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European Journal of Combinatorics 25 (2004) 675–683

European Journal of Combinatorics

www.elsevier.com/locate/ejc

On quotients of posets, with an application to the q-analog of the hypercube

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Received 11 April 2003; accepted 18 September 2003

Abstract

Let G be a finite group having an order preserving and rank preserving action on a finite ranked poset P. Let P/G denote the quotient poset. A well known result in algebraic Sperner theory asserts that an order raising G-linear map on $V(P)$ (the complex vector space with P as basis) satisfying the full rank property induces an order raising linear map on $V(P/G)$, also satisfying the full rank property. In this paper we prove a kind of converse result that has applications to Boolean algebras and their cubical and q-analogs.

For a finite ranked poset P, let L denote the Lefschetz order raising map taking an element to the sum of the elements covering it and let P_i, $0 \le i \le n$, where $n = \text{rank}(P)$, denote the set of elements of rank i. We say that P is unitary Peck (respectively, unitary semi-Peck) if the map

$$L^{n-2i} : V(P_i) \to V(P_{n-i}), \qquad i < n/2$$

is bijective (respectively, injective). We show that the q-analog of the n-cube is unitary semi-Peck.

MSC (2000): Primary 05E25; Secondary 06A07, 06A11

Keywords: Sperner theory; Group actions; Quotients of posets; Fixed points

1. Introduction

Let P be a finite poset (we follow Engel [1] for poset terminology). A *rank function* on P is a function $r : P \to \mathbb{N}$ such that $r(p) = 0$ for some minimal element of P and $r(q) = r(p)+1$ whenever $q \gtrdot p$ (i.e., q covers p). The number $r(P) = \max\{r(p) : p \in P\}$

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doi:10.1016/j.ejc.2003.09.015

is called the *rank* of P. For $0 \le i \le n$, where $n = r(P)$, the ith *level set* of P is defined by $P_i = \{p \in P : r(p) = i\}$ and the number $W_i = |P_i|$ is called the ith *Whitney number*. The sequence $R(P) = (W_0, W_1, \ldots, W_n)$ is called the *rank sequence* of P. We say that P is *rank symmetric* if $W_i = W_{n-i}$, $0 \le i \le n$, and *rank unimodal* if $W_j \ge \min\{W_i, W_k\}$, $0 \le i \le j \le k \le n$. We consider finite ranked posets only.

A ranked poset P is *Sperner* if no antichain (= set of pairwise incomparable elements) of P has cardinality greater than the largest Whitney number. More generally, P is *k-Sperner* if no union of k antichains has cardinality greater than the sum of the k largest Whitney numbers, and is *strongly Sperner* if it is k-Sperner for $1 \le k \le r(P) + 1$. We say that P is a *Peck poset* if it is rank symmetric, rank unimodal, and strongly Sperner.

In Chapter 6 of [1], Engel presents three algebraic techniques for showing that a ranked poset is strongly Sperner: product theorems, finite group actions, and representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. This paper deals with finite group actions. Let us explain this in more detail.

For a finite set S, let $V(S)$ denote the (complex) vector space with S as basis. If P is a rank-n poset, then, for $0 \le i \le n$, $V(P_i)$ denotes the subspace of $V(P)$ generated by the ith level set P_i and we set $V(P_{n+1}) = \{0\}$. A *raising linear map* on $V(P)$ is a linear map $f : V(P) \to V(P)$ satisfying $f(V(P_i)) \subseteq V(P_{i+1})$, $0 \le i \le n$. An *order raising linear map* on $V(P)$ is a raising linear map $f : V(P) \to V(P)$ such that, for $p \in P$ with $r(p) < n$, we have $f(p) = \sum_{q \gtrdot p} c_{pq} q$, for some $c_{pq} \in \mathbb{C}$. The *Lefschetz order raising* map $L : V(P) \to V(P)$ is defined by $L(p) = \sum_{q \gtrdot p} q$, if $r(p) < n$ and $L(p) = 0$ if $r(p) = n$. Let $f : V(P) \to V(P)$ be raising. For $0 \le i \le j \le n$, define $f_{ij} : V(P_i) \to V(P_j)$ to be the map f^{j-i} restricted to $V(P_i)$ (it is clear that the image lies in $V(P_j)$). We say that f has *full rank* if $\operatorname{rank}(f_{ij}) = \min\{W_i, W_j\}$ (i.e., f_{ij} is injective if $W_i \le W_j$ and surjective if $W_i \ge W_j$), for all $0 \le i \le j \le n$.

