Bayesian constraint relaxation

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Prior information often takes the form of parameter constraints. Bayesian methods include such information through prior distributions having constrained support. By using posterior sampling algorithms, one can quantify uncertainty without relying on asymptotic approximations. However, *sharply* constrained priors are (a) unrealistic in many settings; and (b) tend to limit modeling scope to a narrow set of distributions that are tractable computationally. We propose to solve both of these problems via a general class of Bayesian *constraint relaxation* methods. The key idea is to replace the sharp indicator function of the constraint holding with an exponential kernel. This kernel decays with distance from the constrained space at a rate depending on a relaxation hyperparameter. By avoiding the sharp constraint, we enable use of off-the-shelf posterior sampling algorithms, such as Hamiltonian Monte Carlo, facilitating automatic computation in broad models. We study the constrained and relaxed distributions under multiple settings, and theoretically quantify their differences. We illustrate the method through multiple novel modeling examples.

KEY WORDS: Constrained Bayes, Constraint functions, Shrinkage on Manifold, Support Expansion, Ordered Simplex

1 Constraint Relaxation Methodology

Assume that $\theta \in \mathcal{D} \subset \mathcal{R}$ is an unknown parameter, with $\dim(\mathcal{R}) = r < \infty$. The constrained sample space $\mathcal{D}$ is embedded in the r-dimensional Euclidean space $\mathcal{R}$, and can have either zero or positive measure with respect to Lesbesgue measure on $\mathcal{R}$.

The traditional Bayesian approach to including constraints requires a prior density $\pi_{\mathcal{D}}(\theta)$ with support on $\mathcal{D}$. The posterior density of θ given data Y and $\theta \in \mathcal{D}$ is then

$$\pi_{\mathcal{D}}(\theta \mid Y) \quad \propto \quad \pi_{\mathcal{D}}(\theta)\mathcal{L}(\theta; Y). \tag{1}$$

We assume in the sequel that the restricted prior $\pi_{\mathcal{D}}(\theta) \propto \pi_{\mathcal{R}}(\theta)\mathbb{1}_{\mathcal{D}}(\theta)$, with $\pi_{\mathcal{R}}(\theta)\mathbb{1}_{\mathcal{D}}(\theta)$ an initial unconstrained prior on $\mathcal{R}$ and $\mathbb{1}_{\mathcal{D}}(\theta)$ an indicator function that the constraint is satisfied.

As noted in Section 1, there are two primary problems motivating this article. The first is that it is often too restrictive to assume that θ is *exactly* within $\mathcal{D}$ *a priori*, and often is more plausible to assume that θ has high probability of falling within a small neighborhood of $\mathcal{D}$. The second is that the difficulty of posterior sampling from (1) has greatly limited the scope of modeling, and there is a critical need for general algorithms that are tractable for a broad variety of choices of prior, likelihood and constraint.

In attempting to address these problems, we propose to replace (1) with the following *COnstraint RElaxed* (CORE) posterior density:

$$\tilde{\pi}_{\lambda}(\theta) \propto \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta) \exp\left(-\frac{1}{\lambda}\|v_{\mathcal{D}}(\theta)\|\right), \tag{2}$$

where we repress the conditioning on data Y in $\tilde{\pi}_{\lambda}(\theta)$ for concise notation. $\pi_{\mathcal{R}}(\theta)$ is a density related to (often, proportional to) $\pi_{\mathcal{D}}(\theta)$, except with a different support. We assume the the initial prior $\pi_{\mathcal{R}}(\theta)$ is proper, and use $\|v_{\mathcal{D}}(\theta)\|$ as a distance from θ to the constrained space. For example, $\|v_{\mathcal{D}}(\theta)\| = \inf_{x \in \mathcal{D}} \|\theta - x\|_2$ with $\|.\|$ an appropriate metric.

