

PECK EDGE POSETS

DAVID HEMMINGER, AARON LANDESMAN, ZIJIAN YAO

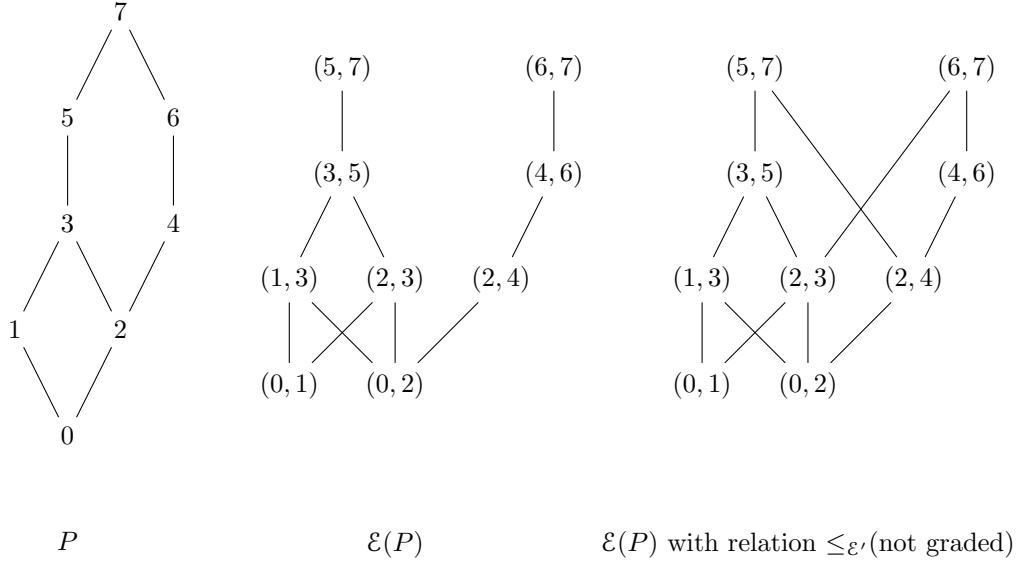
ABSTRACT. For any graded poset P , we define a new graded poset, $\mathcal{E}(P)$, whose elements are the edges in the Hasse diagram of P . For any group, G , acting on the boolean algebra, B_n , we conjecture that $\mathcal{E}(B_n/G)$ is Peck. We prove this conjecture holds for “common cover transitive” actions. We show common cover transitive actions are a fairly large class of actions and are closed under semidirect products. We also prove $\mathcal{E}(B_n/G)$ is unimodal and symmetric in the case that G is the cyclic group or dihedral group.

CONTENTS

List of Todos	2
1. Introduction	3
2. Background	4
3. The Edge Poset Construction	5
3.1. Definition and Basic Properties	6
3.2. Proof of Theorem 1.5	7
3.3. A Generalization of \mathcal{H}, \mathcal{E}	10
4. Common Cover Transitive Actions	10
4.1. Preservation Under Semidirect Products	11
4.2. Examples of Cover Transitive Actions	12
5. Wreath Product of two Symmetric Groups	15
5.1. Recalling Pak and Panova’s Result	15
5.2. The Poset $\mathcal{E}^r(B_n)/G$	15
5.3. Ensuing Conclusions	18
6. Relations to Polya Theory	18
6.1. A Brief Review of Polya Theory	18
6.2. A Rank Generating Function	19
6.3. Further Identities for the Wreath Product	20
7. Unimodal Quotients	23
7.1. Groups of order n .	23
8. Rank unimodality for Non-CCT actions	24
8.1. Quotient by the Cyclic Group	24
8.2. Quotient by the Dihedral Group	28
9. A q -analog	29
10. The object $\mathcal{E}(B_n)$.	30
10.1. Proof that $\mathcal{E}(B_n)$ is unitary Peck	31
11. Odds, Ends, and Failed Attempts	35
11.1. Quotients by Automorphism Groups of Rooted Trees	35
11.2. A Failed Attempt for Showing $\mathcal{E}(B_n/G)$ is unimodal	36
11.3. Further Questions	38

LIST OF TODOS

■ Relegate functoriality to a remark	3
■ Include a section for generalizations of E and include redfield polya generalization	3
■ Include the proposition that if a poset is self dual, then the edge poset is also self dual	3
■ Include a question about q analogs, it is currently at the end, but it should probably be weaved in here somewhere.	4
■ restate this at the appropriate place in the dihedral group subsection	4
■ Obviously come back to this at the end, but since this is the introduction for the REU report, not a journal paper, we should also include descriptions of the sections we don't think will go in the journal paper.	4
■ anyone know how to align these evenly?	8
■ Reference Mo's paper (title is "On Volumes of Permutation Polytopes", and it has the label "permutation polytopes" in References.bib)	13
■ merge sections 7 and 8	22
■ Currently Editing	27
■ This proof needs more work (self)	28

FIGURE 1. The definition of \mathcal{E}

1. INTRODUCTION

Let P be a finite graded poset of rank n . It is natural to study the structure of the edges in the Hasse diagram of P . To this end, define an endofunctor \mathcal{E} on the category finite graded posets with rank-preserving morphisms as follows

Definition 1.1. For \mathcal{P} the category of graded posets and define the *Edge Poset Construction* $\mathcal{E} : \mathcal{P} \rightarrow \mathcal{P}$ as follows. The elements of the graded poset $\mathcal{E}(P)$ are (x, y) where $x, y \in P$, $x \leq_P y$, and $\text{rk}(y) = \text{rk}(x) + 1$. Define the covering relation $\leq_{\mathcal{E}}$ on $\mathcal{E}(P)$ by $(x, y) \leq_{\mathcal{E}} (x', y')$ if $x \leq_P x'$ and $y \leq_P y'$. Then define the relation $\leq_{\mathcal{E}}$ on $\mathcal{E}(P)$ to be the transitive closure of $\leq_{\mathcal{E}}$. Let Q be a finite graded poset of rank n . Given a morphism $f : P \rightarrow Q$, define $\mathcal{E}(f) : \mathcal{E}(P) \rightarrow \mathcal{E}(Q)$ by $\mathcal{E}(f)(x, y) = (f(x), f(y))$.

We will show that \mathcal{E}^r is well-defined in Section 3. Note that an edge in the Hasse diagram of P can be written as $(x, y) \in P \times P$ such that $x < y$. Such edges are in bijection with elements $(x, y) \in \mathcal{E}(P)$.

Example 1.2. It is important we declare the relation $\leq_{\mathcal{E}}$ to be the transitive closure of $\leq_{\mathcal{E}}$. If instead we defined a relation $\leq_{\mathcal{E}'}$ on $\mathcal{E}(P)$ by $(x, y) \leq_{\mathcal{E}'} (a, b)$ if $x \leq a, y \leq b$, then $\mathcal{E}(P)$ would not necessarily be a graded poset. In Figure 1 we give an example of a poset P for which $\mathcal{E}(P)$ is not graded with the relation $\leq_{\mathcal{E}'}$. In the figure it is clear that $\mathcal{E}(P)$ is a graded poset, with $\text{rk}(x, y) = \text{rk}(x)$, but the Hasse diagram on the right represents a poset which does not have a grading.

We observe that when P has a nice structure, $\mathcal{E}(P)$ commonly has a nice structure as well. In particular, let the *boolean algebra of rank n* , denoted B_n , be the poset whose elements are subsets of $\{1, \dots, n\}$ with the relation given by containment, i.e. for all $x, y \in B_n$, $x \leq y$ if x is a subset of y .

Relegate functoriality to a remark

Include a section for generalizations of \mathcal{E} and include redfield polya generalization

Include the proposition that if a poset is self dual, then the edge poset is also self dual

For the remainder of this paper, all group actions on posets $\phi : G \times P \rightarrow P$ are assumed to be order preserving and rank preserving, meaning for all $g \in G, x \leq y \Leftrightarrow gx \leq gy, \text{rk}(gx) = \text{rk}(x)$.

Using Theorem 2.6 and the fact that B_n is unitary peck, if G is any action on B_n , then B_n/G is Peck. We conjecture the following.

Conjecture 1.3. *If $G \subset \text{Aut}(B_n)$, then $\mathcal{E}(B_n/G)$ is Peck.*

We prove this conjecture holds whenever the group action of G on B_n has the following property.

Definition 1.4. A group action of G on P is *common cover transitive*, (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z, y \lessdot z$, and $y \in Gx$, there exists some $g \in \text{Stab}(z)$ such that $g \cdot x = y$.

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

A large number of group actions on B_n have the CCT property. In order to construct a family of such actions we first prove that some basic group actions on B_n are CCT, then show that cover transitivity is preserved under semidirect products. Throughout the paper we let a subgroup $G \subseteq S_n$ act on B_n by letting it act on the elements within subsets of $[n] := \{1, \dots, n\}$, i.e. $g \cdot x = \{g \cdot i : i \in x\}$ for all $g \in G, x \in B_n$. We also embed the dihedral group D_{2n} into S_n by letting it act on the vertices of an n -gon.

Proposition 1.6. *Let p be prime. The following actions are CCT.*

- (1) *The action of S_n on B_n .*
- (2) *The actions of D_{2p} and D_{4p} on B_p and B_{2p} , respectively.*

Proposition 1.7. *Let $G \subseteq \text{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

The paper is organized as follows. In Section 2 we cover the necessary background for posets and Peck posets. In Section 3 we show that \mathcal{E} is well-defined and prove Theorem 1.5 along with various other nice properties of \mathcal{E} . Section 4 contains the proofs of Propositions 1.6 and 1.7 as well as some examples of families of group actions shown to be CCT by these propositions. In Section we prove that $\mathcal{E}(B_n/G)$ is rank-unimodal for certain group actions that are not CCT. Section discusses a q -analog to Conjecture 1.3, including some partial results.

2. BACKGROUND

In this section we review morphisms of graded posets, Peck posets, and a useful theorem about quotients of Posets.

Throughout the paper we write $x \leq_P y$ to denote that x is less than or equal to y under the relation defined on the poset P . When the poset is clear we will omit the P and simply write $x \leq y$.

A poset P is a *graded* poset if there exists a rank function $\text{rk} : P \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the following conditions.

- (1) If $x \in P_i$ and $x \lessdot y$, then $\text{rk}(x) + 1 = \text{rk}(y)$,
- (2) If $x < y$ then $\text{rk}(x) < \text{rk}(y)$

Include a question about q analogs, it is currently at the end, but it should probably be weaved in here somewhere.

restate this at the appropriate place in the dihedral group subsection

Obviously come back to this at the end, but since this is the introduction for the REU report, not a journal paper, we should also include descriptions of the sections we don't think will go in the journal paper.

Then, define the i th rank of P to be $P_i = \{x \in P : \text{rk}(x) = i\}$. Additionally, if for all $x \in P, 0 \leq \text{rk}(x) \leq n$, and there exists x, y with $\text{rk}(x) = 0, \text{rk}(y) = n$, we say P is a graded poset of rank n .

Denote the category of finite graded posets by \mathcal{P} , and let P, Q be finite graded posets of rank n . A map $f: P \rightarrow Q$ is a *morphism* from P to Q if it is rank-preserving and order preserving, i.e. for all $x, y \in P$, $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$ and $\text{rk}(x) = \text{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Remark 2.1. Note that we do not require the implication $f(x) \leq_Q f(y) \Rightarrow x \leq_P y$ in order for f to be a morphism. In particular this means a bijective morphism f need not be an isomorphism, since it will not necessarily have a two-sided inverse.

Write $p_i = |P_i|$. If we have

$$p_0 \leq p_1 \leq \dots \leq p_k \geq p_{k+1} \geq \dots \geq p_n$$

for some $0 \leq k \leq n$, then P is *rank-unimodal*, and if $p_i = p_{n-i}$ for all $1 \leq i \leq n$, then P is *rank-symmetric*. An *antichain* in P is a set of elements in P that are pairwise incomparable. If no antichain in P is larger than the largest rank of P , then P is *Sperner*. More generally, P is *k-Sperner* if no union of k antichains in P is larger than the union of the largest k ranks of P , and P is *strongly Sperner* if it is k -Sperner for all $1 \leq k \leq n$. We then make the following definition.

Definition 2.2. P is *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x : x \in P\}$ and $\{x : x \in P_i\}$ respectively. Note that we will frequently abuse notation and write P and P_i for $V(P)$ and $V(P_i)$ when our meaning is clear. In determining whether P is Peck, it is often useful to consider certain linear transformations on $V(P)$.

Definition 2.3. A map $U: V(P) \rightarrow V(P)$ is an *order-raising operator* if $U(V(P_n)) = 0$ and for all $0 \leq i \leq n-1$, $x \in P_i$ we have

$$U(x) = \sum_{y \succ x} c_{x,y} y$$

for some constants $c_{x,y} \in \mathbb{C}$. We say that U is the *Lefschetz map* if all $c_{x,y}$ on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

Lemma 2.4 (??, Lemma 1.1). *P is Peck if and only if there exists an order-raising operator U such that for all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}: V(P_i) \rightarrow V(P_{n-i})$ is an isomorphism.*

Definition 2.5. If the Lefschetz map satisfies the condition for U in Lemma 2.4, then P is *unitary Peck*.

We say that a group G acts on P if the action defines an embedding $G \hookrightarrow \text{Aut}(P)$, and define the *quotient poset* P/G to be the poset whose elements are the orbits of G , with the relation $\mathcal{O} \leq \mathcal{O}'$ if there exist $x \in \mathcal{O}, x' \in \mathcal{O}'$ such that $x \leq x'$. We will use the following result later in the paper.

Theorem 2.6 (??, Theorem 1). *If P is unitary Peck and $G \subseteq \text{Aut}(P)$, then P/G is Peck.*

3. THE EDGE POSET CONSTRUCTION

In Section 3.1 we show that \mathcal{E} as described in Definition 1.1 is a well-defined functor and prove some useful properties of \mathcal{E} . Section 3.2 is devoted to the proof of Theorem 1.5.

3.1. Definition and Basic Properties. First we prove Proposition 3.4, which states that \mathcal{E} is a well-defined functor. We break the proof of Proposition 3.4 into Lemmas 3.1, 3.2, and 3.3 which follow below. After showing that \mathcal{E} is well-defined we then define a natural G action on $\mathcal{E}(P)$ and define a surjection $\mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ that will be important for the proof of Theorem 1.5.

Notation. When the poset P is clear, we shall simply use $\leq_{\mathcal{E}}, <_{\mathcal{E}}$ to refer to $\leq_{\mathcal{E}(P)}, <_{\mathcal{E}(P)}$. Similarly, in Subsection 3.2, we define an object $\mathcal{H}(B_n)$, and shall use $\leq_{\mathcal{H}}, <_{\mathcal{H}}$ in place of $\leq_{\mathcal{H}(P)}, <_{\mathcal{H}(P)}$.

Lemma 3.1. *The relation $\leq_{\mathcal{E}}$ defines a partial order on $\mathcal{E}(P)$.*

Proof. We have that $(x, y) \leq_{\mathcal{E}} (x, y)$ and that $\leq_{\mathcal{E}}$ is transitive by definition. It remains to be shown that $\leq_{\mathcal{E}}$ is antisymmetric. Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$ and $(x', y') \leq_{\mathcal{E}} (x, y)$. Then $x \leq_P x' \leq_P x$ and $y \leq_P y' \leq_P y$, so $x = x'$ and $y = y'$ by antisymmetry of \leq_P , hence $(x, y) = (x', y')$. \square

Lemma 3.2. *For P a graded poset, the object $\mathcal{E}(P)$ is a graded poset.*

Proof. To show $\mathcal{E}(P)$ is graded, we must show $(x, y) <_{\mathcal{E}} (x', y') \implies \text{rk}(x, y) < \text{rk}(x', y')$ and $(x, y) <_{\mathcal{E}} (x', y') \implies \text{rk}(x, y) + 1 = \text{rk}(x', y')$. Both of these facts follow immediately from the definition of $<_{\mathcal{E}}$ and the definition $\text{rk}_{\mathcal{E}}(x, y) = \text{rk}_P(x)$. \square

Lemma 3.3. *If $f: P \rightarrow Q$ is morphism of finite ordered posets, then $\mathcal{E}(f): \mathcal{E}(P) \rightarrow \mathcal{E}(Q)$ is a morphism of finite ordered posets.*

Proof. First, $\mathcal{E}(f)$ is rank-preserving, since for all $(x, y) \in \mathcal{E}(P)$ we have

$$\text{rk}_{\mathcal{E}}(x, y) = \text{rk}_P(x) = \text{rk}_P(f(x)) = \text{rk}_{\mathcal{E}}(\mathcal{E}(f)(x, y)).$$

Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$. Then $x \leq_P x'$, $y \leq_P y'$, and since f is order-preserving, it follows that $f(x) \leq_P f(x')$, $f(y) \leq_P f(y')$. Hence $\mathcal{E}(f)(x, y) \leq_{\mathcal{E}} \mathcal{E}(f)(x', y')$. Thus $\mathcal{E}(f)$ is order-preserving and hence a morphism of finite ordered posets. \square

Proposition 3.4. *The map $\mathcal{E}(P)$ is an endofunctor on the category of finite graded posets \mathcal{P} .*

Proof. By Lemmas 3.1, 3.2, and 3.3 we have that \mathcal{E} takes objects in \mathcal{P} to objects in \mathcal{P} and morphisms of \mathcal{P} to morphisms of \mathcal{P} . Furthermore it is clear that $\mathcal{E}(\text{id}_P) = \text{id}_{\mathcal{E}(P)}$, so it remains to be shown that for finite graded posets P, Q , and R with morphisms $f: P \rightarrow Q$ and $g: Q \rightarrow R$ that $\mathcal{E}(g \circ f) = \mathcal{E}(g) \circ \mathcal{E}(f)$. This is clear, however, as for all $(x, y) \in \mathcal{E}(P)$ we have $\mathcal{E}(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}(g) \circ \mathcal{E}(f))(x, y)$. \square

Given a group action of G on P , we can easily define a natural group action of G on $\mathcal{E}(P)$ using Proposition 3.4. For all $g \in G$ we have that multiplication by g is an automorphism of P , so it follows that $\mathcal{E}(g)$ is an automorphism of $\mathcal{E}(P)$. Since \mathcal{E} is a functor this gives a well-defined group action.

Definition 3.5. Given a G -action on P , define a G -action on $\mathcal{E}(P)$ by $g \cdot (x, y) = \mathcal{E}(g)(x, y) = (gx, gy)$.

Using Definition 3.5, we have a well-defined quotient poset $\mathcal{E}(P)/G$. It is then natural to ask whether the operation of quotienting out by G commutes with \mathcal{E} , that is, whether $\mathcal{E}(P/G) \cong \mathcal{E}(P)/G$. Unfortunately the two posets are rarely isomorphic, but there is always a surjection $\mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$, and this surjection is also an injection precisely when the G -action on P is CCT.

