SYMMETRIC CHAIN DECOMPOSITION OF NECKLACE POSETS

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Abstract. A finite ranked poset is called a symmetric chain order if it can be written as a disjoint union of rank-symmetric, saturated chains. If $\mathcal{P}$ is any symmetric chain order, we prove that $\mathcal{P}^n/\mathbb{Z}_n$ is also a symmetric chain order, where $\mathbb{Z}_n$ acts on $\mathcal{P}^n$ by cyclic permutation of the factors.

1. Introduction

Let $(\mathcal{P}, <)$ be a finite poset. A *chain* in $\mathcal{P}$ is a sequence of the form $x_1 < x_2 < \cdots < x_n$ where each $x_i \in \mathcal{P}$. For $x, y \in \mathcal{P}$, we say y *covers* x (denoted $x \lessdot y$) if $x < y$ and there does not exist $z \in \mathcal{P}$ such that $x < z$ and $z < y$. A *saturated* chain in $\mathcal{P}$ is a chain where each element is covered by the next. We say $\mathcal{P}$ is *ranked* if there exists a function $\mathrm{rk} : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ such that $x \lessdot y$ implies $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. The *rank* of $\mathcal{P}$ is defined as $\mathrm{rk}(\mathcal{P}) = \max\{\mathrm{rk}(x) \mid x \in \mathcal{P}\} + \min\{\mathrm{rk}(x) \mid x \in \mathcal{P}\}$. A saturated chain $\{x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n\}$ in a ranked poset $\mathcal{P}$ is said to be *rank-symmetric* if $\mathrm{rk}(x_1) + \mathrm{rk}(x_n) = \mathrm{rk}(\mathcal{P})$.

We say that $\mathcal{P}$ has a *symmetric chain decomposition* if it can be written as a disjoint union of saturated, rank-symmetric chains. A *symmetric chain order* is a finite ranked poset for which there exists a symmetric chain decomposition.

A finite product of symmetric chain orders is a symmetric chain order. This result can be proved by induction [1] or by explicit constructions (e.g. [3]). Naturally, this raises the question of whether the quotient of a symmetric chain order under a given group action has a symmetric chain decomposition. For example, if X is a set then $\mathbb{Z}_n$ acts on the set $Map(\mathbb{Z}_n, X) \simeq X^n$. The elements of $X^n/\mathbb{Z}_n$ are called *n-bead necklaces with labels in X*. A symmetric chain decomposition of the poset of binary necklaces was first constructed by K. Jordan [6], building on the work of Griggs-Killian-Savage [4]. There have been recent independent proofs and generalizations of these results [2, 5]. The main result of this paper is the following:

1.1. **Theorem.** If $\mathcal{P}$ is a symmetric chain order, then $\mathcal{P}^n/\mathbb{Z}_n$ is a symmetric chain order.

We give a brief outline of the proof. First, we show that the poset of n-bead binary necklaces is isomorphic to the poset of *partition necklaces*, i.e. n-bead necklaces labeled by positive integers which sum to n. It turns out to be convenient to exclude the maximal and minimal binary necklaces, which correspond to those partitions of n having n parts and 0 parts, respectively. Let $\mathcal{Q}(n)$ denote the poset of partition necklaces with these two elements removed. We decompose $\mathcal{Q}(n)$ into rank-symmetric sub-posets $\mathcal{Q}_\alpha$, running over partition necklaces α where 1 does not appear. This decomposition corresponds to the "block-code" decomposition of binary necklaces defined in [4].

We can also extend this idea to non-binary necklaces. In fact, the poset of n-bead $(m+1)$-ary necklaces embeds into the poset of nm-bead binary necklaces, and the image corresponds to the union of those $\mathcal{Q}_\alpha \subset \mathcal{Q}(mn)$ such that every part of α is divisible by m.

Next, we prove a "factorization property" for $\mathcal{Q}_\alpha \subset \mathcal{Q}(n)$. If P and Q are finite ranked posets, we say that P *covers* Q (or Q is *covered by* P) if there is a morphism of ranked posets from P to Q which is a bijection on the underlying sets. We denote this relation as $P \tilde{\hookrightarrow} Q$. Note that any ranked poset covered by a symmetric chain order is also a symmetric chain order. If α is aperiodic, then $\mathcal{Q}_\alpha$ is covered by a product of symmetric chains. If α is periodic of period d, then $\mathcal{Q}_\alpha$ is covered by the poset of (n/d)-bead necklaces labeled by $\mathcal{Q}_\beta$, for some aperiodic d-bead necklace β.

Finally, if $\mathcal{P}$ is a symmetric chain order, then $\mathcal{P}^n/\mathbb{Z}_n$ has a decomposition into posets which are either products of chains, or posets of d-bead necklaces with labels in a product of chains (where $d < n$), or posets of n-bead $(m+1)$-ary necklaces for some $m \geq 1$. In each case, we apply induction to finish the proof.

