

## Peckness of Edge Posets

David Hemminger · Aaron Landesman · Zijian Yao

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**Abstract** For any graded poset  $P$ , we define a new graded poset,  $\mathcal{E}(P)$ , whose elements are the edges in the Hasse diagram of  $P$ . For any group,  $G$ , acting on the boolean algebra,  $B_n$ , in a rank-preserving fashion we conjecture that  $\mathcal{E}(B_n/G)$  is Peck. We prove that the conjecture holds for “common cover transitive” actions. We give some infinite families of common cover transitive actions and show that the common cover transitive actions are closed under direct and semidirect products.

**Keywords** Boolean Algebra · Edges · Group Actions · Peck Posets · Quotient Posets · Unimodality

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D. Hemminger  
Duke University, 6300 Lakeland Dr., Raleigh, NC 27612 E-mail: dhemminger22@gmail.com

A. Landesman  
Harvard University, 323 Eliot Mail Center, Cambridge, MA 02138 E-mail: aaronlandesman@gmail.com

Z. Yao  
Brown University 7285 Brown University, 69 Brown Street, Providence, RI 02912 E-mail: yaozijian1992@gmail.com

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■ Proof of Lemma 3.9 - your notation $(x,y) \rightarrow (x,y)$ is a bit confusing (and runs into the margin). Perhaps explain in words that you are mapping edges to edges. Aaron: I made these edits suggested. I'm not certain whether it is clearer. What do you think? . . . . .	8
■ Example 3.11: Figure 3.1, not 3.2. – As far as I can tell the reference is right, so I'm not sure why we're getting the wrong figure number on compilation. - David Aaron: Me neither. We can just manually reference this figure if we can't figure out how to fix it. . . . .	8
■ Aaron: Is this sufficient detail, or should we spell it out . . . . .	9
■ Remark 6.1 should be converted to a corollary and should be moved to the introduction, to immediately follow Theorem 1.5. Aaron: I'm fine with moving this to the introduction, but I don't think we should call it a corollary. Should we perhaps make this a separate theorem or proposition, and just refer people to the REU report? . . . . .	19
■ References - check if the Pak and Panova article has appeared. Also, several words should be capitalized in the bibliography - Sperner (twice), Lefschetz, Volume. – I fixed everything except for capitalizing "Volume", which is up to Bibtex's bibliography style. I think this style is the one Order asks for, so we shouldn't change anything? - David Aaron: I think it's fine not to change anything else, just mention what we don't change in the additional document. . . . .	20
List of Todos	

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## 1 Introduction

Let  $P$  be a finite graded poset of rank  $n$ . In this paper we study the structure of the edges in the Hasse diagram of  $P$ . To this end, we define an endofunctor  $\mathcal{E}$  on the category of finite graded posets with rank-preserving morphisms as follows

**Definition 1.1** For  $\mathcal{P}$  the category of graded posets, define the *functor of edges*  $\mathcal{E}: \mathcal{P} \rightarrow \mathcal{P}$  as follows. Given  $P \in \mathcal{P}$ , the elements of the graded poset  $\mathcal{E}(P)$  are pairs  $(x, y)$  where  $x, y \in P$ ,  $x \leq_P y$ , and  $\text{rk}(y) = \text{rk}(x) + 1$ . Define the covering relation  $\leq_{\mathcal{E}}$  on  $\mathcal{E}(P)$  by  $(x, y) \leq_{\mathcal{E}} (x', y')$  if  $x \leq_P x'$  and  $y \leq_P y'$ . Then define the relation  $\leq_{\mathcal{E}}$  on  $\mathcal{E}(P)$  to be the transitive closure of  $\leq_{\mathcal{E}}$ .

Let  $Q$  be a finite graded poset of rank  $n$ . Given a morphism  $f: P \rightarrow Q$ , define  $\mathcal{E}(f): \mathcal{E}(P) \rightarrow \mathcal{E}(Q)$  by  $\mathcal{E}(f)(x, y) = (f(x), f(y))$ .

We will show that  $\mathcal{E}(P)$  is a well-defined graded poset in Section 3. Note that an edge in the Hasse diagram of  $P$  can be identified with a pair  $(x, y) \in P \times P$  such that  $x \leq y$ , and the edges in the Hasse diagram are in bijection with elements  $(x, y) \in \mathcal{E}(P)$  via this identification. With this in mind, we will frequently refer to  $\mathcal{E}(P)$  as the *edge poset* of  $P$ .