The hyperparameter $\lambda > 0$ controls how concentrated the prior is around $\mathcal{D}$, and as $\lambda \to 0$ the kernel $\exp\left(-\lambda^{-1}\|v_{\mathcal{D}}(\theta)\|\right)$ converges to $\mathbb{1}_{\mathcal{D}}(\theta)$ in a pointwise manner, excluding $\theta \in \partial\mathcal{D}$ on the boundary of $\mathcal{D}$. For all $\lambda > 0$, $\tilde{\pi}(\theta)$ has support $\mathcal{R}$. This unrestricted support greatly simplifies the design of posterior sampling algorithms, making it easier to consider a much wider variety of models.

In this article, we assume θ is a continuous random variable with $\pi_{\mathcal{R}}(\theta)$ absolutely continuous with respect to Lebesgue measure $\mu_{\mathcal{R}}$ on $\mathcal{R}$. To construct a relaxed distribution, we first start by considering a d-neighborhood of $\mathcal{D}$

$$\{\theta \in \mathcal{R} : \|v_{\mathcal{D}}(\theta)\| \leq d)\}.$$

Clearly, the definition of distance $\|v_{\mathcal{D}}(\theta)\|$ is critical as it describes how support expansion occurs around $\mathcal{D}$. We elaborate the details in the next section.

1.1 Distance to Constrained Space

To induce a neighborhood surrounding only $\mathcal{D}$, a minimal condition is that $\|v_{\mathcal{D}}(\theta)\|$ is zero for $\theta \in \mathcal{D}$ and positive for $\theta \notin \mathcal{D}$. There are many choices of distances that satisfy this condition; however, for two useful purposes, one may be interested in one of the distances.

(I) The ones directly measuring how far $\theta \in \mathcal{R}$ is from the original constrained space $\mathcal{D}$. This is useful when one hopes to directly control and/or measure the support expansion based on θ. For example, when $\mathcal{D}$ is a symmetric compact manifold, one might be interested in uniformly expanding the support of θ, so that the relaxation does not create any directional asymmetry. In this article, we consider a simple isotropic distance

$$\|v_{\mathcal{D}}(\theta)\| = \inf_{x \in \mathcal{D}} \|\theta - x\|_k, \tag{3}$$

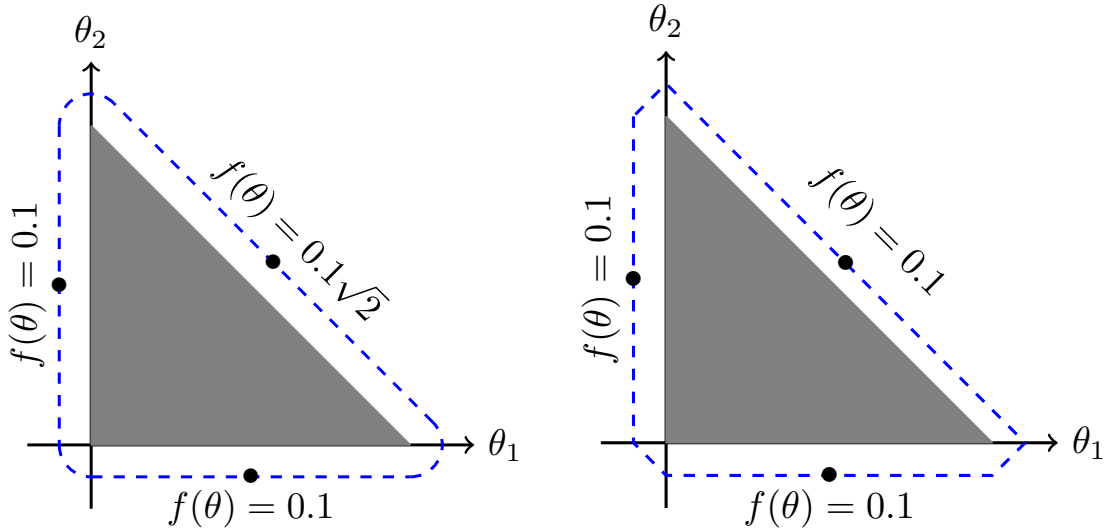
where $\|\cdot\|_k$ denotes a k-norm distance, commonly $k = 1$ or 2.