Proposition 3.6. *The map $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism.*

Proof. Note that q is well defined because if $(x', y') = g(x, y) = (g \cdot x, g \cdot y)$ for some $g \in G$, then $x' \in Gx$ and $y' \in Gy$. Clearly q is rank-preserving and surjective, so it suffices to show that q is order-preserving. Suppose that $G(x, y) \leq G(w, z)$. Then there exist some $(x_0, y_0) \in G(x, y)$, $(w_0, z_0) \in G(w, z)$ such that $x_0 \leq w_0$ and $y_0 \leq z_0$. We then have that $(Gx, Gy) \leq (Gw, Gz)$ by definition, hence q is order-preserving. \square

Lemma 3.7. *Let G be a group acting on a graded poset P . The following are equivalent:*

- (1) *The action of G on P is CCT.*
- (2) *Whenever $x < y, x < z$, and $y \in Gz$, there exists some $g \in \text{Stab}(x)$ with $gx = z$.*
- (3) *The map $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is an bijective morphism (but not necessarily an isomorphism).*

Proof. First, we shall show (1) \Leftrightarrow (3). Observe that q is a bijection exactly when there do not exist distinct orbits $G(x, y) \neq G(x', y')$ with $x' \in Gx, y' \in Gy$. Fix $(x, y), (x', y') \in \mathcal{E}(P)$ such that $x' \in Gx$ and $y' \in Gy$. Pick a $g \in G$ such that $g \cdot y' = y$. Then $(g \cdot x', y) \in G(x', y')$, so $G(x, y) = G(x', y')$ if and only if there exists some $g' \in G$ such that $g' \cdot x = g \cdot x'$ and $g' \cdot y = y$. Hence q is a bijection if and only if the G action is CCT.

Next, we shall show (2) \Leftrightarrow (3). It is quite analogous to the proof of (1) \Leftrightarrow (3), but we shall include the argument for completeness. First, q is a bijection if and only if there do not exist distinct orbits $G(x, y) \neq G(x', y')$ with $x' \in Gx, y' \in Gy$. Fix $(x, y), (x', y') \in \mathcal{E}(P)$ such that $x' \in Gx, y' \in Gy$. Choose $g \in G$ with $gx' = x$. Then, $(x', g \cdot y) \in G(x', y')$. Since $gy' \in Gy$, and q is a bijection, the statement $(x', g \cdot y) \in G(x', y')$ is equivalent to $G(x', y') = G(x, y)$. This, in turn is equivalent to the existence of $g' \in G$ with $g'x = x, g'y = gy'$. So, $g' \in \text{Stab}(x), g'y = (gy')$, which is what was claimed in property (2). \square

Remark 3.8. While q is a bijection if and only if the action of G on P is CCT, it is not true that if the action of G on P is CCT, then q is an isomorphism. For example, take $G = D_{20} \subset S_{10}$ acting by reflections and rotations on $[10]$ and hence acting on B_{10} . We shall see in Lemma 8.15 that this action is CCT. However, consider $x = \{2, 4\}, y = \{1, 2, 4\}, a = \{2, 4, 7\}, b = \{2, 4, 6, 7\}$. Then we may observe $(x, y), (a, b) \in \mathcal{E}(B_{10})$ and $Gx < Ga, Gy < Gb$, so $(Gx, Gy) <_{\mathcal{E}} (Ga, Gb)$. However, it is not true that $G(x, y) <_{\mathcal{E}} G(a, b)$.

3.2. Proof of Theorem 1.5. In this section we prove Theorem 1.5, which we recall here:

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n)/G$ is Peck.*

The proof is largely motivated by the following Lemma.

Lemma 3.9. *If $f: P \rightarrow Q$ is a bijection (but not necessarily an isomorphism) and P is Peck then Q is Peck.*

Proof. Let $\text{rk}(P) = \text{rk}(Q) = n$. Since P is Peck there exists an order-raising operator U such that $U^{n-2i}: P_i \rightarrow P_{n-i}$ is an isomorphism. Since f is a poset morphism, it follows that the map $f \circ U \circ f^{-1}$ is an order-raising operator on Q . We then have that $f \circ U^{n-2i} \circ f^{-1} = (f \circ U \circ f^{-1})^{n-2i}: Q_i \rightarrow Q_{n-i}$ is an isomorphism since $U^{n-2i}: P_i \rightarrow P_{n-i}$ is an isomorphism and f is a bijection. \square

By Lemma 3.9 and Proposition 3.6, in order to prove Theorem 1.5 it suffices to prove that $\mathcal{E}(B_n)/G$ is Peck. One way to do this is to prove that $\mathcal{E}(B_n)$ is unitary Peck and then apply Theorem 2.6. We prove that $\mathcal{E}(B_n)$ is unitary Peck in Section 10, but unfortunately the proof is messy and computational.

Fortunately there is a simpler – albeit less direct – route. In order to avoid showing that $\mathcal{E}(B_n)$ is unitary Peck, we define a graded poset $\mathcal{H}(B_n)$ for all n such that $\mathcal{H}(B_n)$ is easily seen to be unitary Peck in Corollary 3.18. Furthermore, we define $\mathcal{H}(B_n)$ such that a group action of G on B_n induces a group action on $\mathcal{H}(B_n)$ (Lemma 3.15) and there is always a bijective morphism $f: \mathcal{H}(B_n)/G \rightarrow \mathcal{E}(B_n)/G$ (Lemma 3.16). By the above discussion, Theorem 1.5 readily follows.

Definition 3.10. For P a graded poset, define the graded poset $\mathcal{H}(P)$ as follows. Let the elements $(x, y) \in \mathcal{H}(P)$ to be pairs $(x, y) \in P \times P$ such that $x < y$. Define $(x, y) <_{\mathcal{H}} (x', y')$ if $x < x', y < y'$ and $x' \neq y$. Then, define $\leq_{\mathcal{H}}$ to be the transitive closure of $<_{\mathcal{H}}$, and define $\text{rk}_{\mathcal{H}}(x, y) = \text{rk}_P(x)$.

Example 3.11. We give an example of the poset $\mathcal{H}(B_3)$ in Figure 2. Observe that $\mathcal{H}(B_3)$ can be written as a disjoint union of three copies of B_2 . This is a single case of the more general phenomenon proven in Proposition 3.17.

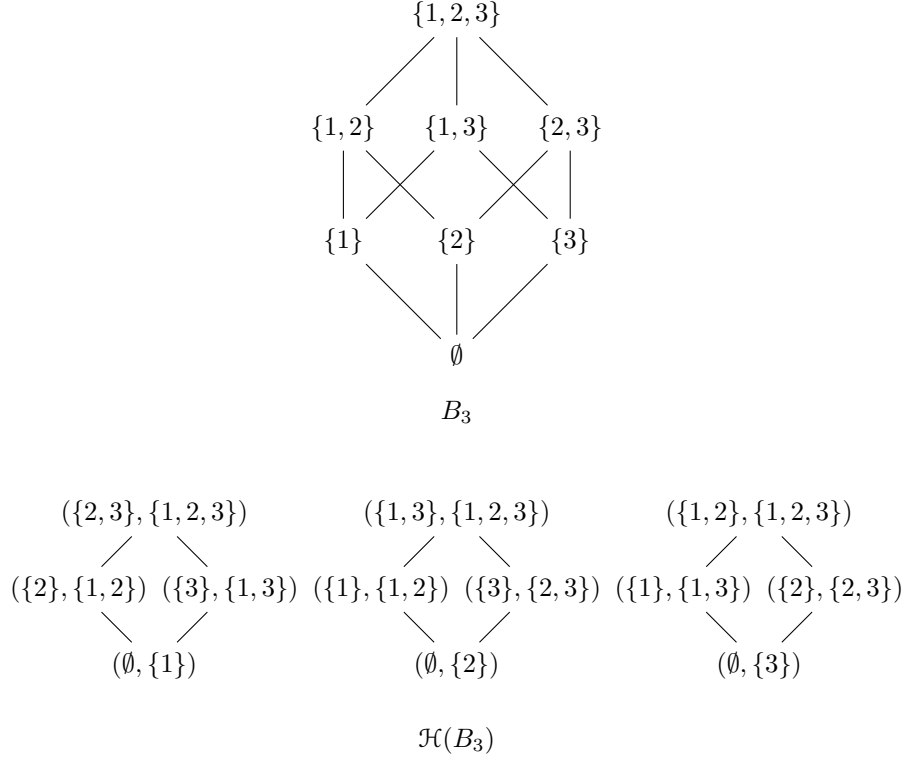


FIGURE 2. B_3 and $\mathcal{H}(B_3)$

Remark 3.12. Note that by definition $(x, y) <_{\mathcal{H}} (x', y')$ precisely when $(x, y) <_{\mathcal{E}} (x', y')$ and $x' \neq y$, hence $(x, y) <_{\mathcal{H}} (x', y') \Rightarrow (x, y) <_{\mathcal{E}} (x', y')$. In other words, $\mathcal{H}(P)$ has the same elements as $\mathcal{E}(P)$ but with a weaker partial order.

Lemma 3.13. For P a graded poset, the object $\mathcal{H}(P)$, as defined in Definition 3.10 is a graded poset.

Proof. This follows immediately from Remark 3.12 and the fact that $\mathcal{E}(P)$ is graded. \square

Remark 3.14. While $\mathcal{E} : \mathcal{P} \rightarrow \mathcal{P}$ is a functor, \mathcal{H} is not a functor. In particular, we have not specified how \mathcal{H} acts on morphisms. If, for $f : P \rightarrow Q$, we attempted to define $\mathcal{H}(f) : \mathcal{H}(P) \rightarrow \mathcal{H}(Q)$, we would quickly run into trouble. For example, suppose we took f mapping a diamond P (see Figure 3.14) to a chain of length three Q . Then, $\mathcal{H}(P)$ has four points, with two pairs of edges, whereas $\mathcal{H}(Q)$ has two disconnected points. It is clear that there can be no order-preserving mapping between these two objects.

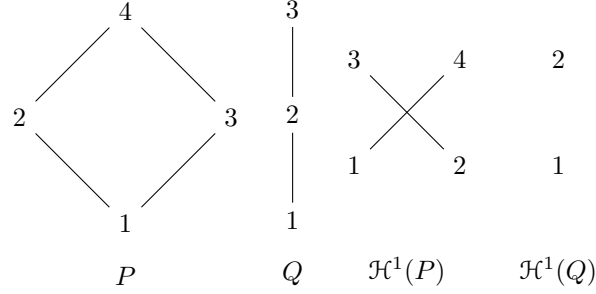


FIGURE 3.

Given an action of a group G on P , we define an action of G on $\mathcal{H}(P)$ as we did for $\mathcal{E}(P)$ by again defining $g \cdot (x, y) = (gx, gy)$ for all $(x, y) \in P$. We will then have a well-defined quotient poset $\mathcal{H}(P)/G$ with the same elements as $\mathcal{E}(P)/G$.

Lemma 3.15. *The function defined by $g \cdot (x, y) = (gx, gy)$ for all $g \in G$, $(x, y) \in \mathcal{H}(P)$ is a well-defined group action of G on $\mathcal{H}(P)$.*

Proof. Let $g \in G$. Since $\leq_{\mathcal{H}}$ is the transitive closure of $\leq_{\mathcal{H}}$ it suffices to show that for all $(x, y), (x', y') \in \mathcal{H}(P)$ we have $(x, y) \leq_{\mathcal{H}} (x', y') \Leftrightarrow g(x, y) \leq_{\mathcal{H}} g(x', y')$. Since g is an automorphism of P , we have $x \leq_P x' \Leftrightarrow gx \leq_P gx'$, $y \leq_P y' \Leftrightarrow gy \leq_P gy'$, and $y \neq x' \Leftrightarrow gy \neq gx'$, so the result follows from the definition of $\leq_{\mathcal{H}}$. \square

Lemma 3.16. *The map $f : \mathcal{H}(P)/G \rightarrow \mathcal{E}(P)/G$ defined by $G(x, y) \mapsto G(x, y)$ is a bijection for any group action of G on P .*

Proof. The elements of $\mathcal{H}(P)/G, \mathcal{E}(P)/G$ are the same by definition. Therefore, as long as f is a morphism, it is automatically a bijection. Since f is clearly rank-preserving, to show f is a morphism, it suffices to show f is order-preserving. This is immediate from Remark 3.12. \square

The remaining ingredient to the proof of Theorem 1.5 is to show that $\mathcal{H}(B_n)$ is unitary Peck, which we do by generalizing Example 3.11 and showing that $\mathcal{H}(B_n)$ is isomorphic to the disjoint union of boolean algebras.

Proposition 3.17. *$\mathcal{H}(B_n)$ is isomorphic to n disjoint copies of B_{n-1} .*

Proof. Suppose we have $(x, y), (x', y') \in \mathcal{H}(B_n)$ with $(x, y) \leq (x', y')$. Let $j \in [n]$ such that $y' = y \cup \{j\}$, and let $i \in [n]$ such that $x' = x \cup \{i\}$. If $i \neq j$, then $x' \leq y$, contradicting the assumption that $(x, y) \leq_{\mathcal{H}} (x', y')$. Thus $x' = x \cup \{i\}$ and $y' = y \cup \{i\}$ for some $i \in [n]$.

Conversely we can easily check that if $i \notin y$, then $(x, y) \leq_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$. It follows that for all subsets $w \subset [n]$ such that $|w| = 1$, there is an isomorphism from the subposet of elements $\{(x, y) : y \setminus x = w\}$ to B_{n-1} defined by $(x, y) \mapsto (x \setminus w, y \setminus w)$. Furthermore if $y \setminus x \neq y' \setminus x'$, then (x, y) and (x', y') are incomparable, so these subposets indexed by w are pairwise disjoint, and $\mathcal{H}(B_n)$ is isomorphic to n copies of B_n . \square

Corollary 3.18. *$\mathcal{H}(B_n)$ is unitary Peck for all $n \geq 0$.*

Proof. This follows immediately from Proposition 3.17 and the fact that B_{n-1} is unitary Peck. The latter is shown, for instance in [?, Theorem 2a] by noting that $B_k = (B_1)^k$ and that B_1 is clearly unitary Peck. \square

Corollary 3.19. *$\mathcal{H}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. This follows from Corollary 3.18 and Theorem 2.6. \square

Corollary 3.20. *$\mathcal{E}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. By Corollary 3.19, $\mathcal{H}(B_n)/G$ is Peck. By Lemma 3.16, the map $f : \mathcal{H}(P) \rightarrow \mathcal{E}(P), G(x, y) \mapsto G(x, y)$ is a bijection. Then, by Lemma 3.9, it follows that $\mathcal{E}(B_n)/G$ is Peck. \square

Proof of Theorem 1.5. By Corollary 3.20, $\mathcal{E}(B_n)/G$ is Peck for any group action of G on B_n . Since the G -action is CCT, there is a bijective morphism from $\mathcal{E}(B_n)/G$ to $\mathcal{E}(B_n/G)$, hence $\mathcal{E}(B_n/G)$ is Peck by Lemma 3.16. \square

3.3. A Generalization of \mathcal{H}, \mathcal{E} . Although for the most part, we shall investigate $\mathcal{E}(P/G), \mathcal{E}(P)/G$, there is a natural generalization of \mathcal{E}, \mathcal{H} , to what we shall call $\mathcal{E}^{\vec{r}}, \mathcal{H}^{\vec{r}}$, where $\vec{r} = r_1, r_2, \dots, r_k$ is an integer valued sequence. The results holding for these generalizations are analogous to those developed above. The purpose of developing these generalizations notion will be to give a more general application to Polya theory than could be given with \mathcal{E} .

Definition 3.21. Let $\vec{r} = r_1, \dots, r_k$. For a graded poset P , define the graded poset $\mathcal{H}^{r_1, \dots, r_k}(P)$, also notated $\mathcal{H}^{\vec{r}}(P)$, whose elements are formal symbols $(x_1, x_2, \dots, x_{k+1})$ such that $rk(x_i) + r_i = rk(x_{i+1})$, for all $i \in [k]$. Say $(x_1, x_2, \dots, x_{k+1}) \leq_{\mathcal{H}^{\vec{r}}} (y_1, y_2, \dots, y_{k+1})$ if $x_i \leq_P y_i, y_i \not\leq_P x_{i+1}$ for all $i \in [k+1]$. Then, define a relation $\leq_{\mathcal{H}^{\vec{r}}}$ on $\mathcal{H}^{r_1, \dots, r_k}(P)$, to be the transitive closure of $\leq_{\mathcal{H}^{\vec{r}}}$. Finally, define $rk_{\mathcal{H}^{\vec{r}}(P)}(x_1, \dots, x_{k+1}) = rk_P(x_1)$

Definition 3.22. Let $\vec{r} = r_1, \dots, r_k$, with $r_i \in \mathbb{N}$ for all $i \in [k]$. Let \mathcal{P} be the category of graded posets. Define the functor $\mathcal{E}^{\vec{r}} : \mathcal{P} \rightarrow \mathcal{P}$, also notated $\mathcal{E}^{r_1, \dots, r_k}$. For a graded poset P , define the graded poset $\mathcal{E}^{r_1, \dots, r_k}(P)$, also notated whose elements are formal symbols $(x_1, x_2, \dots, x_{k+1})$ such that $rk(x_i) + r_i = rk(x_{i+1})$, for all $i \in [k]$. Say $(x_1, x_2, \dots, x_{k+1}) \leq_{\mathcal{E}^{\vec{r}}} (y_1, y_2, \dots, y_{k+1})$ if $x_i \leq_P y_i$ for all $i \in [k+1]$. Then, define a relation $\leq_{\mathcal{E}^{\vec{r}}}$ on $\mathcal{E}^{r_1, \dots, r_k}(P)$, to be the transitive closure of $\leq_{\mathcal{E}^{\vec{r}}}$. Finally, define $rk_{\mathcal{E}^{\vec{r}}(P)}(x_1, \dots, x_{k+1}) = rk_P(x_1)$

Observe that under this notation, $\mathcal{E}^1 = \mathcal{E}, \mathcal{H}^1 = \mathcal{H}$.

Remark 3.23. For all $\vec{r}, \mathcal{E}^{\vec{r}}$ is a functor, for the same reasons that \mathcal{E} is a functor. This generalization is essentially taking the nerves of the poset P . See [?] for some related constructions, although their constructions are not completely analogous.

In the next Lemmas, we cite analogous results which hold for $\mathcal{H}^{\vec{r}}$. The proofs are almost identical to those for \mathcal{H} .