A chain $p_1 < p_2 < \cdots < p_t$ in a ranked poset is *saturated* if p_i covers p_{i-1} for $2 \le i \le t$. Griggs [2] proved that a ranked poset P is rank unimodal and strongly Sperner if and only if, for all $0 \le i \le j \le r(P)$, there exist $\min\{W_i, W_j\}$ pairwise disjoint saturated chains in P starting at some element in the ith level and ending at some element in the jth level. Using this characterization, Stanley [10] showed that a ranked poset P is rank unimodal and strongly Sperner if and only if there exists an order raising linear map on $V(P)$ having full rank. It follows that a ranked poset P, with $r(P) = n$, is Peck if and only if there is an order raising linear map $f : V(P) \to V(P)$ such that, for $0 \le i < n/2$, the linear map $f^{n-2i} : V(P_i) \to V(P_{n-i})$ is bijective (to see this, it is enough to observe that $f_{ij} = f_{kj} \circ f_{ik}$, for $0 \le i \le k \le j \le n$). We say that P is *unitary Peck* if this property holds for the Lefschetz order raising map, i.e., $L^{n-2i} : V(P_i) \to V(P_{n-i})$, $0 \le i < n/2$, is bijective.

A ranked poset P, with $r(P) = n$, is said to be *semi-Peck* (respectively *unitary semi-Peck*) if, for some order raising linear map $f : V(P) \to V(P)$ (respectively for the Lefschetz order raising map $L : V(P) \to V(P)$), the maps $f^{n-2i} : V(P_i) \to V(P_{n-i})$ (respectively $L^{n-2i} : V(P_i) \to V(P_{n-i})$), $0 \le i < n/2$, are injective.

Group actions are a tool for showing that certain order raising linear maps have full rank. Let G be a finite group having an order preserving and rank preserving action on a finite ranked poset P, i.e., $x \le y$ implies $gx \le gy$ and $r(x) = r(gx)$, for all $x, y \in P$, $g \in G$. The *quotient of P under G*, denoted P/G, is the ranked poset whose underlying set is the set of all orbits of the action of G on P and whose order relation is given by: $O_1 \leq O_2$ in P/G if and only if, for some $x \in O_1, y \in O_2$, we have $x \leq y$ in P. It is easily seen that this makes P/G into a ranked poset with rank function given by $r_{P/G}(O) = r(x), x \in O$.

A linear map $f : V(P) \to V(P)$ is said to be G-linear if f commutes with the action of G, i.e., $f(gv) = gf(v), g \in G, v \in V(P)$. Two integer sequences $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ of the same length are said to be of the same *type* provided, for all i, j, we have $a_i \leq a_j$ if and only if $b_i \leq b_j$. The following theorem is due to Harper [3], Pouzet and Rosenberg [8], and Stanley [11] (for an exposition see Theorem 6.1.9 in Chapter 6 of Engel [1]).

Theorem 1.1. *Let G be a finite group having an order preserving and rank preserving action on a finite ranked poset P. Let f be an order raising G-linear map on $V(P)$ having full rank. Then*

(i) *$R(P)$ and $R(P/G)$ have the same type.*
(ii) *There is an order raising linear map f/G on $V(P/G)$ having full rank.*

In the case of Peck/semi-Peck posets, Theorem 1.1 simplifies to the following result.

Theorem 1.2. *Let G be a finite group having an order preserving and rank preserving action on a finite ranked poset P of rank n. Let f be an order raising G-linear map on $V(P)$. Suppose that, for $0 \leq i < n/2$, the map $f^{n-2i} : V(P_i) \to V(P_{n-i})$ is injective (respectively bijective).*

Then there exists an order raising linear map f/G on $V(P/G)$ such that, for $0 \leq i < n/2$, the map $(f/G)^{n-2i} : V((P/G)_i) \to V((P/G)_{n-i})$ is injective (respectively bijective).

