UNIMODALITY IDEAS

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1. Directions to move

(1) Look at generalising p_i^r for general r.
(2) Generalizing to q analog of cyclic group.
(3) Try relating p_i, q_i.
(4) Coding which groups G we have $p_i = q_i$.
(5) When are $p_i = q_i$.
(6) Try to compute q_i.
(7) Look at simple groups, and maybe solvable groups, try quotienting by normal subgroups?
(8) Are there any ways to combine G_1, G_2 where G_i are groups with $p_i = q_i$.
(9) Are there some characterisations of groups with q_i, p_i.
(10) How to use sage, what can we do with groups?
(11) Which edge poset definition do we want? Do we include edges containing y or exclude them?
(12) Look at $B_n(q)$.
(13) Look at generalizing $F_r(B_n)$ to arbitrary posets
(14) Try relating Wilson's Normal Form to our posets?

2. Edge Functor

Remark 2.0.1. *We assume all posets are ranked posets, and G actions are rank preserving, order preserving actions.*

Definition 2.0.2. *A poset is B_k full if whenever it contains a vertex v and p vertices above v, then it contains a p dimensional hypercube containing v.*

Lemma 2.0.3. *B_n is B_k full for all k. Quotients of B_n are B_k full.*

Proof. □

Definition 2.0.4. *Define the poset category $\mathcal{P}_r$, where $P \in \mathcal{P}_r$ is a ranked poset, and the morphisms $Mor(P, Q)$ are rank preserving, order preserving maps, which send all B_{r+1} to other B_{r+1}.*

Definition 2.0.5. *Define the Faces functors (there is one for each r) $\mathcal{F}_r : \mathcal{P}_r \to \mathcal{P}_r$, which takes a poset to the poset of its i faces. That is, for each B_k subalgebra of P_r, we associate a point. We say a point $p \lessdot q$ if p and q are nonintersecting boolean subalgebras, and the bottommost point of the cube representing p is right below the bottommost point of the cube representing q. It takes a map of posets to the induced map on cubes, by definition of the morphisms in $\mathcal{P}_r$. For ease of notation, we shall use $\mathcal{F}$ for $\mathcal{F}_1$.*

3. The Picture for B_n

Theorem 3.0.6. *For B_n the boolean algebra, $\mathcal{F}(B_n)$ is unitary peck.*

Proof. $\mathcal{F}(B_n)$ is actually just a disjoint union of n copies of B_{n-1}, where each copy is indexed as corresponding to the set of pairs $B_{n-1} \cong (\mathcal{F}(B_n))_{(i)} = \{(y,x) | y \gtrdot x, y/x = i\}$, where $i \in [n]$. $\square$

Notation 3.0.7. *Let $\Delta(G) \subset G \times G$ denote the diagonal subgroup. Define $X_G(P) = Ind_{\Delta(G)}^{G\times G}(\mathcal{F}(V(P)))/(G \times G)$*

Notation 3.0.8. *Let $V(P)$ denote the graded vector space with basis $p \in P$. The grading is given by the rank of p.*

Theorem 3.0.9. *We then have an isomorphism of graded vector spaces $\mathcal{F}(V(P)/G) \cong X_G(P)$.*

Proof. The basis for $F(V(P)/G)$ is exactly given by the edges of $V(P)/G$. By definition, we have an edge (Gx, Gy) in $V(P)/G$ if and only if and $\exists g, h \in G$ with $gx \lessdot hy$. Next, $Ind_{\Delta(G)}^{G\times G}(\mathcal{F}(V(P)))$ is precisely the set of all edges of the form gx, hy for $g, h \in G$. And hence, we have a natural $G \times G$ action on it. Then, by definition, if we quotient by the $G \times G$ action, we obtain the exact same set of edges as in $\mathcal{F}(V(P)/G)$. Since ranks are always preserved under these maps, we obtain the claimed isomorphism of graded vector spaces. $\square$

Lemma 3.0.10. *The poset corresponding the the graded vector space $Ind_{\Delta(G\times G)}^{G\times G}(F(V(P)))$ is unitary peck.*

Proof. First, by 3.0.6, we know $\mathcal{F}(V(P))$ is unitary peck. Then, induction simply makes $|G|$ disjoint copies of $F(V(P))$. Therefore, we can take the corresponding blcok diagonal raising operators for each disjoint copy, and they obviously provide isomorphisms from level i to $n-i$. $\square$

Theorem 3.0.11. *(Stanley) The quotient of a unitary peck poset by an order preserving, rank preserving group action G, is peck.*

Corollary 3.0.12. *The $X_G(P)$ are vector spaces with an underlying peck poset structure.*

Proof. By 3.0.10, we have $Ind_{\Delta(G)}^{G\times G}(\mathcal{F}(V(P)))$ is unitary peck. But then, since $X_G(P) = Ind_{\Delta(G)}^{G\times G}(\mathcal{F}(V(P)))/(G\times G)$, by 3.0.11 we obtain the poset corresponding to $X_G(P)$ is peck. $\square$

Corollary 3.0.13. *The poset of edges $\mathcal{F}(V(P)/G)$ is unitary peck.*

Proof. By 3.0.12 we know $X_G(P)$ is peck, but by 3.0.9 we have $\mathcal{F}(V(P)/G) \cong X_G(P)$, and so $\mathcal{F}(V(P)/G)$ is peck as well. $\square$

Notation 3.0.14. *For any poset P, define p_i to be the number of edges from the ith level set P_i to the $i+1$th level set P_{i+1},*

Corollary 3.0.15. *The sequence p_i is unimodal and symmetric.*

Proof. By definition, the rank of the ith level set of $\mathcal{F}(V(P)/G)$ is exactly the number of edges from levels i to i+1 of P. That is $\dim(\mathcal{F}(V(P)/G)_i) = p_i$. Since by 3.0.13, $\mathcal{F}(V(P)/G)$ is peck, it is in particular symmetric and unimodal, and so the p_i are symmetric and unimodal. $\square$