STAT 8004, Assignment 4

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Answer to Question 1

- Let α_i be the fixed effect of the i^{th} temperature level, $i = 1, 2, 3$.
- Let u_{ij} be the random effect of the j^{th} cooler at the i^{th} temperature level, $j = 1, 2, 3, 4$.
- Let ϵ_{ijk} be the error associated with the k^{th} cut for the j^{th} cooler at the i^{th} temperature level, $k = 1, 2$.
- Let y_{ijk} be the scores assigned by the putative meat scoring experts, with i, j, k in the range just mentioned.

The model is then:

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{141} \\ y_{142} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \\ y_{241} \\ y_{242} \\ y_{311} \\ y_{312} \\ y_{321} \\ y_{322} \\ y_{331} \\ y_{332} \\ y_{341} \\ y_{342} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \\ u_{31} \\ u_{32} \\ u_{33} \\ u_{34} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{141} \\ \epsilon_{142} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \\ \epsilon_{241} \\ \epsilon_{242} \\ \epsilon_{311} \\ \epsilon_{312} \\ \epsilon_{321} \\ \epsilon_{322} \\ \epsilon_{331} \\ \epsilon_{332} \\ \epsilon_{341} \\ \epsilon_{342} \end{bmatrix}.$$

Or, to be a bit more space-saving about the whole thing,

$$y_{ijk} = \mu + \alpha_i + u_{ij} + \epsilon_{ijk} \quad \text{for } 1 \le i \le 3, 1 \le j \le 4, 1 \le k \le 2.$$

Question 2

The model $\mathbf{y} = \mathbf{X}\mu + \epsilon$ here is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

With the covariance matrix

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

We see that

$$\text{rank}(X) = 2, \quad \text{so that} \quad n - \text{rank}(X) = 3 - 2 = 1$$

Consider then the following matrix $\mathbf{B}$ that satisfies $\mathbf{B}\,\mathbf{X} = 0$, with $\text{rank}(\mathbf{B}) = 1$:

$$\mathbf{B} = (1, -1, 0)$$

Then, according to the discussion at the end of lecture 6[1], $\mathbf{B}\,\mathbf{y} = (1, -1, 0) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 - y_2$ will be normally distributed with:

$$\begin{aligned} \mathbf{B}\,\mathbf{y} &\sim \mathcal{N}(\,0,\ \mathbf{B}\,\Sigma\,\mathbf{B}^T\,) \\ &\sim \mathcal{N}\left(\,0,\ \sigma^2 (1, -1, 0) \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\,\right) \\ &\sim \mathcal{N}(\,0,\ \sigma^2\,) \end{aligned}$$

Thus

$$\boxed{y_1 - y_2 \sim \mathcal{N}(\,0,\ \sigma^2\,)}$$

Now compute the log-likelihood and maximize it by setting its derivative with respect to σ^2 to zero:

$$\begin{aligned} l(\sigma^2) &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2}\frac{1}{\sigma^2}(y_1 - y_2)^2 \\ \frac{d(l(\sigma^2))}{d\sigma^2} &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(y_1 - y_2)^2 = 0 \end{aligned}$$

Therefore

$$\boxed{\hat{\sigma}^2 = (y_1 - y_2)^2}$$

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[1] For the choice of $\mathbf{M}$ also discussed in the lecture, we could compute the projection matrices:

$$\mathbf{P}_X = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{P}_X = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M} = (1, -1, 0)$$

Then

$$\mathbf{M}(\mathbf{I} - \mathbf{P}_X)\mathbf{y} = (1, -1, 0) \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y} = \mathbf{B}\mathbf{y} = (1, -1, 0) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 - y_2$$