Peckness of Edge Posets

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August 6, 2014

Outline of Talk

Basic Definitions

Definition

Let P be a finite graded poset of rank n, that is:

- Elements of P are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks*
- If $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$
- Define $\mathrm{rk}(x) = k$, where $x \in P_k$.

Definition

A map $f \colon P \to Q$ is a *morphism* from P to Q if $x \leq_P y \implies f(x) \leq_Q f(y)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Peck Posets

Definition

Write $p_i = |P_i|$. P is

- *Rank-symmetric* if $p_i = p_{n-i}$ for all $1 \leq i \leq n$
- *Rank-unimodal* if for some $0 \leq k \leq n$ we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

- *k-Sperner* if no disjoint union of k antichains (sets of pairwise incomparable elements) in P is larger than the disjoint union of the largest k ranks of P
- *Strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$.
- *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Definition

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x|x \in P\}$ and $\{x|x \in P_i\}$

Lemma (Stanley, 1980)

P is Peck if and only if there exists an linear transformation $U\colon V(P) \to V(P)$ such that

- *For every basis element $x \in P$,*

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

- *For all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition

If the Lefschetz map defined by

$$L(x) = \sum_{y \gtrdot x} y$$

satisfies the second condition in the previous lemma, then P is *unitary Peck*.

Outline of Talk

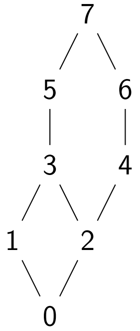
Definition of the Edge Poset

Definition

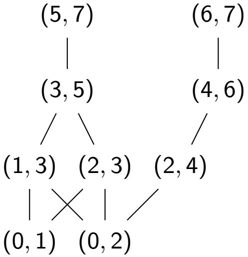
For P a finite graded poset, it's *edge poset* $\mathcal{E}(P)$ is the finite graded poset defined as follows.

- Elements of $\mathcal{E}(P)$ are ordered pairs $(x, y) \in P \times P$ where $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$
- Define $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$
- Define $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$.

Basic Example



P



$\mathcal{E}(P)$

Conjecture on the Peckness of Edge Posets

Definition

The *boolean algebra of rank n* is the poset whose elements are subsets of $[n]$ with order given by containment, i.e. for $x, y \in B_n$, $x \leq y$ if $x \subseteq y$.

Conjecture (Hemminger, Landesman, and Yao 2014)

Let $G \subseteq \mathrm{Aut}(B_n)$. *Then* $\mathcal{E}(B_n/G)$ *is Peck.*

Outline of Talk

Main Result

Definition

A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in \mathrm{Stab}_G(z)$ such that $g \cdot x = y$.

Theorem (Hemminger, Landesman, and Yao 2014)

If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.

Definition

Given a group action of G on P, we define a group action of G on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

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Proposition

The map $q \colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism. Furthermore, q is also injective if and only if the action of G on P is CCT.

Lemma

If $f : P \to Q$ is a bijective morphism and P is Peck then Q is Peck.

Theorem (Stanley, 1984)

If P is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then P/G is Peck.

It would then suffice to show that $\mathcal{E}(B_n)$ is unitary Peck, but our proof for this is complicated. Instead we construct a unitary Peck poset $\mathcal{H}(B_n)$ such that there is a bijective morphism $\mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$.

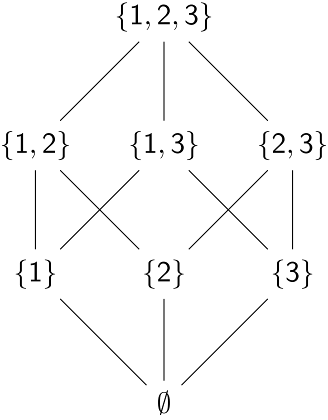
Definition of $\mathcal{H}(P)$

Definition

For P a finite graded poset, define the graded poset $\mathcal{H}(P)$ as follows.