Since the Lefschetz order raising map is G-linear, it follows that the quotient of a unitary Peck poset (respectively unitary semi-Peck poset) is Peck (respectively semi-Peck) (see [11] for an application of this result to the poset of partitions contained in a rectangle). In Theorem 2.3 of Section 2 we give a kind of converse to Theorem 1.2 (a similar converse can be stated for Theorem 1.1, though its formulation is more cumbersome) and in Section 3 we use this result to show that the q-analog of the n-cube is unitary semi-Peck.

2. Group actions

Let G be a finite group acting on the finite sets S and T. Denote the subspace of $V(S)$ consisting of vectors fixed by all elements of G by $F(G, S) = \{v \in V(S) : gv = v \text{ for all } g \in G\}$. Similarly define $F(G, T)$.

Lemma 2.1. *Let $f : V(S) \to V(T)$ be G-linear. Then*

(i) *$f(F(G, S)) \subseteq F(G, T)$.*
(ii) *$f : V(S) \to V(T)$ surjective implies $f : F(G, S) \to F(G, T)$ surjective.*
(iii) *$f : V(S) \to V(T)$ injective implies $f : F(G, S) \to F(G, T)$ injective.*

Proof. The action of G on S and T gives rise to permutation representations of G on $V(S)$ and $V(T)$ respectively. The subspaces $F(G, S)$ and $F(G, T)$ are the isotypical components (in $V(S)$ and $V(T)$ respectively) of the trivial representation of G. The result now follows from Schur's lemma. $\square$

Lemma 2.2. *Let $f : V(S) \to V(T)$ be G-linear.*

(i) *Suppose that, for each $t \in T$, we can find a subgroup $G_t \subseteq G$ fixing t (i.e., $gt = t$ for all $g \in G_t$) such that $f : F(G_t, S) \to F(G_t, T)$ is surjective. Then $f : V(S) \to V(T)$ is surjective.*

(ii) *Suppose that, for each $s \in S$, we can find a subgroup $G_s \subseteq G$ fixing s such that $f : F(G_s, S) \to F(G_s, T)$ is injective. Then $f : V(S) \to V(T)$ is injective.*

Proof. (i) Let $t \in T$. By assumption $t \in F(G_t, T)$ and there exists $v \in F(G_t, S)$ such that $f(v) = t$. It follows that $f(V(S)) = V(T)$.

(ii) This part can be deduced from part(i) using dual spaces and dual representations. We can also give a direct proof as follows.

Let $f(\sum_{y\in S} c_y y) = 0$, for some $c_y \in \mathbb{C}$. We need to show that $c_y = 0$ for all $y \in S$. Fix $s \in S$. Let $S = O_1 \uplus O_2 \uplus \cdots \uplus O_t$ (disjoint union) be the decomposition of S into orbits under the action of G_s on S. Without loss of generality we can assume that $O_1 = \{s\}$. For $2 \le j \le t$, let $w_j = \sum_{y \in O_j} y$. Then a standard fact on group actions says that $s, w_2, \ldots, w_t$ form a basis for $F(G_s, S)$. Since $f(\sum_{y\in S} c_y y) = 0$ and f is G-linear, we have

$$0 = \left(\sum_{g\in G_s} g\right) f\left(\sum_{y\in S} c_y y\right) = f\left(\left(\sum_{g\in G_s} g\right)\left(\sum_{y\in S} c_y y\right)\right)$$
$$= f(|G_s| c_s s + b_2 w_2 + \cdots + b_t w_t),$$

for some $b_2, \ldots, b_t \in \mathbb{C}$. Since f, restricted to $F(G_s, S)$, is injective we have $c_s = 0$. $\square$

Let G be a finite group having an order preserving and rank preserving action on a finite ranked poset P. Let $f : V(P) \to V(P)$ be an order raising G-linear map. The space $V(P/G)$ can be identified with $F(G, P)$ (an orbit O going to the vector $\sum_{p\in O} p$) and thus, by Lemma 2.1, there is a well defined (order raising) linear map $f/G : V(P/G) \to V(P/G)$, called the *induced map*. An application of Lemma 2.2 now gives the following result.