Lemma 3.24. *Given a group action $\phi : G \times P \rightarrow P$ there are well defined group actions $\phi_{\mathcal{E}} : G \times \mathcal{E}^{\vec{r}}(P) \rightarrow \mathcal{E}^{\vec{r}}(P), \phi_{\mathcal{H}} : G \times \mathcal{H}^{\vec{r}}(P) \rightarrow \mathcal{H}^{\vec{r}}(P)$, both given by $g \cdot (x_1, \dots, x_{k+1}) = (g \cdot x_1, \dots, g \cdot x_{k+1})$.*

Proof. The proof is analogous to Lemma 3.15, Lemma 3.15. \square

Lemma 3.25. *The poset $\mathcal{H}^{r_1, \dots, r_k}(B_n)$ is isomorphic to the multinomial coefficient $\binom{n}{r_1, r_2, \dots, r_k}$ disjoint copies of $B_{n - \sum_{i=1}^k r_i}$. Consequently, $\mathcal{H}^{\vec{r}}(B_n)$ is unitary Peck and $\mathcal{H}^{\vec{r}}(B_n)/G$ is Peck.*

Proof. Once again, the proof of the above three statements is analogous to those of 3.17, Corollary 3.18, and Corollary 3.19. The reason there are multinomial coefficients here instead of binomial coefficients, is that an element $(x_1, x_2, \dots, x_{k+1})$ lies in the copy of $B_{n - \sum_{i=1}^k r_i}$ determined by the ordered tuple of subsets $(x_2 \setminus x_1, x_3 \setminus x_2, \dots, x_{k+1} \setminus x_k)$. The first consists of r_1 elements, the next of r_2 elements, up through the last which consists of r_k elements. Since the total number of ways to choose r_1, \dots, r_k in k distinct groups is $\binom{n}{r_1, r_2, \dots, r_k}$, there are exactly this many disjoint copies of $B_{n - \sum_{i=1}^k r_i}$. \square

4. COMMON COVER TRANSITIVE ACTIONS

In this section, we develop the theory of CCT actions ϕ where G is a group, P is a poset, and $\phi : G \times P \rightarrow P$ is an action. Recall Definition 1.4, that ϕ is CCT if whenever $x, y, z \in P, x \leq y, y \leq z, x \in Gy$ then there exists $g \in \text{Stab}(z)$ with $gx = y$. For CCT actions

$\phi : G \times P \rightarrow P$, we shall show that, $\mathcal{E}(P/G)$ is Peck. Additionally, we shall show the CCT property is closed under semidirect products, in the appropriate sense. It is obvious that the action of S_k on B_k is CCT, and we shall also see in Lemma 8.15, that the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

Remark 4.1. Before continuing with the description of cover transitive, it is worth noting that the notion of cover transitivity generalizes to \mathcal{E} and even $\mathcal{E}^{\vec{r}}$. Letting $\vec{r} = r_1, \dots, r_k$, one would call a group action $\phi : G \times P \rightarrow P$, \vec{r} *common cover transitive* if for any chains $(x_0, \dots, x_k), (y_0, \dots, y_k)$, such that there exists $g_i \in G$ with $x_i = g_i y_i$, for all $i, 0 \leq i \leq k$, then there exists a single $g \in G$ with $x_i = g y_i$. It is trivial to see that 1 common cover transitive agrees with our definition of common cover transitive. Furthermore, it is also simple to see that an action is \vec{r} common cover transitive if and only if the natural surjection $q : \mathcal{E}^{\vec{r}}(P)/G \rightarrow \mathcal{E}^{\vec{r}}(P/G)$ is a bijection. Additionally, very many of the properties developed in this section will also hold for this generalized notion of \vec{r} common cover transitive.

Example 4.2. Two rather trivial examples of CCT actions are $\phi : S_n \times B_n \rightarrow B_n, \psi : G \times B_n \rightarrow B_n$ where G is arbitrary, ϕ is the action induced by S_n permuting the elements of $[n]$, and ψ is the trivial action. In the former case, $\mathcal{E}(B_n/S_l)$ is simply a chain with $n-1$ points, and so is $\mathcal{E}(B_n)/S_l$, since all (x, y) are identified under the S_l action. In the latter case, since G acts trivially by ϕ , both $\mathcal{E}(B_n/G) \cong \mathcal{E}(B_n)$ and $\mathcal{E}(B_n)/G \cong \mathcal{E}(B_n)$. So again, ψ is CCT.

Theorem 4.3. *If an action $\phi : G \times P \rightarrow P$ is CCT, and $\mathcal{E}(P)$ is Peck, then $\mathcal{E}(P/G)$ is Peck.*

Proof. By Proposition 3.6, there is a surjection $q : \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$. Additionally, by Lemma 3.6, if ϕ is CCT, q is a bijection. Finally, by Lemma 3.9, since $q : \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ is a bijection and $\mathcal{E}(P)/G$ is Peck, it follows that $\mathcal{E}(P/G)$ is Peck. \square

4.1. Preservation Under Semidirect Products. Recall Proposition 1.7, as stated in the introduction. This Proposition essentially says the CCT property is preserved under semidirect products. We shall use this to obtain simple proofs that CCT actions are also preserved under direct products and wreath products.

Proposition 1.7. *Let $G \subseteq \text{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

Proof. Since $G = H \rtimes K$, every element $g \in G$ can be written uniquely as a product hk for some $h \in H, k \in K$. Let $x, y, z \in P$ such that $x < z, y < z$ and such that there exists some $h_0 k_0 \in G$ such that $h_0 k_0 \cdot x = y$. It suffices to show that there exists some $g \in \text{Stab}(z)$ such that $g \cdot x = y$.

The orbits $Hx, Hy, Hz \in P/H$ satisfy $Hx < Hz, Hy < Hz$ such that $k_0 \cdot Hx = Hy$, so since the action of K on P/H is CCT there exists some $k_1 \in K$ such that $k_1 \in \text{Stab}(Hz)$ and $k_1 \cdot Hx = Hy$. It follows that there exists some $h_1 \in H$ such that $h_1 k_1 h_0 \in \text{Stab}(z)$ and $h_1 k_1 h_0 \cdot x \in Hy$.

Write $x' = h_1 k_1 h_0 \cdot x$. Since the group action of G must be order-preserving by definition, we have that $x' < z$. We already had that $y < z$ and $x' \in Hy$, hence there exists some $h_2 \in \text{Stab}(z)$ such that $h_2 \cdot x' = y$ by the fact that the action of H on P is CCT. Then we have that $h_2 h_1 k_1 h_0 \cdot x = h_2 \cdot x' = y$ and $h_2 h_1 k_1 h_0 \cdot z = h_2 \cdot z = z$, as desired. \square

Proposition 4.4. *For $\phi : G \times P \rightarrow P, \psi : H \times Q \rightarrow Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \rightarrow (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.*

Proof. First, by definition of CCT, if either G or H acts trivially, it is clear the action of $G \times H$ is CCT. To see this, suppose G acts trivially, the case of H acting trivially is similar.

Then, if $(x, y) \leq (a, b)$, $(r, s) \leq (a, b)$, $(x, y) \in (G \times H)(r, s)$, then $x \in Gr$ so $x = r$. There are now two cases, either $b = y = s$ or $x = r = a$. First, if $b = y = s$, then $(e, e) \in \text{Stab}(x, y)$ and $(e, e)(x, y) = (r, s)$. Second, if $x = r = a$, and then $y \leq b$, $s \leq b$, $y \in Hs$, and so by cover transitivity of G , there exists $g \in G$ so that $(e, g)(x, y) = (r, s)$ and $(e, g) \in \text{Stab}(a, b)$.

Next, Observe that $G \times H$ can be viewed as the semidirect product of $(G \times \{e\}) \rtimes (\{e\} \times H)$. Since the action of G on P is CCT, by the above paragraph, the action of $G \times \{e\}$ on $P \times Q$ is CCT. Also, since the action of H on Q is CCT, it follows that the action of $\{e\} \times H$ on $(P/G) \times Q$ is CCT. Therefore, the action of $(G \times \{e\}) \rtimes (\{e\} \times H)$ satisfies the conditions of Proposition 1.7 and so the action of $G \times H$ is CCT. \square

Next, we use Proposition 1.7 to prove in Proposition 4.9 that the CCT property is preserved under wreath products with the symmetric group. First, we need some definitions of wreath product.

Definition 4.5. For G, H groups, with $H \subset S_l$, the *wreath product*, notated $G \wr H$, is the group whose elements are pairs $(g, h) \in G^l \times H$ with multiplication defined by

$$((g'_1, \dots, g'_l), h') \cdot ((g_1, \dots, g_l), h) = ((g'_{h'(1)}g_1, \dots, g'_{h'(l)}g_l), hh')$$

where $h \in H$ acts on $[l]$ by the restricting of the action of S_l on $[l]$ by all permutations to H .

In other words, $G \wr H$ can be viewed as a certain semidirect product of $G^l \rtimes H$.

Notation 4.6. For any group G with a given action $\psi : G \times P \rightarrow P$, we obtain an induced action of $G \wr H$, $\phi : G \wr H \times P^l \rightarrow P^l$ defined by

$$((g_1, \dots, g_l), h)(a_1, \dots, a_l) = (g_{h^{-1}(1)} \cdot a_{h^{-1}(1)}, \dots, g_{h^{-1}(l)} \cdot a_{h^{-1}(l)}).$$

Remark 4.7. Heuristically, one may think of the above action as obtained by first having G act separately on the l distinct copies of P , and then letting H act by permuting the copies.

Lemma 4.8. For P a poset, the action $\phi : S_l \times P^l \rightarrow P^l$, $\sigma \cdot (x_1, \dots, x_l) = (x_{\sigma(1)}, \dots, x_{\sigma(l)})$ is CCT.

Proof. For $a \in P^l$ notate $a = (a_1, \dots, a_l)$. Suppose $x, y, z \in P^l$ with $x \leq z$, $y \leq z$, and $x \in S_l y$. This means there is a unique i such that $x_i \leq z_i$, $x_k = z_k$ for $k \neq i$. Additionally, there is a unique j for which $y_j \leq z_j$, $y_k = z_k$ for $k \neq j$. Since $x \in S_l y$, we obtain the equality of multisets $\{x_1, \dots, x_l\} = \{y_1, \dots, y_l\}$. But for $k \neq i, j$, $x_k = z_k = y_k$, we also obtain equality of sets $\{x_i, x_j\} = \{y_i, y_j\}$. Since $rk(y_j) \leq rk(x_j)$, we obtain $y_j = x_i$, $y_i = x_j$. Then, taking the transposition $\sigma = (ij) \in S_l$, it follows that $\sigma \in \text{Stab}(z)$ and $\sigma \cdot x = y$. \square

Proposition 4.9. If $\psi : G \times P \rightarrow P$ is CCT, then $\phi : G \wr S_l \times P^l \rightarrow P^l$ where ϕ is the induced action defined in Notation 4.6 is also CCT.

Proof. First, the wreath product $G \wr S_l$ can be viewed as the semidirect product $G^l \rtimes S_l$. Observe that since the action of G on P is CCT, using Proposition 4.4, we obtain the action of G^l on P^l is CCT. Explicitly, this action is $(g_1, \dots, g_l)(x_1, \dots, x_l) = (g_1 \cdot x_1, \dots, g_l \cdot x_l)$. Additionally, since $P^l/G^l \cong (P/G)^l$. So, for $\sigma \in S_l$, $x_i \in P/G$, the action $S_l \times (P/G)^l \rightarrow (P/G)^l$, $(\sigma, (x_1, \dots, x_l)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(l)})$ is CCT by Lemma 4.8. So, the action ϕ , satisfies the conditions of Proposition 1.7 and so ϕ is CCT. \square

4.2. Examples of Cover Transitive Actions. In this subsection, we describe several classes of cover transitive actions. First, we show that the automorphism group of any rooted tree is cover transitive. Second, we show linear automorphisms of simplices and octahedra are cover transitive. Third, we show that the left multiplication action is cover transitive if and only if the group is \mathbb{Z}_2^k , and that any action of \mathbb{Z}_2^k on $[n]$ induces a cover transitive action on B_n .

Reference Mo's paper (title is "On Volumes of Permutation Polytopes", and it has the label "permutation polytopes" in References.bib)

4.2.1. An application to rooted trees. In this subsubsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Proposition 4.9 and Proposition 4.4, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT.

Definition 4.10. A graded poset P is a *rooted tree* if there is a unique element $x \in P$ of maximal rank, called the *root*, and for all $x \in P$, other than the root, there exists a unique $y \in P$ with $y \succ x$.

Definition 4.11. For P a rooted tree, an element $x \in P$ is a *leaf* if there is no $z \in P$ with $x \succ z$. Denote the set of all leaves of P by $L(P)$.

Lemma 4.12. Let P be a rooted tree. Then, the action of $\text{Aut}(P)$ on P induces an action of $\text{Aut}(P)$ on $L(P)$. Furthermore, there is also an induced action of $\text{Aut}(P)$ on B_n where $n = |L(P)|$.

Proof. First, we must show that $\text{Aut}(P)$ induces an action on $L(P)$. That is, we must show that for any $g \in \text{Aut}(P)$, $x \in L(P)$, then $gx \in L(P)$. If $x \in L(P)$ but $gx \notin L(P)$, then there exists $y < gx$. However, then $g^{-1}y < x$, contradicting the assumption that x is a leaf. We then obtain the induced action $\text{Aut}(P) \times L(P) \rightarrow L(P)$, $(g, x) \mapsto gx$.

Finally, to obtain the induced action of $\text{Aut}(P)$ on B_n identify $L(P) \cong [n]$ as sets, where $|L(P)| = n$, and from this we get of action of $\text{Aut}(P)$ on B_n . \square

Convention 4.13. For the rest of this section only, fix a rooted tree P and denote by G the group of automorphisms $\text{Aut}(P)$. Let G act on B_n , where $n = |L(P)|$ by the induced action $\phi : G \times P \rightarrow P$, given by restricting $\text{Aut}(P)$ to $\text{Aut}(L(P))$, as described in the proof of Claim 4.12.

Notation 4.14. For $x \in P$, denote $D(x) = \{y \in P : y \leq x\}$.

Proposition 4.15. Let P be a rooted tree with root vertex labeled r . Let $\{A_1, \dots, A_m\}$ denote the set of isomorphism classes of $\{D(x) : x \leq r\}$. For $T \in A_j$, denote $G_j = \text{Aut}(T)$. Then,

$$(4.1) \quad \text{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

In particular, $\text{Aut}(P)$ can be expressed as a sequence of direct products and wreath products of symmetric groups.

Proof. It is clear that if P is rank 0, then $\text{Aut}(P)$ is trivial. So, let us proceed by induction. That is, label the vertices of P by $\{0, 1, \dots, s\}$ such that the root is labeled 0 and the vertices just below the root are labeled $1, \dots, k$. Let A_1, \dots, A_m denote the distinct isomorphism classes of trees in the set $\{D(1), \dots, D(k)\}$. For $T \in A_j$, denote $G_j = \text{Aut}(T)$. Let $T_j = \{t : t \leq 0, D(t) \cong A_j\}$. Then, letting Q_j be the subtree of P whose elements lie in the set $0 \cup (\cup_{t \in T_j} D(t))$, it follows $\text{Aut}(Q_j) \cong G_j \wr S_{i_j}$, because after choosing a permutation of the elements of T_j , we are free to choose any element of G_j to permute each $D(t)$, $t \in T_j$. If $t_1 \leq 0, t_2 \leq 0, g \cdot t_1 = t_2$, then it must be that $D(t_1) = D(t_2)$. Therefore, $\text{Aut}(P)$ must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$(4.2) \quad \text{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

Since each G_j is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (4.2) that so is $\text{Aut}(P)$. \square

Corollary 4.16. For P the automorphism group of a rooted tree, $\text{Aut}(P)$ is CCT.

Proof. By Proposition 4.9, wreath products with symmetric groups preserve the CCT property, and by Proposition 4.4 the direct product of two CCT groups is again CCT. Therefore, by Proposition 4.15, all groups of the form $\text{Aut}(P)$ are built up from these operations, and so $\text{Aut}(P)$ is also CCT. \square

4.2.2. Automorphisms of Polytopes. As another class of CCT actions, we describe several linear automorphism groups of polytopes whose actions are CCT. In particular, we prove that the linear automorphism group of simplices, and octahedra are common cover transitive, although both of these are fairly straightforward. Later in Lemma 8.15, we shall also see that the action of the dihedral group on a regular n -gon, for $n = p, 2p$, is cover transitive. This is the group of linear automorphisms of the regular n -gon. This action gives yet another example of automorphisms of the linear automorphism group of polytopes being CCT.

Definition 4.17. Let M be a polytope, with a particular embedding in \mathbb{R}^n . The group of linear automorphisms of M is the subgroup of GL_n whose elements are $\{g \in GL_n : g \cdot M = M\}$.

Proposition 4.18. Let G be the group linear automorphisms of the $n - 1$ simplex, whose vertices lie at the standard basis vectors in \mathbb{R}^n . The action of G on the $n - 1$ -simplex induces an action on $[n]$, given by identifying $[n]$ with the n vertices of the $n - 1$ simplex. Hence, it induces an action on B_n . This induced action on B_n is CCT.

Proof. The group of linear automorphisms in this case induces the usual action of S_n on B_n , because any permutation matrix defines a linear map on \mathbb{R}^n . However, we know the action of S_n on B_n is CCT from Example 4.2 \square

Proposition 4.19. Let G be the group of linear automorphisms of the n -octahedron, embedded inside \mathbb{R}^n , whose vertices are located at $\pm e_i$, where e_1, \dots, e_n are the standard basis vectors of \mathbb{R}^n . Then, the action of G on the octahedron induces an action of G on the $2n$ vertices of the octahedron, and hence on B_{2n} . This induced action on B_{2n} is common cover transitive.

Proof. It is simple to see that the group of linear automorphisms of the n -octahedron with vertices at $\pm e_i$, where e_1, \dots, e_n is the hyperoctahedral group, since it is generated by the permutation matrices, together with the matrix A with $A_{1,1} = -1, A_{i,i} = 1, A_{j,k} = 0$ where $i \neq 1, j \neq k$. (The hyperoctahedral group is commonly notated B_n , since it is the type B Coxeter group, but we do not use this here to avoid confusing with the boolean algebra.) It is well known that the hyperoctahedral group can be written as $S_2 \wr S_n$. Then, by Proposition 4.9, it follows that $S_2 \wr S_n$ is CCT. \square

4.2.3. Cover Transitive Actions of \mathbb{Z}_2^k . In this subsection, we show that any embedding of \mathbb{Z}_2^k into S_n defines an action on B_n which is CCT. In particular, this implies that the left multiplication action of \mathbb{Z}_2^k , is CCT. However, it turns out that this is the only class of groups for which the left multiplication action is CCT.

Lemma 4.20. Let n, k be arbitrary. Let $G = \mathbb{Z}_2^k$ and pick any group action $\phi : G \times [n] \rightarrow [n]$. Then the induced action $\phi : G \times B_n \rightarrow B_n$ is CCT.

