Orbit Determination by the Method of Gauss

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Based on notes by Adam Rengstorf and others listed in the Preface.

(Last update, Feb 22, 2024.)

Preface.

This document is largely based on the marvelous Orbit Determination (OD) Packet by Adam Rengstorf, and (in previous incarnations) by William Andersen, Tracey Furutani, Agnes Kim, Tom Steiman-Camero, Gary Einhorne, Amy Barr-Mlinar and, probably some others. I learned most of what I know about Orbit Determination from the Rengstorf Packet. My purpose in writing this new document is to supplement the OD packet with (perhaps) simpler arguments. Another goal of mine was to write a shorter document, but in this I failed.

In Section II, Theory of Orbits, I derive all the needed theoretical results using the simplest techniques that I could find in the literature. In particular, I wanted to avoid the use of the vector identity $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$, which most SSP students view as a black box. As much as possible, I want a proof to look like a clear physical argument, rather than a mathematical exercise. But this is not always possible.

Section IV, the Method of Gauss, is a nearly *verbatim* transcript of a PowerPoint presentation that I inherited from previous SSP instructors including Tracy Furutani, William Andersen, Adam Rengstorf, as well as former TA Razvan Ungureanu. What I found most appealing about this particular presentation of MoG, is that it gives a straightforward way to make initial estimates of the position and velocity of the asteroid on the middle night, estimates which are needed to start the iterative MoG calculation. With this method, the (rather complicated) scalar equation of Lagrange is unnecessary.

Appendix II, which is a quick summary of the Method of Gauss, was inspired by a very similar document by Bill Andersen.

I gratefully acknowledge use the of physics/math notes written by my older brother Ronald when he was a graduate student at the University of Wisconsin circa 1972.

Finally, I thank my very conscientious colleague, Prof. Cassandra Fallscheer, for proof-reading and editorial comments on initial drafts of this document.

Section I: Preliminaries

"For in those days I was in the prime of my age for inventions ..."

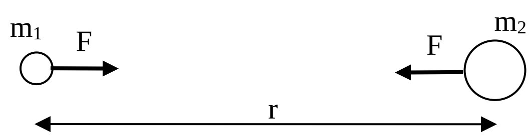
I.A. Newton, Gravity, and the 2-body Problem.

At age 22, Isaac Newton (1643-1727) was forced to leave his work at Cambridge University and spend nearly 2 years at his family's estate in the countryside. England was in lockdown due to an epidemic of bubonic plague. During this long vacation, Newton invented calculus, the science of optics, and what we now call Newtonian Mechanics.

Included in his Mechanics was a statement of Newton's Universal Law of Gravitation: Any two masses m_1 and m_2 exert an attractive gravitational force on each other according to

$$F \;=\; G\,\frac{m_1\;m_2}{r^2} \qquad\qquad (1)$$

Fig. 1



This applies to all masses, not just big ones. "Big G" = G is the universal constant of gravitation $= \;6.67 \times 10^{-11}\ \mathrm{N\,m^2 / kg^2}$. G is very small, so it is very difficult to measure precisely.

Newton showed that the force of gravity must act according to this rule in order to produce the observed motions of the planets around the sun, of the moon around the earth, and of projectiles near the earth. He realized that this same force acts between *all* masses. [That gravity acts between all masses, even small ones, was experimentally verified in 1798 by Cavendish.]

Newton couldn't say *why* gravity acts this way, only *how*. Einstein's (1915) General Theory of Relativity explained why gravity acts like this.

Newton showed that the problem of 2 bodies interacting by gravity could be reduced to a 1-body problem. Consider two masses m_1 and m_2, at positions $\vec{r}_1$ and $\vec{r}_2$, separated by displacement $\vec{r} \;=\; \vec{r}_2 - \vec{r}_1$ interacting by mutual gravitation (Fig.2). The center of the mass of the system, $\vec{R}$, is defined by

$$M\vec{R} \;=\; m_1\,\vec{r}_1 + m_2\,\vec{r}_2 \;, \qquad\qquad (2)$$

where $M = (m_1 + m_2)$ is the total mass of the system. Applying Newton's 2nd law to each body, we have

$$m_1\,\ddot{\vec{r}}_1 \;=\; +\frac{G\,m_1\,m_2}{r^2}\,\hat{r} \qquad \text{and} \qquad m_2\,\ddot{\vec{r}}_2 \;=\; -\frac{G\,m_1\,m_2}{r^2}\,\hat{r} \qquad\qquad (3)$$

$$\ddot{\vec{r}} \;=\; \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \;=\; -\frac{G(m_1 + m_2)}{r^2}\,\hat{r} \qquad . \qquad\qquad (4)$$

From (2), we have

$$\ddot{\vec{R}} = \frac{1}{M}\left(m_1\,\ddot{\vec{r}}_1 + m_2\,\ddot{\vec{r}}_2\right) = \frac{1}{M}\left(\vec{F}_{12} + \vec{F}_{21}\right) = 0 \qquad (5)$$

The last term, $\left(\vec{F}_{12} + \vec{F}_{21}\right) =$ [(force on 1 due to 2) + (force on 2 due to 1)], must be zero, by Newton's 3rd Law, $\vec{F}_{12} = -\vec{F}_{21}$. Therefore, the center-of-mass $\vec{R}$ is not accelerating, and, without loss of generality, we can choose an inertial reference frame in which the center-of-mass is stationary.

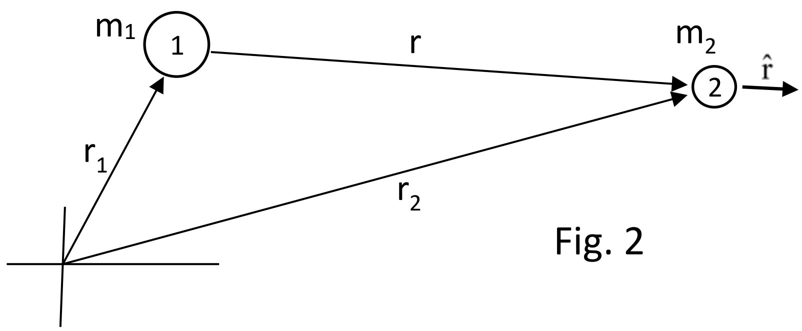


Fig. 2

We define a *reduced mass*, $\mu = \dfrac{m_1\,m_2}{m_1 + m_2}$. Equation (4) can then be written as

$$\frac{m_1\,m_2}{m_1+m_2}\ddot{\vec{r}} = -\frac{G(m_1+m_2)}{r^2}\,\frac{m_1\,m_2}{m_1+m_2}\,\hat{r}, \qquad \text{or}$$

$$\mu\,\ddot{\vec{r}} = -\frac{G\,m_1\,m_2}{r^2}\,\hat{r} \qquad (6)$$

We have managed to replace our 2-body problem (3), with a 1-body problem (6) for a fictitious mass μ.

The case of interest to us is the orbit of a small asteroid about the sun. For our near-earth asteroids (NEA's), the mass m is less than $10^{-11}\,M_\odot$. Since m << M, we have

$\mu = \dfrac{m\;M_\odot}{m+M_\odot} \cong m$, and $G\,(M_\odot + m) \cong G\,M_\odot$.

In orbital mechanics, the symbol μ is usually reserved for something other than the reduced mass; from now on, we follow this convention and define $\mu \equiv G\,(M_\odot + m)$. Confusion with the reduced mass [mM/(m+M)] is a danger. From now on, the symbol μ will mean $\mu \equiv G\,(M_\odot + m)$.

In the limit m << M, the reduced mass becomes m, the quantity $\mu \equiv G\,(M_\odot + m)$ becomes $\mu = G\,M_\odot$, and equation (6) becomes

$$\ddot{\vec{r}} = -\frac{G M_\odot}{r^2}\,\hat{r} \quad . \qquad (7)$$

We will solve this equation of motion in Section II. Incidentally, the 3-body problem can<u>not</u> be similarly reduced to a 2-body problem. The general 3-body problem has no analytic solution; only approximate numerical solutions exist.

I.B. Orbital Elements

To completely specify the orbit of a small mass m gravitationally bound to a large stationary mass M, we need 7 numbers: the 3 components of the initial position $\vec{r}$, the 3 components of the initial velocity $\dot{\vec{r}}$, and the initial time t.

Another way to specify an orbit is to give the *elements of the orbit*:

a = semi-major axis
e = eccentricity
i = inclination
ω = longitude of the ascending node
Ω = argument of perihelion
θ = true anomaly at a given time t (or the mean anomaly M, or the eccentric anomaly E, at t).

The parameters a and e specify the size and shape of the orbital ellipse. To specify the orientation of the ellipse in space, we need 3 angles: i, ω, and Ω. And finally, to specify the position of the asteroid along the ellipse at some time t, we need an angle (either M, or θ, or E).

The angles i, ω, and Ω are shown in Fig. 3. The plane of reference is the plane of the ecliptic. The reference direction is the vernal equinox point, zero hours RA and zero degrees DEC.

The inclination i is the angle between the plane of the orbit and the plane of the ecliptic.

The longitude of ascending node Ω is defined as follows. The line of nodes is the intersection of the orbital plane and the ecliptic plane. The angle Ω is the angle between the direction of the vernal equinox (the reference direction) and the line of nodes on the ascending side of the orbit, with the angle measured counterclockwise as viewed from the north celestial pole.

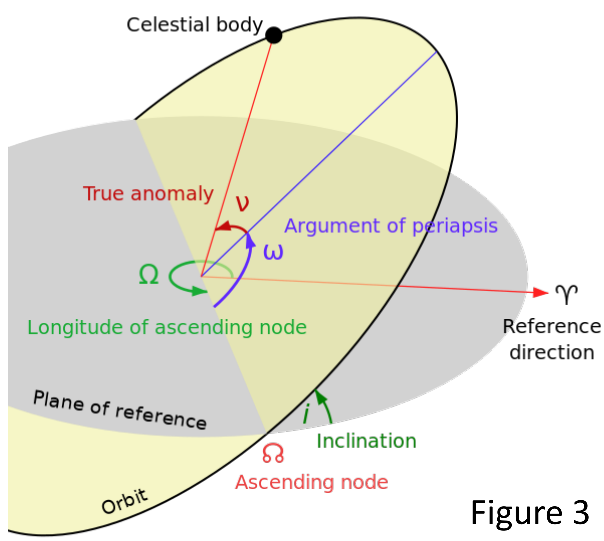


Figure 3

The argument of perihelion ω (the argument of periapsis in the figure,) is the angle between the ascending node line and the line from the sun to the perihelion point.

The true anomaly θ (labeled ν in Fig.3), at a given time t, is the angle between the perihelion line and the line from the sun to the asteroid. We will use the symbols θ and ν (Greek letter nu) for the true anomaly interchangeably. The mean anomaly M and the eccentric anomaly E will be defined in section II.E.

Section II: Theory of Orbits.

In this section, we establish all the theoretical results we will need for both ephemeris generation and orbit determination.

II.A. Kepler's Laws

Using position data from the Danish astronomer Tycho Brahe (1546-1601), the German mathematician/astronomer Johannes Kepler (1571-1630) deduced three laws of planetary motion. ("Brahe" is pronounced BRA-hay.)

KI. The shape of the orbit of a planet around the sun is an ellipse with the sun at one focus.

KII. In its elliptical orbit around the sun, a planet goes faster when it is nearer the sun and slower when it is further, in such a way that a line drawn from the sun to the planet sweeps out equal areas in equal intervals of time.

KIII. The period P of a planet and the semi-major axis a of its elliptical orbit are related by $\frac{P^2}{a^3} = \text{constant}$, where the constant has the same value for all the planets orbiting the sun.

Kepler had no idea why these rules were true. He simply observed that the orbits of the planets obeyed these rules. In 1665, some 35 years after Kepler's death, Isaac Newton was able to *derive* these rules from his second law, $\vec{F}_{net} = m\vec{a}$, and his Universal Law of Gravitation.

Incidentally, the data which Tycho Brahe collected were remarkably precise. Brahe was a naked-eye astronomer; he lived before the invention of the telescope in 1608. Brahe was a consummate engineer and he designed and built huge devices for measuring the angles between objects in the sky. He was able to determine the positions of stars and planets to within 1 or 2 arc minutes (1 arcmin = 1/60 degree) which is the limit of resolution of the human eye. This is an astonishing feat. Equally astonishing, Kepler deduced his laws with pen and paper, a task that took some 9 years for KI and KII and another 10 years for KIII. Kepler might never have discovered KIII, but for the timely invention of logarithms by the Scottish mathematician Napier in 1614.

II.B. Properties of ellipses.

Before deriving Kepler's Laws, let us review the properties of ellipses. There are at least 4 equivalent ways of defining an ellipse.

1. In cartesian coordinates, the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (8)$$

where a is the *semimajor axis* and b is the *semiminor axis*. Here, the origin is at the center of the ellipse.