Theorem 2.3. *Let G be a finite group having an order and rank preserving action on a finite ranked poset P of rank n. Let f be an order raising G-linear map on $V(P)$. Suppose that, for every $0 \le i < n/2$ and every $p \in P_i$, we can find a subgroup $G_p \subseteq G$ fixing p such that $(f/G_p)^{n-2i} : V((P/G_p)_i) \to V((P/G_p)_{n-i})$ is injective. Then $f^{n-2i} : V(P_i) \to V(P_{n-i})$ is injective.* $\square$

The following result provides a triangular criterion for applying Theorem 2.3 when f is the Lefschetz order raising map.

Theorem 2.4. *Let G be a finite group having an order and rank preserving action on a finite ranked poset P of rank n. Suppose that, for every $0 \le i < n/2$ and every $p \in P_i$, we can find a subgroup $G_p \subseteq G$ fixing p, a map $\chi_p : (P/G_p)_i \to (P/G_p)_{n-i}$ and a linear order $<_p$ on $(P/G_p)_i$ such that the following three conditions hold:*

(i) *χ_p is injective.*
(ii) *$O < \chi_p(O)$, for every $O \in (P/G_p)_i$.*
(iii) *$N, O \in (P/G_p)_i$ and $N <_p O$ implies $O \not< \chi_p(N)$.*

Then $L^{n-2i} : V(P_i) \to V(P_{n-i})$, $0 \le i < n/2$ is injective and P is unitary semi-Peck.

Proof. Let $0 \le i < n/2$ and fix $p \in P_i$. Put $k = |(P/G_p)_i|$ and $l = |(P/G_p)_{n-i}|$. Let M be the $k \times l$ matrix, with rows indexed by $(P/G_p)_i$, with columns indexed by $(P/G_p)_{n-i}$, and with row O (where $O \in (P/G_p)_i$) given by $(L/G_p)^{n-2i}(O)$. Assume that the rows are listed in the order given by $<_p$. Assume that the first k columns are indexed, in order, by $\chi_p(O)$, $O \in (P/G_p)_i$ (this is possible, since χ_p is injective). Let N denote the $k \times k$ submatrix of M given by the first k columns. Conditions (ii) and (iii) in the statement imply that N is upper triangular with nonzero diagonals. It follows that N, and hence M, has full row rank. The conclusion now follows from Theorem 2.3. □

3. q-Analog of the hypercube

The main examples to which the theory of Section 2 applies are the Boolean algebras and their cubical and q-analogs. The case of subsets and subspaces has been considered by Kantor [5], so we will concentrate on cubical analogs. Let B_n denote the poset of all subsets of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. The q-analog of B_n, denoted $B_n(q)$, is the poset of subspaces (under inclusion) of an n-dimensional vector space over $\mathbb{F}_q$, the finite field with q elements. B_n is ranked by cardinality and $B_n(q)$ is ranked by dimension.

We now define the cubical analog of B_n. Let $H_n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1\}$ for all $i\}$ denote the discrete hypercube. For $I \subseteq [n]$ and $\alpha : I \to \{0, 1\}$ (when I is empty we take α to be the unique function with empty domain and co-domain $\{0, 1\}$) define

$$F(I, \alpha) = \{(x_1, \ldots, x_n) \in H_n : x_i = \alpha(i) \text{ for } i \in I\}.$$

A *face* of H_n is a subset of H_n of the form $F(I, \alpha)$, for some $I \subseteq [n]$ and $\alpha : I \to \{0, 1\}$. The *cubical poset* C_n is defined as the set of all faces of H_n, ordered by reverse inclusion, i.e., for faces S_1, S_2 of H_n, we have $S_1 \le S_2$ if and only if $S_2 \subseteq S_1$. If $S_1 = F(I, \alpha)$ and $S_2 = F(J, \beta)$ are faces then we have

$$S_1 \le S_2 \qquad \text{if and only if } I \subseteq J \text{ and } \beta(i) = \alpha(i) \text{ for all } i \in I. \tag{1}$$

Clearly, C_n is a rank-n poset with rank function given by $r(F(I, \alpha)) = |I|$.