Proof. First, observe that all elements $g \in G$ have order 2. Suppose $x < z, y < z, x = gy$. We will show $g \in \text{Stab}(z)$. By definition, this would imply that ϕ is common cover transitive.

Let a be the unique element such that $a \in y, a \notin x$. Certainly $g(y \setminus a) \subset gx = y \subset z$. So, to show $g \in \text{Stab}(z)$, it suffices to show that $ga \in x$. Let $b = g^{-1}a$. We know $b \in x$. However, since g has order 2, it must be that $ga = g^2b = b \in x$. Hence, $g \in \text{Stab}(z)$, and ϕ is CCT. \square

Corollary 4.21. Let \mathbb{Z}_2^k act on itself by left multiplication $\psi : \mathbb{Z}_2^k \times \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2^k, (g, h) \mapsto g \cdot h$. By identifying $\mathbb{Z}_2^k \cong [2^k]$ as sets, we obtain an induced action $\phi : \mathbb{Z}_2^k \times B_{2^k} \rightarrow B_{2^k}$ which is cover transitive.

Proof. By Lemma 4.20, any embedding of $\mathbb{Z}_2^k \rightarrow S_n$ defines a CCT action on B_n . So, this particular embedding defines a CCT action. \square

Proposition 4.22. Let G be a finite group, acting on itself by left multiplication, $\psi : G \times G \rightarrow G, (g, h) \mapsto g \cdot h$, with $|G| = n$. By identifying $G \cong [n]$ as sets, ψ induces an action $\phi : G \times B_n \rightarrow B_n$. Then, ϕ is cover transitive if and only if $G \cong \mathbb{Z}_2^k$ for some k .

Proof. If $G \cong \mathbb{Z}_2^k$, then by Corollary 4.21, we know the action $\phi : G \times B_{2^k} \rightarrow B_{2^k}$ is indeed CCT.

Next, we prove the converse. First, let us show $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$. The forward implication is obvious. To see the converse, first note that if $\forall g \in G, g^2 = e$, then G is abelian because $aba^{-1}b^{-1} = abab = (ab)^2 = e$. Then, G is an abelian group, all of whose elements have order two, the structure theorem of finite abelian groups tells us $G \cong \mathbb{Z}_2^k$.

So, Suppose $G \not\cong \mathbb{Z}_2^k$. Then, there exists $g \in G, g^2 \neq e$. Clearly $\{e\} < \{e \cup g\}, \{g\} < \{e \cup g\}, \{g\} \in G\{e\}$. So, to show the induced action $\phi : G \times B_n \rightarrow B_n$ is not cover transitive, it suffices to show there is no $h \in G, h \in \text{Stab}(\{e \cup g\}), h \cdot \{e\} = \{g\}$. If $h \in \text{Stab}(\{e \cup g\})$ then $h \cdot e = e$ or $h \cdot e = g$. In both cases, it is simple to see that $g^2 = e$. Therefore, there does not exist such an h and left multiplication is not CCT. \square

Remark 4.23. One nice way of restating the above results is that for G an abstract group, there exists some n and some embedding $G \subset S_n$ such that the induced action of G on B_n is not CCT if and only if $G \not\cong \mathbb{Z}_2^k$ for some k . If $G \cong \mathbb{Z}_2^k$, we have seen this holds in Lemma 4.20. If $G \not\cong \mathbb{Z}_2^k$, then by Proposition 4.22, the left multiplication action of G on $B_{|G|}$ defines an action which is not CCT.

5. WREATH PRODUCT OF TWO SYMMETRIC GROUPS

In this section, we prove a result similar to that of [?, Theorem 1.1]. We shall construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. It shall give an alternate proof of Theorem 1.1 in the case that $r = 1$.

Notation 5.1. We shall now show For this section, fix l, m with $n = l \cdot m$ and fix $G = S_m \wr S_l$. Let S_m act on B_m by the permutation representation, and then let G act on $B_m^l \cong B_{m \cdot l}$ by the action defined in Notation 4.6.

5.1. Recalling Pak and Panova's Result. We first review the necessary definitions and then state [?, Theorem 1.1]:

A *partition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *partition of n* , notated $\lambda \vdash n$. A *composition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$. That is, it is a partition where order matters. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *composition of n* . Let $P_n(l, m)$ denote the set of partitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, such that $\lambda_1 \leq m, k \leq l$. That is, $P_n(l, m)$ is the partitions which fit inside an $l \times m$ rectangle.

Notation 5.2. [?, Section 1] For λ a partition, let $\nu(\lambda)$ be the number of distinct nonzero part sizes of λ .

Notation 5.3. [?, Section 1] Let $p_k(l, m, r) = \sum_{\lambda \in P_k(l, m)} \binom{\nu(\lambda)}{r}$.

Theorem 5.4. [?, Theorem 1.1] *The sequence $p_r(l, m, r), p_{r+1}(l, m, r), \dots, p_{l \cdot m}(l, m, r)$ is unimodal and symmetric.*

5.2. The Poset $\mathcal{E}^r(B_n)/G$. Now that we have managed to state Pak and Panova's Theorem, we will begin the trek toward showing how the $p_1(l, m, 1)$ are actually ranks of $\mathcal{E}(B_n/G)$. The first step is to describe the Peck poset $\mathcal{E}(B_n)/G$. In particular, we shall obtain an explicit formula for the sizes of its ranks in Theorem 5.19. To do this, we will first describe representatives for the vertices of B_n/G and then analogous representatives for vertices of $\mathcal{E}(B_n)/G$.

Definition 5.5. For $x \subset [l \cdot m]$ we shall say x is *left justified* if whenever $a \in x$, such that $a \not\equiv 1 \pmod{l}$, then $a - 1 \in x$. In other words, when we pick out the boxes of the $l \times m$ rectangle which lie in x the resulting diagram is left justified.

Definition 5.6. For $x \subset [l \cdot m]$, define the *composition of x* , denoted $\text{comp}(x) = \lambda_1, \dots, \lambda_l$ where $\lambda_i = |\{a \in x : \cdot(m-1) < a \leq i \cdot m\}|$. That is, if we pick out the boxes of the $l \times m$ rectangle which lie in x , the composition is just the sequence of the number of boxes in each row. Similarly, define the *partition of x* , notated $\text{part}(x)$ as follows. Let $\pi \in S_l$ be a

permutation such that $\pi(i) \geq \pi(i+1)$ for all $i, 1 \leq i \leq l-1$. Then $\text{part}(x) = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(l)})$. That is, $\text{part}(x)$ is the $\text{comp}(x)$, written in decreasing order.

Lemma 5.7. *For any $g \in G, x \in B_n$, it follows that $\text{part}(x) = \text{part}(gx)$.*

Proof. It suffices to show this holds for any generator g . Then G is generated by elements which permute rows, and elements that swap rows. It is clear that if g only permutes elements in a single row, then $\text{comp}(gx) = \text{comp}(x)$, so in particular, $\text{part}(gx) = \text{part}(x)$. If g swaps two rows, then $\text{comp}(gx)$ is simply a reordering of the parts of $\text{comp}(x)$, and so again $\text{part}(gx) = \text{part}(x)$. \square

Definition 5.8. For $x \in [l \cdot m]$, say x is a *Young Diagram* if x is left justified and $\text{comp}(x)$ is a partition.

Lemma 5.9. *For each $x \in B_{l \cdot m}$ there exists a unique representative $z \in Gx$ such that z is a Young Diagram.*

Proof. Uniqueness is clear, because for any $g \in G$, by Claim 5.7, $\text{part}(gx) = \text{part}(x)$. So, to we only have to show there is some $g \in G$ for which gx is a Young Diagram. Indeed, first, choose $h_1 \in G$ so that h_1x is left justified. This can be done because every permutation of a single row in the $l \times m$ rectangle lies in the wreath product, so, we can take h_1 to be the product of the elements that left justify each individual row. Then, let h_2 be the element that swaps the rows of h_1x so that they increase going down. Finally, taking $g = h_2h_1$, it follows that gx is a Young diagram. \square

Notation 5.10. For $x \in B_n$ denote by \bar{x} the unique element in Gx such that \bar{x} is a Young Diagram.

1

Now that representatives for each G orbit in B_n have been described, we shall move on to describing representatives for each G orbit in $\mathcal{E}(B_n)$.

Notation 5.11. For $\lambda = (\lambda_1, \dots, \lambda_k)$ a partition, introduce the alternate notation $\lambda = a_1^{b_1} \dots a_s^{b_s}$ if the first b_1 parts of λ are equal to a_1 , the next b_2 parts of λ are equal to a_2 , and in general, for $1 \leq h \leq s$, the parts $\lambda_{h+\sum_{i=1}^{h-1} b_i}$ are all equal to a_h . Furthermore, all a_j must be distinct.

Lemma 5.12. *If $(x, y), (w, y) \in \mathcal{E}^r(B_n)_i$ then $(x, y) \in G(w, y)$ if and only if for all $g \in G$ such that gy is a Young diagram, we have $(gx, gy) \in G(gw, gy)$. In particular, if $gy = \bar{y}$ then $(x, y) \in G(w, y)$ if and only if $(gx, \bar{y}) \in (gw, \bar{y})$.*

Proof. Clearly $g(x, y) \in G(g(w, y))$ if and only if $(x, y) \in G(w, y)$. \square

The point of the proceeding Lemma is that in order to determine when two elements are identified, we may assume that y is a young diagram.

Notation 5.13. For y a Young Diagram with $\text{part}(y) = \text{comp}(y) = a_1^{b_1} \dots a_s^{b_s}$, and $x \subset y$, for all $i, 1 \leq i \leq s$, define y_i to be the $a_i \times b_i$ rectangle consisting of the rows $1 + \sum_{j=1}^{i-1} b_j, 2 + \sum_{j=1}^{i-1} b_j, \dots, \sum_{j=1}^i b_j$. That is, y_i is just the rectangle of all parts of y which are of length a_i .

Proposition 5.14. *If y is a Young Diagram so that $\text{part}(y) = \text{comp}(y) = a_1^{b_1} \dots a_s^{b_s}$, and $(x, y), (w, y) \in \mathcal{E}^r(B_n)_j$ then $(x, y) \in G(w, y)$ if and only if for all $i, 1 \leq i \leq s$, it holds that $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$.*

Proof. First, suppose for all $i, 1 \leq i \leq s$, that $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$. We know both $x \subset y, w \subset y$. Further, since y_i is a rectangle, there exists some $g \in \text{Stab}(y_i)$ so that $gx = w$. Then, for each i , there exists some $g_{1,i} \in \text{Stab}(y_i)$ so $g_{1,i}(x \cap y_i) = \overline{x \cap y_i}$, which only interchange elements in y_i . Similarly, there exists $g_{2,i} \in \text{Stab}(y_i)$, $g_{2,i}(w \cap y_i) = \overline{w \cap y_i}$. However, the assumption $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$ precisely means $\overline{x \cap y_i} = \overline{w \cap y_i}$. Therefore, $g_{2,i}^{-1}g_{1,i} \in \text{Stab}(y)$ and $g_{2,i}^{-1}g_{2,i}(x \cap y_i) = (w \cap y_i)$. Applying this same procedure for

all i and multiplying the corresponding group elements together gives an element $g \in \text{Stab}(y)$ with $gx = w$. Therefore, $g(x, y) = (gx, gy) = (gx, y) = (w, y)$, so $(x, y) \in G(w, y)$.

Conversely, note that $(x, y) \in G(w, y)$ is equivalent to the existence of a $g \in \text{Stab}(y)$ with $gx = w$. However, any $g \in \text{Stab}(y)$ can only interchange rows of the same length. Therefore, we obtain the stronger result that we can write $g = h_1 \cdots h_s$ where $h_i \in \text{Stab}(y_i)$, and $h_i(p) = p$ for all $p \notin y_i$. Then, it follows that $h_i(x \cap y_i) = w \cap y_i$ and so $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$, as claimed. \square

Remark 5.15. So, one way of viewing G orbits of an element $(x, y) \in \mathcal{E}^r(B_n)$ is as “outer” Young Diagrams, made up of a sequence of rectangles stacked on top of one another, each one wider than the next, which form the Young diagram \bar{y} . Then, to each such rectangle we associate an “inner” Young Diagram. The inner young diagram corresponds to the elements in that rectangle in y but not in x . Two elements are in the same G orbit if and only if their “outer” Young Diagram and “inner” Young Diagrams are all the same.

The next step is to give explicitly formulas for the ranks of $\mathcal{E}^r(B_n)$.

Notation 5.16. For $p(x) = \sum_{i=0}^N c_i x^i$, a polynomial, define the notation $[p(x)]_r = c_r$.

Now, we briefly introduce notation for q binomial coefficients, so that we can state the next propositions. Let $q \in \mathbb{R}$ and let $[n]_q = \sum_{i=0}^{n-1} q^i$. Then, denote $[n]_q! = \prod_{i=1}^n [i]_q$. Finally, let $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$.

Notation 5.17. For $\lambda = (a_1^{b_1} \cdots a_s^{b_s})$ a partition, denote $\eta(\lambda, r) = \left[\prod_{i=1}^s \binom{a_i + b_i}{a_i}_q \right]_r$.

Proposition 5.18. [?, Proposition 1.3.19] *There is an equality $[(\frac{a+b}{b})_q]_j = |P_j(l, m)|$.*

Theorem 5.19. *The sizes of the ranks of $\mathcal{E}^r(B_n)/G$ can be written as*

$$|(\mathcal{E}^r(B_n)/G)_i| = \sum_{\lambda \in P_i(l, m)} \eta(\lambda, r).$$

Proof. By Proposition 5.14 each orbit (x, y) has a representative such that y is a Young Diagram, with $\text{part}(y) = \prod_{i=1}^s a_i^{b_i}$, and each $x \cap y_i$ is a Young Diagram. Therefore, for a fixed y , the number of orbits (x, y) with $|x| + r = |y|$ is exactly determined by the Young Diagrams of $\overline{x \cap y_i}$, for $1 \leq i \leq s$. Equivalently, it is determined by the Young Diagrams $[y_i \setminus x]$, for $1 \leq i \leq s$. So, we wish to calculate the number of tuples of Young Diagrams of the form $(y_1 \setminus x, y_2 \setminus x, \dots, y_s \setminus x)$ so that $\sum_{i=1}^s |y_i \setminus x| = r$. Now, define j_i by $j_i = |y_i \setminus x|$, still of course with, $\sum_{i=1}^s j_i = r$. Then, by Proposition 5.18, the number of such partitions is $[(\frac{a_i + b_i}{b_i})_q]_{j_i}$. Therefore, the number of elements $G(x, y)$ with $j_i = |y_i \setminus x|$, is simply the product $\prod_{i=1}^s [(\frac{a_i + b_i}{b_i})_q]_{j_i}$. Therefore, since j_i were chosen arbitrarily, only subject to the constraint that $\sum_{i=1}^s j_i = r$, It follows that the total number of elements $G(x, y)$, for y fixed, is equal to

$$\sum_{\substack{(j_1, \dots, j_s), \\ \sum_{i=1}^s j_i = r}} \prod_{i=1}^s \left[\binom{a_i + b_i}{b_i}_q \right]_{j_i} = \left[\prod_{i=1}^s \binom{a_i + b_i}{b_i}_q \right]_r = \eta(\lambda, r).$$

Then, summing this over all Young Diagrams y , gives that

$$|\mathcal{E}^r(B_n)/G|_i = \sum_{\lambda \in P_i(l, m)} \left[\prod_{i=1}^s \binom{a_i + b_i}{b_i}_q \right]_r = \sum_{\lambda \in P_i(l, m)} \eta(\lambda, r).$$

\square

5.3. Ensuing Conclusions. We now reap the fruits of our labor in order to draw some nice results.

Corollary 5.20. *The sequence $\sum_{\lambda \in P_r(l,m)} \eta(\lambda, r), \dots, \sum_{\lambda \in P_n(l,m)} \eta(\lambda, r)$, is the sequence of ranks of the Peck poset $\mathcal{E}^r(B_{l,m})/G$. In particular, the sequence is symmetric and unimodal.*

Proof. By Theorem 5.19, $\sum_{\lambda \in P_i(l,m)} \eta(\lambda, r)$ is the size of the $i - r$ th rank of the Peck poset $\mathcal{E}^r(B_{l,m})/G$. Since Peck posets are unimodal and symmetric, this sequence is as well. \square

We now obtain an easy proof of Theorem 5.4 for the case $r = 1$. Using the fact that $S_m \wr S_l$ is CCT. Namely,

Corollary 5.21. *The sequence $p_1(l, m, 1), p_{r+1}(l, m, 1), \dots, p_{l-m}(l, m, 1)$ is the rank generating function of $\mathcal{E}(B_{l,m})/G$. In particular, the sequence is unimodal and symmetric.*

Proof. First, we know $F^1(B_{l,m})/G$ is Peck, since it is the quotient of a unitary Peck poset by a group action. However, it is easy to see that $\eta(\lambda, 1) = \nu(\lambda)$, since both count the number Young Diagrams of size $|\lambda| - 1$ fitting inside the Young Diagram corresponding to λ . Therefore, $\sum_{\lambda \in P_i(l,m)} \eta(\lambda, r) = \sum_{\lambda \in P_i(l,m)} \nu(\lambda) = p_i(l, m, 1)$. However, by Corollary 5.20, $\sum_{\lambda \in P_r(l,m)} \eta(\lambda, r), \dots, \sum_{\lambda \in P_n(l,m)} \eta(\lambda, r)$, are the ranks of $\mathcal{E}(B_{l,m})/G$. Therefore, $p_i(l, m, 1)$ are also the ranks of $\mathcal{E}(B_{l,m})/G$, hence symmetric and unimodal. \square

6. RELATIONS TO POLYA THEORY

In this section, we obtain summation formulas for the ranks of $\mathcal{E}^{\vec{r}}(B_n)/G$, in the case that the action of G on $\mathcal{E}^{\vec{r}}$ is induced by some action of G on $[n]$. These formulas are given by certain polynomials in Polya theory. In particular, we obtain polynomials from Polya theory which are the rank generating functions of $\mathcal{E}^{\vec{r}}(B_n)/G$, which implies that these polynomials have symmetric, unimodal coefficients. The result we prove is a generalization of [?, Corollary 7.16].

6.1. A Brief Review of Polya Theory. We shall follow the treatment from [?, Chapter 7]. First, we build up some definitions to state Polya's Theorem.

Definition 6.1. Let $G \subset S_n$ act on $[n]$ by the restriction of the action of S_n on $[n]$ by all permutations. For $\pi \in G$, the action of π on $[n]$ can be written in cycle notation so that there are c_i cycles of length i . Define the *cycle indicator* of π to be the monomial $Z_\pi(z_1, \dots, z_n) = z_1^{c_1} z_2^{c_2} \dots z_n^{c_n}$.

