 & 0 & 0 & 2 & 2 & -1 & -1 & -1 & -1 \\ 0 & -2 & 4 & -2 & 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 \\ 0 & -2 & 4 & -2 & 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 \\ 0 & -2 & -2 & 4 & 0 & 0 & -1 & -1 & -1 & -1 & 2 & 2 \\ 0 & -2 & -2 & 4 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & 2 \end{pmatrix}. \tag{14.88}$$

This can be written as the difference

$$\mathbf{L} = \tfrac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

$$-\tfrac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 4 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 4 & 4 & 4 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 4 & 4 & 4 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \tfrac{1}{3}\begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{O} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{0} & \mathbf{A}_{21} & \mathbf{O} & \mathbf{A}_{22} \end{pmatrix} - \tfrac{1}{3}\begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{B}_{11} & \mathbf{O} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{0} & \mathbf{B}_{21} & \mathbf{O} & \mathbf{B}_{22} \end{pmatrix}, \qquad (14.89)$$

where $\mathbf{A}_{11} = 12\mathbf{I}_3$, $\mathbf{B}_{11} = 4\mathbf{j}_3\mathbf{j}_3'$, $\mathbf{B}_{12} = 2\mathbf{j}_3\mathbf{j}_6'$, $\mathbf{B}_{21} = 2\mathbf{j}_6\mathbf{j}_3'$, $\mathbf{B}_{22} = \mathbf{j}_6\mathbf{j}_6'$,

$$\mathbf{A}_{12} = \begin{pmatrix} 6\mathbf{j}_2' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0}' & 6\mathbf{j}_2' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{0}' & 6\mathbf{j}_2' \end{pmatrix}, \quad \mathbf{A}_{21} = \begin{pmatrix} 6\mathbf{j}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 6\mathbf{j}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 6\mathbf{j}_2 \end{pmatrix},$$

$$\mathbf{A}_{22} = \begin{pmatrix} 3\mathbf{j}_2\mathbf{j}_2' & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & 3\mathbf{j}_2\mathbf{j}_2' & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & 3\mathbf{j}_2\mathbf{j}_2' \end{pmatrix}.$$

If we write $\boldsymbol{\beta}$ in (14.4) in the form

$$\boldsymbol{\beta} = (\mu, \boldsymbol{\alpha}', \beta_1, \beta_2, \boldsymbol{\gamma}')',$$

where $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\gamma}' = (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}, \gamma_{31}, \gamma_{32})$, then $\boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta}$ in (14.87) becomes

$$\begin{aligned} \boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta} = {} & \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{A}_{11}\boldsymbol{\alpha} + \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{A}_{12}\boldsymbol{\gamma} + \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{A}_{21}\boldsymbol{\alpha} + \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{A}_{22}\boldsymbol{\gamma} - \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{B}_{11}\boldsymbol{\alpha} \\ & - \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{B}_{12}\boldsymbol{\gamma} - \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{B}_{21}\boldsymbol{\alpha} - \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{B}_{22}\boldsymbol{\gamma}. \end{aligned}$$

Since $\mathbf{A}_{21}' = \mathbf{A}_{12}$ and $\mathbf{B}_{21}' = \mathbf{B}_{12}$, this reduces to

$$\begin{aligned} \boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta} = {} & \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{A}_{11}\boldsymbol{\alpha} + \tfrac{2}{3}\boldsymbol{\alpha}'\mathbf{A}_{12}\boldsymbol{\gamma} + \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{A}_{22}\boldsymbol{\gamma} - \tfrac{1}{3}\boldsymbol{\alpha}'\mathbf{B}_{11}\boldsymbol{\alpha} \\ & - \tfrac{2}{3}\boldsymbol{\alpha}'\mathbf{B}_{12}\boldsymbol{\gamma} - \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{B}_{22}\boldsymbol{\gamma}. \end{aligned}$$