The q-analog of C_n was defined by Harper [4]. First we derive an alternate description of C_n. An interval in B_n is a subset of B_n of the form $[X, Y] = \{Z \subseteq [n] : X \subseteq Z \subseteq Y\}$, for some $X, Y \subseteq [n]$, $X \subseteq Y$. Int(B_n) denotes the poset of all intervals in B_n, ordered by reverse inclusion, i.e., if $[X, Y]$ and $[X_1, Y_1]$ are intervals then

$$[X, Y] \le [X_1, Y_1] \qquad \text{if and only if } X \subseteq X_1 \subseteq Y_1 \subseteq Y. \tag{2}$$

It is easily seen that Int(B_n) is a rank-n poset with rank function given by $r([X, Y]) = n - (|Y| - |X|)$.

Define a map $\Gamma : C_n \to \text{Int}(B_n)$ by $\Gamma(F(I, \alpha)) = [X, Y]$, where $X = \{i \in I : \alpha(i) = 1\}$ and $Y = X \cup ([n] - I)$. Using (1) and (2) we see that Γ is a rank preserving order isomorphism.

We now define $C_n(q)$, the q-analog of C_n, to be $\mathrm{Int}(B_n(q))$, the poset of intervals in $B_n(q)$ ordered by reverse inclusion. In more detail, let $S_n(q)$ denote an n-dimensional vector space over $\mathbb{F}_q$ and let $B_n(q)$ denote the poset of subspaces of $S_n(q)$ (under inclusion). Elements of $C_n(q)$ are intervals in $B_n(q)$ of the form $[U, W] = \{N \in B_n(q) : U \subseteq N \subseteq W\}$, for subspaces U, W of $S_n(q)$, $U \subseteq W$. If $[U, W]$ and $[U_1, W_1]$ are intervals then

$$[U, W] \leq [U_1, W_1] \qquad \text{if and only if } U \subseteq U_1 \subseteq W_1 \subseteq W.$$

It is easily seen that $C_n(q)$ is a rank-n poset with rank function given by

$$r([U, W]) = n - (\dim W - \dim U) = \dim U + \operatorname{codim} W.$$

Harper [4] proved that $C_n(q)$ is a normal poset (for the definition see [1]) with log-concave Whitney numbers. In particular, $C_n(q)$ is rank unimodal and strongly Sperner. It follows that there is an order-raising linear map on $V(C_n(q))$ having the full rank property. The rank-1 vs. rank-n incidence matrix of C_n (i.e., the $2n \times 2^n$ facet-vertex incidence matrix of the hypercube) is not of full rank and thus the Lefschetz order-raising map of C_n (and similarly, $C_n(q)$) is not of full rank. We show below that $C_n(q)$ is unitary semi-Peck (a proof that C_n is unitary semi-Peck is given in [1] using the product theorem).

Let G denote the (finite) group (under composition) of all nonsingular $\mathbb{F}_q$-linear transformations of $S_n(q)$. The group G acts on $C_n(q)$ by

$$g \cdot [X, Y] = [g(X), g(Y)], \qquad g \in G, [X, Y] \in C_n(q).$$

Clearly, this action is rank and order preserving.

For the rest of this section fix $m < n/2$ **and fix** $[X, Y] \in C_n(q)$ **with** $r([X, Y]) = \dim \boldsymbol{X} + \mathbf{codim}\, Y = m$.

Given $[X_1, Y_1] \in C_n(q)$, define the *relative position of* $[X_1, Y_1]$ *with respect to* $[X, Y]$ to be the 4-tuple

$$d([X_1, Y_1]) = (\dim X \cap X_1, \dim Y \vee X_1, \operatorname{codim} X \cap Y_1, \operatorname{codim} Y \vee Y_1).$$

We put the *lexical order* on elements of $\mathbb{N}^4$, i.e., $s = (s_1, s_2, s_3, s_4) <_l t = (t_1, t_2, t_3, t_4)$ if, for some $i = 0, 1, 2, 3$, we have $s_1 = t_1, \ldots, s_i = t_i$, and $s_{i+1} < t_{i+1}$. We write $s \leq_l t$ if $s = t$ or $s <_l t$.

