UNIMODALITY IDEAS

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1. Directions to move

(1) Look at generalising p_i^r for general r.
(2) Generalizing to q analog of cyclic group.
(3) Try relating p_i, q_i.
(4) Coding which groups G we have $p_i = q_i$.
(5) When are $p_i = q_i$.
(6) Try to compute q_i.
(7) Look at simple groups, and maybe solvable groups, try quotienting by normal subgroups?
(8) Are there any ways to combine G_1, G_2 where G_i are groups with $p_i = q_i$.
(9) Are there some characterisations of groups with q_i, p_i.
(10) How to use sage, what can we do with groups?

2. Cyclic group edges

Remark 2.0.1. *All subscripts will be taken* $\pmod n$.

Theorem 2.0.2. *The statistic p_i as Zijian defined are unimodal for the necklace poset.*

Proof. □

Lemma 2.0.3. $q_i = \binom{n-1}{i-1}$.

Proof. □

Lemma 2.0.4. *The difference $p_i - q_i$ is the number of pairs $(x, y), x \lessdot y$, such that there exists σ so that $\sigma x \lessdot y$, but there does not exist g with $gx = \sigma x, gy = y$.*

Proof. □

Definition 2.0.5. *We call (x, y) a special pair if there exists σ so that $\sigma x \lessdot y$, but there does not exist g with $gx = \sigma x, gy = y$.*

Lemma 2.0.6. *$p_i - q_i$ is the number of orbits of special pairs*

Proof. □

Remark 2.0.7. *We are restricting to the cyclic group C_n, and so we are assuming that all elements are generated by the permutation $c = (12 \cdots n)$. We now wish to bound the number of orbits of special pairs.*

Remark 2.0.8. *To compute the number of orbits of special pairs, we can always assume $x = \{t_1, t_2, \ldots, t_{i-1}\}_<$ and $y = \{t_1, t_2, \ldots, t_i\}_<$ by simply composing with an element of C_n in order to make the missing t_i the biggest element of the set. We may as well assume $t_1 = 1$.*

Lemma 2.0.9. *Suppose (x, y) is a special pair, as in 2.0.8 with $\sigma x \lessdot y$ Then, σ is the permutation sending t_1 to t_2 or t_1 to t_i.*

Proof. Suppose otherwise, that it sent t_1 to t_k for $k \neq 2, i$. Then, since the relative ordering of the elements are preserved, we must have t_l is sent to t_{l+k-1}. Since we can only act by elements of C_n, this gives us that the values $t_{l+1} - t_l = t_{k+l} - t_{k+l-1}$. for all $l \in [i]$. However, this means that $\sigma y = y$, (THAT IS THE TRICKIEST PART TO SEE) so x, y is not a special pair. $\square$

Lemma 2.0.10. *Any special pair (x, y) must have $y = \{t_1, t_1 + a, t_1 + 2a, \cdots, t_1 + (i-1)a\}$ for some $a < n$.*

Proof. By 2.0.9 we must have that σ sends t_1 to t_2 or t_i. Let's assume it sends t_1 to t_2, the other case is similar. However, this means we must send t_l to t_{l+1}, which means $t_l - t_{l-1} = t_{l+1} - t_l$, which means that y is of the claimed form. $\square$

Lemma 2.0.11. *The number of orbits of special pairs is between $n-1$ and $\frac{n}{2} - 1$.*

Proof. By the above, there are at most n orbits (as determined by the value of a) which can have special pairs. However, it is clear that $a = i$ and $a = n - i$ lie in the same orbit. Therefore, we have $\frac{n}{2}$ identifications, which tells us $p_i - q_i \geq \frac{n}{2} - 1$. On the other hand, in the worst possible case, we have all n orbits are equivalent, which implies $p_i - q_i \leq n$. $\square$

Lemma 2.0.12. *The p_i statistics are unimodal.*

Proof. By 2.0.3 we know the differences of the q_i on the nose it is obvious that $q_i - q_{i-1} \geq \frac{n}{2}$. However, by 2.0.11 we have $n - 1 \geq (q_i - p_i) \geq \frac{n}{2} - 1$. Therefore, $p_i - p_{i-1} \geq \frac{n}{2} - \frac{n}{2} \geq 0$. $\square$