If we partition $\boldsymbol{\gamma}$ as $\boldsymbol{\gamma}' = (\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2', \boldsymbol{\gamma}_3')$, where $\boldsymbol{\gamma}_i' = (\gamma_{i1}, \gamma_{i2})$, then

$$\begin{aligned} \tfrac{2}{3}\boldsymbol{\alpha}'\mathbf{A}_{12}\boldsymbol{\gamma} &= \tfrac{12}{3}\boldsymbol{\alpha}' \begin{pmatrix} \mathbf{j}_2' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{j}_2' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{0}' & \mathbf{j}_2' \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \boldsymbol{\gamma}_3 \end{pmatrix} \\ &= 4\boldsymbol{\alpha}' \begin{pmatrix} \mathbf{j}_2'\boldsymbol{\gamma}_1 \\ \mathbf{j}_2'\boldsymbol{\gamma}_2 \\ \mathbf{j}_2'\boldsymbol{\gamma}_3 \end{pmatrix} = 4\sum_{i=1}^{3} \alpha_i \gamma_{i.}. \end{aligned}$$

Now, using the definitions of $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, and $\mathbf{B}_{22}$ following (14.89), we obtain

$$\begin{aligned} \boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta} &= 4\boldsymbol{\alpha}'\boldsymbol{\alpha} + 4\sum_{i=1}^{3} \alpha_i \gamma_{i.} + \sum_{i=1}^{3} \boldsymbol{\gamma}_i'\mathbf{j}_2\mathbf{j}_2'\boldsymbol{\gamma}_i - \tfrac{4}{3}\boldsymbol{\alpha}'\mathbf{j}_3\mathbf{j}_3'\boldsymbol{\alpha} \\ &\quad - \tfrac{4}{3}\boldsymbol{\alpha}'\mathbf{j}_3\mathbf{j}_6'\boldsymbol{\gamma} - \tfrac{1}{3}\boldsymbol{\gamma}'\mathbf{j}_6\mathbf{j}_6'\boldsymbol{\gamma} \\ &= 4\sum_{i=1}^{3} \alpha_i^2 + 4\sum_{i=1}^{3} \alpha_i \gamma_{i.} + \sum_{i=1}^{3} \gamma_{i.}^2 - \tfrac{4}{3}\alpha_.^2 - \tfrac{4}{3}\alpha_. \gamma_{..} - \tfrac{1}{3}\gamma_{..}^2. \qquad (14.90) \end{aligned}$$

By expressing $\gamma_{i.}$, $\alpha_.$, and $\gamma_{..}$ in terms of means, (14.90) can be written in the form

$$\boldsymbol{\beta}'\mathbf{L}\boldsymbol{\beta} = 4\sum_{i=1}^{3}(\alpha_i - \bar{\alpha}_. + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 = 4\sum_{i=1}^{3}\alpha_i^{*2} \qquad \text{[by (14.16)]}. \tag{14.91}$$

For an alternative approach leading to (14.91), note that since $E(\mathbf{C}\hat{\boldsymbol{\beta}}) = \mathbf{C}\boldsymbol{\beta}$, (14.86) can be written as

$$E[\mathrm{SS}(\alpha|\mu, \beta, \gamma)] = 2\sigma^2 + [E(\mathbf{C}\hat{\boldsymbol{\beta}})]'[\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{C}']^{-1}E(\mathbf{C}\hat{\boldsymbol{\beta}}). \tag{14.92}$$

By (14.75), $\mathrm{SS}(\alpha|\mu, \beta, \gamma) = \mathrm{SSH} = (\mathbf{C}\hat{\boldsymbol{\beta}})'[\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{C}']^{-1}\mathbf{C}\hat{\boldsymbol{\beta}}$. Thus, by (14.92), we can obtain $E[\mathrm{SS}(\alpha|\mu, \beta, \gamma)]$ by replacing $\mathbf{C}\hat{\boldsymbol{\beta}}$ in $\mathrm{SS}(\alpha|\mu, \beta, \gamma)$ with $\mathbf{C}\boldsymbol{\beta}$ and adding $2\sigma^2$. To illustrate, we replace $\bar{y}_{i..}$ and $\bar{y}_{...}$ with $E(\bar{y}_{i..})$ and $E(\bar{y}_{...})$ in $\mathrm{SS}(\alpha|\mu, \beta, \gamma) = 4\sum_{i=1}^{3}(\bar{y}_{i..} - \bar{y}_{...})^2$ in (14.69). We first find $E(\bar{y}_{i..})$:

$$\begin{aligned}
E(\bar{y}_{i..}) &= E\left(\tfrac{1}{4}\sum_{jk} y_{ijk}\right) = \tfrac{1}{4}\sum_{jk} E(y_{ijk}) \\
&= \tfrac{1}{4}\sum_{jk} E(\mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}) \\
&= \tfrac{1}{4}\sum_{jk} (\mu + \alpha_i + \beta_j + \gamma_{ij}) \\
&= \tfrac{1}{4}\left(4\mu + 4\alpha_i + 2\sum_j \beta_j + 2\sum_j \gamma_{ij}\right) \\
&= \mu + \alpha_i + \bar{\beta}_. + \bar{\gamma}_{i.}.
\end{aligned} \tag{14.93}$$