2. Generalities on necklaces

We begin by recalling some basic facts about $\mathbb{Z}_n$-actions on sets. We will use additive notation for the group operation of $\mathbb{Z}_n$. The subgroups of $\mathbb{Z}_n$ are of the form $\langle d\rangle$ where d is a positive divisor of n, and $\mathbb{Z}_n/\langle d\rangle \simeq \mathbb{Z}_d$. If X is a set with $\mathbb{Z}_n$-action, let $X^{\langle d\rangle}$ denote the set of $\langle d\rangle$-fixed points in X. Equivalently:

$$X^{\langle d\rangle} = \{x \in X \mid \langle d\rangle \subset Stab_{\mathbb{Z}_n}(x)\}.$$

Note that $X^{\langle c\rangle} \subset X^{\langle d\rangle}$ if c is a divisor of d. Next, we define:

$$X^{\{d\}} = \{x \in X \mid \langle d\rangle = Stab_{\mathbb{Z}_n}(x)\}.$$

Of course, we have:

$$X = \bigsqcup_{d|n} X^{\{d\}}$$

and the $\mathbb{Z}_n$ action on $X^{\{d\}}$ factors through $\mathbb{Z}_d$. In other words, we have a bijection:

$$X/\mathbb{Z}_n \simeq \bigsqcup_{d|n} X^{\{d\}}/\mathbb{Z}_d.$$

Now consider the special case where $X = Map(\mathbb{Z}_n, Y)$ for some arbitrary set Y, where $\mathbb{Z}_n$ acts on the first factor. In other words,

$$(af)(b) = f(a+b)$$

for any $a, b \in \mathbb{Z}_n$ and $f : \mathbb{Z}_n \to Y$. Now the previous paragraph implies that:

$$Map(\mathbb{Z}_n, Y) = \bigsqcup_{d|n} Map(\mathbb{Z}_n, Y)^{\{d\}}$$

and

$$Map(\mathbb{Z}_n, Y)/\mathbb{Z}_n = \bigsqcup_{d|n} Map(\mathbb{Z}_n, Y)^{\{d\}}/\mathbb{Z}_d.$$

The elements of $Map(\mathbb{Z}_n, Y)/\mathbb{Z}_n$ are called *n-bead necklaces with labels in Y*.

An element of $Map(\mathbb{Z}_n, Y)^{\{d\}}/\mathbb{Z}_d$ is said to be *periodic of period d*. An element of $Map(\mathbb{Z}_n, Y)^{\{n\}}/\mathbb{Z}_n$ is said to be *aperiodic*. Given a map $g : \mathbb{Z}_n \to Y$, let $[g]$ denote the corresponding necklace in $Map(\mathbb{Z}_n, Y)/\mathbb{Z}_n$. A n-bead necklace with labels in Y can be visualized as a sequence of n elements of Y placed evenly around a circle, where we discount the effect of rotation by any multiple of $\frac{2\pi}{n}$ radians. Given $(y_1, \ldots, y_n) \in Y^n$, let $[y_1, \ldots, y_n]$ denote the corresponding n-bead necklace.

Our first observation is that an n-bead necklace of period d is uniquely determined by any sequence of d consecutive elements around the circle. Moreover, as we rotate the circle, these d elements will behave exactly like an aperiodic d-bead necklace.

2.1. **Proposition.** There is a natural bijection between n-bead necklaces of period d and aperiodic d-bead necklaces.

Proof. Recall the following general fact: if G is a group, H is a normal subgroup of G, and Y is an arbitrary set, then there is an isomorphism of G-sets:

$$Map(G, Y)^H \simeq Map(G/H, Y)$$
$$f \mapsto (gH \mapsto f(g)).$$

Moreover, the action of G on each side factors through G/H. In particular, there is an isomorphism of $\mathbb{Z}_n$-sets:

$$Map(\mathbb{Z}_n, Y)^{\langle d\rangle} \simeq Map(\mathbb{Z}_d, Y)$$

where the $\mathbb{Z}_n$-action factors through $\mathbb{Z}_d$. Looking at elements of period d, we get:

$$Map(\mathbb{Z}_n, Y)^{\{d\}} \simeq Map(\mathbb{Z}_d, Y)^{\{d\}}$$

and so:

$$Map(\mathbb{Z}_n, Y)^{\{d\}}/\mathbb{Z}_d \simeq Map(\mathbb{Z}_d, Y)^{\{d\}}/\mathbb{Z}_d.$$