Note that $[X_1, Y_1] \leq [X_2, Y_2]$, i.e., $X_1 \subseteq X_2 \subseteq Y_2 \subseteq Y_1$ implies

$$d([X_1, Y_1]) \leq_l d([X_2, Y_2]). \tag{3}$$

Let $H = \{g \in G : g(X) = X, g(Y) = Y\}$ denote the subgroup of all elements of G fixing $[X, Y]$. The orbits of $C_n(q)$, under the action of H, will be called *H-orbits*.

Lemma 3.1. *Let $[X_1, Y_1], [X_2, Y_2] \in C_n(q)$. Then they are in the same H-orbit if and only if*

(i) $\dim X_1 = \dim X_2$ *and* $\operatorname{codim} Y_1 = \operatorname{codim} Y_2$.
(ii) $d([X_1, Y_1]) = d([X_2, Y_2])$.

Proof. (Only if). Let $X_2 = g(X_1)$ and $Y_2 = g(Y_1)$ for some $g \in H$. Clearly, condition (i) in the statement of the lemma holds. Now, since g is nonsingular and fixes X, we

have $g(X \cap X_1) = g(X) \cap g(X_1) = X \cap X_2$ and hence $\dim(X \cap X_1) = \dim(X \cap X_2)$. Similarly, we can show the other three equalities and conclude $d([X_1, Y_1]) = d([X_2, Y_2]$.
(if) Consider the following two chains of eight subspaces of $S_n(q)$:

$$X_1 \cap X \subseteq Y_1 \cap X \subseteq X \subseteq X \vee (Y \cap X_1) \subseteq X \vee (Y \cap Y_1) \subseteq Y \subseteq Y \vee X_1 \subseteq Y \vee Y_1 \quad (4)$$
$$X_2 \cap X \subseteq Y_2 \cap X \subseteq X \subseteq X \vee (Y \cap X_2) \subseteq X \vee (Y \cap Y_2) \subseteq Y \subseteq Y \vee X_2 \subseteq Y \vee Y_2. \quad (5)$$

From the hypothesis the dimension of the first three and last three subspaces of chain (4) are equal to the dimensions of the corresponding subspaces in chain (5). Now,

$$\begin{aligned} \dim X_1 &= \dim Y \cap X_1 + \dim Y \vee X_1 - \dim Y \\ &= \dim X \cap X_1 + \dim X \vee (Y \cap X_1) - \dim X + \dim Y \vee X_1 - \dim Y. \end{aligned} \quad (6)$$

We can write down similar formulas for the dimensions of X_2, Y_1 and Y_2. It follows from the hypothesis that the dimensions of the fourth and fifth subspaces are also the same in chains (4) and (5).

Choose a basis $\{u_{ij} : 1 \le i \le 9, 1 \le j \le k_i\}$ of $S_n(q)$ with the following property (it is easily seen that such a basis exists):

(a) $\{u_{1j} : 1 \le j \le k_1\}$ is a basis of $X_1 \cap X$.
(b) $\{u_{ij} : 1 \le i \le 2, 1 \le j \le k_i\}$ is a basis of $Y_1 \cap X$.
(c) $\{u_{ij} : 1 \le i \le 3, 1 \le j \le k_i\}$ is a basis of X.
(d) $\{u_{ij} : 1 \le i \le 4, 1 \le j \le k_i\}$ is a basis of $X \vee (Y \cap X_1)$ and $u_{4j} \in Y \cap X_1$, $1 \le j \le k_4$.
(e) $\{u_{ij} : 1 \le i \le 5, 1 \le j \le k_i\}$ is a basis of $X \vee (Y \cap Y_1)$ and $u_{5j} \in Y \cap Y_1$, $1 \le j \le k_5$.
(f) $\{u_{ij} : 1 \le i \le 6, 1 \le j \le k_i\}$ is a basis of Y.
(g) $\{u_{ij} : 1 \le i \le 7, 1 \le j \le k_i\}$ is a basis of $Y \vee X_1$ and $u_{7j} \in X_1$, $1 \le j \le k_7$.
(h) $\{u_{ij} : 1 \le i \le 8, 1 \le j \le k_i\}$ is a basis of $Y \vee Y_1$ and $u_{8j} \in Y_1$, $1 \le j \le k_8$.