- Elements are pairs $(x, y) \in P \times P$ such that $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{H}} (x', y')$ if $x \lessdot_P x', y \lessdot_P y'$ **and** $\mathbf{y} \neq \mathbf{x}'$
- Define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$
- Define $rk_{\mathcal{H}}(x, y) = rk_P(x)$.

The Boolean Algebra B_3



B_3

$\mathcal{H}(B_3)$ is unitary Peck

$$
\begin{tikzcd}[column sep=tiny]
 & (\{2,3\},\{1,2,3\}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\{2\},\{1,2\}) & & (\{3\},\{1,3\}) \\
 & (\emptyset,\{1\}) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
 & (\{1,3\},\{1,2,3\}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\{1\},\{1,2\}) & & (\{3\},\{2,3\}) \\
 & (\emptyset,\{2\}) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
 & (\{1,2\},\{1,2,3\}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\{1\},\{1,3\}) & & (\{2\},\{2,3\}) \\
 & (\emptyset,\{3\}) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
$$

$\mathcal{H}(B_3)$

Definition

As before, for G acting on P, define $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Remark

Since $\mathcal{E}(P)$ and $\mathcal{H}(P)$ have the same elements and $(x, y) \leq_{\mathcal{H}} (x', y') \implies (x, y) \leq_{\mathcal{E}} (x', y')$, there is a natural bijective morphism $\mathcal{H}(P)/G \to \mathcal{E}(P)/G$.

Proof of Main Result.

$\mathcal{H}(B_n)$ unitary Peck $\implies$ $\mathcal{H}(B_n)/G$ Peck $\implies$ $\mathcal{E}(B_n)/G$ Peck $\implies$ $\mathcal{E}(B_n/G)$ Peck. $\square$

Outline of Talk

CCT actions

Lemma

Let G be a group acting on a graded poset P. The following are equivalent:

1. *The action of G on P is CCT.*
2. *Whenever $x \lessdot y, x \lessdot z$, and $y \in Gz$, there exists some $g \in Stab(x)$ with $gx = z$.*
3. *The map $q\colon \mathcal{E}^r(P)/G \to \mathcal{E}^1(P/G)$ defined by $q(G(x,y)) = (Gx, Gy)$ is an bijective morphism (but not necessarily an isomorphism).*
4. *The map $q\colon \mathcal{E}^r(P)/G \to \mathcal{E}^1(P/G)$ defined by $q(G(x,y)) = (Gx, Gy)$ is an injective morphism.*
5. *For all i there is an equality $|(\mathcal{E}^1(P)/G)_i| = |(\mathcal{E}^1(P/G))_i|$*

Some examples of CCT actions

The direct product

Theorem

For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.

The semi-direct product

Proposition

Let $G \subseteq \mathrm{Aut}(P)$, $H \lhd G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that H acts CC transitively on P and K acts CC transitively on P/H. Then G acts CC transitively on P.

The wreath product

Definition

For G, H groups, with $H \subset S_l$, the *wreath product*, notated $G \wr H$, is the group whose elements are pairs $(g, h) \in G^l \times H$ with multiplication defined by

$$((g'_1, \ldots, g'_l), h') \cdot ((g_1, \ldots, g_l), h) = ((g'_{h'(1)} g_1, \ldots, g'_{h'(l)} g_l), hh')$$

where $h \in H$ acts on $[l]$ by the restriction of the permutation action of S_l to H.

The wreath product

Theorem

If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action is also CCT.

The automorphism of rooted trees

Automorphism of rooted trees

The Dihedral group D_{2p} and D_{4p}

Outline of Talk

Unimodality of ranks of certain edge posets

Outline of Talk

The q analog of the problem

References

Acknowledgements

1. Thanks to Dr. Victor Reiner and Elise DelMas for mentoring and TAing this project.
2. The work for this project took place at the Minnesota at Twin Cities REU. Thanks to Dr. Gregg Musiker and the University of Minnesota School of Mathematics for coordinating and hosting the REU.
3. This research was supported by the RTG grant NSF/DMS-1148634.