(II) The ones measuring how far a function of $\theta \in \mathcal{R}$ is from the constrained value when $\theta \in \mathcal{D}$. This is particularly useful when the interest of relaxation is not on parameter θ, but on the function $f(\theta)$ associated with the constraint. For example, when a model is constrained by assigning an inequality to a function, one may be interested in how much the function deviates from the constrained value. Similar to (i), we also consider a simple distance

$$\|v_{\mathcal{D}}(\theta)\| = \inf_{x \in \mathcal{D}} \|f(\theta) - f(x)\|_k. \tag{4}$$

This distance has a simple closed form, when $f(x)$ is a constant for $\theta \in \mathcal{D}$.

To illustrate their difference, we consider a simple constrained space $\mathcal{D} = \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 > 0, \theta_1 + \theta_2 < 1\}$. For type-I distance, we use 2-norm $\inf_{x \in \mathcal{D}} \|\theta - x\|_2$ around space $\mathcal{D}$. One function of interest is the total violation to the three inequalities $f((\theta)) = (-\theta_1)_+ + (-\theta_2)_+ + (1 - \theta_1 - \theta_2)_+$ where $(x)_+ = x$ if $x > 0$ and 0 otherwise. As $f(\theta) = 0$ when $\theta \in \mathcal{D}$, the associated type-II distance is simply $\|v_{\mathcal{D}}(\theta)\| = f(\theta)$. Figure 1 plots the expanded space of $\{\theta : \|v_{\mathcal{D}}(\theta)\| < 0.1\}$ and the values of $f(\theta)$ along $\|v_{\mathcal{D}}(\theta)\| = 0.1$, using those two distances.



(a) Expanded support using type-I distance

(b) Expanded support using type-II distance

Figure 1: The boundary (blue dashed line) along $\|v_{\mathcal{D}}(\theta)\| = 0.1$ formed by two type of distances, around a space formed by three inequalities (gray area) . Panel(a) shows type-I distance uniformly expands the support of the θ; panel (b) shows type-II distance, while does not uniformly expands the support, retains the same amount of total violation to constraints (quantified in a function $f(\theta)$) along the boundary.

1.2 Constraint Relaxation

Intuitively, the exactly constrained model can be viewed as the result of fixing the distance $\|v_{\mathcal{D}}(\theta)\| = 0$. one could achieve constraint relaxation by assigning a distribution that allows $\|v_{\mathcal{D}}(\theta)\|$ to be slightly greater than 0. However, this raises two important questions: (i) how are the constrained density and relaxed density related to each other? (ii) what are the limiting conditions that apply to this methodology? It turns out the answers are very different depending on whether the constrained space $\mathcal{D}$ has a zero measure with respect to $\mu_{\mathcal{R}}$. Therefore, we discuss them separately in the next two subsections.

1.2.1 Constrained Space with Positive Measure

We start with the case when $\mathcal{D}$ is a subset of $\mathcal{R}$ with positive measure with respect to a unconstrained density $\pi_{\mathcal{R}}(\theta)$, that is $\mu_{\mathcal{R}}(\mathcal{D}) > 0$. Some common examples include linear inequality constraints $a^T\theta < 0$ or non-linear inequality constraints. The sharply constrained prior is simply a space-truncated version of the unconstrained prior $\pi_{\mathcal{R}}(\theta)$, leading to the posterior density

$$\pi_{\mathcal{D}}(\theta \mid Y) = \frac{\mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)\mathbb{1}_{\mathcal{D}}(\theta)}{\int_{\mathcal{D}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta) d\mu_{\mathcal{R}}(\theta)} \propto \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)\mathbb{1}_{\mathcal{D}}(\theta),$$

which is defined with respect to $\mu_{\mathcal{R}}$.

For constraint relaxation, one could simply replace the indicator with the exponential function

$$\tilde{\pi}_\lambda(\theta \mid Y) = \frac{\mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta) \exp\big(-\lambda^{-1}\|v_{\mathcal{D}}(\theta)\|\big)}{\int_{\mathcal{R}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta) \exp\big(-\lambda^{-1}\|v_{\mathcal{D}}(\theta)\|\big) d\mu_{\mathcal{R}}(\theta)} \propto \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta) \exp\big(-\lambda^{-1}\|v_{\mathcal{D}}(\theta)\|\big) \tag{5}$$

which is also absolutely continuous with respect to $\mu_{\mathcal{R}}$.