It follows from (6) that

$$\{u_{ij} : i \in \{1, 4, 7\},\ 1 \le j \le k_i\} \text{ is a basis of } X_1. \quad (7)$$

Similarly we can show that

$$\{u_{ij} : i \in \{1, 2, 4, 5, 7, 8\},\ 1 \le j \le k_i\} \text{ is a basis of } Y_1.$$

Now choose a basis $\{v_{ij} : 1 \le i \le 9, 1 \le j \le k_i\}$ of $S_n(q)$ satisfying properties (a) to (h) above, with u_{ij} replaced with v_{ij}, X_1 replaced by X_2, and Y_1 replaced by Y_2 (this is possible since the dimension sequences of the chains (4) and (5) are identical).

It is now clear that the nonsingular linear transformation $g \in G$ taking u_{ij} to v_{ij} belongs to H and satisfies $g \cdot [X_1, Y_1] = [X_2, Y_2]$. $\square$

For $[A, B] \in C_n(q)$, the orbit under the H-action containing $[A, B]$ will be denoted $\overline{[A, B]}$. Note that whenever $\overline{[A, B]} = \overline{[C, D]}$ we have $\dim A = \dim C$, $\operatorname{codim} B = \operatorname{codim} D$, and $d([A, B]) = d([C, D])$. This will be used tacitly in what follows.

Lemma 3.2. *Consider an H-orbit $\overline{[X_1, Y_1]}$ with $r([X_1, Y_1]) = m$. Then there is an unique H-orbit $\overline{[X_2, Y_2]}$ satisfying*

(i) $r([X_2, Y_2]) = n - m$.
(ii) $\overline{[X_1, Y_1]} \le \overline{[X_2, Y_2]}$.

(iii) $\operatorname{codim} Y_1 = \operatorname{codim} Y_2$.
(iv) $d([X_1, Y_1]) = d([X_2, Y_2])$.

The orbit $\overline{[X_2, Y_2]}$ *is denoted* $\phi(\overline{[X_1, Y_1]})$ *and we have defined a function* $\phi : (C_n(q)/H)_m \to (C_n(q)/H)_{n-m}$.

Proof. (Uniqueness). Let $\overline{[X_2, Y_2]}$ and $\overline{[X_3, Y_3]}$ satisfy the conditions (i) to (iv) of the lemma. From (i) and (iii) we get $\dim X_2 = \dim X_3$ and $\operatorname{codim} Y_2 = \operatorname{codim} Y_3$. Since $d([X_2, Y_2]) = d([X_3, Y_3])$, uniqueness now follows from Lemma 3.1.
(Existence). Consider the chain of subspaces (4). From (6) we have

$$\dim X \vee (Y \cap X_1) = \dim X_1 - \dim X \cap X_1 + \dim X + \dim Y - \dim Y \vee X_1.$$

Similarly

$$\dim X \vee (Y \cap Y_1) = \dim Y_1 - \dim X \cap Y_1 + \dim X + \dim Y - \dim Y \vee Y_1.$$

Subtracting we get

$$\begin{aligned} &\dim X \vee (Y \cap Y_1) - \dim X \vee (Y \cap X_1) \\ &\quad = \dim Y_1 - \dim X_1 + \dim X \cap X_1 - \dim X \cap Y_1 + \dim Y \vee X_1 - \dim Y \vee Y_1 \\ &\quad = n - m + \dim X \cap X_1 - \dim X \cap Y_1 + \dim Y \vee X_1 - \dim Y \vee Y_1 \\ &\quad \geq n - m + 0 - \dim X + \dim Y - n \\ &\quad = n - 2m. \end{aligned} \tag{8}$$

Now choose a basis $\{u_{ij} : 1 \leq i \leq 9, 1 \leq j \leq k_i\}$ of $S_n(q)$ satisfying properties (a) to (h) stated in the proof of Lemma 3.1. By (8) we have $k_5 \geq n - 2m$. Now put $Y_2 = Y_1$ and $X_2 = \operatorname{Span}(X_1 \cup \{u_{5j} : 1 \leq j \leq n - 2m\})$.