□

Now suppose that Y is a disjoint union of non-empty subsets:

$$Y = \bigsqcup_{i\in I} Y_i$$

where I is a finite set. Equivalently, we have a surjective map $\pi : Y \to I$, where $Y_i = \pi^{-1}(i)$ for each $i \in I$. It follows that there is a surjective map:

$$\pi_* : Map(\mathbb{Z}_n, Y) \to Map(\mathbb{Z}_n, I)$$

$$\pi_*(f) = \pi \circ f.$$

Given a map $g : \mathbb{Z}_n \to I$, we define:

$$Map_g(\mathbb{Z}_n, Y) = \pi_*^{-1}(g) = \{f : \mathbb{Z}_n \to Y \mid \pi \circ f = g\}.$$

In other words, $f \in Map_g(\mathbb{Z}_n, Y)$ if and only if $f(a) \in Y_{g(a)}$ for all $a \in \mathbb{Z}_n$. Since π_* is surjective, we have a decomposition:

$$Map(\mathbb{Z}_n, Y) = \bigsqcup_{g \in Map(\mathbb{Z}_n, I)} Map_g(\mathbb{Z}_n, Y).$$

Note that $Map_g(\mathbb{Z}_n, Y)$ is not necessarily stable under the action of $\mathbb{Z}_n$. If $a, b \in \mathbb{Z}_n$ and $f \in Map_g(\mathbb{Z}_n, Y)$, then:

$$a(f)(b) = f(a + b) \in Y_{g(a+b)}$$

so we have a bijection:

$$Map_g(\mathbb{Z}_n, Y) \simeq Map_{ag}(\mathbb{Z}_n, Y)$$

induced by the action of $a \in \mathbb{Z}_n$. We define:

$$Map_{[g]}(\mathbb{Z}_n, Y) = \bigcup_{a \in \mathbb{Z}_n} Map_{ag}(\mathbb{Z}_n, Y).$$

Note that $\mathbb{Z}_n$ acts on $Map_{[g]}(\mathbb{Z}_n, Y)$.

2.2. **Remark.** We recall a basic observation which will make it easier to define maps on sets of necklaces. Suppose S and T are sets equipped with equivalence relations $\sim$ and $\approx$, respectively. Let U be a subset of S which has a non-trivial intersection with each equivalence class in S. Then U inherits the equivalence relation $\sim$ and the natural map from $U/{\sim}$ to $S/{\sim}$ is a bijection. Given a map $f : U \to T$ such that $u_1 \sim u_2 \implies f(u_1) \approx f(u_2)$ for all $u_1, u_2 \in U$, we obtain a map $(S/\sim) \simeq (U/\sim) \to (T/\approx)$.

2.3. **Remark.** If α is a periodic n-bead necklace of period d with labels in I, then:

$$\alpha = [\underbrace{\beta, \ldots, \beta}_{\frac{n}{d} \text{ times}}]$$

where $\beta = (\beta_1, \ldots, \beta_d)$ is a d-tuple of elements in I such that $[\beta]$ is aperiodic.

2.4. **Lemma.** Let $\pi : Y \to I$ be a surjective map where I is finite.

(1) There is a natural decomposition:

$$Map(\mathbb{Z}_n, Y)/\mathbb{Z}_n = \bigsqcup_{d|n} \left(\bigsqcup_{\alpha \in Map(\mathbb{Z}_n, I)^{\{d\}}/\mathbb{Z}_d} Map_\alpha(\mathbb{Z}_n, Y)/\mathbb{Z}_n \right).$$

(2) If $\alpha = [\beta, \ldots, \beta] \in Map(\mathbb{Z}_n, I)^{\{d\}}/\mathbb{Z}_d$, where $\beta = (\beta_1, \ldots, \beta_d)$, then there is a bijection:

$$Map_\alpha(\mathbb{Z}_n, Y)/\mathbb{Z}_n \simeq (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}/\mathbb{Z}_{\frac{n}{d}}.$$

Proof. (1) Since

$$Map(\mathbb{Z}_n, Y) = \bigsqcup_{g \in Map(\mathbb{Z}_n, I)} Map_g(\mathbb{Z}_n, Y)$$

and

$$Map(\mathbb{Z}_n, I) = \bigsqcup_{d|n} Map(\mathbb{Z}_n, I)^{\{d\}}$$

we see that:

$$Map(\mathbb{Z}_n, Y) = \bigsqcup_{d|n} \left(\bigsqcup_{g \in Map(\mathbb{Z}_n, I)^{\{d\}}} Map_g(\mathbb{Z}_n, Y) \right).$$

As noted above, in order to make this an equality of $\mathbb{Z}_n$-sets we need to take the coarser decomposition:

$$Map(\mathbb{Z}_n, Y) = \bigsqcup_{d|n} \left(\bigsqcup_{[g] \in Map(\mathbb{Z}_n, I)^{\{d\}}/\mathbb{Z}_d} Map_{[g]}(\mathbb{Z}_n, Y) \right).$$

Now we simply take the quotient by $\mathbb{Z}_n$ on both sides:

$$Map(\mathbb{Z}_n, Y)/\mathbb{Z}_n = \bigsqcup_{d|n} \left(\bigsqcup_{[g] \in Map(\mathbb{Z}_n, I)^{\{d\}}/\mathbb{Z}_d} Map_{[g]}(\mathbb{Z}_n, Y)/\mathbb{Z}_n \right).$$

Note that we are simply organizing the n-bead Y-labeled necklaces by looking at the periods of the underlying n-bead I-labeled necklaces.