Similarly

$$E(\bar{y}_{...}) = \mu + \bar{\alpha}_. + \bar{\beta}_. + \bar{\gamma}_{..}. \tag{14.94}$$

Then,

$$\begin{aligned}
E[\mathrm{SS}(\alpha|\mu, \beta, \gamma)] &= 2\sigma^2 + 4\sum_{i=1}^{3}[E(\bar{y}_{i..}) - E(\bar{y}_{...})]^2 \\
&= 2\sigma^2 + 4\sum_i (\mu + \alpha_i + \bar{\beta}_. + \bar{\gamma}_{i.} - \mu - \bar{\alpha}_. - \bar{\beta}_. + \bar{\gamma}_{..})^2 \\
&= 2\sigma^2 + 4\sum_i (\alpha_i - \bar{\alpha}_. + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 \\
&= 2\sigma^2 + 4\sum_i \alpha_i^{*2} \qquad \text{[by (14.16)]}.
\end{aligned}$$

PROBLEMS

14.1 Obtain θ_1 and θ_5 in (14.7) from (14.6).

14.2 In a comment following (14.8), it is noted that there are no estimable contrasts in the α's alone or β's alone. Verify this statement.

14.3 Show that $\frac{1}{3}(\theta_3 + \theta_3' + \theta_3'')$ has the value shown in (14.11).

14.4 Verify the following results in (14.15) using the definitions of α_i^*, β_j^*, and γ_{ij}^* in (14.14):

(a) $\sum_i \alpha_i^* = 0$
(b) $\sum_j \beta_j^* = 0$
(c) $\sum_i \gamma_{ij}^* = 0, \quad j = 1, 2, \ldots, b$
(d) $\sum_j \gamma_{ij}^* = 0, \quad i = 1, 2, \ldots, a$

14.5 Verify the following results from (14.15) using the definitions of α_i^*, β_j^*, and γ_{ij}^* in (14.16), (14.17), and (14.18):

(a) $\sum_i \alpha_i^* = 0$
(b) $\sum_j \beta_j^* = 0$
(c) $\sum_i \gamma_{ij}^* = 0, \quad j = 1, 2, \ldots, b$
(d) $\sum_j \gamma_{ij}^* = 0, \quad i = 1, 2, \ldots, a$

14.6 **(a)** Show that $\beta_j^* = \beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..}$ as in (14.17).
(b) Show that $\gamma_{ij}^* = \gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..}$ as in (14.18).

14.7 Show that $\hat{\alpha}_i$ and $\hat{\gamma}_{ij}$ in (14.21) are unbiased estimators of α_i^* and γ_{ij}^* as noted following (14.21).

14.8 **(a)** Show that $\bar{y}_{11.} + \bar{y}_{12.} = 2\bar{y}_{1..}$ and that $\bar{y}_{21.} + \bar{y}_{22.} = 2\bar{y}_{2..}$, as used to obtain (14.22).
(b) Show that $\hat{\alpha}_1 - \hat{\alpha}_2 + \frac{1}{2}(\hat{\gamma}_{11} + \hat{\gamma}_{12}) - \frac{1}{2}(\hat{\gamma}_{21} + \hat{\gamma}_{22}) = \bar{y}_{1..} - \bar{y}_{2..}$ as in (14.22).

14.9 Show that $(\mathbf{X}'\mathbf{X})^-$ in (14.23) is a generalized inverse of $\mathbf{X}'\mathbf{X}$ in (14.5).

14.10 Show that SSE in (14.26) is equal to SSE in (14.25).

14.11 Show that the second equality in (14.34) is equivalent to the second equality in (14.35); that is, $\mu_{21} - \mu_{22} = \mu_{31} - \mu_{32}$ implies $\gamma_{21} - \gamma_{22} - \gamma_{31} + \gamma_{32} = 0$.

