Mean-Field Factorisation Strategy

Mean-field variational inference imposes conditional independence on the posterior to enable tractable inference. The key is to factor parameters according to their coupling structure in the likelihood.

Full joint posterior:

$$p(\beta, u, \tau_u, \tau_e \mid y) \propto p(y \mid \beta, u, \tau_e)\, p(u \mid \tau_u)\, p(\beta)\, p(\tau_u)\, p(\tau_e)$$

Mean-field factorisation:

$$q(\beta, u, \tau_u, \tau_e) = q(\beta, u) \cdot q(\tau_u) \cdot q(\tau_e)$$

This factorisation preserves the coupling between β and u in the likelihood, while separating the precision parameters to admit conjugate Gamma updates.

Coordinate Ascent Updates

Regression block: $p(\beta, u \mid y, \tau_u, \tau_e)$ *is Gaussian, so* $q(\beta, u)$ *is Gaussian:*

$$\begin{aligned}\Sigma_{\beta u}^{\text{new}} &= \left[X^T X\, \mathbb{E}[\tau_e] + \text{diag}(\Gamma_\beta^{-1}, \mathbb{E}[\tau_u]\, \mathbf{1}_u)\right]^{-1} \\ \mu_{\beta u}^{\text{new}} &= \Sigma_{\beta u}^{\text{new}}\, X^T y\, \mathbb{E}[\tau_e]\end{aligned}$$

Residual precision: $p(\tau_e \mid y, \beta, u)$ *is Gamma, so* $q(\tau_e)$ *is Gamma:*

$$\begin{aligned}a_e^{\text{new}} &= a_e + \tfrac{n}{2} \\ b_e^{\text{new}} &= b_e + \tfrac{1}{2}\mathbb{E}[(y - X\beta - Zu)^T (y - X\beta - Zu)]\end{aligned}$$

Random-effects precision: $p(\tau_u \mid u)$ *is Gamma, so* $q(\tau_u)$ *is Gamma:*

$$\begin{aligned}a_u^{\text{new}} &= a_u + \tfrac{Q}{2} \\ b_u^{\text{new}} &= b_u + \tfrac{1}{2}\mathbb{E}[u^T u]\end{aligned}$$

All expectations are computed using current variational parameters. The loop repeats until ELBO convergence.