*Example 1.2* We give an example of an edge poset in Figure 1.1. Note that it is important we declare the relation  $\leq_{\mathcal{E}}$  to be the transitive closure of  $\leq_{\mathcal{E}}$ . If instead we defined a relation  $\leq_{\mathcal{E}'}$  on  $\mathcal{E}(P)$  by  $(x, y) \leq_{\mathcal{E}'} (a, b)$  if  $x \leq a, y \leq b$ , then  $\mathcal{E}(P)$  would not necessarily be a graded poset. In Figure 1.1 we give an example of a poset  $P$  for which  $\mathcal{E}(P)$  is not graded with the relation  $\leq_{\mathcal{E}'}$ . In Figure 1.1 it is clear that  $\mathcal{E}(P)$  is a graded poset, with  $\text{rk}(x, y) = \text{rk}(x)$ , but the Hasse diagram on the right represents a poset which does not have a grading.

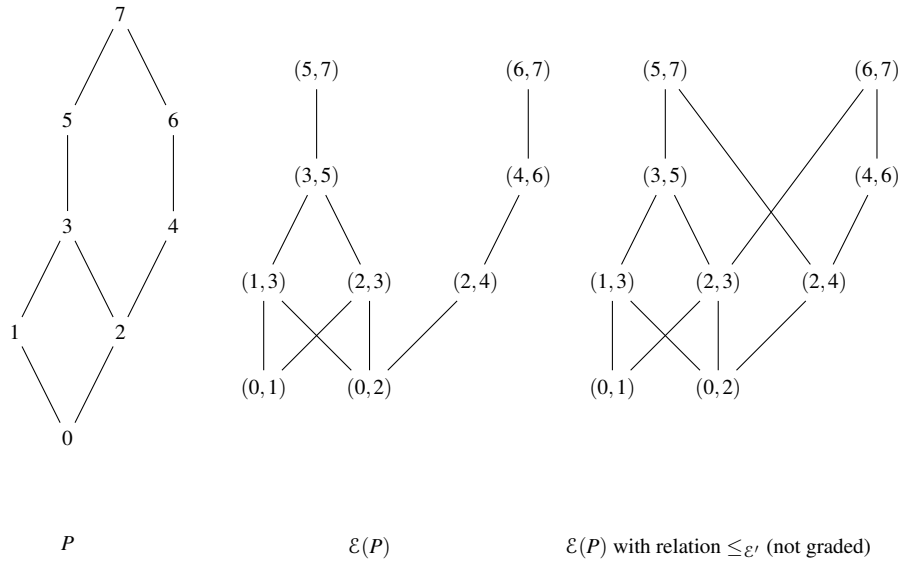


Fig. 1.1 Examples of  $\mathcal{E}$

We observe that when  $P$  has a nice structure,  $\mathcal{E}(P)$  commonly has a nice structure as well. In particular we examine the *boolean algebra of rank  $n$* , denoted  $B_n$ , which is defined to be the poset whose elements are subsets of  $\{1, \dots, n\}$  with the relation given by containment, namely, for all  $x, y \in B_n$ ,  $x \leq y$  if  $x$  is a subset of  $y$ .

Throughout the paper we say that a group  $G$  acts on  $P$  if it acts on the elements of  $P$  and the action is order-preserving and rank-preserving, that is, for all  $g \in G$  we have  $x \leq y \Leftrightarrow gx \leq gy$  and  $\text{rk}(gx) = \text{rk}(x)$ . By Theorem 2.6 (due to Stanley) and the fact that  $B_n$  is unitary peck, if  $G$  is any action on  $B_n$ , then  $B_n/G$  is Peck. We conjecture the following.

**Conjecture 1.3** If  $G \subseteq \text{Aut}(B_n)$ , then  $\mathcal{E}(B_n/G)$  is Peck.

We prove this conjecture holds whenever the group action of  $G$  on  $B_n$  has the *common cover transitive* property, which we introduce in the following definition.

**Definition 1.4** A group action of  $G$  on  $P$  is *common cover transitive* (CCT) if whenever  $x, y, z \in P$  are such that  $x \lessdot z$ ,  $y \lessdot z$ , and  $y \in Gx$ , then there exists some  $g \in \text{Stab}(z)$  such that  $g \cdot x = y$ .