(2) Let $g \in Map(\mathbb{Z}_n, I)^{\{d\}}$ and let $a \in \mathbb{Z}_n$. By definition, $ag = (a+x)g$ if and only if $x \in \langle d \rangle$. So:

$$Map_{ag}(\mathbb{Z}_n, Y) = Map_{(a+x)g}(\mathbb{Z}_n, Y)$$

if $x \in \langle d \rangle$. On the other hand, if

$$h \in Map_{ag}(\mathbb{Z}_n, Y) \cap Map_{(a+x)g}(\mathbb{Z}_n, Y)$$

for some $x \in \mathbb{Z}_n$, then $\pi \circ h = ag = (a+x)g$, which implies that $x \in \langle d \rangle$. The upshot is that we can actually write $Map_{[g]}(\mathbb{Z}_n, Y)$ as a *disjoint* union over $\mathbb{Z}_d$:

$$Map_{[g]}(\mathbb{Z}_n, Y) = \bigsqcup_{a \in \mathbb{Z}_d} Map_{ag}(\mathbb{Z}_n, Y).$$

Now consider the sequence of values $g(a)$ for $a \in \mathbb{Z}_n$. This sequence is of the form $(\beta, \dots, \beta)$, where $\beta = (\beta_1, \dots, \beta_d)$. Therefore:

$$Map_g(\mathbb{Z}_n, Y) \simeq (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}$$

and so:

$$Map_{[g]}(\mathbb{Z}_n, Y) \simeq \bigsqcup_{j=0}^{d-1} (Y_{\beta_{j+1}} \times \cdots \times Y_{\beta_d} \times Y_{\beta_1} \times \cdots \times Y_{\beta_j})^{\frac{n}{d}}.$$

Let us apply Remark 2.2 to the following sets:

$$S = \bigsqcup_{j=0}^{d-1} (Y_{\beta_{j+1}} \times \cdots \times Y_{\beta_d} \times Y_{\beta_1} \times \cdots \times Y_{\beta_j})^{\frac{n}{d}} \quad \text{and} \quad T = (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}.$$

The equivalence relations on S and T are defined by group actions: $\mathbb{Z}_n$ acts on $S \simeq Map_{[g]}(\mathbb{Z}_n, Y)$ and $\mathbb{Z}_{\frac{n}{d}}$ acts on T by cyclic permutation of the factors. Let U be the subset of S corresponding to the $j = 0$ component:

$$U = (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}.$$

Each element of S is equivalent to an element of U, and the restricted equivalence relation on U is given by the action of the subgroup $\langle d \rangle$ which is exactly the same as the action of $\mathbb{Z}_{\frac{n}{d}}$ by cyclic permutation of the factors. Therefore:

$$S/\mathbb{Z}_n \simeq U/\langle d \rangle \simeq T/\mathbb{Z}_{\frac{n}{d}}.$$

□

2.5. **Remark.** We can visualize the above result as follows: we choose a place to "cut" an n-bead Y-labeled necklace in order to get an n-tuple of elements of Y. We can always rotate the original necklace so that the underlying I-labeled necklace has a given position with respect to the cut. Moreover, if the underlying I-labeled necklace has period d, then we can break the n-tuple into segments of size d so that the corresponding I-labeled d-bead necklaces are aperiodic. As we rotate the original necklace by multiples of $\frac{2\pi}{d}$ radians, we will permute these segments among each other.

3. Partition necklaces

Let n be a positive integer. Consider the set of ordered partitions of n into r positive parts:

$$\mathcal{P}(n,r) = \{(a_1, \ldots, a_r) \in \mathbb{Z}^r_{>0} \mid \sum_{i=1}^{r} a_i = n\}$$

Define:

$$\mathcal{P}(n) = \bigsqcup_{r=1}^{n-1} \mathcal{P}(n,r)$$

In other words, $\mathcal{P}(n)$ is the set of non-empty ordered partitions of n into positive parts, where at least one part is greater than 1. Note that refinement of partitions defines a partial order on $\mathcal{P}(n)$, and the rank of a partition is given by the number of parts.