Definition 6.2. The *cycle indicator* for a group $G \subset S_n$, is the polynomial

$$Z_G(z_1, \dots, z_k) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi(z_1, \dots, z_k).$$

Definition 6.3. A *coloring* of a set S by the colors $R = \{r_1, \dots, r_k\}$ is a map $S \rightarrow R$. Heuristically, a coloring of S can be thought of as an assignment of a “color” from the set R to each of the elements of S .

Notation 6.4. For A, B sets, let $A^B = \text{Hom}_{\text{sets}}(B, A)$. Let $G \subset S_n$ act on B_n . Then, G acts on $R^{B_n} = \text{Hom}_{\text{sets}}(B_n, R)$ by $g \cdot f(S) = f(g \cdot S)$ where $g \in G, S \in B_n, f \in R^{B_n}$. Then, let R^{B_n}/G denote the quotient of R^{B_n} by the action of G defined above.

Suppose $R = \{r_1, \dots, r_k\}$, and define the map $c : R^S \rightarrow \mathbb{Z}^k$, by sending $f \in R^S$ to $c(f) = (i_1, \dots, i_k)$ if $|\{s \in S : f(s) = r_j\}| = i_j$ for $1 \leq j \leq k$. Observe that if $f \in Gh$ then $c(f) = c(h)$, so the map *color* descends to a map on G orbits.

Define

$$\kappa(i_1, \dots, i_k) = |\{Gf \in R^{B_n}/G \text{ such that } c(f) = (i_1, \dots, i_k)\}|.$$

Finally, let

$$F_G(r_1, \dots, r_k) = \sum_{i_1, \dots, i_k} \kappa(i_1, \dots, i_k) r_1^{i_1} \dots r_k^{i_k}.$$

Remark 6.5. The above notation may seem extremely cumbersome. It is simply a formal way of saying that $\kappa(i_1, \dots, i_k)$ is the number of inequivalent colorings of subsets of B_n , under the G action. Once again, see [?, Chapter 7] for a more lengthy exposition.

Theorem 6.6. (*Polya's Theorem*) Let $G \subset S_n$ act on B_n . With F_G, Z_G as defined above, the following equality holds.

$$F_G(r_1, \dots, r_k) = Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right).$$

6.2. A Rank Generating Function. Next, we produce rank generating functions for $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$. The reason for starting at i_2 and ending at i_{k-1} is that we shall include additional numbers i_1 and i_k to be colored by the first color and the k th color. Given an element $x_2, \dots, x_k \in \mathcal{E}^{i_2, \dots, i_{k-1}}$, the first color will be used to color the elements in x_2 and the k th color will be used to color the elements in $[n] \setminus x_k$.

Notation 6.7. Let $[f(x_1, \dots, x_k)]_{x_{j_1}^{i_1} \dots x_{j_l}^{i_l}}$ denote the coefficient of $x_{j_1}^{i_1} \dots x_{j_l}^{i_l}$ in $f(x_1, \dots, x_k)$, which may itself be a polynomial.

Lemma 6.8. The number $\kappa(i_1, \dots, i_k)$ is equal to $|(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|$. Consequently,

$$\left[Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

Proof. Define a map $m : (\mathcal{E}^{r_2, \dots, r_{k-1}}(B_n))_{i_1} \rightarrow \{f \in R^{B_n} : c(f) = (i_1, \dots, i_k)\}$ as follows. For $(x_2, \dots, x_k) \in \mathcal{E}^{r_2, \dots, r_{k-1}}(B_n)$, let $m(x_2, \dots, x_k) = f$, where

$$f(t) = \begin{cases} r_1 & \text{if } t \in x_2 \\ r_j & \text{if } t \in x_{j+1} \setminus x_j \\ r_k & \text{if } t \notin x_k \end{cases}.$$

Observe that m is in fact a bijection, as we can easily define an inverse map by sending a coloring f to (x_2, \dots, x_k) , where x_j is the set of all elements $t \in [n]$ for which there exists $l < j$ with $f(t) = r_l$. Next, the two group actions were defined so that so that,

$$g(x_1, \dots, x_{k-1}) = (y_2, \dots, y_{k-1}) \Leftrightarrow g \cdot m(x_1, \dots, x_{k-1}) = m(y_2, \dots, y_{k-1})$$

Therefore, m descends to a bijection $m^G : (\mathcal{E}^{r_2, \dots, r_{k-1}}(B_n))_{i_1}/G \rightarrow R^{B_n}/G$. This implies $\kappa(i_1, \dots, i_k)$ is equal to $|(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|$.

Then, by Poly's Theorem 6.6,

$$\left[Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{r_1^{i_1} \dots r_k^{i_k}} = \kappa(i_1, \dots, i_k) = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

□

Now we arrive at our main result of the section. The following result provides a rank generating function for $\mathcal{E}^{\vec{r}}(B_n)/G$.

Proposition 6.9. Let $z_j = \sum_{j=1}^k r_j^j$. Let

$$Z_G^{i_2, \dots, i_{k-1}}(r_1) = [Z_G(z_1, \dots, z_{k-1}, 1)]_{r_2^{i_2} \dots r_{k-1}^{i_{k-1}}} = \sum_t c_t r_1^t,$$

be a polynomial in r_1 . Then, the coefficients c_t are the ranks of the Peck poset $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$. In particular, the coefficients form a symmetric, unimodal sequence.

Proof. In Lemma 6.8, we saw

$$[Z_G(z_1, \dots, z_k)]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

However, since $\sum_{j=1}^k i_j = n$, it follows that i_k is determined by the numbers i_1, \dots, i_{k-1} , and so

$$[Z_G(z_1, \dots, 1)]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

Next, write $Z_G^{i_2, \dots, i_{k-1}}(r_1) = \sum_t c_t(r_1)^t$. We have just shown that $c_t = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|$. Since $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$ is a Peck poset by Lemma 3.25, its rank sizes form a symmetric and unimodal sequence. Therefore, the coefficients of $Z_G^{i_2, \dots, i_{k-1}}(r_1)$ form a symmetric, unimodal sequence. \square

Remark 6.10. Note that in the case $\vec{r} = 0$, the above result precisely becomes [?, Corollary 7.16].

6.3. Further Identities for the Wreath Product. In this section, we draw on the methods developed earlier in this section to write down some interesting generating functions for the case that $G = S_m \wr S_l$. First, we shall use Proposition 6.9 to obtain an explicit generating function for $p_i(l, m, 1)$, and then we shall relate the sum $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}$ to counting certain types of set partitions, which also count $|(\mathcal{E}^{\vec{r}}(B_n)/G)_0|$, where \vec{r} is of the form $\vec{r} = 1, \dots, 1$.

Notation 6.11. For this section only, fix $m, l \in \mathbb{N}$ and fix $G = S_m \wr S_l$. Additionally, fix $n = m \cdot l$.

6.3.1. A Generating Function for $p_i(l, m, 1)$.

Proposition 6.12. *Let c_i be the number of i cycles in π and define*

$$W_\pi(z_1, \dots, z_n) = \begin{cases} \frac{Z_\pi(z_1, \dots, z_n)}{z_1} & \text{if } c_1 > 0 \\ 0 & \text{if } c_1 = 0 \end{cases}.$$

Then, there is an equality

$$\sum_{i=0}^n p_i(l, m, 1) r_1^i r_3^{n-i-1} = \sum_{\pi \in G} |\text{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

Proof. Recall from Corollary 5.21 that $p_i(l, m, 1)$ is the rank generating function of $\mathcal{E}(B_{l \cdot m})/G$.

As was seen in Proposition 6.9, it is also the case that

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}$$

is the rank generating function of $|\mathcal{E}^1(B_n)/G|$. Therefore,

$$p_i(l, m, 1) = [Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}.$$

So, to complete the proof, it suffices to show

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2} = \sum_{\pi \in G} |\text{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To show this, it further suffices to show that for all $\pi \in G$,

$$Z_\pi(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)_{r_2} = |\text{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To see, we may note that r_2 must have a nonzero coefficient in the expansion of r_2 in Z_π . First, if r_2 has a nonzero coefficient, π must have some 1-cycle, as otherwise, no variable could appear in the expansion of Z_π raised only to the first power. Second, if π has some 1-cycle, then the coefficient of r_2 in

$$Z_\pi(r_1 + r_2 + r_3, r_1^2 + r_2^2 + r_3^2, \dots, r_1^n + r_2^n + r_3^n) = \prod_{i=1}^n (r_1^i + r_2^i + r_3^i)^{c_i}$$

is precisely

$$c_1 \cdot W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n) = |\text{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n)$$

because c_1 is the number of 1-cycles in π , which is by definition the number of fixed points of π . This is exactly what we wanted to show. \square

6.3.2. Bounded Partition Sizes. This subsubsection is an application of Polya theory, and is moderately tangential to the rest of the paper. However, it relates to the functor $\mathcal{E}^{\vec{\tau}}$, in that it counts the $|(\mathcal{E}^{\vec{\tau}}(B_n)/G)_0| = |(\mathcal{E}^{\vec{s}}(B_n)/G)_1|$, as described in Remark 6.16.

Notation 6.13. Let $P_{[t]}[l, m]$ denote the set of partitions of the set $[t]$ into at most l blocks, such that each block has size at most m . Let $P[l, m] = \cup_{t \in \mathbb{N}} P_{[t]}[l, m]$. Denote $p_{[t]}[l, m] = |P_{[t]}[l, m]|$, and $p[l, m] = |P[l, m]|$.

Remark 6.14. The numbers $p_{[t]}[l, m]$ satisfy the recursive formula

$$p_{[t]}[l, m] = \sum_{k=1}^l \binom{t-1}{k-1} p_{[t-k]}[l-1, m].$$

This formula can be shown by splitting up the set $P_{[t]}[l, m]$ based on how many elements appear in the first row. The reason for including $t-1$ instead of t is that 1 must always appear in the first row.

Proposition 6.15. *There is an equality $p_{[t]}[l, m] = \frac{t!}{l!(m!)^l} \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}$.*

Proof. Using Polya's Theorem 6.6, we know

$$F_G(r_1, \dots, r_k) = \frac{1}{|G|} Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right).$$

In particular,

$$[F_G(r_1, \dots, r_k)]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = \left[\frac{1}{|G|} Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}}.$$

Of course, by definition of F_G , we know

$$[F_G(r_1, \dots, r_k)]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = \kappa(1, 1, \dots, 1, m \cdot l - t, 0, \dots, 0)$$

where there are t 1's in the above expression. By definition, κ is just the number of inequivalent ways to distinctly color t numbers in an $l \times m$ rectangle, up to the action of $G = S_m \wr S_l$. For any such coloring, let square a_i be colored with color r_i . Each G orbit of colorings has a unique representative such that the colors are sorted in increasing order along each row, and in increasing order down the first column. Then, the colorings defined above are in bijection with partitions of $[t]$ so that this partition has at most l blocks, and each block has at most m elements. The set of such partitions is exactly $P_{[t]}[l, m]$. Therefore, the number of such partitions is $p_{[t]}[l, m]$. Then, it follows that

$$p_{[t]}[l, m] = \kappa(1, 1, \dots, 1, m \cdot l - t, 0, \dots, 0) = \left[\frac{1}{|G|} Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}}.$$

Of course, $\frac{1}{|G|} = \frac{1}{l!(m!)^l}$ when $G = S_m \wr S_l$. So, to complete the proof, we just need to show that

$$(6.1) \quad \left[Z_G \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = t! \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}.$$

We reduce this problem to showing Z_G it is the sum over all the cycle indicators $\sum_{\pi \in G} Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)$. So, to show Equation (6.1) holds, it suffices to show that

$$(6.2) \quad \left[Z_\pi \left(\sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m-l-t}} = t! \binom{\text{Fix}(\pi)}{t}$$

This is now apparent, because $Z_\pi = \prod_{i < m \cdot l} (\sum_j r_j^i)^{c_i}$, where c_i is the number of i cycles in π . The only way we can obtain a monomial of the form $r_1 r_2 \dots r_t r_{t+1}^{m-l-t}$ is if $c_1 \geq t$. Then, the coefficient of such a term will exactly be the number of ways to choose an ordered set of t elements from c_1 terms. That is, it is precisely $t! \binom{c_1}{t}$. However, c_1 is the number of 1-cycles, that is $c_1 = |\text{Fix}(\pi)|$. Hence, (6.2) holds. \square

Remark 6.16. Thanks to Proposition 6.9, an equivalent way to state the above Proposition 6.15 is that $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} = |(\mathcal{E}^{\vec{r}}(B_n)/G)_0|$, if \vec{r} is the vector consisting of t ones. Also, letting \vec{s} be the vector with $t-1$ ones, we may note $|(\mathcal{E}^{\vec{r}}(B_n)/G)_0| = |(\mathcal{E}^{\vec{s}}(B_n)/G)_1|$, since one may think of elements of $\mathcal{E}^{\vec{s}}(B_n)/G$ as tuples of $t-1$ elements, each contained in the next, whose lowest element is of rank 1, and one may think of elements of $(\mathcal{E}^{\vec{r}}(B_n)/G)_0$ as tuples of t elements, each contained in the next, whose lowest element is of rank 0. There is an obvious bijection between these two sets, given by adding or removing the element \emptyset or rank 0 in B_n . Therefore, we also obtain $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} = |(\mathcal{E}^{\vec{s}}(B_n)/G)_1|$.

We can also obtain a formula for the size of the whole set $p[l, m]$, by simply summing over all possible values of t .

Proposition 6.17. Define the function $f : \mathbb{N} \cup 0 \rightarrow \mathbb{N}$, by

$$f(x) = \begin{cases} \lfloor e \cdot n! \rfloor & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

Then,

$$p[l, m] = \frac{1}{l!(m!)^l} \sum_{\pi \in G} f(|\text{Fix}(\pi)|).$$

Proof. First, note that $\sum_{i=1}^k i! \binom{k}{i} = f(k)$. This can be seen fairly easily, because $\sum_{i=1}^k i! \binom{k}{i} = k! \sum_{i=0}^k \frac{1}{i!}$, while $k! \cdot e = k! \sum_{i=0}^{\infty} \frac{1}{i!}$. For $k > 1$, it is easy to bound the difference $\sum_{i=k+1}^{\infty} \frac{1}{i!} < \frac{1}{k!}$, which implies $\sum_{i=1}^k i! \binom{k}{i} = \lfloor k! \cdot e \rfloor = f(k)$ for $k > 1$.

By definition, $p[l, m] = \sum_{t=0}^{l \cdot m} p_{[t]}(l, m)$. Therefore, by Proposition 6.15,

$$\begin{aligned} p[l, m] &= \sum_{t=0}^{l \cdot m} p_{[t]}(l, m) \\ &= \sum_{t=0}^{l \cdot m} \frac{t!}{l!(m!)^l} \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{t=0}^{l \cdot m} \sum_{\pi \in G} t! \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} \sum_{t=0}^{l \cdot m} t! \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} f(|\text{Fix}(\pi)|). \end{aligned}$$

\square

7. UNIMODAL QUOTIENTS

In this section, show that several important posets $\mathcal{E}(B_n/G)$ are rank symmetric and rank unimodal. First, we examine left multiplication actions, showing that for $|G| = n$, the action of G on itself by left multiplication induces an action of G on B_n such that $|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}$. Using this result, we show that the left multiplication action of the cyclic group, C_n on itself, which can also be thought of as C_n acting on the vertices of an n -gon by rotation, induces an action on B_n such that $\mathcal{E}(B_n/C_n)$ is rank unimodal and symmetric. Then, letting the dihedral group act on the vertices of an n -gon by rotations and reflections, we obtain an induced action on B_n so that $\mathcal{E}(B_n/D_{2n})$ is rank unimodal and symmetric. Furthermore, in the case that $n = p, 2p$ for p a prime, the action of D_{2n} on B_n is CCT. Finally, we develop a q -analog the cyclic group $C_n(q)$ acting on the q -analog of the boolean algebra, $B_n(q)$ and show $|(\mathcal{E}(B_n(q)/C_n(q)))_i| = \binom{n-1}{i}_q$.

7.1. Groups of order n . In this subsection, we shall show that for any group G such that $|G| = n$, with a transitive action on B_n , then

$$|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}.$$

The most important example of such an action is the left multiplication action

$$\phi : G \times G \rightarrow G, (g, h) \mapsto g \cdot h.$$

By identifying $G \cong [n]$ as sets, this induces an action of G on B_n . We can then use a simple result in group theory to calculate the ranks of $\mathcal{E}(B_n)/G$ for all transitive actions $\phi : G \times P \rightarrow P$, where G is abelian.

For the rest of this section, fix a group G with $|G| = n$, and a transitive action of G on $[n]$, which induces an action of G on B_n .

Lemma 7.1. *For G a group acting transitively on $[n]$, it follows that $\text{Stab}(x) = \{e\}$.*

Proof. Since G acts transitively on n we have the number of orbits $|[n]/G| = 1$. Therefore, using Burnside's Lemma, together with the observation that $e \in \text{Stab}(x)$,

$$|G| = \sum_{x \in X} |\text{Stab}(x)| \geq \sum_{x \in X} 1 = |X| = |G|.$$

Then, the above string of inequalities tells us that we must have $|\text{Stab}(x)| = 1$ for all x , or in other words, $\text{Stab}(x) = \{e\}$. \square

Proposition 7.2. *For any element $(x, y) \in \mathcal{E}(B_n)$, then $\text{Stab}(x, y) = e$.*

Proof. From Lemma 7.1 for any $x \in B_n$ with $|x| = 1$, we have $\text{Stab}(x) = \{e\}$. Then, by definition $\text{Stab}(x, y) = \text{Stab}(x) \cap \text{Stab}(y) = \text{Stab}(x) \cap \text{Stab}(y \setminus x)$. However, $\text{Stab}(y \setminus x)$ is by assumption a 1 element set, because $|y| = |x| + 1, x \subset y$. This implies that $\text{Stab}(x, y) = \text{Stab}(x) \cap \text{Stab}(y \setminus x) = \text{Stab}(x) \cap \{e\} = \{e\}$, for all $(x, y) \in \mathcal{E}(B_n)$, as claimed. \square

Lemma 7.3. *The number $|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}$.*

Proof. Since there are $\binom{n}{i}$ elements $x \in (B_n)_i$, each with $n - i$ elements $y \in (B_n)_{i+1}$ with $x \subset y$, there are a total of $\binom{n}{i} \cdot (n - i)$ elements $(x, y) \in \mathcal{E}^r(B_n)_i$. By Proposition 7.2, we have $|\text{Stab}(x, y)| = 1$. Therefore, by the Orbit Stabilizer Theorem, it follows $|G(x, y)| = n$. So, all orbits have the same size. From this,

$$|\mathcal{E}(B_n)_i/G| = |\mathcal{E}(B_n)_i|/|G| = \frac{\binom{n}{i} \cdot (n - i)}{n} = \binom{n-1}{i}.$$

\square

Lemma 7.4. *For G a group let $H \subset G$ be the subgroup such that $hx = x$ for all $x \in X, h \in H$. Then, H is a normal subgroup of G and if $|G/H| = n$, then $\mathcal{E}(B_n)/G = \mathcal{E}(B_n)/(G/H)$.*

Proof. First, we check H is normal. For any $g \in G, x \in x$ it follows $ghg^{-1}x = gh(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = x$, and so $ghg^{-1} \in H$. Hence, G/H is a group.