The exponential function can be view as part of a folded Laplace kernel

$$\frac{1}{\lambda}\exp(-\frac{w}{\lambda})$$

for random variable $w = \|v_{\mathcal{D}}(\theta)\|$ with scale λ. Laplace enjoys sharp sharp concentration near zero with small λ and is routinely used in shrinkage literature. This is useful when one wants to allow only a small relaxation from $\|v_{\mathcal{D}}(\theta)\| = 0$ with a fixed small λ, or apply shrinkage towards a constrained space by estimating λ posteriori. Naturally, one could consider other kernel such as folded generalized Pareto (REFS). In this article, we focus on the folded Laplace case for its simplicity.

To address the two motivating questions at the beginning, provided $\|v(\theta)\| = 0$ if and only if $\theta \in \mathcal{D}$, the constrained density is obviously the pointwise limit of relaxed density when $\lambda \to 0$, and this relaxation method is generally applicable with little restriction.

As shown in the last section, the choice of distance has an impact on the geometry of the posterior support. Therefore, care must be taken to ensure the choice is coherent with the prior belief. In the previous example of triangular constrained region, assuming $\pi_{\mathcal{R}}(\theta)$ is a uniform density over a compact $\mathcal{R}$, choosing type-I distance reflects the belief that the parameter θ can depart from $\mathcal{D}$ at equal distance with same prior probability; choosing type-II distance reflects the belief that the prior probability would drop at a rate depending on the total violation to inequalities. On the other hand, if λ is arbitrarily small, their difference vanishes, we will quantify this behavior in the theory section.

For now, we illustrate a Gaussian with a relaxed inequality. Suppose θ is the normal mean of the data, with the prior of θ follows a weakly informative Gaussian

$$y_i \overset{iid}{\sim} \mathrm{No}(\theta, 1) \text{ for } i = 1, \ldots, n, \qquad \theta \sim \mathrm{No}(0, 1000)$$

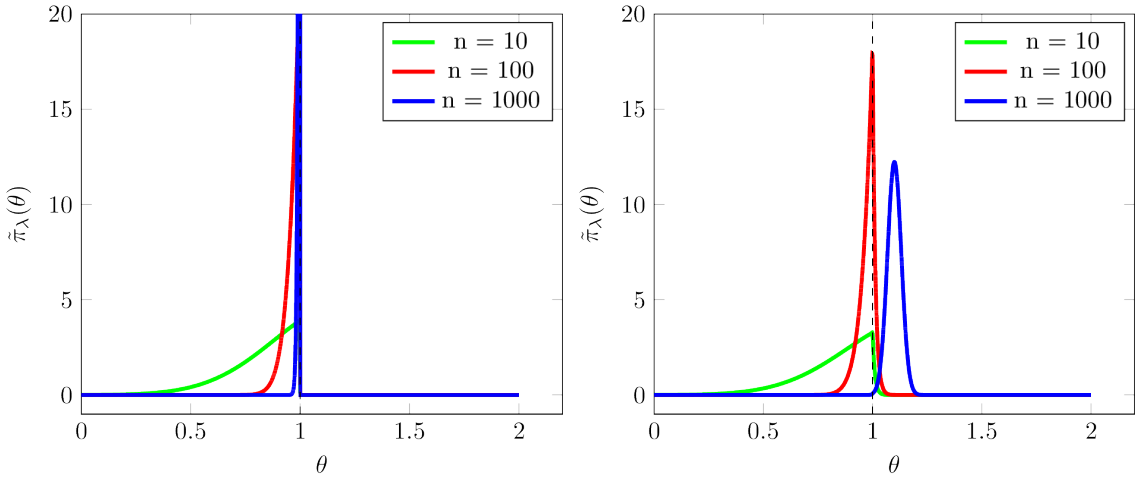
with an prior belief of inequality $\theta < 1$. The posterior under a sharply constrained model is