Let $\mathcal{Q}(n)$ denote the set of necklaces associated to $\mathcal{P}(n)$:

$$\mathcal{Q}(n) = \bigsqcup_{i=1}^{n-1} \mathcal{P}(n,r)/\mathbb{Z}_r$$

In other words:

$$\mathcal{Q}(n) = \{[a_1, \ldots, a_r] \in \mathbb{Z}^r_{>0}/\mathbb{Z}_r \mid 1 \le r \le n-1, \sum_{i=1}^{r} a_i = n\}$$

where $[a_1, \ldots, a_r]$ denotes the $\mathbb{Z}_r$-orbit of $(a_1, \ldots, a_r)$.

The elements of $\mathcal{Q}(n)$ are called *partition necklaces.* Note that $\mathcal{Q}(n)$ inherits the structure of a ranked poset from $\mathcal{P}(n)$.

Let $\mathcal{N}(n, 1)$ denote the set of n-bead binary necklaces with the necklaces $[0, \ldots, 0]$ and $[1, \ldots, 1]$ removed.

3.1. **Proposition.** For any $n \geq 1$, there is an isomorphism of ranked posets:

$$\psi_n : \mathcal{N}(n, 1) \simeq \mathcal{Q}(n).$$

Proof. Given a non-empty n-bead binary necklace β of rank r, let $\psi_n(\beta)$ be the necklace whose entries are given by the number of steps between consecutive non-zero entries of β. More precisely, ψ_n is given by:

$$[1, 0^{c_1}, 1, 0^{c_2}, \ldots, 1, 0^{c_r}] \mapsto [c_1 + 1, \ldots, c_r + 1]$$

Note that the right hand side is the necklace of a partition of n into r positive parts. The inverse of ψ_n is given by:

$$[a_1, \ldots, a_r] \mapsto [1, 0^{a_1-1}, 1, 0^{a_2-1}, \ldots, 1, 0^{a_r-1}].$$

Moreover, changing a "zero" to a "one" in a binary necklace corresponds to a refinement of the corresponding partition necklace, so the above bijection is compatible with the partial orders and rank functions on each poset. $\square$

An ordered partition $(a_1, \ldots, a_r)$ and the corresponding partition necklace $[a_1, \ldots, a_r]$ are said to be *fundamental* if each $a_i \geq 2$. Let $\mathcal{F}(n)$ denote the set of fundamental partition necklaces in $\mathcal{Q}(n)$.

Now we apply Remark 2.2 to the case where $S = \mathcal{P}(n)$ and T is the subset of $\mathcal{P}(n)$ consisting of fundamental partitions. Equip each set with the necklace equivalence relation, so $(S/\sim) = \mathcal{Q}(n)$ and $(T/\approx) = \mathcal{F}(n)$. Define the subset:

$$U = \{(1^{n_1}, m_1, 1^{n_2}, m_2, ..., 1^{n_k}, m_k) \in \mathcal{P}(n) \mid n_i \geq 0, m_i \geq 2 \text{ for all } 1 \leq i \leq k\}$$

Since we have excluded $(1, \ldots, 1)$ from $\mathcal{P}(n)$, we see that any element of $\mathcal{P}(n)$ is equivalent to some element in U. Now define:

$$f : U \to T$$
$$(1^{n_1}, m_1, 1^{n_2}, m_2, ..., 1^{n_k}, m_k) \mapsto (m_1 + n_1, \ldots, m_k + n_k).$$

Since f is compatible with the respective equivalence relations, we obtain a map:

$$\pi_n : \mathcal{Q}(n) \to \mathcal{F}(n)$$
$$[1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k] \mapsto [m_1 + n_1, m_2 + n_2, \ldots, m_k + n_k].$$

Note that π_n restricts to the identity on $\mathcal{F}(n)$. In particular, π_n is surjective. Therefore, we get a decomposition of $\mathcal{Q}(n)$:

$$\mathcal{Q}(n) = \bigsqcup_{\alpha \in \mathcal{F}(n)} \mathcal{Q}_\alpha$$

where $\mathcal{Q}_\alpha = \pi_n^{-1}(\alpha)$. This decomposition is the same as the decomposition for binary necklaces defined in [4]. Indeed, the map $\pi_n \circ \psi_n$ is essentially the necklace version of the "block-code" construction.

If $m \geq 1$, a fundamental partition necklace $[a_1, \ldots, a_r] \in \mathcal{F}(n)$ is said to be *divisible* by m if each a_i is divisible by m. Define the following sub-poset of $\mathcal{Q}(n)$:

$$\mathcal{Q}(n,m) = \{\alpha \in \mathcal{Q}(n) \mid \pi_n(\alpha) \text{ is divisible by } m\} = \bigsqcup_{\substack{\alpha \in \mathcal{F}(n) \\ m \mid \alpha}} \mathcal{Q}_\alpha.$$

Let $\mathcal{N}(n,m)$ denote the set of n-bead $(m+1)$-ary necklaces with the necklaces $[0, \ldots, 0]$ and $[m, \ldots, m]$ removed. We have the following generalization of Proposition 3.1.