**Theorem 1.5** If a group action of  $G$  on  $B_n$  is CCT, then  $\mathcal{E}(B_n/G)$  is Peck.

We have found many group actions on  $B_n$  that have the CCT property. We first prove that some basic group actions on  $B_n$  are CCT. Throughout the paper we let a subgroup  $G \subseteq S_n$  act on  $B_n$  by letting it act on the elements within subsets of  $[n] := \{1, \dots, n\}$ , i.e.  $g \cdot x = \{g \cdot i : i \in x\}$  for all  $g \in G$ ,  $x \in B_n$ . We also embed the dihedral group  $D_{2n}$  into  $S_n$  by letting it act as rotations and reflections on the vertices of an  $n$ -gon.

**Proposition 1.6** Let  $n$  be a positive integer and  $p$  be prime. The following actions are CCT.

1. The action of  $S_n$  on  $B_n$ ,
2. The action of  $D_{2p}$  on  $B_p$ ,
3. The action of  $D_{4p}$  on  $B_{2p}$ ,

We further show that cover transitivity is preserved under semidirect products, allowing us to describe several large families of CCT actions in Subsection 4.2.

**Proposition 1.7** Let  $G \subseteq \text{Aut}(P)$ ,  $H \triangleleft G$ ,  $K \subset G$  such that  $G = H \rtimes K$ . Suppose that the action of  $H$  is on  $P$  is CCT and the action of  $K$  on  $P/H$  is CCT. Then the action of  $G$  on  $P$  is CCT.

Proposition 4.3 - should call it Proposition 1.7

The paper is organized as follows. In Section 2 we cover the necessary background for posets and Peck posets. In Section 3 we show that  $\mathcal{E}$  is well-defined and prove Theorem 3.4 regarding CCT actions along with various other nice properties of  $\mathcal{E}$ . Section 4 contains the proofs of Propositions 1.6 and 1.7 as well as some examples of families of group actions shown to be CCT by these propositions. In Section 5, we obtain a very different proof of [3, Theorem 1.1], in the case that  $r = 1$ .

## 2 Background

In this section we review necessary background for this paper. As references for this material, see [8, Chapter 4] and [6].

A *graded poset*  $P$  is a poset with a rank function  $\text{rk}: P \rightarrow \mathbb{Z}_{\geq 0}$  satisfying the following conditions.

1. If  $x \in P$  and  $x \lessdot y$ , then  $\text{rk}(x) + 1 = \text{rk}(y)$ ,
2. If  $x < y$  then  $\text{rk}(x) < \text{rk}(y)$

Note that the second condition follows from the first in the case that the poset  $P$  is finite.

We denote the  $i$ th rank of  $P$  by  $P_i = \{x \in P : \text{rk}(x) = i\}$ . Additionally, if for all  $x \in P$ , we have  $0 \leq \text{rk}(x) \leq n$ , and there exists  $x, y$  with  $\text{rk}(x) = 0, \text{rk}(y) = n$ , we say that  $P$  is a graded poset of *rank*  $n$ .

Throughout the paper we write  $x \leq_P y$  to denote that  $x$  is less than or equal to  $y$  under the relation defined on the poset  $P$ . When the poset is clear we omit the  $P$  and simply write  $x \leq y$ .

Let  $P, Q \in \mathcal{P}$  be two finite graded posets. A map  $f: P \rightarrow Q$  is a *morphism* from  $P$  to  $Q$  if it is rank-preserving and order preserving, in other words, for all  $x, y \in P$ ,  $x \leq_P y$  implies  $f(x) \leq_Q f(y)$  and  $\text{rk}(x) = \text{rk}(f(x))$ . We say that  $f$  is *injective/surjective/bijective* if it is an injection/surjection/bijection from  $P$  to  $Q$  as sets.

**Remark 2.1** Note that we do not require the implication that  $f(x) \leq_Q f(y)$  implies  $x \leq_P y$  in order for  $f$  to be a morphism. In particular this means that a bijective morphism  $f$  need not be an isomorphism, since it will not necessarily have a two-sided inverse.