Then, observe $Gx = (G/H)x$ because H acts trivially. Therefore, $\mathcal{E}(B_n)/G \cong \mathcal{E}(B_n)/(G/H)$, since $G, G/H$ have exactly the same orbits. \square

Lemma 7.5. *All faithful transitive abelian groups A acting on the set $[n]$ have $|A| = n$.*

Proof. This is a well known, easy to prove, result. See for instance [?] \square

Corollary 7.6. *Any abelian group A which acts transitively on $[n]$ has*

$$|(\mathcal{E}(B_n)/A)_i| = \binom{n-1}{i}.$$

Proof. By Lemma 7.4, we can assume A acts faithfully as well. By Lemma 7.5, $|A| = n$.

Then, by Lemma 7.3, $|(\mathcal{E}(B_n)/A)_i| = \binom{n-1}{i}$. \square

8. RANK UNIMODALITY FOR NON-CCT ACTIONS

8.1. Quotient by the Cyclic Group. In this section, we show that for $G = C_n$ the cyclic group of order n , the poset $\mathcal{E}(B_n/G)$ is rank symmetric and rank unimodal. The rank symmetric part is obvious. From Corollary 7.6, we know that for $G = C_n$, the size of the i^{th} rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|(\mathcal{E}(B_n)/C_n)_i| = \binom{n-1}{i}.$$

We will bound $|(\mathcal{E}(B_n/C_n))_i|$ by $|(\mathcal{E}(B_n)/C_n)_i|$ to obtain the unimodality.

Notation 8.1. We fix an arbitrary n to start with, set

$$Q(n) = \mathcal{E}(B_n)/C_n, \quad P(n) = \mathcal{E}(B_n/C_n)$$

and let

$$q(i, n) = |Q(n)_i|, \quad p(i, n) = |P(n)_i|.$$

When there is no ambiguity of which n we are referring to, we denote $Q(n), P(n)$ respectively by Q, P , and write $q_i = q(i, n), p_i = p(i, n)$.

The quotient B_n/C_n is the well-studied necklace poset. The elements in the poset are represented by n filled or empty beads ordered cyclically. Label the positions of the beads in the necklace by $1, 2, \dots, n$. More specifically, since each element $x \in B_n$ is represented by a sequence of empty or filled beads, an element $G(x) \in B_n/C_n$ is represented by such a sequence up to rotational equivalence. Similarly, an element $(Gx, Gy) \in (\mathcal{E}(B_n)/C_n)_i$ where $x \leq y$ can be regarded as a necklace where $i+1$ beads out of n are filled (which represent y), and 1 of the filled bead is distinct from all others (so the other i beads represent x).

Notation 8.2. We fix $\sigma_0 = (1 \ 2 \ \dots \ n)$, the generator of the group C_n .

Notation 8.3. We shall abuse notation by writing (Gx, Gy) in place of $(Gx, Gy) \cap \mathcal{E}(B_n)$ as sets, and hence, viewing $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}(B_n) \subset \mathcal{E}(B_n)$. Namely,

$$(Gx, Gy) = \{(\sigma x, \tau y) \in \mathcal{E}(B_n) : \sigma, \tau \in G; \sigma x \leq \tau y\}.$$

Similarly, we view

$$G(x, y) = \{(gx, gy) \in \mathcal{E}(B_n) : g \in G\}$$

Definition 8.4. Let $\sigma \in G$. We call each subset $s \subset x \in B_n$ that is fixed by σ a *full cycle* (under the action of σ). Let \mathcal{S} be the set of all full cycles in x , we call $x \setminus (\cup_{s \in \mathcal{S}} s)$ a *tail cycle* (under the action of σ).

Lemma 8.5. *Suppose that $\sigma_0^r x \leq y$ for some (x, y) , where $x \leq y$ and $r < n/2$. Also suppose that there is no $g \in G$ such that $gx = \sigma_0^r(x)$ and $gy = y$. Then r is the only integer $1 \leq r < n/2$ such that $\sigma_0^r(x) \leq y$.*

Proof. By assumption, x has a tail cycle under the action of σ_0^r , for otherwise x is fixed by σ_0^r , because then $g = id$ satisfies $gx = \sigma_0^r x$, $gy = y$. Additionally, y must have some tail cycle under the action of σ_0^r , for otherwise y is fixed under σ_0^r and we can take $g = \sigma_0^r$. Call γ the tail cycle of y , so γ consists of some filled beads in the necklace, one of which is the element $z = y \setminus x$. Let σ be any rotation such that $\sigma x \leq y$. then σ has to take a full cycle of x (under σ_0^r) to another full cycle (under σ_0^r), so $\sigma(\gamma \setminus z) \leq \gamma$. Recall that $\sigma_0^r(\gamma \setminus z) \leq \gamma$ by assumption, so the elements in γ are r -positions away from each other, namely $\gamma = \{z, z-r, z-2r, \dots, z-lr\}$, but $z+r \notin \gamma$, since γ a tail cycle. Since $\sigma(\gamma \setminus z) \leq \gamma$, there is some m for which $\sigma(z-mr) = z$. But then, $z - (m-1)r \in x$, and $\sigma(z - (m-1)r) = z+r \notin y$, a contradiction. \square

Lemma 8.6. *Let $G = C_n$, and $q_i = q(i, n)$, $p_i = p(i, n)$ as usual, then*

$$q_i - p_i = |\{G(x, y) : \exists \sigma_0^r x \leq y \text{ where } r < n/2, \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^r x, gy = y\}|.$$

Proof. From definition, q_i counts number of the disjoint subsets $G(x, y)$ in $\mathcal{E}(B_n)$ and p_i counts the number of disjoint subsets (Gx, Gy) . Hence difference of $q_i - p_i$ is counting the difference of the number of G -orbits $G(x, y)$ and the number of $(Gx, Gy) \in \mathcal{E}(B_n/C_n)$. Note that as subsets of $\mathcal{E}(B_n)$, $G(x, y) \subset (Gx, Gy)$, so the problem of counting $q_i - p_i$ is the same as that of counting the number of distinct G -orbits $G(x, y)$ with non-identity $\sigma \in G$ such that $\sigma x < y$.

It is clear that the G -orbits correspond to the (rotational equivalence classes of) fillings of the necklaces, where we fill $i+1$ elements with 1 of them being marked as the distinct element. Lemma 8.5 tells us that each subset $(Gx, Gy) \subset \mathcal{E}(B_n)$, contains at most two G -orbits (also regarded as subsets of $\mathcal{E}(B_n)$). The number of (Gx, Gy) containing exactly two G orbits is precisely the number of pairs of G -orbits $(G(x, y), G(\sigma x, y))$ (where $x \leq y$ and $\sigma x < y$) with $G(x, y) \neq G(\sigma x, y)$. The number of such pairs is then the number of $G(x, y)$ such that $\exists r, r < \frac{n}{2}, G(\sigma_0^r x, y) \neq G(x, y), \sigma_0^r x < y$. We restrict $r < n/2$ to avoid double counting pairs. This proves the Lemma. \square

For the unimodality of p_i , we first consider a special case where $n = p$ is a prime, for which we have a simple formula for p_i based on the formula for q_i .

Lemma 8.7. *Let $G = C_p$ where p is a prime, then $q_i - p_i = (p-1)/2$ for $1 \leq i \leq p-2$.*

Proof. By Lemma 8.6, we need to prove that the number of G -orbits $G(x, y)$ such that there is some $r < n/2$ with $\sigma_0^r x \leq y$ but there does not exist $g \in \text{stab}(y)$ such that $gx = \sigma_0^r x$ is $(p-1)/2$.

We consider the action of G on B_p . Note that p being a prime guarantees that the action of any nontrivial $\sigma \in G$ has no fixed points, since σ is always an p -cycle in its cycle decomposition. Now suppose $(x, y) \in \mathcal{E}(B_p)_i$ is an element such that $\sigma x = y$ for some $\sigma \neq e$. Then, there is no $g \in C_p$ such that $gx = \sigma x$ and $gy = y$, since $gx = \sigma x \Rightarrow g = \sigma$ and $gy = y \Rightarrow g = e$. Therefore, in the case where p is a prime, we have:

$$q_i - p_i = |\{G(x, y) : \exists \sigma_0^r x \leq y \text{ where } 1 \leq r < n/2\}|.$$

We claim that for each r such that $1 \leq r < p/2$, there is precisely one G orbit $G(x, y)$ such that $\sigma_0^r x \leq y$.

First consider $r = 1$, so $\sigma = \sigma_0$. Suppose there is $(x, y) \in \mathcal{E}(B_n)$ such that $x \leq y$ and $\sigma x \leq y$, i.e., we take out the distinct filled bead from y in the necklace, and rotate y by a 1-click rotation, then the remaining i filled beads remain in σy . It is clear that the only possibility for this to occur is if we have $i+1$ consecutive filled beads, which gives the same G -orbit as $G(x, y)$.

The same construction apply to all cases where $1 \leq r < p/2$, so for each $r = 1, 2, \dots, (p-1)/2$, there is a precisely one orbit $G(x_r, y_r)$ such that $x_r \leq y_r$ and $\sigma_0^r x < y$. This proves the lemma. \square

Corollary 8.8. *Let $G = C_n$ where n is a prime, and p_i as defined above. Then the sequence p_i is unimodal.*

Proof. By the Proposition and the symmetry of p_i , we only need to show $p_0 \leq p_1$, but $p_0 = 1$, so the claim holds. \square

A similar method can be used to bound $q_i - p_i$ for n not necessarily prime. Let $\lambda_i = q_i - p_i$, our next goal in the section is to show that, for sufficiently large n , the difference $\lambda_{i+1} - \lambda_i$ is bounded above by $q_{i+1} - q_i$. (cf. Lemma 8.12).

Note that this suffices to show that the p_i are unimodal, since

$$p_{i+1} - p_i = (q_{i+1} - \lambda_{i+1}) - (q_i - \lambda_i) = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0.$$

Definition 8.9. Fix $r < n/2$ to be an integer and let $G(x, y)$ be an G -orbit of edges. We call $G(x, y)$ an *isolated pair* if $\sigma_0^r x < y$, but there does not exist $g \in C_n$ such that $gx = \sigma_0^r x$, and $gy = y$.

First we prove several sublemmas.

Lemma 8.10. $\lambda_i = p_i - q_i$ is equal to the number of $G(x, y)$ such that there exists $r < n/2$ with $\sigma_0^r x < y$ and y has a tail cycle under the action of σ_0^r .

Proof. Let

$$S_1 = \{G(x, y) : \exists \sigma_0^r x < y \text{ where } r < n/2, \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^r x, gy = y\}$$

and

$$S_2 = \{G(x, y) : \exists \sigma_0^r x < y \text{ where } r < n/2, \text{ and } y \text{ has a tail cycle}\}.$$

From Lemma 8.6 we know that, $\lambda_i = q_i - p_i = |S_1|$. We want to show that $S_1 = S_2$. It is clear that $S_2 \subset S_1$, so it remains to show that $S_1 \subset S_2$.

For each $r < n/2$ such that $(r, n) = 1$, there is precisely one such G -orbit for each i , since in this case σ_0^r is a full n -cycle and the argument in Lemma 8.7 applies. In this case, suppose that $\sigma_0^r x < y$ then y is automatically a tail cycle.

Now consider $r < n/2$ such that $(r, n) > 1$. In this case, σ_0^r fixes elements of the form $\{s, s+r, s+2r, \dots\}$ for any starting point s , which are proper subsets of $[n]$. For $y \in B_n$, one may write y as the union of full cycles and at most one tail cycle under the action of σ_0^r . Now, if $y \in B_n$ has no tail cycles under the action of σ_0^r then $(\sigma_0^r x, \sigma_0^r y) = (\sigma_0^r x, y) \in G(x, y)$. Hence $G(x, y) \notin S_1$, and this proves the lemma. \square

Lemma 8.11.

$$\lambda_{i+1} - \lambda_i \leq |\{G(x, y) : |y| = i+1, \text{ and } y \text{ does not contain a tail cycle}\}|$$

Proof. So, $q_i - p_i$ is equal to the number of $G(x, y)$ such that there exists r with $x < y$, $\sigma_0^r x < y$ and y has a tail cycle under the action of σ_0^r . Now let us focus on the difference of $\lambda_{i+1} - \lambda_i$. Note that the difference only occurs when we count $r \leq n/2$ and $(r, n) > 1$. Let $G(x, y)$ be a G -orbit where $x < y$ such that $|y| = i+1$, and let $\sigma_0^r(x) < y$. Suppose that there is no g such that $gx = \sigma_0^r(x)$ and $gy = y$. From the discussion above, we know that there a tail cycle of y , written $w := \{s, s+r, s+2r, \dots, s+lr\}$ for some l , and $x = y \setminus \{l\}$. Suppose that $y \cup \{s-r\}$ (mod n whenever necessary) still has a tail cycle $w \cup \{(s-r)\}$, with respect to σ_0^r . Then, if we let $y' = y \cup \{(s-r)\}$ and $x' = x \cup \{(s-r)\}$, the orbit $G(x, y')$ where $|y'| = i+2$ is also an isolated pair, thus does contribute 1 when we count λ_{i+1} . In this case, our counting process for λ_{i+1} and λ_i both increase by 1, so such isolated pair does not contribute to $\lambda_{i+1} - \lambda_i$. Now suppose that adding $s-r$ completes the sequence z into a tail cycle, namely $\{s, s+r, \dots, s-r, s\}$. Now y' will not be an isolated pair anymore, since σ_0^r fixes y' . In this situation, we increase 1 for λ_i only. If we choose to ignore this scenario, we will get an upper bound for $\lambda_{i+1} - \lambda_i$. \square

Lemma 8.12. Let $G = C_n$ and $q(i, m)$ be as defined above, then

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n, i+1) \\ 3 \leq k}} q\left(\frac{(i+1)}{k}, \frac{n}{k}\right)$$

Proof. Now return to the situation where $G(x, y)$ is not an isolated pair, but there exists $s \in [n]$ so that $G(y, y \cup s)$ is an isolated pair, where $y' = y \cup \{s'\}$ for any $s' \notin y$. We need to count how many times this happens. First we note that this happens when $d|n$, and $(n/d)|(i+1)$ where $d = (r, n)$. Note that, for different r such that (r, n) remains the same, there is only family of such isolated pairs, since the x part will be d -rotationally symmetric. We claim that, this is at most counting, over all $d|n$, where $1 < d < n$ and $(n/d)|i+1$, the sum $\sum q\left(\frac{(i+1)d}{n}, d\right)$. This is because we have at most $q\left(\frac{(i+1)d}{n}, d\right)$ ways to insert the tail cycle (with precisely 1 distinct bead) in the block of d elements.

Currently Editing

We can introduce $k = n/d$ and rewrite the sum as

$$\sum_{\substack{k|(i+1, n), \\ 1 < k < n}} q\left(\frac{i+1}{k}, \frac{n}{k}\right).$$

Note that in our sum, k can never be 2, since otherwise the tail cycle of x has precisely 1 element, and then y is fixed (consisting of full cycles). So this proves the assertion that

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n, i+1) \\ 3 \leq k}} q\left(\frac{i+1}{k}, \frac{n}{k}\right).$$

□

Theorem 8.13. *The poset $\mathcal{E}(B_n/C_n)$ is rank symmetric and rank unimodal.*

Proof. We need to show that for $G = C_n$, the p_i are unimodal. So, we only need to bound

$$\sum_{\substack{k|(n, i+1) \\ 3 \leq k < n/2}} q\left(\frac{i+1}{k}, \frac{n}{k}\right). \text{ It is clear that for sufficiently large } n, \text{ say for } n \geq 9,$$

$$q((i+1)/k, n/k) = \binom{n/k - 1}{(i+1)/k - 1} \leq \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil}.$$

Since $k \leq i$, and $i+1 \geq n/2$, we coarsely bound the sum by

$$\frac{n}{2} \cdot \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil}.$$

We want to show that this is smaller than the difference of

$$q_i - q_{i-1} = \frac{(n-1)!}{(i)!(n-i)!} - \frac{(n-1)!}{(i-1)!(n-i+1)!} = \binom{n}{i} \frac{n-2i}{n} \geq \binom{n}{i} \frac{2}{n}.$$

Namely, we want to show that, for sufficiently large n ,

$$\left(\frac{n}{2}\right)^2 \cdot \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil} \leq \binom{n}{i}.$$

This bound works for $n \geq 12$. For smaller n the claim can be easily checked.

Now, by Lemma 8.12, we know that

$$p_{i+1} - p_i = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0,$$

which proves that the p_i are increasing for $i < n/2$. By symmetry, the p_i are unimodal.

□

8.2. Quotient by the Dihedral Group. In this subsection, we show a similar result for the Dihedral groups that act naturally on B_n . Namely, for $G = D_{2n}$ the dihedral group of order $2n$, the poset $\mathcal{E}(B_n/G)$ is rank symmetric and rank unimodal.

Notation 8.14. In this section, we fix an arbitrary n again, set

$$Q = \mathcal{E}(B_n)/D_{2n}, \quad P = \mathcal{E}(B_n/D_{2n})$$

and let

$$q_i = |(\mathcal{E}(B_n)/D_{2n})_i|, \quad p_i = |(\mathcal{E}(B_n/D_{2n}))_i|.$$

Lemma 8.15. *Let $G = D_{2p}$ or $G = D_{4p}$ where p is prime, and p_i, q_i as defined above. Then $p_i = q_i$. In other words, the action of D_{2p} (respectively D_{4p}) on B_p (respectively B_{2p}) is CCT.*

Proof. The proof of the Lemma is similar to the proof of Lemma 8.7. For example, the group $G = D_{2n}$ contains C_n as a subgroup and is generated by C_n with arbitrary reflections. The additional reflections guarantee that there are no special orbits $G(x, y)$ for any $r < n/2$, which implies $p_i = q_i$. Similarly, D_{4n} is generated by the rotational subgroup C_{2n} and reflections, and it is not hard to see that for any pair of elements $(x, z), (y, z) \in \mathcal{E}(B_n)$ such that x and y differ by a rotation, which does not fix z , then there always exists a reflection τ such that $\tau x = y$ and τ fixes z . Following the proof of 8.7, we know that the action of D_{4n} on B_n is CCT. \square

Remark 8.16. Let G be as in Lemma 8.15. We remark that, in addition to unimodality, Theorem 1.5 implies that the poset $\mathcal{E}(B_n/G)$ is Peck.