3.2. **Lemma.** For any $n, m \geq 1$, there is an isomorphism of ranked posets:

$$\psi_{n,m} : \mathcal{N}(n,m) \simeq \mathcal{Q}(mn, m).$$

Proof. Given an n-bead $(m+1)$-ary necklace, we construct an mn-bead binary necklace via the substitution: $j \mapsto 1^j 0^{m-j}$, and then we apply the map ψ_{mn} from Proposition 3.1. This composition is clearly a morphism of ranked posets. Here is an explicit formula for $\psi_{n,m}$:

$$[b_1, 0^{c_1}, b_2, 0^{c_2}, \ldots, b_r, 0^{c_r}] \mapsto [1^{b_1-1}, m(c_1+1) - b_1 + 1, \ldots, 1^{b_r-1}, m(c_r+1) - b_r + 1]$$

where each $b_i \geq 1$ and $c_i \geq 0$. The sum of the terms in the partition necklace is:

$$\sum_{i=1}^{r} (b_i - 1 + m(c_i+1) - b_i + 1) = m(r + \sum_{i=1}^{r} c_i) = mn$$

as desired. Let us check that $\pi_{mn} \circ \psi_{n,m}(\alpha)$ is divisible by m for all $\alpha \in \mathcal{N}(n,m)$. Consider the element:

$$\alpha = [b_1, 0^{c_1}, b_2, 0^{c_2}, \ldots, b_r, 0^{c_r}].$$

If $c_i > 0$ or $b_i < m$, then the terms 1^{b_i-1} and $m(c_i+1) - b_i + 1$ in $\psi_{m,n}(\alpha)$ merge together under π_{mn} to give $m(c_i+1)$. On the other hand, whenever $b_i = m$ and $c_i = 0$, we will get a 1^m term in $\psi_{m,n}(\alpha)$. Applying π_{mn} will result in adding m to the next occurrence of $m(c_j+1)$, where $c_j > 1$. In other words:

$$\pi_{mn}(\psi_{n,m}(\alpha)) = [me_1, \ldots, me_s]$$

where $\pi_n(c_1+1, \ldots, c_r+1) = [e_1, \ldots, e_s]$, and this result is indeed divisible by m.

By reversing the above process, we get a formula for the inverse of $\psi_{n,m}$. An arbitrary element of $\mathcal{Q}(mn, m)$ is of the form:

$$[1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k]$$

where each $m_i \geq 2$, each $m_i + n_i$ is divisible by m, and $\sum_{i=1}^{k}(m_i + n_i) = mn$. The corresponding mn-bead binary necklace is:

$$[1^{n_1+1}, 0^{m_1-1}, \ldots, 1^{n_k+1}, 0^{m_k-1}].$$

Now we need to apply the substitution $1^j 0^{m-j} \mapsto j$. Since $m_i + n_i$ is divisible by m, we can apply this to each block $(1^{n_i+1}, 0^{m_i-1})$ separately. Furthermore, we should break each block into segments of size m and apply the substitution to each segment. Therefore, $(1^{n_i+1}, 0^{m_i-1})$ looks like:

$$(\underbrace{1^m, 1^m, \dots, 1^m}_{q_i \text{ times}}, 1^{r_i}, 0^{m-r_i}, 0^{m_i-1-(m-r_i)}).$$

where q_i is the quotient of the division of $n_i + 1$ by m and r_i is the remainder. Note that $m_i - 1 - (m - r_i) = m_i - 1 - m + (n_i + 1 - mq_i) = m_i + n_i - mq_i - m$, which is divisible by m. Therefore, the inverse of $\psi_{n,m}$ is given by the following formula:

$$[1^{n_1}, m_1, 1^{n_2}, m_2, \dots, 1^{n_k}, m_k] \mapsto [m^{q_1}, r_1, 0^{t_1} \dots, m^{q_k}, r_k, 0^{t_k}]$$

where:

$$n_i + 1 = mq_i + r_i \text{ such that } 0 \leq r_i < m$$

and

$$t_i = \frac{m_i + n_i}{m} - q_i - 1.$$

Note that the number of beads in the above necklace is:

$$\sum_{i=1}^{k} \left(q_i + 1 + \frac{m_i + n_i}{m} - q_i - 1 \right) = \frac{1}{m} \sum_{i=1}^{k} (m_i + n_i) = \frac{mn}{m} = n$$

as desired. □

3.3. **Lemma.** Let $\alpha = [a_1, \dots, a_r] \in \mathcal{F}(n)$. If α is aperiodic, then:

$$\mathcal{Q}_{[a_1]} \times \dots \times \mathcal{Q}_{[a_r]} \xrightarrow{\sim} \mathcal{Q}_\alpha.$$