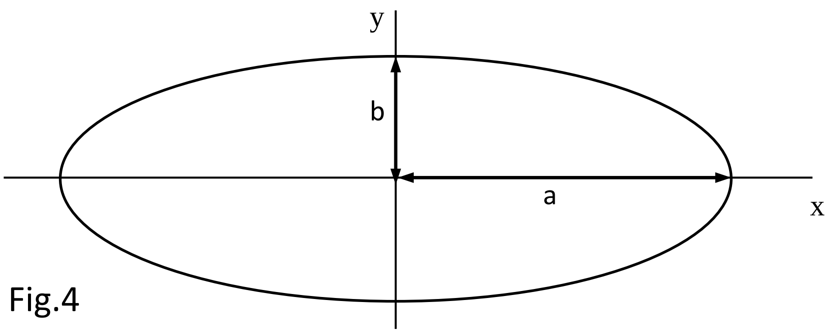


Fig.4

2. An ellipse is the locus of points for which the sum of the distances to two fixed points, the foci, is a constant:

$$r + r' = \text{constant} = 2a \qquad (9)$$

where a is the semimajor axis. The two foci are each at a distance a·e from the center, where e is the *eccentricity* (e < 1).

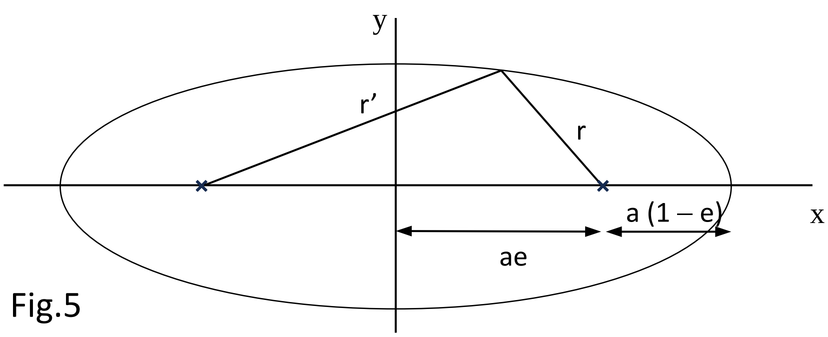


Fig.5

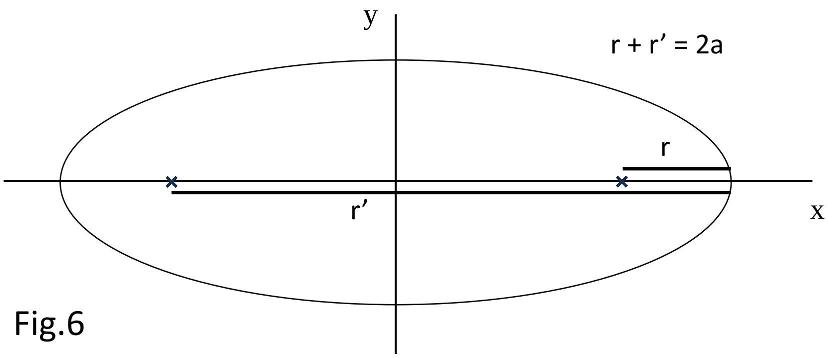


Fig.6

3. In polar coordinates, with the origin at one focus, the equation of an ellipse is

$$r = \frac{\ell}{1 + e\cos\theta} \tag{10}$$

where e is the eccentricity and ℓ is a constant, amusingly called the *semi-latus rectum*. Notice that r = ℓ when $\theta = \pi/2$.

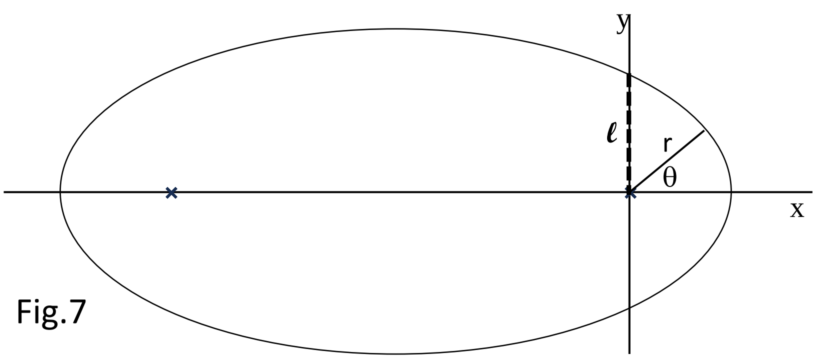
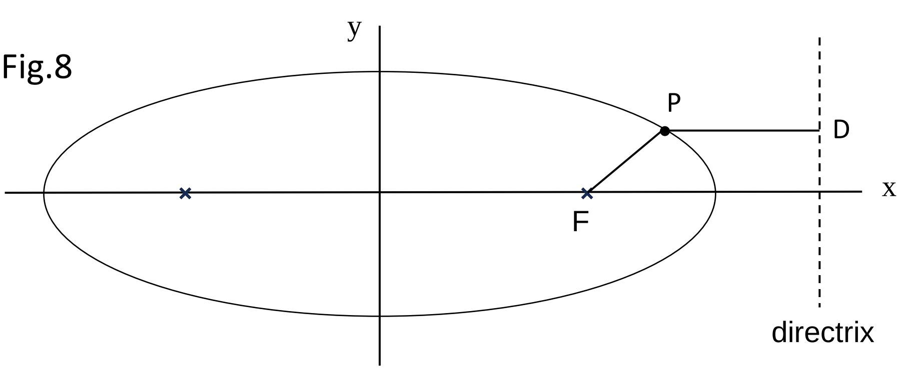


Fig.7

4. An ellipse is the locus of points P such that the ratio of the distance from a fixed point F (the focus) over the distance to a fixed line (the directrix) is a constant, the eccentricity:

$$\frac{FP}{DP} = e. \tag{11}$$

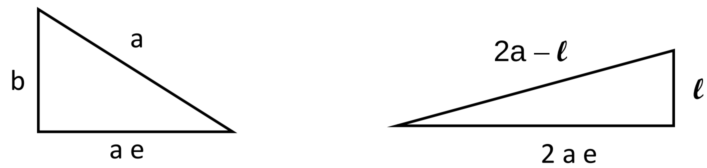
Fig.8



We will not have occasion to use this 4th definition. I include it just for completeness.

From the figures above, we can draw some useful right triangles:

Fig.9



From these right triangles, we get two useful relations:

$$b = a\sqrt{1-e^2} \tag{12}$$

$$\ell = a(1-e^2) \tag{13}$$

From any one of these 4 definitions of an ellipse, one can derive the other three. For instance, consider Fig.5 with the angle θ indicated.

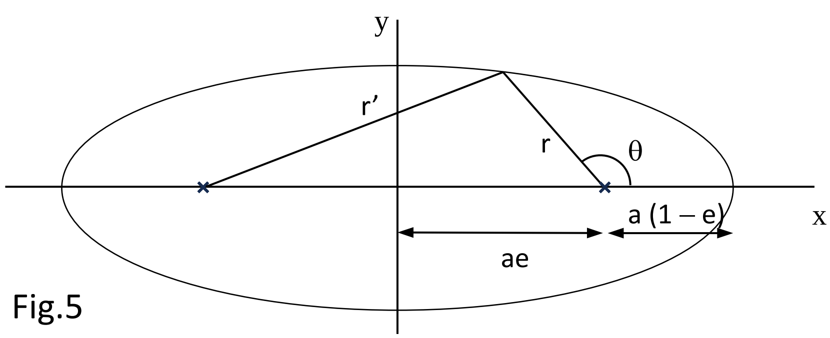


Fig.5

From the Law of Cosines, we can write

$$(2ae)^2 + r^2 + 4ae\, r\cos\theta = r'^2 \quad .$$

From definition 2, we have

$$r' = 2a - r \quad .$$

If we combine these last two equations to eliminate r', and solve for r, we get definition 3:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{\ell}{1+e\cos\theta} \quad .$$

II.C. Derivation of Kepler's Laws.

We are now able to derive Kepler's Laws. We consider the case of a small satellite of mass m in orbit around a large stationary mass M, with m << M. This is a very good approximation for a near-earth asteroid orbiting the Sun, for which $m/M \cong 10^{-11}$. But it is not such a good approximation for earth orbiting the Sun, for which $m/M \cong 10^{-6}$.

The acceleration of the satellite is (by Newton's 2nd Law) $\vec{a} = \dfrac{\vec{F}_{grav}}{m}$. From Newton's Law of Gravitation, this becomes (eqn 7)

$$\ddot{\vec{r}} \;=\; -\frac{G\,M_\odot}{r^2}\,\hat{r} \;=\; -\frac{\mu}{r^2}\,\hat{r} \qquad (14)$$

From here on we will use the notation $\mu = G\,M_\odot$. More precisely, μ is defined as $G\,(M_\odot + m)$ but we are assuming m << M. We also will use the notation $k = \sqrt{GM_\odot} = \sqrt{\mu}$.

Our task is to solve the equation of motion (14) and show that the solution is an ellipse of the form (10). We begin by expanding the acceleration vector $\ddot{\vec{r}}$. In spherical coordinates

$$\vec{r} \;=\; r\,\hat{r} \qquad (15)$$

$$\dot{\vec{r}} \;=\; \dot{r}\,\hat{r} + r\,\dot{\hat{r}} \;=\; \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} \qquad (16)$$

We have used $\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}$ in the last step.

$$\ddot{\vec{r}} \;=\; \ddot{r}\,\hat{r} \;+\; \dot{r}\,\dot{\hat{r}} \;+\; \dot{r}\,\dot{\theta}\,\hat{\theta} \;+\; r\,\ddot{\theta}\,\hat{\theta} \;+\; r\,\dot{\theta}\,\dot{\hat{\theta}}$$

Using $\dot{\hat{\theta}} = -\dot{\theta}\,\hat{r}$, this becomes

$$\ddot{\vec{r}} \;=\; \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} \;+\; \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\theta} \qquad (17)$$

Comparing this with eqn (14), we have

$$\ddot{r} - r\dot{\theta}^2 \;=\; -\frac{\mu}{r^2} \qquad (18)$$

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0 \qquad (19)$$

Equation (19), $r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0$, is a statement of the conservation of angular momentum. The angular momentum of a particle at position $\vec{r}$ with linear momentum $\vec{p} = m\,\vec{v}$ is defined as

$$\vec{L} \;=\; \vec{r} \times \vec{p} \;=\; \vec{r} \times m\,\vec{v} \qquad (20)$$

We define the specific angular momentum, the angular momentum per mass m, as

$$\vec{h} = \vec{r} \times \dot{\vec{r}} \qquad (21)$$

The magnitude of the vector $\vec{h}$ is given by

$$h = |\vec{r}|\,|\dot{\vec{r}}| \sin\theta = r^2\,\dot{\theta}\,. \qquad (22)$$

See Fig.18 on page 20, if the last equation is puzzling.

Now, the angular momentum vector of the asteroid is constant in time.

$$\vec{h} = \text{constant}\,.$$

We can see this in two ways: the physics way or the math way. The physics way is to recall that torque is the time rate of change of angular momentum:

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \qquad (23)$$

The force of gravity from the sun on the asteroid is a central force, so the torque is zero. Hence angular momentum is constant.

The math way is to take the time derivative of equation (22):

$$\dot{h} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \qquad (24)$$

By equation (19), this last quantity is zero; hence the magnitude h is constant. The direction of the vector $\vec{h}$ is constant, because the vectors $\vec{r}$ and $\dot{\vec{r}}$ are always in the fixed plane of the orbit, and $\vec{h} = \vec{r} \times \dot{\vec{r}}$ is perpendicular to that fixed plane.

We will have frequent occasion to exploit the fact that

$$h = r^2\,\dot{\theta} = \text{constant} \,. \qquad (25)$$

Returning now to the issue at hand, we wish to solve equation (18), which can now be written in terms of the constant angular momentum:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \qquad (26)$$

We want to rewrite this equation to replace the time derivative $\dfrac{d}{dt}$ with a theta derivative

$$\frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\dot{\theta} = \frac{d}{d\theta}\frac{h}{r^2}\,.$$

This will give us a differential equation in r = r (θ) rather than r = r(t).

We begin with a clever change of variable, $u = \frac{1}{r}$, which leads to

$$r = \frac{1}{u} \quad \Rightarrow \quad \dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{1}{u^2}\frac{du}{d\theta}hu^2 = -\frac{du}{d\theta}h$$

$$\Rightarrow \quad \ddot{r} = -\frac{d^2u}{d\theta^2}\dot{\theta}h = -\frac{d^2u}{d\theta^2}h^2u^2$$

Substituting into equation (26) gives $\quad -\frac{d^2u}{d\theta^2}h^2u^2 - h^2u^3 = -\mu u^2 \quad \Rightarrow$

$$\frac{d^2u}{d\theta^2} = -u + \frac{\mu}{h^2} \qquad . \qquad (27)$$

The solution of this differential equation is

$$u = \frac{1}{r} = A\cos(\theta-\theta_0) + \frac{\mu}{h^2} \qquad (28)$$

where A and θ_0 are constants of integration, set by the initial conditions. One can see that (28) is the solution of (27) in one of two ways. You can substitute the proposed solution (28) into (27) and see that it satisfies the differential equation. Or, if you recall your first course in differential equations, note that $u_h = A\cos(\theta-\theta_0)$ is the solution to the homogeneous equation $\frac{d^2u}{d\theta^2} = -u$, and then note that $u_p = \frac{\mu}{h^2}$ is a particular solution to (27). The most general solution is then $u = u_h + u_p$.

