Notes on distance between intrinsic and extrinsic priors

0.1 Property of Extrinsic Prior

We now study the properties of the extrinsic prior. One important task is to quantify the difference between extrinsic and intrinsic priors. We first focus on the first case when $\int_{\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)d\theta > 0$.

Remark 1. *Let $M_1 = \int_{\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)d\theta$ and $M_2 = \int_{\mathcal{R}} \pi_{0,\mathcal{R}}(\theta)\mathcal{K}(\theta;\mathcal{D})d\theta$, when $M_1 > 0$, the total variation distance between the measures of extrinsic and intrinsic prior*

$$||\pi_{0,\mathcal{D}}(\theta), \tilde{\pi}_{0,\mathcal{D}}(\theta)||_{TV} = 1 - \frac{M_1}{M_2} \leq \frac{\int_{\theta\in\mathcal{R}\setminus\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)\mathcal{K}(\theta;\mathcal{D})d\theta}{M_1}$$

.

proof: via definition of total variation distance and $K(\theta;\mathcal{D}) = 1$ when $\theta \in \mathcal{D}$, 0 otherwise.

In the case of exponential smoothing function, we have:

Corollary 1. *Let $M_1 = \int_{\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)d\theta > 0$ and $\mathcal{K}(\theta;D) = \prod_{k=1}^{m} \exp(-v_k(\theta)/\lambda_k)$, one sufficient condition to have*

$$\lim_{all\ \lambda_k\to 0} ||\pi_{0,\mathcal{D}}(\theta), \tilde{\pi}_{0,\mathcal{D}}(\theta)||_{TV} = 0$$

is that $\pi_{0,\mathcal{R}}(\theta)$ is proper, $\int_{\mathcal{R}} \pi_{0,\mathcal{R}}(\theta)d\theta < \infty$.

proof: via dominated convergence theorem

Rewriting $\mathcal{K}(\theta;D) = \exp(-v(\theta)/\lambda)$ with $\lambda = \sup_k \lambda_k$, $v(\theta) = \lambda \sum_{k=1}^{m} \frac{v_k(\theta)}{\lambda_k}$, we obtain the convergence rate:

Remark 2. *Assuming $M_3 = \int_{\mathcal{R}\setminus\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)d\theta < \infty$, and $f(v)$ be the density of $v(\theta)$ as the transform of $\pi_{0,\mathcal{R}}(\theta)/M_3$. If there exists $t > 0$ such that $f(v) < \infty$ for $v < t$,*

$$\int_0^{\infty} \pi_{0,\mathcal{R}}(\theta)exp(-\frac{v(\theta)}{\lambda})d\theta \leq 2M_3 \exp(-\frac{t}{\lambda}) + M_3 \sup_{t^*\in(0,t)} f(t^*)\lambda$$

proof:

$$
\begin{aligned}
\int_0^\infty f(v)\exp(-\frac{v}{\lambda})dv =& \int_0^t f(v)\exp(-\frac{v}{\lambda})dv + \int_t^\infty f(v)\exp(-\frac{v}{\lambda})dv \\
\leq& F(t)\exp(-\frac{t}{\lambda}) + \frac{1}{\lambda}\int_0^t F(v)\exp(-\frac{v}{\lambda})dv + \exp(-\frac{t}{\lambda}) \\
=&(F(t)+1)\exp(-\frac{t}{\lambda}) + \frac{1}{\lambda}\int_0^t f(v^*)v\exp(-\frac{v}{\lambda})dv \\
\leq&(F(t)+1)\exp(-\frac{t}{\lambda}) + \sup_{t^*\in(0,t)} f(t^*)\int_0^t \frac{1}{\lambda}v\exp(-\frac{v}{\lambda})dv \\
\leq& 2\exp(-\frac{t}{\lambda}) + \sup_{t^*\in(0,t)} f(t^*)\lambda
\end{aligned} \tag{1}
$$

where $F(v)=\int_0^v f(x)dx$ and the third step is based on mean value theorem with $v^*\in(0,v)$. Rearranging term yields the result. ■

That is, for λ small, the total varation distance converge to 0 in $O(\lambda)$.

We now examine the second case when $\int_{\mathcal{D}} \pi_{0,\mathcal{R}}(\theta)d\theta = 0$.

Remark 3. *Let $M_1(\mathcal{D}^+)=\int_{\mathcal{D}^+}\pi_{0,\mathcal{R}}(\theta)d\theta$ and $M_2=\int_{\mathcal{R}}\pi_{0,\mathcal{R}}(\theta)\mathcal{K}(\theta;\mathcal{D})d\theta$, with $\mathcal{D}^+$ chosen such that $M_1(\mathcal{D}^+)>0$ and $M_1(\mathcal{D}^+)<M_2$, the total variation distance between the measures of extrinsic and intrinsic prior*

$$
\begin{aligned}
||\pi_{0,\mathcal{D}}(\theta),\tilde{\pi}_{0,\mathcal{D}}(\theta)||_{TV} =& 1-\frac{\lim_{\mathcal{D}^+\supset\mathcal{D}}\int_{\theta\in\mathcal{D}^+}\pi_{0,\mathcal{R}}(\theta)\mathcal{K}(\theta;\mathcal{D})d\theta}{M_2} \\
=&\frac{\lim_{\mathcal{D}^+\supset\mathcal{D}}\int_{\theta\in\mathcal{R}\setminus\mathcal{D}^+}\pi_{0,\mathcal{R}}(\theta)\mathcal{K}(\theta;\mathcal{D})d\theta}{M_2}
\end{aligned} \tag{2}
$$