By (7) we have $\dim X_2 = \dim X_1 + n - 2m$ and thus $r([X_2, Y_2]) = m + n - 2m = n - m$. Clearly $\overline{[X_1, Y_1]} \leq \overline{[X_2, Y_2]}$. We are now left to check that $d([X_1, Y_1]) = d([X_2, Y_2])$.

Let $v \in X \cap X_2$. Since $\{u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k_i\}$ is a basis of X and $\{u_{ij} : i \in \{1, 4, 7\}, 1 \leq j \leq k_i\} \cup \{u_{5j} : 1 \leq j \leq n - 2m\}$ is a basis of X_2, we must have $v \in \operatorname{Span}\{u_{1j} : 1 \leq j \leq k_1\} = X_1 \cap X$. It follows that $X_1 \cap X = X_2 \cap X$. Since $\{u_{5j} : 1 \leq j \leq k_5\} \subseteq Y$, it follows that $Y \vee X_1 = Y \vee X_2$. Since $Y_1 = Y_2$, we now have $d([X_1, Y_1]) = d([X_2, Y_2])$. $\square$

Lemma 3.3. (i) *ϕ is injective.*
(ii) *Let* $\overline{[X_1, Y_1]}, \overline{[X_2, Y_2]} \in (C_n(q)/H)_m$. *Then*

$$d(\overline{[X_1, Y_1]}) <_l d(\overline{[X_2, Y_2]}) \text{ implies } \overline{[X_2, Y_2]} \not< \phi(\overline{[X_1, Y_1]}).$$

Proof. (i) Let $\overline{[X_3, Y_3]} = \phi(\overline{[X_1, Y_1]}) = \phi(\overline{[X_2, Y_2]})$. By Lemma 3.2 we have $\dim Y_3 = \dim Y_1 = \dim Y_2$, $\dim X_3 = \dim X_1 + n - 2m = \dim X_2 + n - 2m$, and $d([X_3, Y_3]) = d([X_1, Y_1]) = d([X_2, Y_2])$. It now follows from Lemma 3.1 that $\overline{[X_1, Y_1]} = \overline{[X_2, Y_2]}$.
(ii) Assume that $\overline{[X_2, Y_2]} < \phi(\overline{[X_1, Y_1]})$. Then we have $d([X_2, Y_2]) \leq_l d([X_1, Y_1])$, by (3) and Lemma 3.1. This is a contradiction. Thus $\overline{[X_2, Y_2]} \not< \phi(\overline{[X_1, Y_1]})$. $\square$

Lemma 3.3 and condition (ii) of Lemma 3.2 now show that the map ϕ satisfies the hypothesis of Theorem 2.4. We thus have the following result.

Theorem 3.4. *$C_n(q)$ is unitary semi-Peck.*

Finally, we would like to state the following problem: Consider the rank-n lattice $\Pi(n+1)$ of partitions of $[n+1]$, ordered by reverse refinement. In Section 6.3 of [1], Engel conjectures (at the bottom of p. 253) that every geometric lattice is semi-Peck. For partition lattices this was proved in Loeb et al. [7] and [9] by constructing a covering of the bottom half of $\Pi(n+1)$ by symmetric chains. We can ask whether partition lattices are unitary semi-Peck. In an important paper, Kung [6] shows that $L^{j-i} : V(\Pi(n+1)_i) \rightarrow V(\Pi(n+1)_j)$, $i \leq j \leq \lceil n/2 \rceil$, is injective. We conjecture that partition lattices are unitary semi-Peck. The symmetric group S_{n+1} acts on $\Pi(n+1)$ by substitution and therefore Theorem 2.3 is applicable in principle. We do not know whether a quotient argument can be used in some way to prove that partition lattices are unitary semi-Peck.

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