If α is periodic of period d and $\alpha = [\underbrace{\beta, \dots, \beta}_{\frac{r}{d} \text{ times}}]$, then:

$$\mathcal{Q}_{[\beta]}^{\frac{r}{d}} / \mathbb{Z}_{\frac{r}{d}} \xrightarrow{\sim} \mathcal{Q}_\alpha.$$

Proof. If $m \geq 2$, note that $\mathcal{Q}_{[m]}$ is a chain with $m - 1$ vertices. We will apply Lemma 2.4 to the following set:

$$\mathcal{Q} = \bigsqcup_{m=2}^{n} \mathcal{Q}_{[m]}.$$

Note that our indexing set is $I = \{2, \dots, n\}$. Let $\alpha = [a_1, \dots, a_r] \in \mathcal{F}(n)$. Since $a_1 + \dots + a_r = n$, we know that each $a_i \leq n$, which implies that α is labeled by elements of I. If α is aperiodic, it follows from part (2) of Lemma 2.4 that we have a rank-preserving bijection:

$$Map_\alpha(\mathbb{Z}_r, \mathcal{Q})/\mathbb{Z}_r \simeq \mathcal{Q}_{[a_1]} \times \dots \times \mathcal{Q}_{[a_r]}.$$

On the other hand, if $\alpha = [\beta, \dots, \beta] \in Map(\mathbb{Z}_r, I)^{\{d\}}/\mathbb{Z}_d$, where $\beta = (\beta_1, \dots, \beta_d)$, then we have rank-preserving bijections:

$$Map_\alpha(\mathbb{Z}_r, \mathcal{Q})/\mathbb{Z}_r \simeq (\mathcal{Q}_{[\beta_1]} \times \dots \times \mathcal{Q}_{[\beta_d]})^{\frac{r}{d}} / \mathbb{Z}_{\frac{r}{d}} \simeq \mathcal{Q}_{[\beta]}^{\frac{r}{d}} / \mathbb{Z}_{\frac{r}{d}}$$

where the second bijection exists due to the fact that $[\beta]$ is aperiodic. It remains to check that the poset relations are preserved. Indeed, any covering relation among two necklaces labeled by $\mathcal{Q}_{[\beta_1]} \times \cdots \times \mathcal{Q}_{[\beta_d]}$ will correspond to a covering relation within a chain $\mathcal{Q}_{[\beta_i]}$ for some i, which will also be a covering relation among the corresponding $\mathcal{Q}$-labeled necklaces. □

3.4. **Remark.** The above Lemma provides an explanation of why it is easier to find a symmetric chain decomposition of n-bead binary necklaces if n in prime [4]. Indeed, in this case all non-trivial necklaces are aperiodic, so each $\mathcal{Q}_\alpha$ is covered by a product of symmetric chains and we can apply the Greene-Kleitman rule.

4. Proof of the theorem

4.1. **Theorem.** If $\mathcal{P}$ is a symmetric chain order, then $\mathcal{P}^n/\mathbb{Z}_n$ is a symmetric chain order.

Proof. The statement is trivial for $n = 1$. Assume that the theorem is true for any $n' < n$. Let $C_1, \ldots, C_r$ denote the chains in a symmetric chain decomposition of $\mathcal{P}$. We may assume that:

$$\mathcal{P} = \bigsqcup_{i=1}^{r} C_i.$$

If we let $I = \{1, 2, \ldots, r\}$ and apply part (1) of Lemma 2.4 to $\mathcal{P}$, we obtain:

$$Map(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n = \bigsqcup_{d|n} \left(\bigsqcup_{\alpha \in Map(\mathbb{Z}_n, I)^{\{d\}}/\mathbb{Z}_d} Map_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n \right).$$

Now we apply part (2) of Lemma 2.4. If $\alpha = [a_1, \ldots, a_n]$ is an aperiodic n-bead necklace with labels in I, then:

$$C_{a_1} \times \cdots \times C_{a_n} \tilde{\hookrightarrow} Map_\alpha(\mathbb{Z}_n, \mathcal{P}).$$

Since $C_{a_1} \times \cdots \times C_{a_n}$ is a symmetric chain order, it follows that $Map_\alpha(\mathbb{Z}_n, \mathcal{P})$ is a symmetric chain order. Also note that $C_{a_1} \times \cdots \times C_{a_n}$ is a centered subposet of $Map(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$. On the other hand, if $\alpha = [\beta, \ldots, \beta]$ is a periodic n-bead necklace with labels in I, where $\beta = (\beta_1, \ldots, \beta_d)$, then:

$$(C_{\beta_1} \times \cdots \times C_{\beta_d})^{\frac{n}{d}}/\mathbb{Z}_{\frac{n}{d}} \tilde{\hookrightarrow} Map_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n.$$

Again, note that this poset is a centered subposet of $Map(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$ since it is a cyclic quotient of a centered subposet of $\mathcal{P}^n$.