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In what follows, let  $P$  be a poset of rank  $n$ , and write  $p_i = |P_i|$ . If we have

$$p_0 \leq p_1 \leq \dots \leq p_k \geq p_{k+1} \geq \dots \geq p_n$$

for some  $0 \leq k \leq n$ , then  $P$  is *rank-unimodal*. If  $p_i = p_{n-i}$  for all  $1 \leq i \leq n$ , then  $P$  is *rank-symmetric*. An *antichain* in  $P$  is a set of elements in  $P$  that are pairwise incomparable. If no antichain in  $P$  is larger than the largest rank of  $P$ , then  $P$  is *Sperner*. More generally,  $P$  is *k-Sperner* if no union of  $k$  disjoint antichains in  $P$  is larger than the union of the largest  $k$  ranks of  $P$ . We say that  $P$  is *strongly Sperner* if it is  $k$ -Sperner for all  $1 \leq k \leq n$ .

**Definition 2.2**  $P$  is *Peck* if  $P$  is rank-symmetric, rank-unimodal, and strongly Sperner.

Let  $V(P)$  and  $V(P_i)$  be the complex vector spaces with bases  $\{x : x \in P\}$  and  $\{x : x \in P_i\}$  respectively. Note that we will frequently abuse notation and write  $P$  and  $P_i$  for  $V(P)$  and  $V(P_i)$  when what we mean is clear from context. In determining whether  $P$  is Peck, it is often useful to consider certain linear transformations on  $V(P)$ .

**Definition 2.3** A linear map  $U : V(P) \rightarrow V(P)$  is an *order-raising operator* if  $U(V(P_n)) = 0$  and for all  $0 \leq i \leq n-1$ ,  $x \in P_i$  we have

$$U(x) = \sum_{y \succ x} c_{x,y} y$$

for some constants  $c_{x,y} \in \mathbb{C}$ . We say that  $U$  is the *Lefschetz map* if all  $c_{x,y}$  on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

**Lemma 2.4 ([5], Lemma 1.1)**  $P$  is Peck if and only if there exists an order-raising operator  $U$  such that for all  $0 \leq i < \frac{n}{2}$ , the map  $U^{n-2i} : V(P_i) \rightarrow V(P_{n-i})$  is an isomorphism.

**Definition 2.5** If the Lefschetz map satisfies the condition for  $U$  in Lemma 2.4, then  $P$  is *unitary Peck*.

Note that a group  $G$  acts on  $P$  if the action defines an embedding  $G \hookrightarrow \text{Aut}(P)$ . We define the *quotient poset*  $P/G$  to be the poset whose elements are the orbits of  $G$ , with the relation  $\mathcal{O} \leq \mathcal{O}'$  if there exist  $x \in \mathcal{O}$ ,  $x' \in \mathcal{O}'$  such that  $x \leq_P x'$ . We will use the following result in the paper.

**Theorem 2.6 ([6], Theorem 1)** If  $P$  is unitary Peck and  $G \subseteq \text{Aut}(P)$ , then  $P/G$  is Peck.

### 3 The Edge Poset

In Subsection 3.1 we show that  $\mathcal{E}$  as described in Definition 1.1 is well-defined and prove some useful properties of  $\mathcal{E}$ . In Subsection 3.2 we prove that  $\mathcal{E}$  sends self-dual posets to self-dual posets. In Subsection 3.3 we give several equivalent definitions for CCT actions, and Subsection 3.4 is devoted to the proof of Theorem 3.4.

#### 3.1 Functoriality of $\mathcal{E}$ and Group Actions

First we show that  $\mathcal{E}$  is well-defined in Lemmas 3.1, 3.2, and 3.3. After showing that  $\mathcal{E}$  is well-defined we then define a natural  $G$  action on  $\mathcal{E}(P)$  and define a surjection  $\mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$  that will be important for the proof of Theorem 3.4.

When the poset  $P$  is clear, we will use  $\leq_{\mathcal{E}}$  and  $\prec_{\mathcal{E}}$  to refer to  $\leq_{\mathcal{E}(P)}$  and  $\prec_{\mathcal{E}(P)}$ . Similarly, in Subsection 3.4, we define posets  $\mathcal{H}(B_n)$ , and will use  $\leq_{\mathcal{H}}$  and  $\prec_{\mathcal{H}}$  in place of  $\leq_{\mathcal{H}(B_n)}$  and  $\prec_{\mathcal{H}(B_n)}$ .