Lemma 8.17. *We have an explicit formula for $q_i = |(\mathcal{E}(B_n)/D_{2n})_i|$:*

$$q_i = \frac{1}{2} \left(\binom{n-1}{i} + \frac{1}{2} [(-1)^{n(i+1)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1} \right)$$

Proof. Consider an element $(x, y) \in \mathcal{E}(B_n)$ where $x < y$. Note that any element τ that fixes both x and y has to fix the difference $y \setminus x$, which is a one-element set. Since $G = D_{2n}$, we know that $1 \leq |\text{Stab}(x, y)| \leq 2$. Let μ_1 be the number of G -orbits with the trivial stabilizer and μ_2 be the number of G -orbits with the stabilizer of size 2, which contains the identity and a reflection. By the Orbit-Stabilizer Theorem, each orbit with the trivial stabilizer is of size $|D_{2n}|/1 = 2n$, while all other orbit is of size $|D_{2n}|/2 = n$. Therefore

$$\mu_1 \cdot 2n + \mu_2 \cdot n = |\{(x, y)\}| = \binom{n}{i+1} \cdot \binom{i+1}{1} = n \binom{n-1}{i}.$$

Our goal is to calculate $q_i = \mu_1 + \mu_2$, so it remains to calculate μ_2 , which counts the number of G -orbits such that the reflection also fixes x . Without loss of generality, we may assume that $y \setminus x = \{1\}$. We break into cases:

- (1) If n is odd and $i+1$ is even, then it is clear that no reflections fix x for any x , so $\mu_2 = 0$.
- (2) For all other cases, there are precisely $\lceil n/2 \rceil - 1$ places to insert $\lceil (i+1)/2 \rceil - 1$ elements of $[n]$ to form x .

This gives the desired formula $\mu_2 = \frac{1}{2} [(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1}$. Therefore,

$$q_i = \frac{1}{2} \left(\binom{n-1}{i} + \frac{1}{2} [(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1} \right)$$

\square

Remark 8.18. An alternate proof of Lemma 8.17 follows directly from Polya's Theorem 6.6, by counting the number of ways to color n elements with three colors, so that there are i of the first color, 1 of the second color, and $n - i - 1$ of the third color.

Corollary 8.19. *The poset $\mathcal{E}(B_n/D_{2n})$ is rank symmetric and rank unimodal.*

Proof. The proof is similar to the proof of Lemma 8.12 and Corollary 8.13. In fact, we can bound the difference of $\lambda_{i+1} - \lambda_i$ by precisely the same bound used in Lemma 8.12, since the difference we get here (where $G = D_{2n}$) is smaller than the previous difference (where $G = C_n$). Then, a similar proof shows that the difference of $q_{i+1} - q_i$ obtained in Lemma 8.15 is significantly larger than the upper bound. \square

9. A q -ANALOG

In this subsection we consider a q -analog of the Boolean algebra and prove Lemma 9.4, which is a q -analog to Lemma 7.3.

Let q be a prime and $B_n(q)$ be the poset of all \mathbb{F}_q -subspaces of $V_n(q) := (\mathbb{F}_q)^n$, graded naturally by dimensions.

Let $C_n(q)$ be the multiplicative subgroup of the finite field \mathbb{F}_{q^n} , so $C_n(q)$ acts \mathbb{F}_q -linearly on \mathbb{F}_{q^n} by multiplication, which is isomorphic to $V_n(q)$ as an \mathbb{F}_q vector space. The action of $C_n(q)$ on $V_n(q)$ induces a group action on the poset $B_n(q)$. The poset $B_n(q)/C_n(q)$ is the q -analog of the necklace poset.

The following Lemma is a standard result. For a proof of the lemma, we refer the reader to, for example, [?].

Lemma 9.1. *The number $|B_n(q)_i|$, that is to say, the number of i dimensional subspace in $V_n(q)$, is $\binom{n}{i}_q$.*

Notation 9.2. Define the graded poset $T = \mathcal{E}(B_n(q))/C_n(q)$ and let $t_i = |T_i|$.

Lemma 9.3. *For any $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$, so that $(V_x, V_y) \in \mathcal{E}(B_n(q))/C_n(q)$, then $\text{Stab}_{C_n(q)}(V_x, V_y) \cong \mathbb{F}_q^\times$. In particular, $|\text{Stab}_{C_n(q)}(V_x, V_y)| = q - 1$.*

Proof. We claim that if $\tau \in C_n(q)$ such that $\tau V_x = V_x$ and $\tau V_y = V_y$, then $\tau \in \mathbb{F}_q^\times$. First, it is clear that any scalar $c \in \mathbb{F}_q^\times$ fixes any $V \in B_n(q)$. Now assume $V_x \subset V_y$ and $\tau V_x = V_x, \tau V_y = V_y$ as in the claim. Then, in particular τ permutes all the elements in V_x , i.e., $\tau : V_x \rightarrow V_x$ is an isomorphism of \mathbb{F}_q -vector spaces. Pick a nonzero element $a \in V_x$, it is clear that

$$\tau^{q^i}(a) = \tau^{q^i}a = \tau a.$$

The latter equation implies that $\tau^{q^i} - \tau = 0$, so τ satisfies the polynomial $X^{q^i} - X \in \mathbb{F}_q[X]$. Similarly, since $\tau V_y = V_y$, τ is a root of the polynomial $X^{q^{i+1}} - X \in \mathbb{F}_q[X]$. Let $f(X)$ be a monic irreducible polynomial of τ over \mathbb{F}_q , then $\deg f(X) | i$ and $\deg f(X) | (i+1)$. This shows that $\deg f(X) = 1$, in another word, τ is a \mathbb{F}_q -scalar. \square

Lemma 9.4. *Let q be a prime, then*

$$|(\mathcal{E}(B_n(q))/C_n(q))_i| = \binom{n-1}{i}_q.$$

Proof. Let $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$ so that $(V_x, V_y) \in \mathcal{E}(B_n(q))$. By Lemma 9.3, for any $(V_x, V_y) \in \mathcal{E}^1(B_n(q))_i, |\text{Stab}_{C_n(q)}(V_x, V_y)| = q - 1$. By the Orbit-Stabilizer Theorem, we know that the size of each orbit in $\mathcal{E}((B_n(q))_i)$ under the action of $C_n(q)$ is

$$|C_n(q)|/(q-1) = \frac{q^n - 1}{q - 1} = (n)_q.$$

Now we can calculate the number q_i , which is the total number of elements in $(B_n(q))_i$ divided by the size of each orbit:

$$q_i = \frac{\binom{n}{i+1}_q \binom{i+1}{1}_q}{(n)_q} = \binom{n-1}{i}_q.$$

\square

10. THE OBJECT $\mathcal{E}(B_n)$.

In this section, we explicitly compute the raising operators corresponding to both $\mathcal{H}(B_n)$, $\mathcal{E}(B_n)$ and explicitly show that $\mathcal{E}(B_n)$ is unitary Peck by showing certain raising maps are invertible.

Notation 10.1. For this section, we will use $M^{n-2i-1} = \mathcal{E}(U)^{n-2i-1}$, where $\mathcal{E}(U)^{n-2i-1} : \mathcal{E}(B_n)_i \rightarrow \mathcal{E}(B_n)_{n-i-1}$ is the Lefschetz map, as defined in 2.3.

Notation 10.2. For the remainder of this section, we shall take $|a| = n-i-1$, $|b| = n-i$, $|x| = i$, $|y| = i+1$. Let $k = n-2i-1$, that is, $k = |a| - |x|$. Additionally, whenever we write an expression of the form (x, y) or (a, b) , it is assumed that $x \leq y$, $a \leq b$.

Theorem 10.3. *Defining L to be the Lefschetz map $\mathcal{H}(B_n) \rightarrow \mathcal{H}(B_n)$, we have $L^{n-2i-1} : \mathcal{H}^1(B_n)_i \rightarrow \mathcal{H}^1(B_n)_{n-i-1}$, and explicitly*

$$L^{n-2i-1}(x, y) = k! \sum_{\substack{y \not\subset a, \\ x \subset a, \\ y \subset b}} (a, b).$$

Proof. Note that the conditions $y \not\subset a, x \subset a, y \subset b$ are equivalent to $(a, b) >_{\mathcal{H}(B_n)} (x, y)$. Clearly, if $(a, b) \not>_{\mathcal{H}(B_n)} (x, y)$, then the coefficient of (a, b) in $L^{n-2i-1}(x, y)$ is 0. So, to complete the proof, it suffices to show that if $(a, b) >_{\mathcal{H}(B_n)} (x, y)$ then the coefficient of (a, b) in $L^{n-2i-1}(x, y)$ is $k!$. However, this coefficient is precisely the number of sequences $(x, y) = (x_0, y_0) \leq_{\mathcal{H}(B_n)} (x_1, y_1) \leq_{\mathcal{H}(B_n)} \cdots \leq_{\mathcal{H}(B_n)} (x_k, y_k) = (a, b)$. By definition of \mathcal{H} , we must have that $y_k \setminus x_k = y \setminus x$ for all k . Therefore, the number of such sequences is equal to the number of sequences $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$, since choosing the x_i determine y_i because $y_i = x_i \cup (y \setminus x)$. Finally, the number of such sequences $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$, is equivalent to the number of ways to add the elements in $a \setminus x$ to x . This is because each sequence $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$ is determined uniquely by the singletons $x_{i+1} \setminus x_i$, $1 \leq i \leq k$. Since in total we are adding k elements to x in order to obtain a , there are $k!$ ways to do this. Therefore, coefficient of (a, b) in $L^{n-2i-1}(x, y)$ is $k!$ \square

Theorem 10.4. *The map M satisfies*

$$M^{n-2i-1}(x, y) = (2^k - 1)(k-1)! \sum_{y \subset a} (a, b) + k! \sum_{\substack{y \not\subset a, \\ x \subset a, \\ y \subset b}} (a, b)$$

Proof. For a particular $(x, y), (a, b)$, if either $y \subset a$ or $y \not\subset a, x \subset a, y \subset b$, then $(a, b) \not>_{\mathcal{H}(B_n)} (x, y)$, and so the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ is 0.

Clearly, we cannot have both $y \subset a$ and $y \not\subset a, x \subset a, y \subset b$, hold at the same time. So, suppose $y \not\subset a, x \subset a, y \subset b$. This implies that $b \setminus a = y \setminus x$. Then, the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ is precisely the number of sequences $(x, y) = (x_0, y_0) \leq_{\mathcal{E}(B_n)} (x_1, y_1) \leq_{\mathcal{E}(B_n)} \cdots \leq_{\mathcal{E}(B_n)} (x_k, y_k) = (a, b)$. However, since $y \setminus x = b \setminus a$, it must be that $y_k \setminus x_k = y \setminus x$ as well. Therefore, the number of such sequences is equal to $k!$, as was shown in the proof of Theorem 10.3.

To complete the proof, we need to show that if $y \subset a$ then the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ is $(2^k - 1)(k-1)!$. Equivalently, we need to show that the number of sequences

$$(10.1) \quad (x, y) = (x_0, y_0) \leq_{\mathcal{E}(B_n)} (x_1, y_1) \leq_{\mathcal{E}(B_n)} \cdots \leq_{\mathcal{E}(B_n)} (x_k, y_k) = (a, b)$$

is $(2^k - 1)(k-1)!$.

For the moment, fix j and consider the set of all sequences of the form in Equation (10.1) such that $(a \setminus b) \cup y_{j-1} = y_j$. First, let us show the number of such with this j fixed is $(k-1)!2^j$. To do this, start by considering the number of sequence y_0, \dots, y_k such that $y = y_0 \leq_{B_n} y_1 \leq_{B_n} \cdots \leq_{B_n} y_k = b$. Since we enforce $y_j = y_{j-1} \cup (a \setminus b)$, the number of such

sequences is precisely the number of ways to order the elements $a \setminus y$. Since $|y| - |a| = k - 1$, there are $(k - 1)!$ such ways. Additionally, for $l < j$ we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$ or $x_{l+1} = x_l \cup (y_l \setminus x_l)$. Either of these is possible at every step. Additionally, for $l \geq j$, we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$. So for any fixed sequence of y_k with so that j is minimal with $(a \setminus b) \in y_j$, there are precisely 2^{j-1} possible sequences $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \dots \leq_{B_n} x_k = a$ so that $(x, y) = (x_0, y_0) \leq_{\mathcal{E}(B_n)} (x_1, y_1) \leq_{\mathcal{E}(B_n)} \dots \leq_{\mathcal{E}(B_n)} (x_k, y_k) = (a, b)$.

So, in total, there are $2^j (k - 1)!$ sequences of the form in (10.1) with $(a \setminus b) \cup y_{j-1} = y_j$. Now, there clearly must be some such j , $1 \leq j \leq k$. Therefore, the coefficient we are looking for is $\sum_{j=1}^k 2^{j-1} (k - 1)! = (2^k - 1)(k - 1)!$, as claimed. \square

10.1. Proof that $\mathcal{E}(B_n)$ is unitary Peck. In this subsection, we will show the rows of M form a basis by showing we can make a change of basis to a map which takes M^{n-2i-1} to L^{n-2i-1} . Since we know L^{n-2i-1} is an isomorphism, it will follow that M^{n-2i-1} is as well.

Notation 10.5. Let $\beta = \frac{2^k - 1}{k}$. Denote

$$v_{(a,b)} = \beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y).$$

Note that $v_{(a,b)}$ are simply the rows of M^{n-2i-1} , each divided by the constant $k!$.

Notation 10.6. For any set s of size at least $n - i$,

$$z_s = \frac{1}{\beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}} \sum_{b \subset s, a \subset b} v_{(a,b)}$$

Lemma 10.7. *For any set s of size at least $n - i$, we have $z_s = \sum_{y \subset s} (x, y)$. In particular, $\sum_{y \subset s} (x, y)$ lies in the span of $v_{(a,b)}$*

Proof. We have

$$\begin{aligned} & \left(\beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1} \right) z_s \\ &= \sum_{b \subset s, a \subset b} v_{(a,b)} \\ &= \sum_{b \subset s, a \subset b} \left(\beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y) \right) \\ &= \sum_{b \subset s, a \subset b} \beta \sum_{y \subset a} (x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\ &= \beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) \sum_{y \subset s} (x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\ &= \beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) \sum_{y \subset s} (x, y) + \binom{|s|-i-1}{n-2i-1} \sum_{y \subset s} (x, y) \\ &= \left(\frac{1}{\beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}} \right) \sum_{y \subset s} (x, y). \end{aligned}$$

In going between the fourth line and the fifth line, one needs to count the number of y satisfying $y \subset a \subset b \subset s$. If we fix s, a there are $(|s| - n + i + 1)$ choices for the element b , since $b = a \cup \{s\}$ for $s \notin a$. Then, we need to count the number of a with $y \subset a \subset s$. This is exactly $\binom{|s|-i-1}{n-2i-2}$. Hence, the total of number of such y is the product $\binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1)$.

In going from the fifth line to the sixth line, for y, s fixed, we count the number of b with $y \subset b \subset s$. This is exactly $\binom{|s|-i-1}{n-2i-1}$. \square

Notation 10.8. For any set s of size at least $n - i$, let $w_s = \sum_{a \subset s, t \notin s} v_{(a, a \cup \{t\})}$.

Lemma 10.9. *We have*

$$w_s = \sum_{t \notin s} \left(\beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, x \cup \{t\}) + \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right).$$

Proof.

$$\begin{aligned} w_s &= \sum_{a \subset s, t \notin s} v_{(a, a \cup \{t\})} \\ &= \sum_{a \subset s, t \notin s} \left(\beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\ &= \sum_{t \notin s} \left(\sum_{a \subset s} \beta \sum_{y \subset a} (x, y) + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\ &= \sum_{t \notin s} \left(\beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y) + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\ &= \sum_{t \notin s} \left(\beta \sum_{y \subset s} (x, y) + \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right) \end{aligned}$$

The equalities between lines three and four, and four and five hold for similar reasons as the equalities between lines four and five, and five and six in 10.7 \square

Notation 10.10. Let $u_s = \frac{w_s - (n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} z_s}{\binom{|s| - i}{n - 2i - 1}} + z_s$.

Lemma 10.11. *We have $u_s = \sum_{x \subset s, y \supset x} (x, y)$. In particular, $\sum_{x \subset s, y \supset x} (x, y)$ lies in the span of $v_{(a, b)}$.*

Proof. By 10.9 and 10.7 we have

$$\begin{aligned} u_s &= \frac{w_s - (n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} z_s}{\binom{|s| - i}{n - 2i - 1}} + z_s \\ &= \frac{\sum_{t \notin s} \left(\beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y) + \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right)}{\binom{|s| - i}{n - 2i - 1}} \\ &\quad - \frac{(n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y)}{\binom{|s| - i}{n - 2i - 1}} + \sum_{y \subset s} (x, y) \\ &= \frac{\sum_{t \notin s} \left(\binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right)}{\binom{|s| - i}{n - 2i - 1}} + \sum_{y \subset s} (x, y) \\ &= \sum_{t \notin s} \left(\binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right) + \sum_{y \subset s} (x, y) \\ &= \sum_{x \subset s, y \supset x} (x, y) \end{aligned}$$

The penultimate line is equal to the ultimate line because any element (x, y) with $x \subset s$ must either have $y \subset s$ or else $y \subset s \cup \{t\}$ for $t \notin s$. The two terms on the penultimate line cover precisely these two cases. \square

Notation 10.12. For any $s \subset [n]$, with $|s| \leq n - i$, define $h_s = z_{[n]} - u_s$.

Lemma 10.13. *With h_s as defined above $h_s = \sum_{x \not\subset s} (x, y)$*

Proof. Using 10.7 and 10.11

$$\begin{aligned} h_s &= z_{[n]} - u_s \\ &= \sum_{x \subset y} (x, y) - \sum_{x \subset s, x \subset y} (x, y) \\ &= \sum_{x \not\subset s} (x, y) \end{aligned}$$

□

Notation 10.14. For $I \subset [n]$, let $I^c = [n] \setminus I$, the complement of I in $[n]$.

Lemma 10.15. *Let I be fixed. If l_J lies in the span of $v_{(a,b)}$ for $J \subset I$, then so does l_I .*

Proof. Let $I = \{i_1, \dots, i_s\}$. Let $A_k = \{x: i_k \in x\}$. Then, $\cap_{k=1}^s A_k = \{x: I \subset x\}$. Using the principle of inclusion exclusion, we can write $\sum_{x \in \cap_{k=1}^s A_k} (x, y)$ as a sum of terms $\pm \sum_{x \in \cap_{k \in J} A_k} (x, y)$, for $J \subset I$. Since we are assuming $\pm \sum_{x \in \cap_{k \in J} A_k} (x, y)$, lie in the span of $v_{(a,b)}$ it follows that $l_I = \sum_{x \in \cap_{k=1}^s A_k} (x, y)$ does as well.

