Peckness of Edge Posets

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Outline of Talk

1. Background
2. Edge Poset Construction
3. Main Result
4. CCT Actions
5. Non-CCT actions
6. Final remarks

Basic Definitions

Definition

Let P be a finite graded poset of rank n. That is:

- Elements of P are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks*
- If $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$
- Define $\mathrm{rk}(x) = k$, where $x \in P_k$.

Definition

A map $f\colon P \to Q$ is a *morphism* from P to Q if $x \leq_P y \implies f(x) \leq_Q f(y)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Peck Posets

Definition

Write $p_i = |P_i|$. P is

- *Rank-symmetric* if $p_i = p_{n-i}$ for all $1 \leq i \leq n$
- *Rank-unimodal* if for some $0 \leq k \leq n$ we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

- *k-Sperner* if no disjoint union of k antichains (sets of pairwise incomparable elements) in P is larger than the disjoint union of the largest k ranks of P
- *Strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$.
- *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Definition

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x|x \in P\}$ and $\{x|x \in P_i\}$

Lemma (Stanley, 1982)

P is Peck if and only if there exists an linear transformation $U\colon V(P) \to V(P)$ such that

- *For every basis element $x \in P$,*

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

- *For all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition

If the Lefschetz map defined by

$$L(x) = \sum_{y \gtrdot x} y$$

satisfies the second condition in the previous lemma, then P is *unitary Peck*.

Outline of Talk

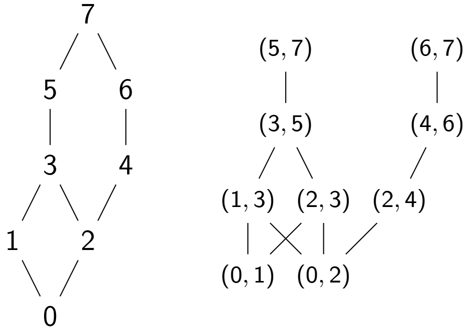
Definition of the Edge Poset

Definition

For P a finite graded poset, its *edge poset* $\mathcal{E}(P)$ is the finite graded poset defined as follows.

- Elements of $\mathcal{E}(P)$ are ordered pairs $(x, y) \in P \times P$ where $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$
- Define $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$
- Define $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$.

Basic Example



P $\mathcal{E}(P)$

Conjecture on the Peckness of Edge Posets

Definition

The *boolean algebra of rank n* is the poset whose elements are subsets of $[n]$ with order given by containment, i.e. for $x, y \in B_n$, $x \leq y$ if $x \subseteq y$.

Conjecture (Hemminger, Landesman, and Yao 2014)

Let $G \subseteq \mathrm{Aut}(B_n)$. *Then* $\mathcal{E}(B_n/G)$ *is Peck.*

Outline of Talk

Main Result

Definition

A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in \mathrm{Stab}_G(z)$ such that $g \cdot x = y$.

Theorem (Hemminger, Landesman, and Yao 2014)

If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.

Definition

Given a group action of G on P, define a group action of G on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

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Proposition

The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism. Furthermore, q is also injective if and only if the action of G on P is CCT.

Lemma

If $f : P \to Q$ is a bijective morphism and P is Peck then Q is Peck.

Theorem (Stanley, 1984; Harper, 1984; Pouzet and Rosenberg, 1986)

If P is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then P/G is Peck.

It suffices to show that $\mathcal{E}(B_n)$ is unitary Peck. Our proof of this is complicated. Instead, we construct a unitary Peck poset $\mathcal{H}(B_n)$ such that there is a bijective morphism $\mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$.

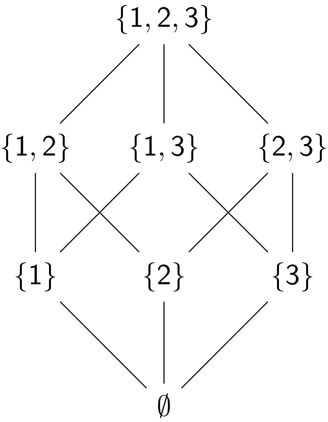
Definition of $\mathcal{H}(P)$

Definition

For P a finite graded poset, define the graded poset $\mathcal{H}(P)$ as follows.

- Elements are pairs $(x, y) \in P \times P$ such that $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{H}} (x', y')$ if $x \lessdot_P x', y \lessdot_P y'$ **and** $\mathbf{y \neq x'}$
- Define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$
- Define $rk_{\mathcal{H}}(x, y) = rk_P(x)$.

The Boolean Algebra B_3



B_3

$\mathcal{H}(B_3)$ is unitary Peck

$$
\begin{tikzcd}[arrows=dash]
 & (\{2,3\},\{1,2,3\}) & \\
(\{2\},\{1,2\}) \arrow[ur] & & (\{3\},\{1,3\}) \arrow[ul] \\
 & (\emptyset,\{1\}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash]
 & (\{1,3\},\{1,2,3\}) & \\
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 & (\emptyset,\{3\}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$\mathcal{H}(B_3)$

Definition

As before, for G acting on $\mathcal{H}(P)$, define $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Remark

Since $\mathcal{E}(P)$ and $\mathcal{H}(P)$ have the same elements and $(x, y) \leq_{\mathcal{H}} (x', y') \implies (x, y) \leq_{\mathcal{E}} (x', y')$, there is a natural bijective morphism $\mathcal{H}(P)/G \to \mathcal{E}(P)/G$.

Proof of Main Result.