Equation (28) can be rewritten as

$$r(\theta) = \frac{(h^2/\mu)}{1 + \left(Ah^2/\mu\right)\cos(\theta-\theta_0)} \qquad (29)$$

Comparing this with equation (10), we see that (29) is the equation of our elliptical orbit with

$$\ell = \frac{h^2}{\mu} \qquad (30)$$

and with $A\frac{h^2}{\mu} = e$, $\theta_0 = 0$.

Using eqn (13), eqn (30) becomes

$$a\left(1 - e^2\right) \;=\; \frac{h^2}{\mu} \qquad (31)$$

This relation will be useful when we discuss orbital energy.

From eqn (31), eqn (29) is often written in the form

$$r \;=\; \frac{a\,(1-e^2)}{1 + e\cos\theta} \qquad (32)$$

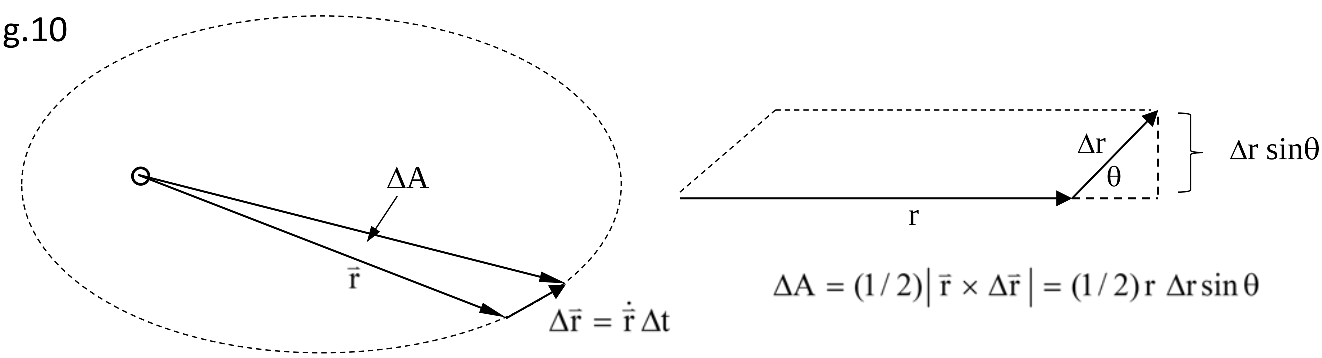
This concludes the proof of KI, Kepler's 1st Law. On to KII.

Kepler's 2nd Law is an immediate consequence of the conservation of angular momentum. In a short time interval Δt, a line from the sun to the asteroid sweeps out an area ΔA, where $\Delta A \;=\; \tfrac{1}{2}\left|\,\vec{r} \times \Delta\vec{r}\,\right|$. Dividing by Δt gives $\dfrac{\Delta A}{\Delta t} \;=\; \tfrac{1}{2}\left|\,\vec{r} \times \dfrac{\Delta\vec{r}}{\Delta t}\right| \quad \Rightarrow$.

$$\frac{dA}{dt} \;=\; \frac{h}{2} \;=\; \text{constant} \qquad (33)$$

Done!

Fig.10



To prove Kepler's 3rd Law, we integrate (33) around one complete orbit: $\int dA \;=\; \dfrac{h}{2}\int dt$. This gives:

$$\pi\, a\, b \;=\; \frac{h}{2}\,P \qquad (34)$$

where P is the period, and the full area of the ellipse is $\pi a b$. From eqn (12), the area can be written as $\pi a^2\sqrt{1 - e^2}$. And, with eqn (31), $h \;=\; \sqrt{\mu a\;(1-e^2)}$, eqn (34) becomes

$$\pi a^2 = \frac{1}{2}\sqrt{\mu a}\, P \qquad \Rightarrow \qquad \frac{P^2}{a^3} = \frac{4\pi^2}{\mu} \tag{35}$$

We have not only proved Kepler's 3rd Law, but we have also shown that the constant in KIII is $\frac{4\pi^2}{G M_\odot}$. Isaac Newton had a brain the size of a planet.

II.D. Energy in orbits

In this section, we prove the *Vis-Viva Equation*

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{36}$$

which gives the speed v of the asteroid as a function of its distance r from the sun. *Vis-Viva* is Latin for "living force"; it is an obsolete term from the early history of mechanics, before the concept of energy was understood.

We begin by writing the equation of energy conservation for the asteroid in its orbit:

$$KE + PE = \frac{1}{2} m v^2 - \frac{GMm}{r} = E_{tot} = \text{constant} \quad \Rightarrow$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon = \text{constant} \tag{37}$$

where ε is the energy per mass, E_{tot}/m .

We can solve for ε by considering the position and velocity of the asteroid at perihelion, where $h = r_p\, v_p$. At the perihelion point,

$$r_p = a - ae = a(1-e), \quad \text{and} \quad v_p = \frac{h}{r_p}$$

Recall eqn (31), $a\left(1-e^2\right) = \frac{h^2}{\mu}$, from which we can write

$$v_p^{\,2} = \frac{h^2}{r_p^{\,2}} = \frac{\mu a (1-e^2)}{a^2 (1-e)^2} = \frac{\mu}{a}\frac{(1+e)}{(1-e)}$$

Eqn (37) then becomes, at the perihelion point,

$$\frac{\mu}{2a}\frac{(1+e)}{(1-e)} - \frac{\mu}{a(1-e)} = \frac{\mu}{2a}\left(\frac{(1+e)}{(1-e)} - \frac{2}{(1-e)}\right) = -\frac{\mu}{2a} = \varepsilon \ .$$

A negative total energy means that we have a bound orbit. Zero total energy occurs when the speed is the escape speed.

With $\varepsilon = -\frac{\mu}{2a}$, eqn (37) becomes $\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$, and this leads immediately to the Vis-Viva Equation (36).

II.E. The Eccentric Anomaly

The angle θ of the asteroid measured from the perihelion point is called the *true anomaly* (see Fig.7). The angle θ and the distance r are complicated functions of time. We need a way to determine θ and r for the asteroid at any given time t. The *eccentric anomaly* E provides a way.

We first define the *mean anomaly* M as

$$M = \frac{2\pi}{P}(t - t_{peri}) = n(t - t_{peri}) \qquad (38)$$

where t_{peri} is the time at which the asteroid is at perihelion, t is the time, and P is the period of the orbit. The constant $n = \frac{2\pi}{P}$ can be computed from Kepler's 3rd Law. From eqn (35),

$$n = \frac{2\pi}{P} = \frac{\sqrt{\mu}}{a^{3/2}} = \frac{k}{a^{3/2}} \qquad (39)$$

Recall that the constant k is defined as $k = \sqrt{GM_\odot} = \sqrt{\mu}$. At perihelion passage, M = 0 and M increases by 2π with each full orbit. Note that n is the average rate of change of the angle θ (in rad), so M is the angle θ the asteroid would have if it moved at this average rate. M would be the same as the true anomaly θ if the orbit was a perfect circle with e = 0.

The *eccentric anomaly* E is defined as follows. The ellipse of the orbit, with semimajor axis *a*, is circumscribed by an *auxiliary circle*, of radius *a* centered on the center of the ellipse. As shown in Fig.11, a vertical line through the position of the asteroid intersects the auxiliary circle at y_a. The eccentric anomaly E is the angle on the auxiliary circle from θ = 0 to the point y_a. We show below that the angles M and E are related by *Kepler's Equation*

$$M = E - e\sin E \qquad (40)$$

Given M, Kepler's equation can only be solved for E numerically.

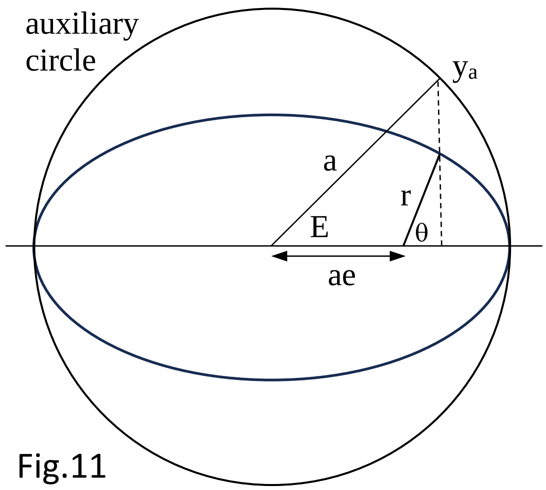


Fig.11

Next, we show that the orbital ellipse can be regarded as a squashed auxiliary circle, compressed along the y-direction by a factor $\frac{b}{a} < 1$. Consider a line of constant x that intersects the ellipse at y_e and intersects the auxiliary circle at y_a, as in Fig.12. We show that

$$\frac{y_e}{y_a} = \frac{b}{a} \quad . \qquad (41)$$

The xy coordinates of points on the auxiliary circle are $x_a = a\cos\theta, \quad y_a = a\sin\theta$

The coordinates of points on the ellipse are $x_e = a\cos\theta, \quad y_e = b\sin\theta$, a fact you can verify by squaring and adding x/a and y/b. You will get the equation of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Observe that $\frac{y_e}{y_a} = \frac{b}{a}$. Done.

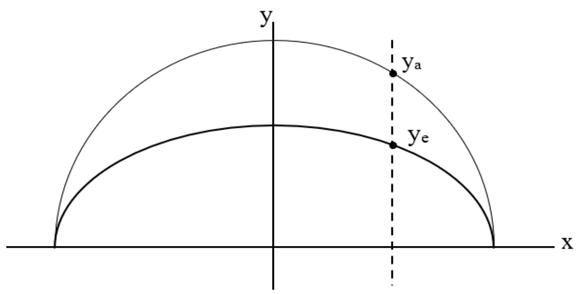


Fig.12

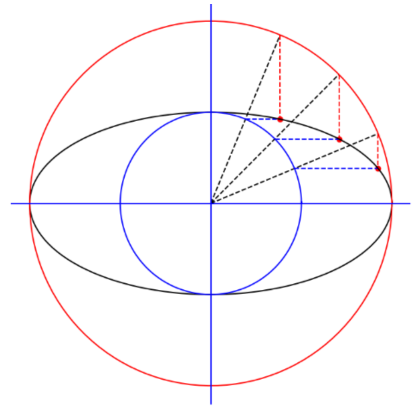


Fig.13. Points on the ellipse are [x,y] = [a cosθ, b sinθ]

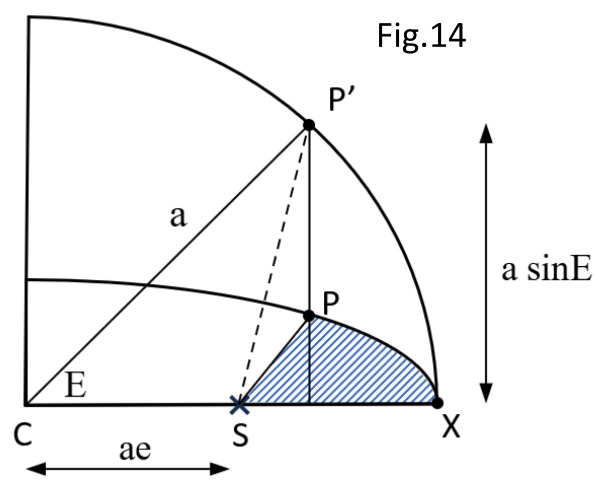
We this “squashing relation” in hand, and referring to Fig.11, we can write the position vector in terms of E as

$$\vec{r} = (a\cos E - ae)\,\hat{i} + \left[\frac{b}{a}(a\sin E)\right]\hat{j} = (a\cos E - ae)\,\hat{i} + \left(a\sqrt{1-e^2}\,\sin E\right)\hat{j} \qquad (42)$$

And from the last equation, one can show that $r = |\vec{r}| = a(1 - e\cos E)$.

Thus, if you know E, you can compute the position of the asteroid.

Now, to prove Kepler’s Equation (40), we refer to Fig. 14. The shaded area is A = SPX. By the “squashing relation” (41), $A = SPX = \left(\frac{b}{a}\right) SP'X$. Also, $SP'X = CP'X - CP'S$. The sector CP’X is the section of a circle of radius a, and its area is $\frac{1}{2}a^2 E$ (with E in radians). The triangle CP’S has area $\frac{1}{2}(a\,e)(a\sin E)$. Putting all this together, we have



$$A = \frac{b}{a}\left(\frac{1}{2}a^2 E - \frac{1}{2}a^2 e\sin E\right) = \frac{1}{2}a\,b\,(E - e\sin E) \qquad (43)$$

Now, by Kepler’s 2nd Law, we can also write the area A as

$$A = \pi a\,b\left(\frac{t - t_{peri}}{P}\right) \qquad (44)$$

Eqn (38) for the mean anomaly is $M = \frac{2\pi}{P}(t - t_{peri})$. Hence, we can write

$$A = \frac{1}{2} a\, b\, M \qquad . \tag{45}$$

Comparing eqns (43) and (45), we have Kepler's Equation: $M = E - e \sin E$.