□

Lemma 10.16. *For all $J \subset [n]$, with $|J| \leq i$, we have l_J lies in the span of $v_{(a,b)}$.*

Proof. We shall prove this by induction on the size of I . First, we know $h_{[n]}$ lies in the span of $v_{(a,b)}$ and so have completed the base case $|I| = 0$ since $h_{[n]} = l_\emptyset$. So, to complete the proof, it suffices to show that if we know this Lemma holds for all $|s| > j$ then it holds for $|s| = j$. This is exactly what 10.15 proves.

□

Lemma 10.17. *For $|x| = i$, we have $l_x = \sum_{y \supset x} (y, x)$.*

Proof. By definition, $l_{\bar{x}} = \sum_{\bar{x} \subset x, \bar{x} \cap x = \bar{x}, x \subset y} (x, y) = \sum_{\bar{x} \subset y} (\bar{x}, y)$. Replacing \bar{x} by x gives the result.

□

Notation 10.18. Define $m_a = \sum_{x \subset a} l_x$.

Lemma 10.19. *With m_a as defined above, we have $m_a = \sum_{x \subset a} (x, y)$.*

Proof. By 10.17, we obtain

$$m_a = \sum_{x \subset a} l_x = \sum_{x \subset a} \sum_{y \supset x} (y, x) = \sum_{x \subset a} (x, y)$$

□

Notation 10.20. Denote $r_a = \sum_{b, b \subset a} v_{(a,b)}$.

Lemma 10.21. *With r_a as defined above, we have*

$$r_a = ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y).$$

Proof.

$$\begin{aligned}
r_a &= \sum_{b \supset a} v_{(a,b)} \\
&= \sum_{b \supset a} \left(\beta \sum_{y \subset a} (x, y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \right) \\
&= \sum_{b \supset a} \beta \sum_{y \subset a} (x, y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \\
&= (i+1)\beta \sum_{y \subset a} (x, y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \\
&= (i+1)\beta \sum_{y \subset a} (x, y) + \sum_{x \subset a, y \not\subset a} (x, y) \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \left(\sum_{x \subset a, y \not\subset a} (x, y) + \sum_{y \subset a} (x, y) \right) \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y).
\end{aligned}$$

□

Notation 10.22. Assuming we do not have $(i+1)\beta = 1$, define $t_a = \frac{r_a - m_a}{(i+1)\beta - 1}$

Lemma 10.23. *With t_a as defined above, $t_a = \sum_{y \subset a} (x, y)$*

Proof. By 10.21 and 10.19, we have

$$\begin{aligned}
t_a &= \frac{r_a - m_a}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y) - \sum_{x \subset a} (x, y)}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x, y)}{(i+1)\beta - 1} \\
&= \sum_{y \subset a} (x, y)
\end{aligned}$$

□

Notation 10.24. Assuming $\beta(i+1) \neq 1$, let $q_{a,b} = v_{(a,b)} - \beta t_a$.

Lemma 10.25. *We have $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)$.*

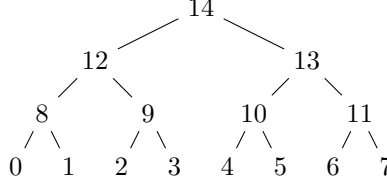


FIGURE 4.

Proof. Using 10.23

$$\begin{aligned}
 q_{a,b} &= v_{(a,b)} - \beta t_a \\
 &= \beta \sum_{y \subset a} (x, y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) - \beta \sum_{y \subset a} (x, y) \\
 &= \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)
 \end{aligned}$$

□

Theorem 10.26. *For $n > 2$, the matrix M^{n-2i-1} is invertible.*

Proof. We saw that if $\beta(i+1) \neq 1$, we have that $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)$ lie in the span of $v_{(a,b)}$. It is always the case that $\beta \geq 1, i \geq 0$. The only time in which $\beta = 1, i = 0$ is when $n = 2$. Therefore, $q_{a,b}$ are always defined for $n > 2$. However, $q_{a,b}$ are exactly the rows of L^{n-2i-1} as defined in Corollary 3.18. Using Theorem ??, we know $\mathcal{H}^1(B_n)$ is unitary Peck, and therefore the rows of L^{n-2i-1} are independent. Therefore, the rows of M^{n-2i-1} span an independent set in a vector space of the same dimension. So, the rows of M^{n-2i-1} are independent. Hence, M^{n-2i-1} is indeed an isomorphism, when $n > 2$. □

Corollary 10.27. *The poset $\mathcal{E}(B_n)$ is unitary Peck.*

Proof. By definition, a poset $P, \text{rk}(P) = t$ is unitary peck if and only if the Lefschetz map $L : P \rightarrow P, x \mapsto \sum_{y \succ x} y$, satisfies $L^{t-2i} : P_i \rightarrow P_{t-i}$ is an isomorphism. In the case $P = \mathcal{E}(B_n), \text{rk}(P) = n-1$, we have that M is the Lefschetz map and by Theorem 10.26, $M^{n-1-2i} : \mathcal{E}(B_n)_i \rightarrow \mathcal{E}(B_n)_{n-1-i}$ is indeed an isomorphism. Hence, $\mathcal{E}(B_n)$ is unitary Peck. □

11. ODDS, ENDS, AND FAILED ATTEMPTS

In this section, we include various remarks, as well as failed attempts, which don't particularly belong in other sections.

11.1. Quotients by Automorphism Groups of Rooted Trees. In Subsection 4.2.1, we saw that the full automorphism groups of rooted trees are CCT. It is well known that the quotient $B_{ml}/(S_m \wr S_l)$ is a distributive lattice. Hence, we were curious whether B_{ml}/G is a distributive lattice as well, where G is the automorphism group of a rooted tree. The answer is a definitive no, as can be seen in the case $G = (S_2 \wr S_2) \wr S_2$, where G acts on a the binary rooted tree P with $|L(P)| = 8$, whose vertices are labeled $0, \dots, 7$ from left to right. In this case, it is easy to see that $G\{0, 5, 6\} \triangleleft G\{0, 1, 5, 6\}, G\{0, 5, 6\} \triangleleft G\{0, 1, 2, 5\}, G\{0, 1, 5\} \triangleleft G\{0, 1, 5, 6\}, G\{0, 1, 5\} \triangleleft G\{0, 1, 2, 5\}$. Hence, $G\{0, 5, 6\}, G\{0, 1, 5\}$ do not have a well defined join, and so B_{ml}/G is not a lattice.

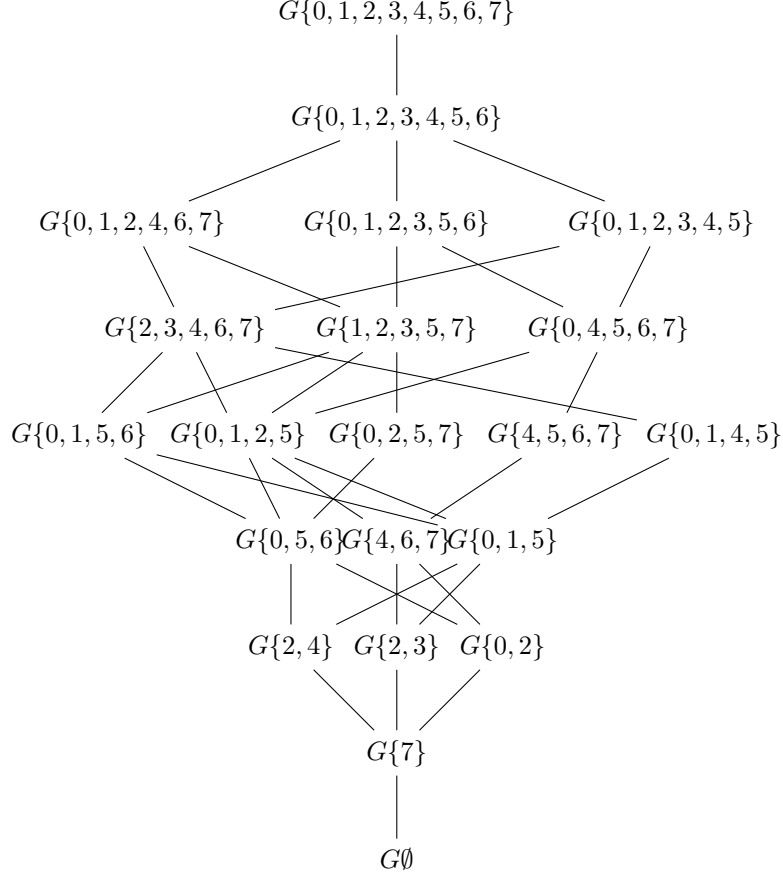


FIGURE 5.

11.2. A Failed Attempt for Showing $\mathcal{E}(B_n/G)$ is unimodal. In this section, we describe one path we were pursuing in order to show $\mathcal{E}(B_n/G)$ is unimodal. We were attempting to do this by trying to show there were injective order raising maps $U_i : \mathcal{E}(B_n/G)_i \rightarrow \mathcal{E}(B_n/G)_{i+1}$.

We shall now define several maps, so that we can draw a certain commuting diagram in Remark 11.2

Notation 11.1. Let $U_i : P_i \rightarrow P_{i+1}$ be the raising operator for the poset P . Then, we obtain an induced map

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \rightarrow P_{i+1} \otimes P_{i+1}, (x, y) \mapsto U(x) \otimes U(y).$$

We also have the natural inclusions

$$\begin{aligned} k_i : \mathcal{E}(P)_i &\rightarrow P_i \otimes P_{i+1}, \\ (x, y) &\mapsto (x, y) \\ k_i^{G \times G} : \mathcal{E}(P/G)_i &\rightarrow (P/G)_i \otimes (P/G)_{i+1}, \\ (Gx, Gy) &\mapsto (Gx, Gy), \end{aligned}$$

where we have $x \leq y$ and $Gx \leq Gy$. The maps above are defined on a basis, and are extended by linearity.

Next, we define the map

$$j_i : (P/G)_i \otimes (P/G)_{i+1} \rightarrow P_i \otimes P_{i+1},$$

$$(Gx, Gy) \mapsto \left(\frac{1}{|G|} \sum_{g \in G} gx, \frac{1}{|G|} \sum_{h \in G} hy \right)$$

where x is an arbitrary representative of Gx and y is an arbitrary representative of Gy

Then, define the map

$$p_i : P_i \otimes P_{i+1} \rightarrow (P/G)_i \otimes (P/G)_{i+1},$$

$$(x, y) \mapsto (Gx, Gy).$$

Further, define the map

$$(U_i \otimes U_{i+1})^{G \times G} : P_i \otimes P_{i+1} \rightarrow P_{i+1} \otimes P_{i+1},$$

$$(Gx, Gy) \mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx, Gy).$$

We also have the projections inclusions

$$\pi_i : P_i \otimes P_{i+1} \rightarrow \mathcal{E}(P)_i,$$

$$(x, y) \mapsto \begin{cases} (x, y), & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_i^{G \times G} : (P/G)_i \otimes (P/G)_{i+1} \rightarrow \mathcal{E}(P/G)_i,$$

$$(Gx, Gy) \mapsto \begin{cases} (Gx, Gy), & \text{if } Gx \leq Gy \\ 0 & \text{otherwise} \end{cases},$$

where we have $x \leq y$ and $Gx \leq Gy$. The maps above are defined on a basis, and are extended by linearity.

Finally, denote

$$\mathcal{E}(U)_i : \mathcal{E}(P)_i \rightarrow \mathcal{E}(P)_{i+1}$$

$$(x, y) \mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x, y)$$

$$\mathcal{E}(U)_i^{G \times G} : \mathcal{E}(P/G)_i \rightarrow \mathcal{E}(P/G)_{i+1}$$

$$(Gx, Gy) \mapsto k_i^{G \times G} \circ (U \otimes U)^{G \times G} \circ \pi_{i+1}^{G \times G}(Gx, Gy)$$

where it is defined above on a basis and we extend to the whole space by linearity.

Remark 11.2. For $i < \frac{n}{2}$ we obtain the following (almost commuting, but $j_{i+1} \circ p_{i+1} \neq \text{id.}$) diagram

$$\begin{array}{ccccccc}
 & & & \mathcal{E}(U)_i & & & \\
 & & & \curvearrowright & & & \\
 \mathcal{E}(P)_i & \xrightarrow{k_i} & P_i \otimes P_{i+1} & \xrightarrow{U_i \otimes U_{i+1}} & P_{i+1} \otimes P_{i+2} & \xrightarrow{\pi_{i+1}} & \mathcal{E}(P)_{i+1} \\
 & & \uparrow j_i & & \downarrow p_{i+1} & & \uparrow j_{i+1} \\
 \mathcal{E}(P/G)_i & \xrightarrow{k_i^{G \times G}} & (P/G)_i \otimes (P/G)_{i+1} & \xrightarrow{(U_i \otimes U_{i+1})^{G \times G}} & (P/G)_{i+1} \otimes (P/G)_{i+2} & \xrightarrow{\pi_{i+1}^{G \times G}} & \mathcal{E}(P/G)_{i+1} \\
 & & & \curvearrowleft & & & \\
 & & & \mathcal{E}(U)_i^{G \times G} & & &
 \end{array}$$

Unfortunately, in general, with the above definitions of the maps,

$$\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \not\subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}).$$

However, if we did have

$$\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}),$$

using the fact that $\mathcal{E}(U)_i, U_i, U_{i+1}$ are all injective, a fairly simple diagram chase would reveal $\mathcal{E}(U)_i^{G \times G}$ is injective. This, in turn, would imply $\mathcal{E}(B_n/G)_i$ is symmetric, unimodal, and sperner. We have tried several variations on these exact maps, but were never quite able to obtain the desired $\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G})$.

11.3. Further Questions.

11.3.1. Category Theoretic Properties. First, it is worth noting that the functor $\mathcal{E} : \mathcal{P} \rightarrow \mathcal{P}$ is neither essentially surjective nor fully faithful. Let P be the poset with points a, b, c, d and relations $a < c, b < c, a < d$. It is easy to see there is no poset A with $\mathcal{E}(A) = P$. This shows \mathcal{E} is not essentially surjective. Next, let Q be the poset whose elements are e, f, g with $e < f, e < g$, and let R be the poset whose elements are h, i, j, k with $h < i, j < k$. It is clear that $\mathcal{E}(Q) \cong \mathcal{E}(R)$ are both a disjoint union of two points, but $Q \not\cong R$. So, \mathcal{E} is also not fully faithful.

However, there are many aspects of the functor \mathcal{E} , and indeed $\mathcal{E}^{\vec{\tau}}$ which are still unexplored. First, note that it is worth noting that \mathcal{E} also determines an endofunctor on the category of (ungraded) posets, which we shall now notate \mathcal{Q} , as well as an endofunctor on the category of graded posets \mathcal{P} . In general, the points of a poset P can be identified with the set $\text{Hom}_{\mathcal{Q}}(B_0, P)$, and the edges of P can be identified with the set $\text{Hom}_{\mathcal{Q}}(B_1, P)$. It is simple to see additionally that the points of $\mathcal{E}(P)$ are $\text{Hom}_{\mathcal{Q}}(B_0, \mathcal{E}(P)) \cong \text{Hom}_{\mathcal{Q}}(B_1, P)$. Additionally, the edges of $\mathcal{E}(P)$ are $\text{Hom}_{\mathcal{Q}}(B_1, \mathcal{E}(P)) \cong \text{Hom}_{\mathcal{Q}}(B_2, P)$. These relations may be a reasonable alternate way to think of the functor of points, and are further suggestive of the existence of an adjoint functor.

Question 11.3. Does the functor \mathcal{E} have a left or right adjoint?

11.3.2. Unitary Peck Generalizations. Additionally, there are several posets which we would like to be unitary Peck, as this would allow us to apply much of the theory developed in this paper. They would also make several of the generalizations of \mathcal{E} potentially more interesting. Two posets we are particularly interested in are $\mathcal{E}^{\vec{\tau}}(B_n)$ and its q-analog, $\mathcal{E}^{\vec{\tau}}(B_n(q))$.

Question 11.4. Is $\mathcal{E}^{\vec{\tau}}(B_n)$ unitary Peck?

As mentioned in the introduction, a natural extension of Conjecture 1.3 would be an analogous result for q-analogs. We suspect that a proof similar to that of Corollary 10.27 may also prove Question 11.5.

Question 11.5. Is $\mathcal{E}(B_n(q))$ unitary peck? What about $\mathcal{E}^{\vec{\tau}}(B_n(q))$?

11.3.3. A q-analog of Conjecture 1.3. In particular, if the answer to the previous Question 11.5 is affirmative, it would immediately hold that $\mathcal{E}(B_n(q))/G$ is Peck, and if G is CCT then $\mathcal{E}(B_n(q))/G$ is Peck. Hence, we pose the followin question:

Question 11.6. For G a group with a CCT action on $B_n(q)$ is $\mathcal{E}(B_n(q))/G$ Peck?

Even more generally, we wonder if the q-analog of Conjecture 1.3 holds.

Question 11.7. For G a group acting on $B_n(q)$, is $\mathcal{E}(B_n(q))/G$ Peck? If not, is $\mathcal{E}(B_n(q))/G$ rank unimodal?

11.3.4. *Additional CCT Actions.* We also found several additional interesting examples of CCT actions, and we are curious whether they generalize. Once such action is the linear automorphism of the n -cube. Using computers, we found that for $n \leq 3$, the linear automorphisms of the n -cube is CCT. We wonder if this generalizes.

Question 11.8. Does the group of linear automorphism of an n -cube in \mathbb{R}^n , whose vertices lie at $(\pm 1, \dots, \pm 1)$ induce a common cover transitive action on \mathbb{R}^n ?

Also, we found using computers that the group of invertible linear maps on \mathbb{F}_2^3 acting on the the seven nonzero points of \mathbb{F}_2^3 induce an action on B_7 which is CCT. We wonder if this generalizes to other groups of invertible linear maps on finite fields.

Question 11.9. Is the action of $GL_n(\mathbb{F}_q)$ on B_{q^n-1} (induced by the action of $GL_n(\mathbb{F}_q)$ on $(\mathbb{F}_q^n)^*$) CCT? What about the action of $PGL_n(\mathbb{F}_q)$ on B_{q^n-1} ? If not, what about the action of $PGL_n(\mathbb{F}_2)$ on B_{2^n-1} ?

DAVID HEMMINGER, DUKE UNIVERSITY
E-mail address: david.hemminger@duke.edu

AARON LANDESMAN, HARVARD UNIVERSITY
E-mail address: aaronlandesman@gmail.com

ZIJIAN YAO, BROWN UNIVERSITY
E-mail address: zijian.yao@brown.edu