14.12 Show that $\sum_i (\bar{y}_{i..} - \bar{y}_{...}) y_{i..} = \sum_i y_{i..}^2/bn - y_{...}^2/abn$ and that $\sum_{ij} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) y_{ij.} = \sum_{ij} y_{ij.}^2/n - \sum_i y_{i..}^2/bn - \sum_j y_{.j.}^2/an + y_{...}^2/abn$, as in (14.40).

14.13 **(a)** In a comment following (14.41), it was noted that the use of $\hat{\boldsymbol{\beta}}$ from (14.24) would produce the same result as in (14.41), namely, $\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} = \sum_{ij} y_{ij.}^2/n$. Verify this.

(b) Show that $\text{SSE} = \sum_{ijk} y_{ijk}^2 - n\sum_{ij} \bar{y}_{ij.}^2$ in (14.25) is equal to $\text{SSE} = \sum_{ijk} y_{ijk}^2 - \sum_{ij} y_{ij.}^2/n$ in (14.46).

14.14 Show that $(a-1)(b-1)$ is the number of independent γ_{ij}^* terms in H_0: $\gamma_{ij}^* = 0$ for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$, as noted near the end of Section 14.4.1.2.

14.15 Show that $\text{SS}(\gamma|\mu, \alpha, \beta) = n\sum_{ijk} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$ in (14.47) is the same as $\text{SS}(\gamma|\mu, \alpha, \beta)$ in (14.45).

14.16 Show that $\text{SSE} = \sum_{ijk} (y_{ijk} - \bar{y}_{ij.})^2$ in (14.48) is equal to $\text{SSE} = \sum_{ijk} y_{ijk}^2 - \sum_{ij} y_{ij.}^2/n$ in (14.46).

14.17 Using $\mathbf{X}'\mathbf{X}$ in (14.5) and $(\mathbf{X}'\mathbf{X})^-$ in (14.23), show that $\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ has the form given in (14.50).

14.18 **(a)** Show that $(\mathbf{X}_1'\mathbf{X}_1)^-$ in (14.53) is a generalized inverse of $\mathbf{X}_1'\mathbf{X}_1$ in (14.42).

(b) Show that $\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}_1'$ has the form given by (14.54).

14.19 Show that $\frac{1}{6}\sum_{j=1}^{2} y_{.j.}^2$ can be written in the matrix form given in (14.59).

14.20 Show that $\frac{1}{2}\mathbf{A} - \frac{1}{4}\mathbf{B} - \frac{1}{6}\mathbf{C} + \frac{1}{12}\mathbf{D}$ has the value shown in (14.61).

14.21 Show that H_0: $\alpha_1^* = \alpha_2^* = \alpha_3^*$ in (14.63) is equivalent to H_0: $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$ in (14.64).

14.22 Obtain $\text{SS}(\mu, \alpha, \gamma)$ and show that $\text{SS}(\beta|\mu, \alpha, \gamma) = \sum_{j=1}^{b} y_{.j.}^2/bn - y_{...}^2/abn$ as in (14.68).

14.23 Prove Theorem 14.4b for the special case $a = 3$, $b = 2$, and $n = 2$.

14.24 **(a)** Using $\mathbf{C}$ in (14.74), $(\mathbf{X}'\mathbf{X})^-$ in (14.23), and $\mathbf{X}$ in (14.4), show that $\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is the 2×12 matrix given in (14.77).

(b) Using $\mathbf{C}$ in (14.74) and $(\mathbf{X}'\mathbf{X})^-$ in (14.23), show that $\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{C}'$ is the 2×2 matrix shown in (14.78).

(c) Show that the matrix $\mathbf{A} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{C}']^{-1}\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ has the form shown in (14.79).

14.25 For the B main effect, formulate a hypothesis H_0: $\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ and obtain $\text{SS}(\beta|\mu, \alpha, \gamma)$ using SSH in (14.75).

14.26 Using assumptions 1, 2, and 3 in Section 14.1, show that $E(\varepsilon_{ijk}^2) = \sigma^2$ for all i, j, k and $E(\varepsilon_{ijk}\varepsilon_{rst}) = 0$ for $(i, j, k) \neq (r, s, t)$, as in (14.82) and (14.82).

14.27 Show that $E(\sum_{i=1}^{a} y_{i..}^2) = ab^2n^2\mu^{*2} + b^2n^2\sum_{i=1}^{a}\alpha_i^{*2} + abn\sigma^2$ as in (14.84).

14.28 **(a)** Show that $E(\sum_{j=1}^{b} y_{.j.}^2) = a^2bn^2\mu^{*2} + a^2n^2\sum_{j=1}^{b}\beta_j^{*2} + abn\sigma^2$.