$$\pi_{\mathcal{D}}(\theta \mid Y) \propto \sigma^{-1}\phi(\frac{\theta - \mu}{\sigma})\mathbb{1}_{\theta<1}, \quad \mu = \frac{\bar{y}n}{1/1000 + n}, \quad \sigma^2 = \frac{1}{1/1000 + n},$$

where ϕ denotes the density of standard Gaussian. However, suppose the average data $\bar{y} = 1.2$ and let n grow. The posterior become increasingly concentrated on the boundary and the sharp inequality becomes much less plausible (Figure 2(a)). In contrast, consider the posterior under relaxed constraint:

$$\pi_\lambda(\theta \mid Y) \propto \sigma^{-1}\phi(\frac{\theta-\mu}{\sigma})\exp(-\frac{(\theta-1)_+}{\lambda}), \quad \mu = \frac{\bar{y}n}{1/1000+n}, \quad \sigma^2 = \frac{1}{1/1000+n}$$

where $(\theta-1)_+$ is the type-I distance to constrained space. With $\lambda = 10^{-2}$, at $n = 10$ and 100, the relaxed posteriors are similar to the sharply constrained ones; however, $\bar{y} = 1.2$ with a large sample size $n = 1000$ leads to result that show the true mean is very likely outside the presumed constrained region.



(a) Sharply constrained posterior

(b) Constraint relaxed posterior

Figure 2: Density of posteriors for Gaussian mean under sharp and relaxed constraints. The relaxed constraint allow strong evidence of large sample with mean $\bar{y} = 1.2$ to overpower the constraint, generating estimated mean outside the constrained region.

1.3 Constrained Space with Zero Measure

In the second case, we consider when $\mathcal{D}$ is a measure zero subset of $\mathcal{R}$, i.e. $\int_{\mathcal{D}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)d\mu_{\mathcal{R}}(\theta) = 0$. Recall that $\mathcal{R}$ have dimensionality r, we now restrict ourselves to the setting where $\mathcal{D}$ can be represented implicitly as the solution set of a consistent system of equations $\{v_j(\theta) = 0\}_{j=1}^s$, so that $\mathcal{D} = \{\theta \mid v_j(\theta) = 0,\ j = 1, \ldots, s\}$ is a $(r-s)$-dimensional submanifold of $\mathcal{R}$. While we impose some restrictions, the result applies on many common constraints (e.g. $\sum_i \theta_i = 1$, $\theta^T\theta = I$).

Due to the zero measure, one cannot obtain conditional probability as before by simply re-normalizing $\left[\int_{\mathcal{D}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}} d\mu_{\mathcal{R}}(\theta)\right]^{-1}$. Instead, we resort to the generalized definition of conditional probability, named *regular conditional probability* (r.c.p.) (**?**) to derive a constrained density coherent with $\pi_{\mathcal{R}}$.

More technical definition of the r.c.p. is provided in the appendix. For now, the following intuition is sufficient. While $\mathcal{D}$ has zero r-dimensional volume (i.e. zero Lebesgue measure), it has a positive $(r-s)$-dimensional 'surface area', formally known as normalized Hausdorff measure, denoted by $\bar{\mathcal{H}}^{(r-s)}$. We can use it as the normalizing constant to obtain a r.c.p density:

$$\pi_{\mathcal{D}}(\theta \mid Y) = \frac{\mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)J^{-1}(\nu_{\mathcal{D}}(\theta))\mathbb{1}_{\mathcal{D}}(\theta)}{\int_{\mathcal{D}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)J^{-1}(\nu_{\mathcal{D}}(\theta))d\bar{\mathcal{H}}^{(r-s)}(\theta)} \propto \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)J^{-1}(\nu_{\mathcal{D}}(\theta))\mathbb{1}_{\mathcal{D}}(\theta).$$

where $J(\nu_{\mathcal{D}}(\theta)) = \sqrt{(D\nu_{\mathcal{D}})'(D\nu_{\mathcal{D}})}$ is the Jacobian of $\nu_{\mathcal{D}}$, which we assume is positive. This term is introduced as we fix $\{v_j(\theta) = 0\}_{j=1}^{s}$. The density is defined with respect to $\bar{\mathcal{H}}^{(r-s)}$.