The utility of the eccentric anomaly is that it allows one to compute the position of the asteroid at any time t. Knowing time t, one can compute the mean anomaly M from eqn (38). Knowing M, one can compute E from Kepler's Equation (40), and knowing E, one can compute the position from eqn (42).

Section III. From $\vec{r}$, $\dot{\vec{r}}$ to Orbital Elements

The Method of Gauss, explained in the next section, starts with the positions of the asteroid in the sky on 3 nights, and from these data, produces the position and velocity $\vec{r}$, $\dot{\vec{r}}$ of the asteroid on the middle night.

Given $\vec{r}$ and $\dot{\vec{r}}$, (in ecliptic coordinates, in units of AU and AU/gaussian day or AU/day) at time t (t in Julian days), we wish to compute the orbital elements: a, e, i, Ω, ω, and T. To compute the time of perihelion passage T, we will need to compute ν, E, and M at the time of observation t. Notice that we are starting with 7 numbers (the x-, y-, and z-components of the vectors $\vec{r}$ and $\dot{\vec{r}}$, and time t) and from these 7 numbers, we compute 6 orbital elements. But T is actually encodes two numbers: it is the time at which ν = 0. So, all is well, from 7 input numbers, we compute 7 output numbers: a, e, i, Ω, ω, T, and ν (where ν = 0).

A comment about units: Recall $\mu = GM_{\odot} = k^2$. [Actually, $\mu = G(M_{\odot} + m_{asteroid})$, but $m_{asteroid} \ll M_{\odot}$]. When time is in days and distance is in AU, then k = 0.0172020989484. To convert from a time interval Δt in days to a time interval Δτ in
Gaussian days, $\Delta\tau = k\,\Delta t$. With distance in AU, and time is in gaussian days, then k = 1, μ = 1.

Semi-major axis a

We compute the semi-major axis a from the Vis-Viva equation (36): $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$,

Notice that $\mu = 1$ if time is measured in Gaussian days. Distance is always measured in AU.

$$\boxed{a = \frac{1}{\left(\frac{2}{r} - \frac{v^2}{\mu}\right)}} \quad \text{where } r = |\vec{r}|,\ v^2 = \dot{\vec{r}} \bullet \dot{\vec{r}}$$

Eccentricity e

Getting the eccentricity e is easy. From eqn (31), $h^2 = \mu a(1-e^2)$, we have

$$\boxed{e = \sqrt{1 - \frac{h^2}{\mu a}}} \quad \text{where } \vec{h} = \vec{r} \times \dot{\vec{r}}\,.$$

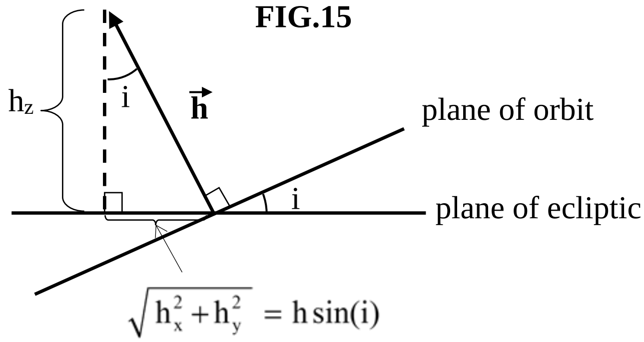
Inclination i

We need an appropriate diagram:

$$\tan i = \frac{\sqrt{h_x^2 + h_y^2}}{h_z}, \text{ or alternatively,}$$

$$\boxed{\cos i = \frac{h_z}{h}} \quad h = \left|\vec{h}\right| = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

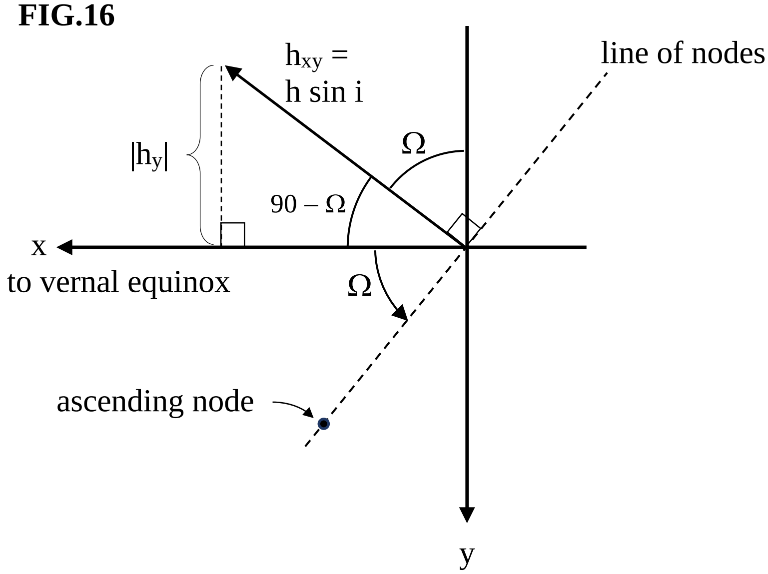
FIG.15



Longitude of ascending node Ω

Start with an instructive diagram (xy = plane of ecliptic) :

FIG.16



The projection of $\vec{h}$ onto the ecliptic plane (xy plane) has magnitude $h_{xy} = h\sin i$ (see Fig.16). Therefore,

$$h_x = (h \sin i) \cos(90 - \Omega) = h \cdot \sin i \cdot \sin \Omega$$

$$h_y = -(h \sin i) \sin(90 - \Omega) = -h \cdot \sin i \cdot \cos \Omega$$

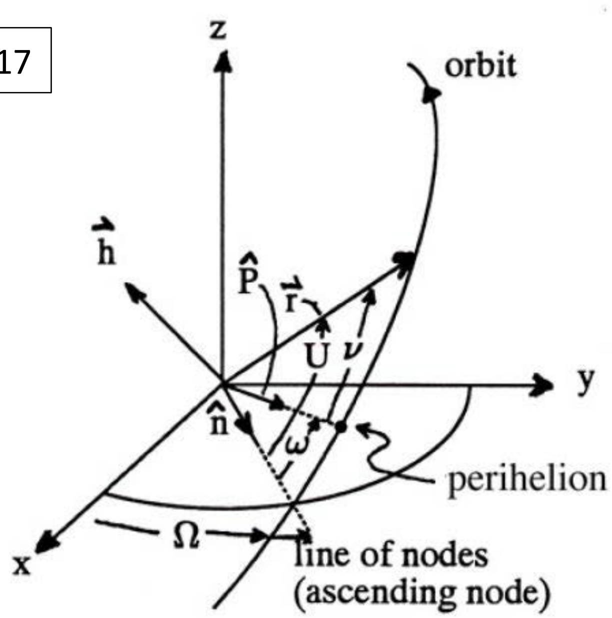
.. from which we can solve for Ω:

$$\left.\boxed{\begin{aligned} \sin\Omega &= \frac{h_x}{h \sin i} \\ \cos\Omega &= -\frac{h_y}{h \sin i} \end{aligned}}\ \right\} \text{ need both for quadrant check}$$

Argument of perihelion ω

Getting ω is rather involved. We begin with the complicated Fig.17 (xy is the plane of the ecliptic):

Fig.17



U is the angle between the line of ascending nodes and the position vector $\vec{r}$.

ν (also called θ) is the true anomaly, the angle between (line from sun to perihelion point) and position $\vec{r}$.

$$\omega = U - \nu$$

We will first compute U and ν, from which we get ω. We compute U first:

$\hat{n}$ = unit vector for the origin (Sun) to ascending node

$$\hat{n} = \hat{x}\cos\Omega + \hat{y}\sin\Omega$$

$$\bar{r} \bullet \hat{n} = (x\,\hat{x} + y\,\hat{y}) \bullet (\cos\Omega\,\hat{x} + \sin\Omega\,\hat{y}) = x\cos\Omega + y\sin\Omega = r\cos U$$

So we have $\cos U = \dfrac{x\cos\Omega + y\sin\Omega}{r}$, but we also need sinU for quadrant check.

To find sinU, we start by writing

$$\hat{n} \times \hat{r} = \hat{n} \times \left(\frac{\bar{r}}{r}\right) = \frac{1}{r}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\Omega & \sin\Omega & 0 \\ x & y & z \end{vmatrix}$$

. The x-component of this is $\dfrac{z}{r}\sin\Omega$.

Note that $\hat{n} \times \hat{r}$ is parallel to $\vec{h}$ and recall that $\left|\vec{A} \times \vec{B}\right| = A\,B\cos\theta$. So..

$\hat{n} \times \hat{r} = \hat{h}\sin U$, since $U = \measuredangle\,\hat{n},\hat{r}$.

The projection of $\hat{h}\sin U$ onto the xy plane has magnitude (sin U · sin i) [See Fig.15] , and the x-component of this projected vector is (sin U · sin i) sin Ω [See Fig.16]. This is the x-component of $\hat{n} \times \hat{r}$. So we can write..

$\dfrac{z}{r}\sin\Omega = \sin U \cdot \sin i \cdot \sin\Omega$. Finally, we have..

$$\left.\boxed{\begin{aligned} \sin U &= \frac{z}{r\sin i} \\[2ex] \cos U &= \frac{x\cos\Omega + y\sin\Omega}{r} \end{aligned}}\right\} \text{need both for quadrant check}$$

Now we need ν to get ω = U – ν. Recall the equation of an ellipse (32), $r = \dfrac{a\,(1-e^2)}{1+e\cos\nu}$

So, we have $e\cos\nu = \left[\dfrac{a\,(1-e^2)}{r} - 1\right]$. This gives us cos ν; now we need sin ν to determine ν.

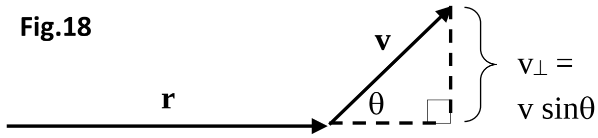
Taking the time derivative d/dt of the previous equation gives: $-e\,\dot{\nu}\sin\nu = -\dfrac{a\,(1-e^2)}{r^2}\dot{r}$ or

$$e \cdot \sin \nu \;=\; \frac{a(1-e^2)}{r^2\,\dot{\nu}}\,\dot{r} \qquad (46)$$

A short aside:

Fig.18

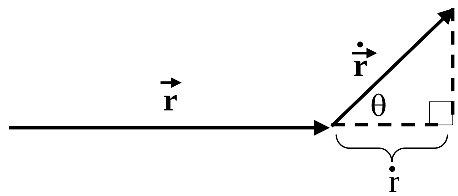
1) $h = \left|\vec{r}\times\vec{v}\right| = r\,\underbrace{v\sin\theta}_{v_\perp = r\dot{\nu}} = r^2\,\dot{\nu}$



2) $\vec{r}\bullet\vec{v} = \vec{r}\bullet\dot{\vec{r}} = (r)(\dot{r})$

Proof: $\dot{\vec{r}} = \vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$, $\vec{r} = r\,\hat{r} \;\Rightarrow$

$\vec{r}\bullet\dot{\vec{r}} = r\,\dot{r}$



Using these results (1) $h = r^2\,\dot{\nu}$ and (2) $\dot{r} = \dfrac{\vec{r}\bullet\dot{\vec{r}}}{r}$, eqn (46) becomes

$e\sin\nu \;=\; \dfrac{a(1-e^2)}{h}\,\dfrac{\vec{r}\bullet\dot{\vec{r}}}{r}$. So finally, we have enough to compute ν:

$$\boxed{\begin{aligned} \cos\nu &= \frac{1}{e}\left[\frac{a(1-e^2)}{r} - 1\right] \\[2ex] \sin\nu &= \frac{a(1-e^2)}{h\cdot e}\,\frac{\vec{r}\bullet\dot{\vec{r}}}{r} \end{aligned}} \quad \Bigg\}\ \text{both needed for quandrant check}$$

Now that we have both U and ν, we can compute $\boxed{\omega = U - \nu}$.

Time of perihelion passage T

Given ν at time t, we can compute the eccentric anomaly E at time t, the mean anomaly M at time t, and the time of perihelion passage T.

From Fig.11, it is not difficult to show that

$$\left.\boxed{\begin{array}{l} \cos E \;=\; \dfrac{a\,e + r\cos\nu}{a} \\[2em] \sin E \;=\; \dfrac{r\sin\nu}{a\sqrt{1-e^2}} \end{array}}\right\} \quad \text{need both for quadrant check}$$

Given E, we determine M from Kepler's equation,

$$\boxed{M \;=\; E - e\sin E}$$

Finally, we compute T from M:

$$\boxed{T \;=\; t - \frac{M}{n}\,,}\quad \text{where} \quad n = \frac{2\pi}{P}$$ and P = period of asteroid is from Kepler's 3rd Law:

$$n \;=\; \frac{2\pi}{P} \;=\; \frac{\sqrt{G M_{\odot}}}{a^{3/2}} \;=\; \frac{k}{a^{3/2}}\,.$$ Be careful with units. Here, distance is in AU, time t is in Julian days. In these units, k has the value k = 0.0172020989484.

