Constructing a Gibbs sampler for a linear model

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I)$$

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Likelihood:

$$p(y \mid \beta, \sigma^2; X) = \prod_{i=1}^{n} p(y_i \mid \beta, \sigma^2; x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_i - x_i\beta)^2/2\sigma^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\sum_i (y_i - x_i\beta)^2/2\sigma^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(y - X\beta)^T(y - X\beta)/2\sigma^2}$$

Priors:

$p(\beta) \propto 1$ (Flat prior, note this is improper)

~~pt~~ Work with $\tau = (\sigma^2)^{-1}$

$p(\tau) \propto \tau^{-1}$ (Note this is a Gamma $(0, 0)$ prior, which is also improper)

Joint distribution $p(y \mid \beta, \tau; X)\, p(\beta)\, p(\tau)$

$$p(y, \beta, \tau \mid X) = \left(\frac{\tau}{2\pi}\right)^{n/2} e^{-\tau(y - X\beta)^T(y - X\beta)/2} \times \cancel{1} \times \tau^{-1}$$

$$= (2\pi)^{-n/2}\, \tau^{n/2 - 1}\, e^{-\tau(y - X\beta)^T(y - X\beta)/2}$$

- Now we need to find conditional posteriors.

- $\beta$ :

The component of the joint distribution that is $f(\beta)$ is :

$$e^{-\tau(y-X\beta)^T(y-X\beta)/2}$$

expanding out

$$e^{-\tau(\beta^T y - \beta^T X^T y - y^T X\beta + \beta^T X^T X\beta)/2}$$

$$\propto e^{-\tau(\beta^T X^T X\beta - \beta^T X^T y - y^T X\beta)/2}$$

$$= e^{-(\beta^T \underbrace{(\tau X^T X)}_{\Sigma^{-1}}\beta - \beta^T \underbrace{\tau(X^T X)}_{\Sigma^{-1}}\underbrace{(X^T X)^{-1}X^T y)}_{\mu} - y^T X (X^T X)^{-1}(X^T X \tau)\beta)/2}$$

~~Note~~ MV Normal

$$e^{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2}$$

$$\cancel{e^{-(x^T\Sigma^{-1}x -}}$$

$$= e^{-(x^T\Sigma^{-1}x - x'\Sigma^{-1}\mu - \mu'\Sigma^{-1}x + \mu'\Sigma^{-1}\mu)/2}$$

$$\propto e^{-(x^T\Sigma^{-1}x - x'\Sigma^{-1}\mu - \mu'\Sigma^{-1}x)/2}$$

$\therefore$

$$p(\beta|\tau, X, y) = N\left(\beta \,\middle|\, (X^T X)^{-1}X^T y, \frac{(X^T X)^{-1}}{\tau}\right)$$

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$\rightarrow$: The component of the joint distribution that is $f(\tau)$ is:

$$\underbrace{\tau^{n/2-1}}_{\alpha} \quad e^{-\tau \underbrace{(y-X\beta)^T(y-X\beta)/2}_{\gamma}}$$

Gamma distribution

$$\frac{\gamma^{\alpha}}{\Gamma(\alpha)} \underbrace{x^{\alpha-1} e^{-\gamma x}}$$

Kernel of gamma

$$x^{\alpha-1} e^{-\gamma x}$$

$\therefore$

$$p(\tau \mid \beta, y, X) := Ga\left(\tau \mid \frac{n}{2}, \frac{(y-X\beta)^T(y-X\beta)}{2}\right)$$