If $d > 1$, then $\frac{n}{d} < n$ and $(C_{\beta_1} \times \cdots \times C_{\beta_d})$ is a symmetric chain order, so

$$(C_{\beta_1} \times \cdots \times C_{\beta_d})^{\frac{n}{d}}/\mathbb{Z}_{\frac{n}{d}}$$

is a symmetric chain order by induction.

If $d = 1$, then:

$$C^n/\mathbb{Z}_n \tilde{\hookrightarrow} Map_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$$

where C is a chain with $m+1$ vertices, for some $m \geq 1$. It suffices to consider the centered subposet $\mathcal{N}(n,m)$. By Lemma 3.2, we have:

$$\mathcal{N}(n,m) \simeq \mathcal{Q}(mn,m).$$

If $\mathcal{Q}_\alpha \subset \mathcal{Q}(mn,m)$, then $\alpha = [ma_1, \ldots, ma_s]$, where $a_1 + \cdots + a_s = n$. In particular, note that $s \leq n$. By Lemma 3.3, there are two possibilities for $\mathcal{Q}_\alpha$. If α is aperiodic, $\mathcal{Q}_\alpha$ is a product of chains, so it is a symmetric chain order. If α is periodic of period d, then:

$$\mathcal{Q}_{[\beta]}^{\frac{s}{d}}/\mathbb{Z}_{\frac{s}{d}} \xhookrightarrow{\sim} \mathcal{Q}_\alpha$$

where $[\beta]$ is a d-bead aperiodic necklace. In particular, $\mathcal{Q}_{[\beta]}$ is itself a product of chains (hence a symmetric chain order). We know that $\beta = (mc_1, \ldots, mc_d)$, where $c_1 + \cdots + c_d = \frac{dn}{s}$. There are three possible cases:

(i) If $d > 1$, then $\frac{s}{d} < n$. Since $\mathcal{Q}_{[\beta]}$ is a symmetric chain order, by induction we conclude that

$$\mathcal{Q}_{[\beta]}^{\frac{s}{d}}/\mathbb{Z}_{\frac{s}{d}}$$

is a symmetric chain order.

(ii) If $d = 1$ and $s < n$ then $Q_{[\beta]}$ is a single chain, so $Q_{[\beta]}^s/\mathbb{Z}_s$ is a symmetric chain order by induction.

(iii) If $d = 1$ and $s = n$, then $\beta = (m)$ and $\alpha = [m, \ldots, m]$. In this case:

$$\mathcal{Q}_{[m]}^n/\mathbb{Z}_n \xhookrightarrow{\sim} \mathcal{Q}_\alpha.$$

Since $\mathcal{Q}_{[m]}$ is a chain with $m-1$ vertices, we see that we have returned to the case of the $\mathbb{Z}_n$-quotient of the n-fold power of a single chain. However, note that the we have managed to decrease the length of the chain by two, i.e. from $m+1$ vertices to $m-1$ vertices. Now we can again apply Lemma 3.2 and Lemma 3.3 to the centered subposet $\mathcal{N}(n, m-2)$, etc.

Eventually, after we go through this argument enough times, we will eventually reach the case of:

$$C^n/\mathbb{Z}_n$$

where C is a chain with one or two vertices. If $|C| = 1$, there is nothing to show. So we are left with the case where C is a chain with two vertices, i.e. the poset of binary necklaces. It suffices to look at the centered subposet $\mathcal{N}(n,1)$. By Proposition 3.1,

$$\mathcal{N}(n,1) \simeq \mathcal{Q}(n).$$

Again, we consider the subposets $\mathcal{Q}_\alpha$. As usual, if α is aperiodic then $\mathcal{Q}_\alpha$ is covered by a product of symmetric chains. If $\alpha = [\beta, \ldots, \beta]$ is periodic of period d then

$$\mathcal{Q}_{[\beta]}^{\frac{n}{d}}/\mathbb{Z}_{\frac{n}{d}} \xhookrightarrow{\sim} \mathcal{Q}_\alpha$$

where $[\beta]$ is an aperiodic d-bead necklace and $\mathcal{Q}_{[\beta]}$ is a product of chains. If $d > 1$, then $\frac{n}{d} < n$ so

$$\mathcal{Q}_{[\beta]}^{\frac{n}{d}}/\mathbb{Z}_{\frac{n}{d}}$$

is a symmetric chain order by induction. Finally, if α is periodic of period $d = 1$ then α is an n-bead partition necklace of period 1 whose entries sum to n, so $\alpha = [1, 1, \ldots, 1]$, but this element was explicitly excluded from the set $\mathcal{Q}(n)$. $\square$

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