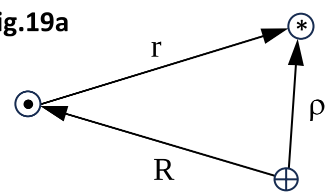
T is the time of perihelion passage and t is the time of observation. The time of perihelion passage is only determined modulo P, so our computed value may differ from the value given in JPL Horizons by an integer multiple of P.

Section IV. The Method of Gauss (MoG)

IV.A. The Method

The basic problem of orbit determination is this: Given 3 asteroid locations on the sky at 3 different times: $(\alpha_1, \delta_1, t_1)$, $(\alpha_2, \delta_2, t_2)$, $(\alpha_3, \delta_3, t_3)$, and given the Sun vector $\vec{R}$ at those times, we

Fig.19a



seek to determine the position and velocity $\vec{r}$, $\dot{\vec{r}}$ of the asteroid at the middle time t_2. From $\vec{r}$, $\dot{\vec{r}}$ at time t_2, we can compute the elements of the orbit of the asteroid.

The input to our calculation is 9 numbers (3 RAs, 3 DECs, 3 times). The output is the position and velocity of the asteroid at a time t_2, which is only 7 numbers (r_x, r_y, r_z, v_x, v_y, v_z, t). So, in principle, we have more than enough information to find a unique solution.

A little history: The Italian priest and astronomer, Giuseppe Piazzi (1746-1826), discovered the first known asteroid, 1 Ceres, on Jan 1, 1801, then lost it on 11 Feb, due to Ceres' proximity to the Sun. Was there a way to predict where it would be later, when it emerged from the Sun's glare?

The great German mathematician, Carl Friedrich Gauss (1777-1855), was intrigued by the problem and wrote *Theoria motus corporum coelestium in sectionibus conicis solem ambientum* (published 1809) as the solution to Piazzi's problem. His predictions were so accurate that when Hungarian astronomer Franz Xaver von Zach rediscovered Ceres on 31 Dec 1801, Gauss was only a half a degree off.

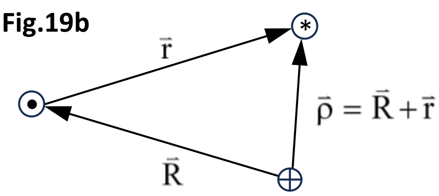
The Gaussian method begins with a geometric construction from Kepler's 2nd Law (equal areas in equal times). Consider three observations of an asteroid made at times t_1, t_2 and t_3 (early, middle, late). What we measure are $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, the directions in the sky.

We seek to solve for $\vec{r}_2$ and $\dot{\vec{r}}_2$, the middle position and velocity. Since all three position vectors lie in a plane (the plane of the orbit), we can write $\vec{r}_2$ as a linear combination of $\vec{r}_1$ and $\vec{r}_3$

$$\vec{r}_2 = a_1 \vec{r}_1 + a_3 \vec{r}_3 \qquad (47)$$

where a_1 and a_3 are scalars to be determined.

Fig.19b



If we knew a_1 and a_3, we could solve for the ρ's like so:

$$\left.\begin{aligned} \vec{r} &= \vec{\rho} - \vec{R} = \rho\hat{\rho} - \vec{R} \\ \vec{r}_2 &= a_1 \vec{r}_1 + a_3 \vec{r}_3 \end{aligned}\right\} \text{ combine to give..}$$

$$\left(\rho_2 \hat{\rho}_2 - \vec{R}_2\right) = a_1\left(\rho_1 \hat{\rho}_1 - \vec{R}_1\right) + a_3\left(\rho_3 \hat{\rho}_3 - \vec{R}_3\right)$$, and

rearrange to give

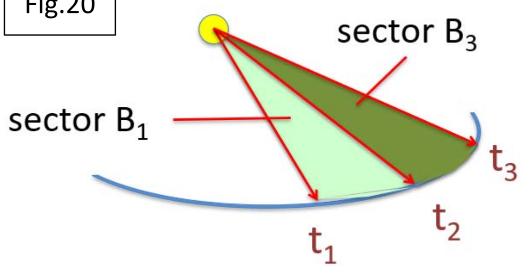
$$a_1 \rho_1 \hat{\rho}_1 - \rho_2 \hat{\rho}_2 + a_3 \rho_3 \hat{\rho}_3 = a_1 \vec{R}_1 - \vec{R}_2 + a_3 \vec{R}_3 \qquad (48)$$

This is a *vector* equation; so it is actually 3 *scalar* equations. The unknowns are the two a's and the three ρ's. (The $\hat{\rho}$'s are known.) If we knew a_1, a_3, we could solve for the ρ's, since we have 3 eqns in 3 unknowns (Cramer's Rule).

Once we have the ρ's, we can compute the r's, via $\vec{r} = \rho\hat{\rho} - \vec{R}$. And once we have the **r**'s, we can compute the velocity at $t = t_2$ from $\vec{v} = \dot{\vec{r}} \simeq \dfrac{\Delta\vec{r}}{\Delta t}$. (See Appendix 1 for how to compute the middle velocity $\dot{\vec{r}}_2$ from $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.) So… how do we get a_1, a_3? Gauss has a way, or at least, an approximate way.

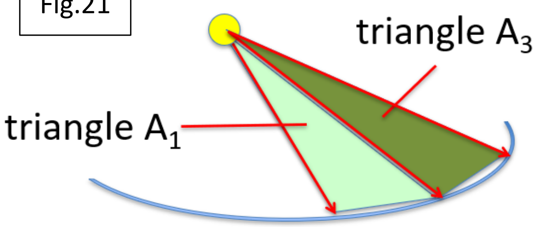
Define sectors as the "swept-out" areas (pie wedges) of each time interval between observations. Let sector B_0 = sector B_1 + sector B_3 .

Fig.20



The sector areas are hard to calculate, so approximate them as triangles by drawing chords, and define the triangle areas A_1 and A_3 like so:

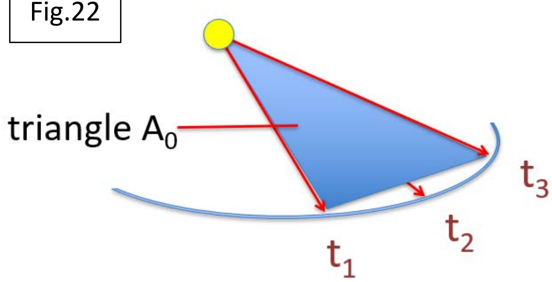
Fig.21



This is a good approximation if the time interval t_1 to t_3 is not too large.)

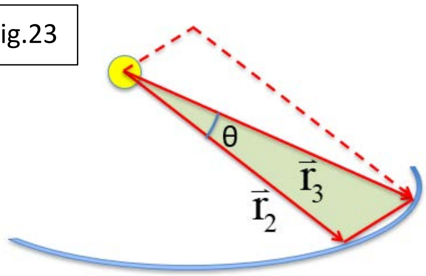
We define the triangle A_0 as shown.

Fig.22



The area of triangle A_3 is half of the size of the outlined parallelogram,

Fig.23



which is the magnitude of the cross product $\left|\vec{r}_2 \times \vec{r}_3\right| = 2A_3$. Now we can solve for a_1:

$$A_3 = \frac{1}{2}\left|\vec{r}_2 \times \vec{r}_3\right| = \frac{1}{2}\left|\left(a_1 \vec{r}_1 + a_3 \vec{r}_3\right) \times \vec{r}_3\right| = \frac{1}{2} a_1 \left|\vec{r}_1 \times \vec{r}_3\right| = \frac{1}{2} a_1 2A_0 = a_1 A_0$$

So, $a_1 = \dfrac{A_3}{A_0}$. Similarly, $a_3 = \dfrac{A_1}{A_0}$. Now, triangle area $A \cong$ sector B, so $a_1 \simeq \dfrac{B_3}{B_0}$ and $a_3 \simeq \dfrac{B_1}{B_0}$.
By Kepler's 2nd Law (equal areas in equal times), $\dfrac{B_3}{B_0} = \dfrac{t_3 - t_2}{t_3 - t_1}$ and $\dfrac{B_1}{B_0} = \dfrac{t_2 - t_1}{t_3 - t_1}$. So we have

$$a_1 \simeq \frac{t_3 - t_2}{t_3 - t_1} \text{ and } a_3 \simeq \frac{t_2 - t_1}{t_3 - t_1} . \qquad (49)$$

These are reasonable estimates which will be the starting point of an iterative process to get much more precise values of a_1 and a_3.

We pause to discuss units. Our units of distance are AU, and, so far, our units of time , t, have been days. We record the times of observation in Julian days. Recall that k is defined as $k \equiv \sqrt{G M_\odot}$. For notational and calculational convenience we will switch to units in which

$$GM_\odot \equiv k^2 = 1.$$

So $\mu \equiv G(M_\odot + m) \cong GM_\odot = 1$

The units of length are still AU, but we have new units of time, called Gaussian days, designated by the Greek letter τ. Recall that

$$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = -\frac{\mu}{r^2}\hat{r} = -\frac{k^2}{r^2}\hat{r} \qquad \text{and} \qquad \frac{P^2}{a^3} = \frac{4\pi^2}{\mu} = \frac{4\pi^2}{k^2} .$$

The value of $\sqrt{G M_\odot} = k$ is quite well known. In units of AU and days, its accepted value is

$$k = 0.0172020989484 .$$

We define the new Gaussian time variable τ as $\tau = k\,t$, so $\Delta\tau = k\,\Delta t$, and $\dfrac{d^2\vec{r}}{dt^2} = k^2 \dfrac{d^2\vec{r}}{d\tau^2} = -\dfrac{k^2}{r^2}\hat{r}$. The equation of motion then simplifies to $\dfrac{d^2\vec{r}}{d\tau^2} = -\dfrac{1}{r^2}\hat{r}$. With time in Gaussian days, k =1, and we have $\dfrac{P^2}{a^3} = 4\pi^2$. With a = 1 AU, we have $P = 2\pi$ Gaussian days. A Gaussian day is the time it takes for a small mass m in circular orbit of radius

1 AU to travel 1 radian about the sun. One Gaussian day equals $\dfrac{365.25}{2\pi} = 58.131$ Julian days.

From here on, we use Gaussian days as our time units.

We define the Gaussian time intervals for the 3 observations:

$$\begin{aligned} \tau_1 &= k\,(t_1 - t_2) \quad \text{(negative!)} \\ \tau_0 &= k\,(t_3 - t_1) \quad \text{(positive)} \\ \tau_3 &= k\,(t_3 - t_2) \quad \text{(positive)} \end{aligned} \qquad (50)$$

Units of distance in our OD calculations will always be AU.

Now, back to the main problem, solving for the ρ's from (48)

$$a_1\,\rho_1\,\hat{\rho}_1 - \rho_2\,\hat{\rho}_2 + a_3\,\rho_3\,\hat{\rho}_3 = a_1\vec{R}_1 - \vec{R}_2 + a_3\vec{R}_3$$

We now have good (but not great) estimates for the a's. We rewrite the vector equation (48) in xyz component form, using the notation $\hat{\rho} = \hat{\rho}_x\,\hat{i} + \hat{\rho}_y\,\hat{j} + \hat{\rho}_z\,\hat{k}$

$$\begin{aligned} \left(a_1\,\hat{\rho}_{1x}\right)\rho_1 - \left(\hat{\rho}_{2x}\right)\rho_2 + \left(a_3\,\hat{\rho}_{3x}\right)\rho_3 &= \left(a_1\,R_{1x} - R_{2x} + a_3\,R_{3x}\right) \\ \left(a_1\,\hat{\rho}_{1y}\right)\rho_1 - \left(\hat{\rho}_{2y}\right)\rho_2 + \left(a_3\,\hat{\rho}_{3y}\right)\rho_3 &= \left(a_1\,R_{1y} - R_{2y} + a_3\,R_{3y}\right) \\ \left(a_1\,\hat{\rho}_{1z}\right)\rho_1 - \left(\hat{\rho}_{2z}\right)\rho_2 + \left(a_3\,\hat{\rho}_{3z}\right)\rho_3 &= \left(a_1\,R_{1z} - R_{2z} + a_3\,R_{3z}\right) \end{aligned} \qquad (51)$$

We can use Cramer's Rule to solve for ρ_1, ρ_2, ρ_3. We would begin by working with ..

$$D = \begin{vmatrix} a_1\,\hat{\rho}_{1x} & -\left(\hat{\rho}_{2x}\right) & a_3\,\hat{\rho}_{3x} \\ a_1\,\hat{\rho}_{1y} & -\left(\hat{\rho}_{2y}\right) & a_3\,\hat{\rho}_{3y} \\ a_1\,\hat{\rho}_{1z} & -\left(\hat{\rho}_{2z}\right) & a_3\,\hat{\rho}_{3z} \end{vmatrix}$$, and 3 other determinants.

