List of Todos

1. Notation and Definitions

Let $G \subset \mathfrak{S}_n$, let $x, y \in 2^{[n]}$, and let $1 \leq r \leq n$. For $r \leq i \leq n$, let $V_i^{(r)}$ be the $\mathbb{R}$-vector space generated by the basis

$$\{e_{(x,y)}\}_{x \subset y, |y|=i, |x|=i-r}$$

Note that G acts on this space with the action $\sigma e_{(x,y)} = e_{(\sigma x, \sigma y)}$. Let $\left(V_i^{(r)}\right)^G$ be the subspace of $V_i^{(r)}$ that is invariant under this action. Write $q_i^r = \dim\left(\left(V_i^{(r)}\right)^G\right)$. When r is understood to be 1, we will often simply write q_i.

Let p_i be defined as by Pak and Panova, that is

$$p_i = \sum_{Gy, |y|=i} \nu(Gy)$$

what's the latex command for the dot?

where $\nu(Gy)$ is the number of covering relations $Gy \cdot > Gx$.

2. Current Goals

Our main goal is describe the groups $G \subset \mathfrak{S}_n$ such that for $2^{[n]}/G$ we have $q_i = p_i$. So far we know that this is true for $G = \{e\}$, $G = \mathfrak{S}_n$, and $G = \mathfrak{S}_k \wr \mathfrak{S}_l$ (this is equivalent to Proposition 4.2). Any other groups for which it holds would be good to know about. In fact, if someone wanted to write a program to compute the sequence q_i for all subgroups of $\mathfrak{S}_4$, $\mathfrak{S}_5$, $\mathfrak{S}_6$, $\mathfrak{S}_7$ and such, that would be awesome!

3. Conjectures

Put any conjectures, no matter how wild, here.

4. Results

Proposition 4.1. *For all $G \subset \mathfrak{S}_n$, $1 \leq r \leq n$, the sequence $q_r^r, q_{r+1}^r, \ldots, q_n^r$ is unimodal and symmetric about $\frac{n+r}{2}$.*

Proposition 4.2. *For $G = \mathfrak{S}_k \wr \mathfrak{S}_l$ and $r = 1$, $q_i = p_i$ for all $1 \leq i \leq n$.*

Lemma 4.3. *$U_i^{(r)}$ is injective for all $i < \frac{n+r}{2}$.*

Lemma 4.4. *For all $\sigma \in G, e_{(x,y)} \in V_i^{(r)}$, we have*

$$U_i^{(r)}(\sigma(e_{(x,y)})) = \sigma(U_i^{(r)}(e_{(x,y)}))$$

Lemma 4.5. *For any group $G \subset \mathfrak{S}_n$ with an action on an $\mathbb{R}$-vector space V with basis $\{v_i\}_{1 \leq i \leq k}$, the G-invariant subspace V^G of V has basis*

$$\sum_{v_i \in Gv} v_i$$

where the sum is taken over the orbits Gv of the group action.

5. Failed Attempts

Have an idea that failed? Show it off here!