$\mathcal{H}(B_n)$ unitary Peck $\implies \mathcal{H}(B_n)/G$ Peck $\implies \mathcal{E}(B_n)/G$ Peck $\implies \mathcal{E}(B_n/G)$ Peck. $\square$

Outline of Talk

CCT actions

Lemma

Let G be a group acting on a graded poset P. The following are equivalent:

1. *The action of G on P is CCT.*
2. *Whenever $w \lessdot x, w \lessdot y$, and $x \in Gy$, there exists some $g \in Stab(w)$ with $gx = y$.*
3. *The map $q \colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x,z)) = (Gx, Gz)$ is a bijective morphism (but not necessarily an isomorphism).*
4. *For all i there is an equality $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$*

Some examples of CCT actions

The building blocks

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3. The group D_{2n} acting on B_n when $n = p$ or $n = 2p$, and p is a prime;

Some examples of CCT actions

The building blocks

1. The trivial group;
2. The group S_n acting on B_n;
3. The group D_{2n} acting on B_n when $n = p$ or $n = 2p$, and p is a prime;
4. The elementary 2-group $(\mathbb{Z}/2\mathbb{Z})^k$ with any action on B_n induced by an action on $[n]$.

The direct product

Lemma

For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.

The semi-direct product

Proposition

Let $G \subseteq \mathrm{Aut}(P)$, $H \lhd G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.

The wreath product

Corollary

If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action is also CCT.

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Corollary

The action $S_m \wr S_l \times B_n \to B_n$ is CCT, where $n = ml$.

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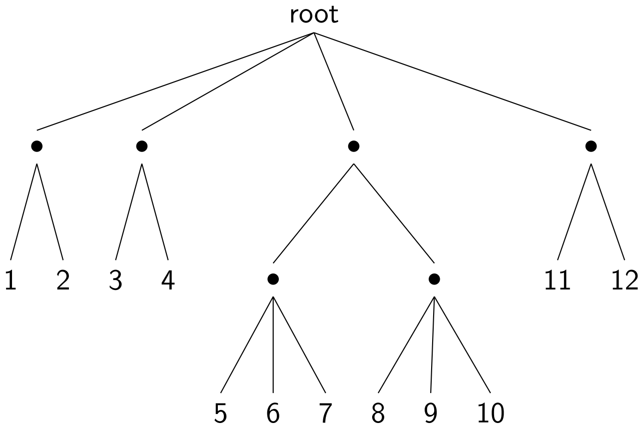
Corollary

The action $S_m \wr S_l \times B_n \to B_n$ is CCT, where $n = ml$.

Remark

This recovers a special case of a theorem obtained by Pak & Panova:
The poset $\mathcal{E}(B_n/S_m \wr S_l)$ is rank symmetric and rank unimodal.
(Furthermore, it is Peck!)

The automorphism of rooted trees



Automorphism of rooted trees

Proposition

Let P be a rooted tree. Then,

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

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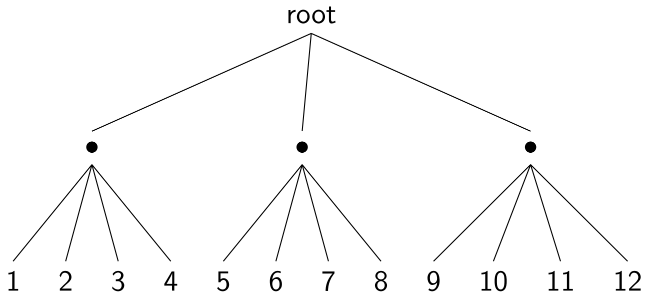
Corollary

Let P be a rooted tree with leaves $L(P)$, and let $n = |L(P)|$, then the action of $Aut(P)$ on B_n induced from the action of $Aut(P)$ on $L(P)$ is CCT.

Rooted trees and the wreath product



Outline of Talk

Unimodality of ranks of certain edge posets

Lemma

Let C_n be the cyclic group which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|\mathcal{E}(B_n)/C_n|_i = \binom{n-1}{i}.$$

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Lemma

Let C_p be the cyclic group with prime order p, then

$$|(\mathcal{E}(B_p)/C_p)_i| - |\mathcal{E}(B_p/C_p)_i| = \frac{p-1}{2}.$$

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Proposition

The poset $\mathcal{E}(B_n/C_n)$ is rank symmetric and rank unimodal.

Unimodality of ranks of certain edge posets

Lemma

Let D_{2n} be the dihedral group of size $2n$ which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/D_{2n}$ is

$$|\mathcal{E}(B_n)/D_{2n}|_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+1)}+1]\cdot\binom{\lceil n/2\rceil - 1}{\lceil (i+1)/2\rceil - 1}\right)$$

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Proposition

The poset $\mathcal{E}(B_n/D_{2n})$ is rank symmetric and rank unimodal.

Outline of Talk

The q analog of the problem

The q-Boolean algebra

Let $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, and $G < Gl_n(\mathbb{F}_q)$. We consider $\mathcal{E}(B_n(q))/G$ and $\mathcal{E}(B_n(q)/G)$.

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Lemma

Let q be a prime, then

$$|\mathcal{F}^1(B_n(q))/C_n(q)|_i = \binom{n-1}{i}_q.$$

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Lemma

Let q be a prime, then

$$|\mathcal{F}^1(B_n(q))/C_n(q)|_i = \binom{n-1}{i}_q.$$

Questions

Is $\mathcal{E}\big(B_n(q)/G\big)$ Peck? or more weakly, is it rank unimodal?

Some Generalizations of $\mathcal{E}$ and Remarks

$\mathcal{E}^r(P)$

Similarly we can define the $\mathcal{E}^r(P)$ on a graded poset P. The elements of $\mathcal{E}^r(P)$ are (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + r$. Define the covering relation $\lessdot_{\mathcal{E}}$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$.

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Other generalizations

$\mathcal{H}^r(P)$; $\mathcal{E}^{\vec{r}}(P)$; $\mathcal{H}^{\vec{r}}(P)$.

References

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