We could assign this as a homework problem. Or we could just tell you the answer:

$$\rho_1 = \frac{a_1\ (\vec{R}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3 - (\vec{R}_2 \times \hat{\rho}_2) \bullet \hat{\rho}_3 + a_3\ (\vec{R}_3 \times \hat{\rho}_2) \bullet \hat{\rho}_3}{a_1\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3}$$

$$\rho_2 = \frac{a_1\ (\hat{\rho}_1 \times \vec{R}_1) \bullet \hat{\rho}_3 - (\hat{\rho}_1 \times \vec{R}_2) \bullet \hat{\rho}_3 + a_3\ (\hat{\rho}_1 \times \vec{R}_3) \bullet \hat{\rho}_3}{-\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3} \qquad (52)$$

$$\rho_3 = \frac{a_1\ (\hat{\rho}_2 \times \vec{R}_1) \bullet \hat{\rho}_1 - (\hat{\rho}_2 \times \vec{R}_2) \bullet \hat{\rho}_1 + a_3\ (\hat{\rho}_2 \times \vec{R}_3) \bullet \hat{\rho}_1}{a_3\ (\hat{\rho}_2 \times \hat{\rho}_3) \bullet \hat{\rho}_1}$$

But notice, this calculation will fail if $(\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3 = 0$, and this occurs when the 3 positions on the sky form a straight line, that is, part of a great circle. In practice, this almost never occurs. This would occur, for instance, if the orbit of the asteroid happened to be exactly in the plane of the ecliptic.

It seems like we are well on the way to a solution. But remember, we have only crude estimates of a_1 and a_3. From these, we would get crude estimates of the ρ's and the r's. And from our crude r's, we would get a poor estimate of $\dot{\vec{r}}_2$, the middle velocity. Let's call these initial estimates the "crummy estimates" of $\vec{r}_2, \dot{\vec{r}}_2$. The orbital elements determined from these crummy, sad values of $\vec{r}_2, \dot{\vec{r}}_2$, would be so inaccurate as to be nearly worthless. Gauss determined a better way, and it involves expanding the vector $\vec{r}_2$ as a *Taylor series*.

As we shall see below, in the Method of Gauss, our initial crummy estimates of $\vec{r}_2, \dot{\vec{r}}_2$ are used to start an iterative calculation, which converges on excellent values of $\vec{r}_2, \dot{\vec{r}}_2$

Recall the general form of the Taylor series expansion of a scalar function

$$f(x) = f(x_0) + f'(x_0) \cdot \Delta x + \frac{f''(x_0)}{2} \cdot \Delta x^2 + \frac{f'''(x_0)}{3!} \cdot \Delta x^3 + \ldots$$

We can perform a similar expansion for the position vector $\vec{r} = \vec{r}(t)$ of the asteroid,

$$\vec{r}(t) = \vec{r}_0 + \dot{\vec{r}}_0\, t + \ddot{\vec{r}}_0 \frac{t^2}{2} + \dddot{\vec{r}}_0 \frac{t^3}{6} + \ldots \qquad (53)$$

Now, can replace the acceleration and jerk terms with central force quantities:

$$\ddot{\vec{r}}_0 = -\mu \frac{\vec{r}_0}{r_0^{\ 3}}\ . \qquad (54)$$

Taking the derivative with respect to t, and using the chain rule, we have

$$\dddot{\vec{r}}_0 = -\mu\left(\frac{r_0^{\,3}\,\dot{\vec{r}}_0 - \vec{r}_0\,(3r_0^{\,2})\dot{r}_0}{r_0^{\,6}}\right) = -\mu\left(\frac{r_0^{\,2}\,\dot{\vec{r}}_0 - 3(\vec{r}_0\bullet\dot{\vec{r}}_0)\vec{r}_0}{r_0^{\,5}}\right) \qquad (55)$$

Plugging (54) and (55) into (53) yields…

$$\vec{r}(t) = \vec{r}_0 + \dot{\vec{r}}_0\,t - \mu\,\frac{\vec{r}_0}{r_0^3}\,\frac{t^2}{2} - \mu\left(\frac{r_0^2\,\dot{\vec{r}}_0 - 3(\vec{r}_0\bullet\dot{\vec{r}}_0)\,\vec{r}_0}{r_0^5}\right)\frac{t^3}{6} + ..$$

Now, regrouping terms with $\vec{r}_0$ and $\dot{\vec{r}}_0$, we have

$$\vec{r}(t) = \left[1 - \mu\frac{t^2}{2r_0^3} + \frac{\mu(\vec{r}_0\bullet\dot{\vec{r}}_0)t^3}{2r_0^5}\right]\vec{r}_0 + \left(t - \frac{\mu\,t^3}{6r_0^3}\right)\dot{\vec{r}}_0 + \ldots \qquad (56)$$

which has the form

$$\vec{r}(t) = f\,\vec{r}_0 + g\,\dot{\vec{r}}_0 \qquad (57)$$

The expressions in front of $\vec{r}_0$ and $\dot{\vec{r}}_0$ are called the "f series" and the "g series".

Notice that vectors $\vec{r}_0$ and $\dot{\vec{r}}_0$ are in the same plane, the plane of the orbit. So, any position vector $\vec{r}$ in this plane is some linear combination of $\vec{r}_0$ and $\dot{\vec{r}}_0$

We will use length in units of **AU**, and time in units of **Gaussian days**, in which case, $\mu = 1$, and t becomes τ. Also, we will be expanding about the middle position $\vec{r}_2$ in the Taylor series, so all our "0" subscripts become "2" subscripts. Using time units of τ, and computing the f and g series to **4th order** in τ, we get…

$$f(\tau) = \left[1 - \frac{\tau^2}{2r_2^{\,3}} + \frac{(\vec{r}_2\bullet\dot{\vec{r}}_2)\tau^3}{2r_2^{\,5}} + \frac{\tau^4}{24r_2^{\,3}}\left(3\left(\frac{\dot{\vec{r}}_2\bullet\dot{\vec{r}}_2}{r_2^{\,2}} - \frac{1}{r_2^{\,3}}\right) - 15\left(\frac{\vec{r}_2\bullet\dot{\vec{r}}_2}{r_2^{\,2}}\right)^2 + \frac{1}{r_2^{\,3}}\right)\right]$$

$$g(\tau) = \left[\tau - \frac{\tau^3}{6r_2^{\,3}} + \frac{(\vec{r}_2\bullet\dot{\vec{r}}_2)\tau^4}{4r_2^{\,5}}\right] \qquad (58)$$

We leave the derivation of the 4^{th} order terms to the energetic student.

Don't forget: the time derivatives are now τ derivatives: $\dot{\vec{r}}_0 = \dfrac{d\vec{r}_0}{d\tau}$. Note that f is dimensionless, and g has units of time (gaussian days), as they must, from equation (57).

Assuming that $\vec{r}_2$ and $\dot{\vec{r}}_2$ are known (we have estimates), we can evaluate $\vec{r}$ at the earlier time τ_1 and the later time τ_3 (as defined in eqns 50):

$$\vec{r}(\tau_1) = \vec{r}_1 = f(\tau_1)\vec{r}_2 + g(\tau_1)\dot{\vec{r}}_2 = f_1\vec{r}_2 + g_1\dot{\vec{r}}_2$$
$$\vec{r}(\tau_3) = \vec{r}_3 = f(\tau_3)\vec{r}_2 + g(\tau_3)\dot{\vec{r}}_2 = f_3\vec{r}_2 + g_3\dot{\vec{r}}_2 \qquad (59)$$

Recall that τ_1 is negative, meaning, at an earlier time.

We can now combine the equations (59) to eliminate $\dot{\vec{r}}_2$. Multiply the 1st equation by g_3 and the 2nd by g_1, then subtract and solve for $\vec{r}_2$.

$$\vec{r}_2 = \left[\frac{g_3}{f_1 g_3 - f_3 g_1}\right]\vec{r}_1 - \left[\frac{g_1}{f_1 g_3 - f_3 g_1}\right]\vec{r}_3 \qquad (60)$$

Comparing this with our earlier equation (47): $\vec{r}_2 = a_1\vec{r}_1 + a_3\vec{r}_3$, we see that

$$a_1 = \left[\frac{g_3}{f_1 g_3 - f_3 g_1}\right], \qquad a_3 = -\left[\frac{g_1}{f_1 g_3 - f_3 g_1}\right] \qquad (61)$$

Similarly, we can combine equations (59) to eliminate $\vec{r}_2$. Multiply the 1st equation by f_3 and the 2nd by f_1, then subtract and solve for $\dot{\vec{r}}_2$.

$$\dot{\vec{r}}_2 = \left[\frac{f_3}{f_3 g_1 - f_1 g_3}\right]\vec{r}_1 - \left[\frac{f_1}{f_3 g_1 - f_1 g_3}\right]\vec{r}_3 \qquad (62)$$

Note the similarities between equations (60) and (62). These can be slightly rearranged to give

$$\vec{r}_2 = \left[\frac{g_3\vec{r}_1 - g_1\vec{r}_3}{f_1 g_3 - f_3 g_1}\right], \qquad \dot{\vec{r}}_2 = \left[\frac{f_3\vec{r}_1 - f_1\vec{r}_3}{f_3 g_1 - f_1 g_3}\right] \qquad (63)$$

We can now start an iterative process, to get better and better estimates of $\vec{r}_2$ and $\dot{\vec{r}}_2$.

Begin with our "crummy estimates" of $\vec{r}_2$ and $\dot{\vec{r}}_2$. From these, using (58), we get initial estimates of f_1, f_3, g_1, g_3. From these f's and g's, we use (61) to get new values of a_1, a_3. And now, we enter a loop..

Step 1: Given a_1 and a_3, solve for ρ_1, ρ_2, ρ_3 by solving…

$$\left(\rho_2\hat{\rho}_2 - \vec{R}_2\right) = a_1\left(\rho_1\hat{\rho}_1 - \vec{R}_1\right) + a_3\left(\rho_3\hat{\rho}_3 - \vec{R}_3\right)$$

[Recall the solutions (52) , $\rho_1 = \dfrac{a_1\,(\vec{R}_1\times\hat{\rho}_2)\bullet\hat{\rho}_3 - (\vec{R}_2\times\hat{\rho}_2)\bullet\hat{\rho}_3 + a_3\,(\vec{R}_3\times\hat{\rho}_2)\bullet\hat{\rho}_3}{a_1\,(\hat{\rho}_1\times\hat{\rho}_2)\bullet\hat{\rho}_3}$, $\quad \rho_2 =$ etc...]

Step 2: Given ρ_1, ρ_2, ρ_3, solve for the position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ using

$$\vec{r}_1 = \rho_1 \hat{\rho}_1 - \vec{R}_1 \ , \quad \vec{r}_2 = \rho_2 \hat{\rho}_2 - \vec{R}_2 \ , \quad \vec{r}_3 = \rho_3 \hat{\rho}_3 - \vec{R}_3$$

Step 3: Now use our improved values of $\vec{r}_1$ and $\vec{r}_3$ to get refined values of $\vec{r}_2, \dot{\vec{r}}_2$ from (63)

$$\vec{r}_2 = \left[\frac{g_3 \vec{r}_1 - g_1 \vec{r}_3}{f_1 g_3 - f_3 g_1} \right] \ , \qquad \dot{\vec{r}}_2 = \left[\frac{f_3 \vec{r}_1 - f_1 \vec{r}_3}{f_3 g_1 - f_1 g_3} \right]$$

Step 4: With new $\vec{r}_2, \dot{\vec{r}}_2$ values, get refined values of f_1, f_3, g_1, g_3 from (58)

$$f(\tau) = \left[1 - \frac{\tau^2}{2 r_2^{\,3}} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^3}{2 r_2^{\,5}} + \frac{\tau^4}{24 r_2^{\,3}} \left(3\left(\frac{\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2}{r_2^{\,2}} - \frac{1}{r_2^{\,3}} \right) - 15 \left(\frac{\vec{r}_2 \bullet \dot{\vec{r}}_2}{r_2^{\,2}} \right)^2 + \frac{1}{r_2^{\,3}} \right) \right]$$

$$g(\tau) = \left[\tau - \frac{\tau^3}{6 r_2^{\,3}} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^4}{4 r_2^{\,5}} \right]$$

Step 5: With refined f and g values, get refined values of a_1, a_3 from (61)

$$a_1 = \left[\frac{g_3}{f_1 g_3 - f_3 g_1} \right] \ , \qquad a_3 = -\left[\frac{g_1}{f_1 g_3 - f_3 g_1} \right]$$

Now, with refined a_1, a_3 values, go back to **Step 1**, and repeat until $\vec{r}_2, \dot{\vec{r}}_2$ converge. In practice, convergence is achieved when the magnitude of $\vec{r}_2$ stops changing in the 10th place:

$$\left| \frac{r_{2,n} - r_{2,n-1}}{r_{2,n}} \right| < 10^{-10}$$

This corresponds to a distance change of just a few meters, much smaller than the size of the asteroid.