**Lemma 3.1** The relation  $\leq_{\mathcal{E}}$  defines a partial order on  $\mathcal{E}(P)$ .

*Proof.* We have that  $(x, y) \leq_{\mathcal{E}} (x, y)$  and that  $\leq_{\mathcal{E}}$  is transitive by definition. Suppose  $(x, y) \leq_{\mathcal{E}} (x', y')$  and  $(x', y') \leq_{\mathcal{E}} (x, y)$ . Then  $x \leq_P x' \leq_P x$  and  $y \leq_P y' \leq_P y$ , so  $x = x'$  and  $y = y'$  by antisymmetry of  $\leq_P$ , hence  $(x, y) = (x', y')$ .  $\square$

**Lemma 3.2** For  $P$  a graded poset, the object  $\mathcal{E}(P)$  is a graded poset.

*Proof.* To show  $\mathcal{E}(P)$  is graded, we must show that  $(x, y) \leq_{\mathcal{E}} (x', y') \implies \text{rk}(x, y) + 1 = \text{rk}(x', y')$ . This fact follows immediately from the definition of  $\leq_{\mathcal{E}}$  and the definition  $\text{rk}_{\mathcal{E}}(x, y) = \text{rk}_P(x)$ .  $\square$

Proof of Lemma 3.3. Insert 'rank-' after 'order' and before 'preserving'. The E subscripts in Part 1 should be written as  $\mathcal{E}(P)$  or  $\mathcal{E}(Q)$ . Here there are two 'P's that should be 'Q's. Aaron: I made all the edits here, except the one about inserting 'rank-'. Do either of you understand what this means? does he want us to call it "order-rank-preserving"?

**Lemma 3.3** *Let  $f: P \rightarrow Q$  be a morphism of finite graded posets, and define a map  $\mathcal{E}(f): \mathcal{E}(P) \rightarrow \mathcal{E}(Q)$  by  $\mathcal{E}(f)(x, y) = (f(x), f(y))$  for all  $(x, y) \in \mathcal{E}(P)$ . Then*

1.  $\mathcal{E}(f)$  is a morphism of finite graded posets
2.  $\mathcal{E}(\text{id}_P) = \text{id}_{\mathcal{E}(P)}$
3. If  $g: Q \rightarrow R$  is a morphism of finite graded posets, then  $\mathcal{E}(g \circ f) = \mathcal{E}(g) \circ \mathcal{E}(f)$ .

*Proof.*

First, we show 1. First,  $\mathcal{E}(f)$  is rank-preserving, since for all  $(x, y) \in \mathcal{E}(P)$  we have

$$\text{rk}_{\mathcal{E}(P)}(x, y) = \text{rk}_P(x) = \text{rk}_Q(f(x)) = \text{rk}_{\mathcal{E}(Q)}(\mathcal{E}(f)(x, y)).$$

Suppose  $(x, y) \leq_{\mathcal{E}(P)} (x', y')$ . Then  $x \leq_P x'$ ,  $y \leq_P y'$ , and since  $f$  is order-preserving, it follows that  $f(x) \leq_Q f(x')$ ,  $f(y) \leq_Q f(y')$ . Hence  $\mathcal{E}(f)(x, y) \leq_{\mathcal{E}(Q)} \mathcal{E}(f)(x', y')$ . Since  $\leq_{\mathcal{E}(Q)}$  is the transitive closure of  $\leq_{\mathcal{E}(Q)}$ , we similarly obtain  $\mathcal{E}(f)$  is order-preserving and hence a morphism of finite graded posets.

Next, 2 is trivial.

Finally, we show 3. For all  $(x, y) \in \mathcal{E}(P)$  we have

$$\mathcal{E}(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}(g) \circ \mathcal{E}(f))(x, y).$$

$\square$

**Remark 3.4** By Lemmas 3.1, 3.2, and 3.3, the edge poset construction  $\mathcal{E}$  defines an endofunctor on the category of finite graded posets with rank-preserving morphisms.

Given a group action of  $G$  on  $P$ , we can now easily define a natural group action of  $G$  on  $\mathcal{E}(P)$  using Lemma 3.3. For all  $g \in G$  we have that application of  $g$  is an automorphism of  $P$ , so it follows that  $\mathcal{E}(g)$  is an automorphism of  $\mathcal{E}(P)$  and furthermore that this gives a well-defined group action by Lemma 3.3.

