STAT 8004, Assignment 8

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Answer to Question 1

The problem data:

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Answer to Question 2

In this model, for the k^{th} piece of the j^{th} roll on the i^{th} machine we have

$$y_{ijk} = \mu + \alpha_i + u_{ij} + \epsilon_{ijk},$$

where all effects except μ are random.

This is a *nested* model where j^{th} piece of roll is nested in the i^{th} machine.

We have $a = 3$ machines (indexed by i), $b = 5$ rolls (indexed by j) from each machine, and $c = 5$ pieces (indexed by k) from each roll:

Table 1: Data for Question 2, Notation Used

Source	SS	df	(notation for `df`)	Mean Squares
Machines	1966	$3-1=2$	$=a-1$	$1966/2 = 983 = \text{MSA}$
Rolls	644	$3\cdot(5-1)=12$	$=a(b-1)$	$644/12 = 53.667 = \text{MSB(A)}$
Pieces	280	$3\cdot 5\cdot(5-1)=60$	$=ab(c-1)$	$280/60 = 4.667 = \text{MSE}$
Total	2890	74	$abc-1$	

(a) Based on the lecture, we know the following to be true (page 3, lecture 8):

$$\begin{aligned}
\mathbf{E}\,(\text{MSE}) &= \sigma_\epsilon^2 \\
\mathbf{E}\,(\text{MSB(A)}) &= \sigma_\epsilon^2 + 5\cdot\sigma_u^2 \\
\mathbf{E}\,(\text{MSA}) &= \sigma_\epsilon^2 + 5\cdot\sigma_u^2 + 5\cdot 5\cdot\sigma_\alpha^2
\end{aligned}$$

Based on these we obtain the *estimates of the variance components:*

$$\begin{aligned}
\hat{\sigma}_\epsilon^2 &= 4.667 \\
\hat{\sigma}_u^2 &= \frac{1}{5}(53.667 - 4.667) = 9.8 \\
\hat{\sigma}_\alpha^2 &= \frac{1}{25}(983 - 53.667) = 37.1733
\end{aligned}$$

(b)

- 95% Confidence Interval for σ_ϵ^2 :

  We make use of the fact that $\frac{60\cdot\text{MSE}}{\sigma_\epsilon^2}$ is χ^2 distributed with 60 degrees of freedom, and compute the 95% confidence limits for σ_ϵ^2:

```
SSE <- 280
alpha <- .05
lower.limit <- SSE / qchisq(1 - alpha/2, df=60)
upper.limit <- SSE / qchisq(alpha/2, df=60)
CI_error <- sqrt(c(lower.limit, upper.limit))
```

  Thus:

$$\boxed{\text{95\% Confidence Interval for } \sigma_\epsilon = (1.833423, 2.629961)}$$

- 95% Confidence Interval for σ_u^2 :

  We first use the Cochran – Saterthwaite procedure to compute the needed degrees of freedom. Using the results in part a, we have:

$$\hat{\nu} = \frac{(\hat{\sigma}_u^2)^2}{\frac{(\text{MSB(A)}/5)^2}{12} + \frac{(-\text{MSE}/5)^2}{60}} = 9.988675$$

  The computation of confidence limits is similar to the one shown above, where this $\hat{\nu}$ is used for the degrees of freedom for the relevant χ^2 distribution.

  The full `R` script that was used to obrain the results is attached in the appendix.

  Thus:

$$\boxed{\text{95\% Confidence Interval for } \sigma_u = (2.186967, 5.496051)}$$

- 95% Confidence Interval for σ_α^2 :
  We do a computation similar to the above. For the degrees of freedom we use:

$$\hat{v} = \frac{(\hat{\sigma}_\alpha^2)^2}{\frac{(\text{MSA}/25)^2}{2} + \frac{(-\text{MSB(A)}/25)^2}{12}} = 1.786695$$

  Thus:

$$\boxed{\text{95\% Confidence Interval for } \sigma_\alpha = (3.097864, 46.297890)}$$

(c) The 95% Confidence Interval for μ is

$$\left(\bar{y}_{...} - \texttt{qt(0.975,df=2)} \sqrt{MSA/75},\ \bar{y}_{...} + \texttt{qt(0.975,df=2)} \sqrt{MSA/75} \right)$$

Which, given $\bar{y}_{...} = 1/75 \sum_{ijk} y_{ijk} = 35$ and MSA from the table, is:

$$\boxed{\text{95\% Confidence Interval for } \mu = (19.42305, 50.57695)}$$

1 Appendix

```
##################################################################
########################### Question 2 ###########################
##################################################################
# from the table:
MSE <- 280/60
MSB_A <- 644/12
MSA <- 1966/2

############################# part(a)#############################
sigma2_c <- MSE                    # ans: 4.666667
sigma2_u <- 1/5 * (MSB_A - MSE)    # ans: 9.8
sigma2_a <- 1/25 * (MSA - MSB_A)   # ans: 37.17333

############################# part(b)#############################
# for sigma2_c:
alpha <- .05
lower.limit <- 60 * MSE / qchisq(1 - alpha/2, df=60)
upper.limit <- 60 * MSE / qchisq(alpha/2, df=60)
CI_error <- sqrt(c(lower.limit, upper.limit))

# for sigma2_u:
v <- (sigma2_u)^2 / ( ((-MSE/5)^2 / 60)  + (MSB_A/5)^2 / 12 )
lower.limit <- (v*sigma2_u) / qchisq(1 - alpha/2, df=v)
upper.limit <- (v*sigma2_u)/ qchisq(alpha/2, df=v)
CI_u <- sqrt(c(lower.limit, upper.limit))

# for sigma2_a:
v <- (sigma2_a)^2 / ( ((MSA/25)^2 / 2)  + (-MSB_A/25)^2 / 12 )
lower.limit <- (v*sigma2_a) / qchisq(1 - alpha/2, df=v)
upper.limit <- (v*sigma2_a)/ qchisq(alpha/2, df=v)
CI_a <- sqrt(c(lower.limit, upper.limit))

############################# part(c)#############################
# 95\% Confidence Interval for \mu:
qt(0.975,df=2)
CI_mu <- c( 35 - qt(0.975,df=2) * sqrt( MSA / 75 ), 35 + qt(0.975,df=2) * sqrt
CI_mu
```