As a handy reference, there is a concise outline of the MoG algorithm in Appendix 2.

If your code converges, you have precise values of $\vec{r}_2, \dot{\vec{r}}_2$, in *equatorial coordinates.* Remember, we started with sky positions in RA and DEC, so we have been in equatorial coordinate the whole time. Before proceeding to compute the orbital elements from $\vec{r}_2, \dot{\vec{r}}_2$, we must convert $\vec{r}_2, \dot{\vec{r}}_2$ into *ecliptic coordinates* with an appropriate rotation matrix. (I forgot to do this the first time through, many years ago. I was puzzled that I was getting ballpark values for the elements – not wildly incorrect, just incorrect.)

Unfortunately, the Method of Gauss is designed to work with exactly 3 positions of the asteroid. I do not know of any elegant or obvious way to generalize the method to include data from more than 3 nights.

IV.B Correction for speed-of-light travel time

There is one simple correction that we can make to improve our MoG results. When we photograph the asteroid, we are not seeing where it is in the sky “now”, that is, at the time of observation. Instead, we are seeing the asteroid where it was when light left the asteroid on its way to our camera. Our camera records where the asteroid was, a light-travel time ago. The sun is 8 light-minutes away. Asteroids near the earth are perhaps 1 or 2 or 3 light minutes away. Most of the NEAs we study move a small, but measurable distance on the sky in 3 minutes, so the effect is non-negligible.

If the distance to the asteroid is ρ, the light travel time between the asteroid and us on earth is

$$\Delta t = \frac{\rho}{c} \qquad (64)$$

where c is the speed of light. Be careful with units here. If you measure distance in AU and time in days, then c has units of AU/day. If you are measuring time in Gaussian days, then c is in units of AU/Gaussian day. Just know what you are doing, which is good advice in all aspects of life.

With the light travel time correction (64), the actual time, t_{actual}, when the asteroid was at the location shown in our camera image, is earlier than the t_{obs}, the time of observation.

$$t_{actual} = t_{obs} - \frac{\rho}{c} \qquad (65)$$

This results in a correction for our τ_1, τ_3 values in Step 4 of our Method of Gauss loop. The times of observation t_1, t_2, t_3 , when corrected for light travel time, become $t_{actural\_1}$, t_{actual_2}, t_{actual_3}, and the τ’s must be revised according $\Delta\tau = k\,\Delta t$. So for instance, τ_1, which was originally given by $k\,(t_{obs1} - t_{obs2})$, must be revised to $\tau_1 = k\,(t_{actual1} - t_{actual2})$.

To implement this correction in your code, we must add a step in the MoG loop which, after update of the ρ’s, updates the t’s according to (65) , yielding new τ’s, which are then used when updating the f- and g-series. I find that this correction is rather small, producing an improvement of the orbital elements of less than 0.2%.

Should we now go back and get new sun vectors for our new light-corrected observation times? The answer is no, I believe. The “geocentric” JPL Horizons sun vectors are already corrected for light travel time and stellar aberration. That is, the sun vector gives the location of the sun “now”, not the apparent location of the sun in the sky, which differs from the “now” location due to a combination of light travel time and stellar aberration. When we correct for light travel time, we are getting $\vec{\rho}$, the asteroid's position, "now"; the sun vector already has the "now" value, so when we close the earth/sun/asteroid triangle, we are getting the correct "now" value of $\vec{r}$.

Quoting from the JPL Horizons manual (https://naif.jpl.nasa.gov/pub/naif/toolkit_docs/C/cspice/spkezp_c.html):

```
We use the adjectives "geometric," "uncorrected," or "true" to refer to an
actual position or state of a target at a specified epoch. When a geometric
position or state vector is modified to reflect how it appears to an
observer, we describe that vector by any of the terms "apparent,"
"corrected," "aberration corrected," or "light time and stellar aberration
corrected."
```

IV.C Visual representation of the MoG convergence

Fig. 24 shows how the MoG converges on the correct position and velocity of an asteroid. The figure on the left shows the elliptical orbit of asteroid 2002 KL6 and the nearly circular orbit of earth. The 3 dots on the orbits show the locations of earth and the asteroid on 3 nights separated by 1-week intervals: June 23, June 30, and July 7, 2023. These data are from JPL Horizons.

The second part of Fig.24 is a zoomed-in view of the first. The x's show the computed locations $\vec{r}_2$ of the asteroid in the first 11 cycles of the MoG loop. The code converged after a total of 108 cycles. The arrows in the figure show the computed velocity $\dot{\vec{r}}_2$ of the asteroid at 5 selected cycles. (The figure would be too busy if all the velocity arrows were displayed.)

Notice that all the x's are on a line from the earth's middle position to the asteroid's middle position. The one thing that the algorithm knows for sure is the direction $\hat{\rho}_2$ of the asteroid as seen from the earth, so all the computed $\vec{r}_2$'s must be along that line.

Next, notice how very crummy the initial "crummy estimates" of $\vec{r}_2, \dot{\vec{r}}_2$ are. The initial estimate of the position is the first x in the lower right-hand corner. This initial distance r from the sun is about 25% larger than the true value (shown by the black dot on the asteroid's orbit), and the initial value of ρ, the distance from the earth, is too large by a factor of 3! The initial crummy velocity vector (the lower most arrow on the figure) has a speed 40% larger than the true, final velocity (the upper most arrow on the figure). The Method of Gauss is remarkably forgiving regarding the initial estimates of $\vec{r}_2, \dot{\vec{r}}_2$. Crummy estimates are not a problem; the method of Gauss zeroes in on the exact solution, regardless. More sophisticated and complicated methods of estimating starting values are of questionable value, since crummy estimates are good enough.

Finally, notice that, as iterations proceed and we get closer to the final value of $\vec{r}_2$, subsequent changes are smaller and smaller. The first 11 iterations got 95% of the way from the initial crummy estimate of position to the final true position. Another 97 iterations were required to go the last 5% and converge on the final value.

Sometimes, the Method of Gauss fails to converge to a reasonable value. I find that it always fails if asteroid is far from the ecliptic, with high declination (roughly, $\delta > 20^o$) in the summer months. It also sometimes fails in cases where the orbit appears completely ordinary. For instance, in summer 2023, SSP students found that for asteroid 1929 SH (1627 Ivar), MoG was extremely sensitive to initial conditions. It converged using data for some nights, but not for other nearby nights. And when the RA and DEC of the position data was changed by just 0.3 arcsec, MOG would converge on wildly incorrect answers (like r = 35 AU).

Fig. 24

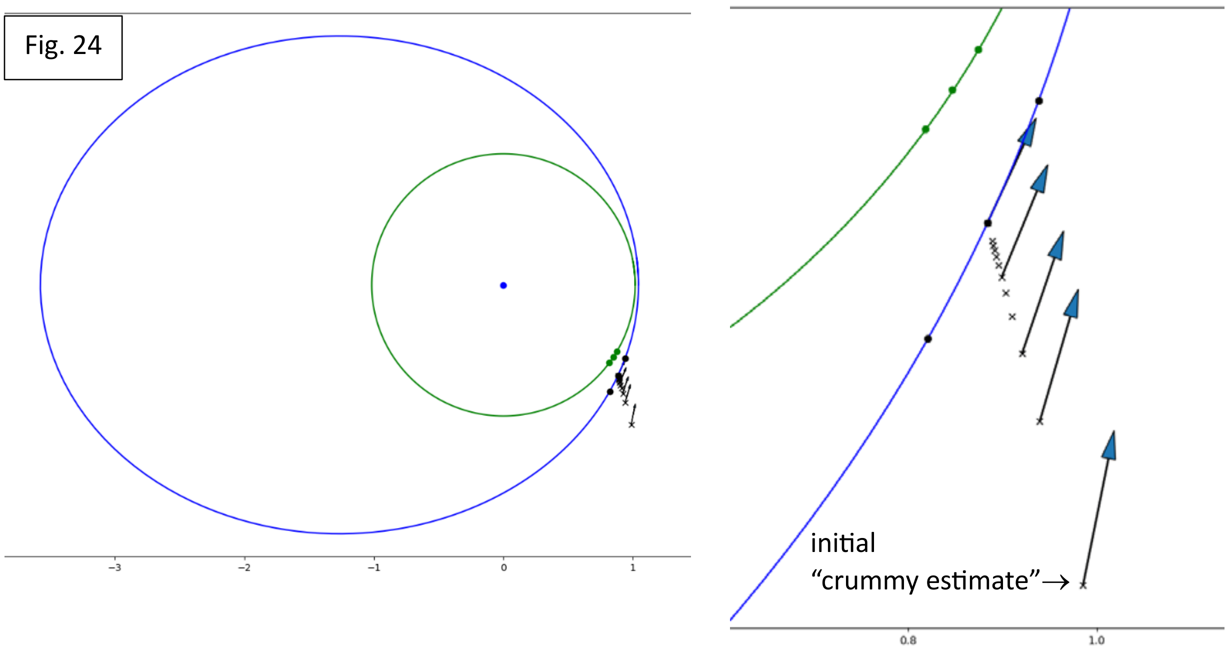


Fig.24. Orbits of 2002 KL6 (blue) and earth (green). The scale is in AU. Dots show locations of earth and the asteroid on the nights of Jun 23, 30, and July 7, 2023. The x’s mark estimates of the asteroid position from the first 11 iterations of the MoG.

Section V. Ephemeris Generation

Goal: We want to predict the position (RA and Dec) of an asteroid at some future time t, given its known orbital elements: a, e, i, ω, Ω, and the value of M at some known time t_o.

a = semi-major axis

e = eccentricity

i = inclination

ω = longitude of the ascending node

Ω = argument of perihelion

M = mean anomaly

E = eccentric anomaly

ν = true anomaly

The mean anomaly M is a linear function of time: $M(t) = M(t_o) + n\,(t - t_o)$, where $n = \dfrac{2\pi}{P}$, and P is the period. The quantity n is the average rate of increase of the true anomaly (averaged over one period P).

Since the semi-major axis, a, is known, we can get the period T (and hence, n) from Kepler's 3rd Law:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G\,M_\odot}\text{, which is often written } n = \frac{2\pi}{P} = \frac{\sqrt{G\,M_\odot}}{a^{3/2}} = \frac{\sqrt{\mu}}{a^{3/2}} = \frac{k}{a^{3/2}}.$$

k and μ are defined by $k^2 = \mu = G\,M_\odot$. [Technically, $\mu = G\left(M_\odot + m_{asteroid}\right)$, but $m_{asteroid} \ll M_\odot$.]

Once we have n, we can compute M(t) at any other time t from $M(t) = M(t_o) + n\,(t - t_o)$. Often, t_o is the time of perihelion passage, $t_o = t_{peri}$, in which case, $M(t_{peri}) = 0$, and $M(t) = n\,(t - t_{peri})$.

Once we have M at time t, we get E at time t from Kepler's equation: $M = E - e\sin E$, which can only be solved numerically. Note that we have to use the known eccentricity e here.

Knowing the E, we can now compute the position vector **r** of the asteroid, relative to the Sun, in orbital coordinates, eqn (42):

$$\vec{r} = (a\cos E - a\,e)\,\hat{i} + (a\sqrt{1-e^2}\,\sin E)\,\hat{j} + 0\,\hat{k}$$

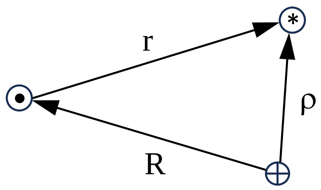
Now we need to convert position vector **r** from orbital coordinates (x, y, z) to ecliptic coordinates (x_{ec}, y_{ec}, z_{ec}) with a sequence of three rotations by the angles ω, i, and Ω.

$$\begin{pmatrix} x_{ec} \\ y_{ec} \\ z_{ec} \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And finally, to convert from ecliptic coordinates (x_{ec}, y_{ec}, z_{ec}) to celestial, equatorial coordinates (x_{eq}, y_{eq}, z_{eq}), we need one more rotation by the obliquity of the ecliptic $\varepsilon \cong 23.4^o$ (look up the exact value for your current date).

$$\begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x_{ec} \\ y_{ec} \\ z_{ec} \end{pmatrix}$$

Now, we want the position of the asteroid relative to our observing post on the Earth, not relative to the Sun, so we use the fundamental triangle and the known Sun vector at our date to compute the asteroid's position $\boldsymbol{\rho}$.



$$\vec{\rho}_{eq} = \vec{R}_{eq} + \vec{r}_{eq}$$

All we really need is the direction of the asteroid in the observer's sky: $\hat{\rho} = (\,\hat{\rho}_x\,,\ \hat{\rho}_y\,,\ \hat{\rho}_z\,) = \dfrac{\vec{\rho}}{\rho}$.