Similar to the constrained space with positive measure, we face the same modeling and computing challenges here as well. Additionally, although Hausdorff measure is a standard tool in geometric measure theory (**?**), the available distributions are scarce in Bayesian literature.

We now take a similar strategy by considering $\|\nu_{\mathcal{D}}(\theta)\|$ as the distance from θ to $\mathcal{D}$. Thus, $\|v_{\mathcal{D}}(\theta)\| = 0$ implies that $\theta \in \mathcal{D}$, otherwise $\|v_{\mathcal{D}}(\theta)\|_1 > 0$ implies $\theta \notin \mathcal{D}$. To construct a relaxed density, we expand support in the neighborhood of $\{\theta : \|v_{\mathcal{D}}(\theta)\| = 0\}$ by replacing the indicator with $\exp(-\lambda^{-1}\|v(\theta)\|)\mathbb{1}_{\mathcal{X}}(v(\theta))$. The truncation of the image of $v(.)$ to $\mathcal{X} \subset v(\mathcal{R})$ serves two purpose: (i) to make sure $\{\theta : v_{\mathcal{D}}(\theta) = x\}$ still has dimension $(r - s)$ for any $x \in \mathcal{X}$; (ii) to make sure that $v(.)$ is applicable in the following transformation (details to be discussed in next section).

To derive the density, we first consider a Lebesgue measure over a Borel set $\mathcal{F} \subset \mathcal{R}$:

$$\int_{\mathcal{F}} \tilde{\pi}_\lambda(\theta)d\mu(\theta) = \frac{\int_{\mathcal{X}} \left[\int_{\{\theta: v(\theta)=x\}\cap\mathcal{F}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)J^{-1}(\nu_{\mathcal{D}}(\theta))d\bar{\mathcal{H}}^{(r-s)}(\theta)\right] \exp\left(-\lambda^{-1}\|x\|\right)dx}{\int_{\mathcal{X}} \left[\int_{\{\theta: v(\theta)=x\}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)J^{-1}(\nu_{\mathcal{D}}(\theta))d\bar{\mathcal{H}}^{(r-s)}(\theta)\right] \exp\left(-\lambda^{-1}\|x\|\right)dx}$$

Using co-area formula (**?**), we can now transform double integrals to single integral. Omitting the integral over $\mathcal{F}$, this simplifies to a density:

$$\tilde{\pi}_\lambda(\theta) = \frac{\mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)\exp\left(-\frac{1}{\lambda}\|\nu_{\mathcal{D}}(\theta)\|\right)\mathbb{1}_{\mathcal{X}}(v(\theta))}{\int_{\mathcal{R}} \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)\exp\left(-\frac{1}{\lambda}\|\nu_{\mathcal{D}}(\theta)\|\right)\mathbb{1}_{\mathcal{X}}(v(\theta))d\mu_{\mathcal{R}}(\theta)} \propto \mathcal{L}(\theta; Y)\pi_{\mathcal{R}}(\theta)\exp\left(-\frac{1}{\lambda}\|\nu_{\mathcal{D}}(\theta)\|\right)\mathbb{1}_{\mathcal{X}}(v(\theta)), \tag{6}$$

which is defined with respect to $\mu_{\mathcal{R}}$. Note the Jacobian term vanishes and the density is with respect to common Lebesgue measure. The relaxed density is very similar to (5) in the last section.

Much like the positive measure case, $\exp\left(-\lambda^{-1}\|\nu_{\mathcal{D}}(\theta)\|\right)$ converges pointwise to $\mathbb{1}_{\mathcal{D}}(\theta)$. As $\lambda \to 0^+$, this multiplicative factor is concentrating the probability to a small layer around the constrained space. As a result, for small λ, one could expect that

$$\int_{\mathcal{R}} g(\theta)\tilde{\pi}_\lambda(\theta)d\mu_{\mathcal{R}}(\theta) \approx \int_{\mathcal{D}} g(\theta)\pi_{\mathcal{D}}(\theta)d\bar{\mathcal{H}}^{(r-s)}(\theta)$$

We provide details of this result and suitable class of functions in the next section. We first provide another example to illustrate.