(b) Show that $E(\sum_{ij} y_{ij.}^2) = abn^2\mu^{*2} + bn^2\sum_i \alpha_i^{*2} + an^2\sum_j \beta_j^{*2} + n^2\sum_{ij}\gamma_{ij}^{*2} + abn\sigma^2$.

(c) Show that $E[\text{SS}(\beta|\mu, \alpha, \gamma)/(b-1)] = \sigma^2 + an\sum_j \beta_j^{*2}/(b-1)$.

(d) Show that $E[\text{SS}(\gamma|\mu, \alpha, \beta)/(a-1)(b-1)] = \sigma^2 + n\sum_{ij}\gamma_{ij}^{*2}/(a-1)(b-1)$.

TABLE 14.6 Lactic Acid[a] at Five Successive Time Periods for Fresh and Wilted Alfalfa Silage

	Period				
Condition	1	2	3	4	5
Fresh	13.4	37.5	65.2	60.8	37.7
	16.0	42.7	54.9	57.1	49.2
Wilted	14.4	29.3	36.4	39.1	39.4
	20.0	34.5	39.7	38.7	39.7

[a]In mg/g of silage.

14.29 Using $\mathbf{C}$ in (14.74) and $(\mathbf{X}'\mathbf{X})^-$ in (14.23), show that $\mathbf{L} = \mathbf{C}'[\mathbf{C}(\mathbf{X}'\mathbf{X})^-\mathbf{C}']^{-1}\mathbf{C}$ has the form shown in (14.88).

14.30 Expand $\sum_{i=1}^{3} (\alpha_i - \bar{\alpha}_. + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2$ in (14.91) to obtain (14.90).

14.31 **(a)** Show that $E(\bar{y}_{...}) = \mu + \bar{\alpha}_. + \bar{\beta}_. + \bar{\gamma}_{..}$ as in (14.94).

(b) Show that $E(\bar{y}_{.j.}) = \mu + \bar{\alpha}_. + \beta_j + \bar{\gamma}_{.j}$.

(c) Show that $E(\bar{y}_{ij.}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$.

14.32 Obtain the following expected values using the method suggested by (14.92) and illustrated at the end of Section 14.5.2. Use the results of Problem 14.31b, c.

(a) $E[\mathrm{SS}(\beta|\mu, \alpha, \gamma)] = \sigma^2 + 6\sum_j \beta_j^{*2}$

(b) $E[\mathrm{SS}(\gamma|\mu, \alpha, \beta)] = 2\sigma^2 + 2\sum_{ij} \gamma_{ij}^{*2}$

TABLE 14.7 Hemoglobin Concentration (g/mL) in Blood of Brown Trout[a]

Rate:	1		2		3		4	
Method:	A	B	A	B	A	B	A	B
	6.7	7.0	9.9	9.9	10.4	9.9	9.3	11.0
	7.8	7.8	8.4	9.6	8.1	9.6	9.3	9.3
	5.5	6.8	10.4	10.2	10.6	10.4	7.8	11.0
	8.4	7.0	9.3	10.4	8.7	10.4	7.8	9.0
	7.0	7.5	10.7	11.3	10.7	11.3	9.3	8.4
	7.8	6.5	11.9	9.1	9.1	10.9	10.2	8.4
	8.6	5.8	7.1	9.0	8.8	8.0	8.7	6.8
	7.4	7.1	6.4	10.6	8.1	10.2	8.6	7.2
	5.8	6.5	8.6	11.7	7.8	6.1	9.3	8.1
	7.0	5.5	10.6	9.6	8.0	10.7	7.2	11.0

[a]After 35 days of treatment at the daily rates of 0, 5, 10, and 15 g of sulfamerazine per 100 lb of fish employing two methods for each rate.

14.33 A preservative was added to fresh and wilted alfalfa silage (Snedecor 1948). The lactic acid concentration was measured at five periods after ensiling began. There were two replications. The results are given in Table 14.6. Let factor A be condition (fresh or wilted) and factor B be period. Test for main effects and interactions.

14.34 Gutsell (1951) measured hemoglobin in the blood of brown trout after treatment with four rates of sulfamerazine. Two methods of administering the sulfamerazine were used. Ten fish were measured for each rate and each method. The data are given in Table 14.7. Test for effect of rate and method and interaction.