Now, at last, we can solve for the RA α and Dec δ of the asteroid, using

$$\hat{\rho}_z = \sin\delta$$

$$\hat{\rho}_x = \cos\delta\,\cos\alpha$$

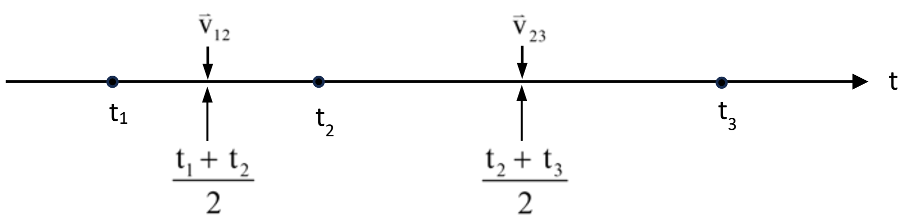
$$\hat{\rho}_z = \cos\delta\,\sin\alpha$$

APPENDIX 1. Estimating the middle velocity

We need estimates of the middle position $\vec{r}_2$ and middle velocity $\vec{v}_2$, to get the MoG iteration started. In this document, we estimate the middle velocity vector $\vec{v}_2$ from the three position vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. We do not assume that the positions are evenly spaced in time. The position vectors can be approximately computed from the times of observation t_1, t_2, and t_3, using the Kepler 2nd Law argument.

Here we go. We start with the 3 position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ at times t_1, t_2, t_3. We begin by computing the average velocity between times t_1 and t_2 and the average velocity between times t_2 and t_3.

$$\vec{v}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \qquad \vec{v}_{23} = \frac{\vec{r}_3 - \vec{r}_2}{t_3 - t_2}$$

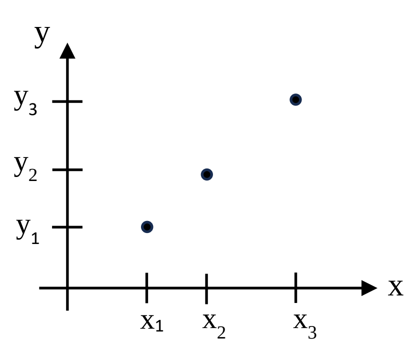


We show below that a linear interpolation gives the following expression for the velocity $\vec{v}_2$ at time t_2.

$$\boxed{\vec{v}_2 = \frac{(t_3 - t_2)\vec{v}_{12} + (t_2 - t_1)\vec{v}_{23}}{(t_3 - t_1)}}$$

Consider 3 points along a line in the xy plane.

The values x_1, x_2, and x_3 are all known. The values y_1 and y_3 are known, but the value y_2 is unknown and we wish to compute y_2 by a linear interpolation between y_1 and y_3.



Note that $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$. Solving for y_2 gives

$$y_2 = y_1 + (x_2 - x_1)\left(\frac{y_3 - y_1}{x_3 - x_1}\right) .$$

Looking back at our earlier problem of v_{12}, v_2, and v_{23} at times $(t_1+t_2)/2$, t_2, and $(t_2+t_3)/2$, we translate y's into v's and x's into t's.

$$y_2 = y_1 + (x_2 - x_1)\left(\frac{y_3 - y_1}{x_3 - x_1}\right) \quad \text{becomes}$$

$$\vec{v}_2 = \vec{v}_{12} + \left(t_2 - \frac{t_1 + t_2}{2}\right)\left(\frac{\vec{v}_{23} - \vec{v}_{12}}{\dfrac{t_2 + t_3}{2} - \dfrac{t_1 + t_2}{2}}\right)$$

(now some algebra..)

$$\vec{v}_2 = \vec{v}_{12} + \left(\frac{t_2 - t_1}{2}\right)\cdot 2\left(\frac{\vec{v}_{23} - \vec{v}_{12}}{t_3 - t_1}\right) = \vec{v}_{12} + \left(\frac{t_2 - t_1}{t_3 - t_1}\right)\left(\vec{v}_{23} - \vec{v}_{12}\right)$$

$$= \frac{(t_3 - t_1)\vec{v}_{12} + (t_2 - t_1)\left(\vec{v}_{23} - \vec{v}_{12}\right)}{(t_3 - t_1)} = \frac{(t_3 - t_2)\,\vec{v}_{12} + (t_2 - t_1)\,\vec{v}_{23}}{(t_3 - t_1)}$$

Done.

Appendix 2. Method of Gauss Outline

Our goal is to start with celestial coordinates (RA and DEC) of our asteroid on 3 nights and, from these data, compute the position and velocity vectors of the asteroid on the middle night.

Given α_1, δ_1, α_2, δ_2, and α_3, δ_3 at times (t_1, t_2, t_3) and the Sun vectors in AU at those times ($\vec{R}_1, \vec{R}_2, \vec{R}_3$), we will compute $\vec{r}_2$ and $\dot{\vec{r}}_2$ (from which we can then compute the elements of the orbit).

Almost all tasks should be written as functions in your code.

A) Create input statements so that the user can easily read in these data from a text file. Comment your work copiously.

B) Decimalize RA and DEC values, in degrees: 0 to 360 for RA, -90 to +90 for DEC

C) Get the topocentric unit vectors $\hat{\rho}$ from α and δ for each observation:

$$\hat{\rho}_i = \begin{bmatrix} \cos\alpha_i \cos\delta_i \\ \sin\alpha_i \cos\delta_i \\ \sin\delta_i \end{bmatrix}$$

Check that the magnitudes of your $\hat{\rho}$ vectors are actually equal to 1!

D) Compute the gaussian time intervals : $\tau_1 = k(t_1 - t_2)$, $\tau_0 = k(t_3 - t_1)$, $\tau_3 = k(t_3 - t_2)$. Note that τ_1 is negative.

E) Compute initial values of a_1 and a_3 (defined by $\vec{r}_2 = a_1\vec{r}_1 + a_3\vec{r}_3$): $a_1 = \frac{\tau_3}{\tau_0}$, $a_3 = -\frac{\tau_1}{\tau_0}$.

F) Given a_1 and a_3, compute initial values of the ρ's from

$$\rho_1 = \frac{a_1\ (\vec{R}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3 - (\vec{R}_2 \times \hat{\rho}_2) \bullet \hat{\rho}_3 + a_3\ (\vec{R}_3 \times \hat{\rho}_2) \bullet \hat{\rho}_3}{a_1\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3}$$

$$\rho_2 = \frac{a_1\ (\hat{\rho}_1 \times \vec{R}_1) \bullet \hat{\rho}_3 - (\hat{\rho}_1 \times \vec{R}_2) \bullet \hat{\rho}_3 + a_3\ (\hat{\rho}_1 \times \vec{R}_3) \bullet \hat{\rho}_3}{-\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3}$$

$$\rho_3 = \frac{a_1\ (\hat{\rho}_2 \times \vec{R}_1) \bullet \hat{\rho}_1 - (\hat{\rho}_2 \times \vec{R}_2) \bullet \hat{\rho}_1 + a_3\ (\hat{\rho}_2 \times \vec{R}_3) \bullet \hat{\rho}_1}{a_3\ (\hat{\rho}_2 \times \hat{\rho}_3) \bullet \hat{\rho}_1}$$

G) Compute an initial value of the three r vectors from $\vec{r} = \rho\hat{\rho} - \vec{R}$. The units are AU. And from these initial r vectors, compute an initial value of $\dot{\vec{r}}_2$.

$$\dot{\vec{r}}_2 = \vec{v}_2 = \frac{(t_3 - t_2)\vec{v}_{12} + (t_2 - t_1)\vec{v}_{23}}{(t_3 - t_1)} \text{, where } \vec{v}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \qquad \vec{v}_{23} = \frac{\vec{r}_3 - \vec{r}_2}{t_3 - t_2}$$

(See Appendix 1 for details of the deviation.) Remember, you should be working in distance units of AU and time units of gaussian days, so convert time in days to time in gaussian days: $\Delta\tau = k\,\Delta t$.

H) Now with your initial values of $\vec{r}_2$ and $\dot{\vec{r}}_2$, you can compute initial values of f and g terms:

$$f(\tau) = \left[1 - \frac{\tau^2}{2 r_2^3} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^3}{2 r_2^5} + \frac{\tau^4}{24 r_0^3}\left(3\left(\frac{\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2}{r_2^2} - \frac{1}{r_2^3}\right) - 15\left(\frac{\vec{r}_2 \bullet \dot{\vec{r}}_2}{r_2^2}\right)^2 + \frac{1}{r_2^3}\right)\right]$$

$$g(\tau) = \left[\tau - \frac{\tau^3}{6 r_2^3} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^4}{4 r_2^5}\right]$$

$$f_1 = f(\tau_1),\ g_1 = g(\tau_1),\ f_3 = f(\tau_3),\ g_3 = g(\tau_3)$$

I) Now recalculate a_1 and a_3 :

$$a_1 = \frac{g_3}{f_1 g_3 - f_3 g_1}, \quad a_3 = -\frac{g_1}{f_1 g_3 - f_3 g_1}$$

Now we are ready to enter a **loop** (next page)

MAIN LOOP

STEP 1: Given refined values of a_1 and a_3, recompute values of the ρ's as in part F above.

$$\rho_1 = \frac{a_1\ (\vec{R}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3 - (\vec{R}_2 \times \hat{\rho}_2) \bullet \hat{\rho}_3 + a_3\ (\vec{R}_3 \times \hat{\rho}_2) \bullet \hat{\rho}_3}{a_1\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3}$$

$$\rho_2 = \frac{a_1\ (\hat{\rho}_1 \times \vec{R}_1) \bullet \hat{\rho}_3 - (\hat{\rho}_1 \times \vec{R}_2) \bullet \hat{\rho}_3 + a_3\ (\hat{\rho}_1 \times \vec{R}_3) \bullet \hat{\rho}_3}{-\ (\hat{\rho}_1 \times \hat{\rho}_2) \bullet \hat{\rho}_3}$$

$$\rho_3 = \frac{a_1\ (\hat{\rho}_2 \times \vec{R}_1) \bullet \hat{\rho}_1 - (\hat{\rho}_2 \times \vec{R}_2) \bullet \hat{\rho}_1 + a_3\ (\hat{\rho}_2 \times \vec{R}_3) \bullet \hat{\rho}_1}{a_3\ (\hat{\rho}_2 \times \hat{\rho}_3) \bullet \hat{\rho}_1}$$

STEP 2: Given refined ρ_1, ρ_2, ρ_3, solve for the position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ using

$$\vec{r}_1 = \rho_1 \hat{\rho}_1 - \vec{R}_1\ ,\quad \vec{r}_2 = \rho_2 \hat{\rho}_2 - \vec{R}_2\ ,\quad \vec{r}_3 = \rho_3 \hat{\rho}_3 - \vec{R}_3$$

STEP 3: Use $\vec{r}_1$ and $\vec{r}_3$ to get refined values of $\vec{r}_2$ and $\dot{\vec{r}}_2$

$$\vec{r}_2 = \left[\frac{g_3\,\vec{r}_1 - g_1\,\vec{r}_3}{f_1\,g_3 - f_3\,g_1}\right] \qquad \dot{\vec{r}}_2 = \left[\frac{f_3\,\vec{r}_1 - f_1\,\vec{r}_3}{f_3\,g_1 - f_1\,g_3}\right]$$

STEP 4: With the refined values of $\vec{r}_2$ and $\dot{\vec{r}}_2$, get refined values of

$$f_1 = f\left(\tau_1\right),\ f_3 = f\left(\tau_3\right),\ g_1 = g\left(\tau_1\right),\ g_3 = g\left(\tau_3\right)$$

$$f(\tau) = \left[1 - \frac{\tau^2}{2\,r_2^{\ 3}} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^3}{2\,r_2^{\ 5}} + \frac{\tau^4}{24 r_0^{\ 3}}\left(3\left(\frac{\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2}{r_2^{\ 2}} - \frac{1}{r_2^{\ 3}}\right) - 15\left(\frac{\vec{r}_2 \bullet \dot{\vec{r}}_2}{r_2^{\ 2}}\right)^2 + \frac{1}{r_2^{\ 3}}\right)\right]$$

$$g(\tau) = \left[\tau - \frac{\tau^3}{6\,r_2^{\ 3}} + \frac{(\vec{r}_2 \bullet \dot{\vec{r}}_2)\tau^4}{4\,r_2^{\ 5}}\right]$$

STEP 5: With the refined f, g values, get refined values of a_1, a_3

$$a_1 = \frac{g_3}{f_1\,g_3 - f_3\,g_1},\quad a_3 = -\frac{g_1}{f_1\,g_3 - f_3\,g_1}$$

STEP 6: With refined a_1, a_3, return to **step 1** and repeat until both $\left|\vec{r}_2\right|$ converges to

$$\left|\frac{r_{2,n} - r_{2,n-1}}{r_{2,n}}\right| < 10^{-10}$$

When this is all working, refine your code to include the correction for time-of-light travel time. Including a time-of-light correction should get you closer to the JPL Horizons value of $|\vec{r}_2|$.

We have been working in *equatorial coordinates*, since we started with RA and DEC. We need to switch to *ecliptic coordinates* for $\vec{r}_2$ and $\dot{\vec{r}}_2$ with appropriate rotations, BEFORE computing the orbital elements.