Example: Constrained Gaussian on Unit Circle

Let $\theta = (\theta_1, \theta_2)$ be a bivariate Gaussian parameterized by mean $\mu \in \mathbb{R}^2$ and covariance matrix, $\Sigma = \sigma^2 I_2$, $\sigma > 0$, except it is constrained to the unit circle $\mathcal{D} = \{(\theta_1, \theta_2) \mid \theta_1^2 + \theta_2^2 = 1\}$. Since the unit circle is one-dimensional and $\theta = (\theta_1, \theta_2)$ is two-dimensional, we use a (2-1)=1-dimensional constraint function

$$v_{\mathcal{D}}(\theta_1, \theta_2) = \theta_1^2 + \theta_2^2 - 1.$$

Then $v_{\mathcal{D}}(\theta_1, \theta_2) = 0 \ \forall \theta \in \mathcal{D}$. Otherwise, $v_{\mathcal{D}}$ is non-zero. Furthermore, $J(\nu_{\mathcal{D}}(\theta)) = 2\|\theta\|_2 = 2$ for $\theta \in \mathcal{D}$. The constrained density of θ given that $\theta \in \mathcal{D}$ is then,

$$\begin{aligned}
\pi_{\mathcal{D}}(\theta_1, \theta_2) = &= \frac{\exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} \right) \mathbb{1}_{\mathcal{D}}(\theta)}{\int_{\mathcal{D}} \exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} \right) d\bar{\mathcal{H}}^1(\theta)} \\
&\propto \exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} \right) \mathbb{1}_{\theta_1^2 + \theta_2^2 = 1} \\
&\propto \exp\left(\frac{\theta' \mu}{\sigma^2} \right) \mathbb{1}_{\theta_1^2 + \theta_2^2 = 1}.
\end{aligned}$$

This density can be interpreted with respect to the normalized Hausdorff-1 measure on the unit circle which coincides with arclength in this case. Observe this is the von Mises–Fisher distribution on the unit circle with location $\mu / \|\mu\|_2$ and concentration $\|\mu\|_2 / \sigma^2$.

We make the relaxed space compact by $\mathcal{R} = (-a, a)^2$, with $a > 1$; and $\mathcal{X} = [-1, 0) \cup (0, 2a^2 - 1]$. Clearly, $\{\theta_1^2 + \theta_2^2 = x\}$ still has dimension 1 for all $x \in \mathcal{X}$.

The relaxed density $\tilde{\pi}_\lambda(\theta)$ is

$$\begin{aligned}
\tilde{\pi}_\lambda(\theta) &= \frac{\exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} - \frac{1}{\lambda} \|v_{\mathcal{D}}(\theta)\| \right) \mathbb{1}_{\mathcal{X}}(v_{\mathcal{D}}(\theta))}{\int_{\mathbb{R}^2} \exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} - \frac{1}{\lambda} \|v_{\mathcal{D}}(\theta)\| \right) \mathbb{1}_{\mathcal{X}}(v_{\mathcal{D}}(\theta)) d\theta_2 d\theta_1} \\
&\propto \exp\left(- \frac{\|\theta - \mu\|^2}{2\sigma^2} \right) \exp\left(- \frac{1}{\lambda} |\theta_1^2 + \theta_2^2 - 1| \right) \mathbb{1}_{\mathcal{X}}(\theta_1^2 + \theta_2^2 - 1).
\end{aligned} \tag{7}$$

Figure **??** depicts a few plots of the relaxed density as λ decreases. For $\lambda = 10^{-2}$ the constraint along the circle is clear. While the relaxed density still places some small probability outside of the constrained region, the rightmost plot becomes similar to the von Mises–Fisher distribution on the circle plotted in two dimensions.