**Definition 3.5** Given a  $G$ -action on  $P$ , define a  $G$ -action on  $\mathcal{E}(P)$  by  $g \cdot (x, y) = \mathcal{E}(g)(x, y) = (gx, gy)$ .

We then have a well-defined quotient poset  $\mathcal{E}(P)/G$ . It is natural to ask whether the operation of quotienting out by  $G$  commutes with  $\mathcal{E}$ , that is, whether  $\mathcal{E}(P/G) \cong \mathcal{E}(P)/G$ . Unfortunately the two posets are rarely isomorphic, but there is always a surjection  $\mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ , and this surjection is also an injection precisely when the  $G$ -action on  $P$  is CCT, as will be shown in Lemma 3.14.

**Proposition 3.6** *The map  $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$  defined by  $q(G(x, y)) = (Gx, Gy)$  is a surjective morphism.*

*Proof.* Note that  $q$  is well defined because if  $(x', y') = g(x, y) = (g \cdot x, g \cdot y)$  for some  $g \in G$ , then  $x' \in Gx$  and  $y' \in Gy$ . Clearly  $q$  is rank-preserving and surjective, so it suffices to show that  $q$  is order-preserving. Suppose that  $G(x, y) \leq_{\mathcal{E}(P)/G} G(w, z)$ . Then there exist some  $(x_0, y_0) \in G(x, y)$ ,  $(w_0, z_0) \in G(w, z)$  such that  $x_0 \leq_P w_0$  and  $y_0 \leq_P z_0$ . We then have that  $(Gx, Gy) \leq_{\mathcal{E}(P/G)} (Gw, Gz)$  by definition. Since  $\leq_{\mathcal{E}(P/G)}$  is the transitive closure of  $\leq_{\mathcal{E}(P/G)}$ ,  $q$  is order-preserving.  $\square$

### 3.2 The Opposite Functor and Self Dual Posets

Next, we introduce the notion of a dual poset, given by applying the opposite functor,  $\text{op}$ , to a graded poset. We will show that  $\text{op}$  commutes with  $\mathcal{E}$ . This will imply that  $\mathcal{E}(P)$  is self-dual if  $P$  is, which, in turn, will imply that  $\mathcal{E}(B_n/G)$  is self-dual for any group action of  $G$  on  $B_n$ .

**Definition 3.7** Let  $\mathcal{P}$  be the category of graded posets and let  $\text{op}: \mathcal{P} \rightarrow \mathcal{P}$  be the opposite functor, defined on posets as follows. For  $P$  a poset, the elements of  $P^{\text{op}}$  are the same as those of  $P$  with order relation  $\leq_{P^{\text{op}}}$  defined by  $x \leq_{P^{\text{op}}} y \Leftrightarrow x \geq_P y$ . Induced maps on morphisms are given as follows: for  $P, Q$  graded posets with  $f: P \rightarrow Q$ , then  $f^{\text{op}}: P^{\text{op}} \rightarrow Q^{\text{op}}$  is defined by  $f^{\text{op}}(x) = f(x)$ . The poset  $P^{\text{op}}$  is called the *dual* poset of  $P$ . A poset  $P$  is *self-dual* if there is an isomorphism of posets  $P \cong P^{\text{op}}$ .

**Remark 3.8** Note that it is easy to check  $\text{op}: \mathcal{P} \rightarrow \mathcal{P}$  is indeed a covariant functor. In more abstract terms, if we view  $P$  as a category,  $P^{\text{op}}$  is the opposite category. Additionally,  $\text{op}$  as defined in this way is actually an endofunctor on the category of all finite posets, which restricts to a functor on the subcategory of graded posets.

**Lemma 3.9** *The functor  $\text{op}: \mathcal{P} \rightarrow \mathcal{P}$  commutes with the functor  $\mathcal{E}: \mathcal{P} \rightarrow \mathcal{P}$ . That is,  $\mathcal{E}(P^{\text{op}}) \cong \mathcal{E}(P)^{\text{op}}$ .*

Proof of Lemma 3.9  
- your notation  $(x,y)$   
-  $\mathcal{E}(x,y)$  is a bit confusing (and runs into the margin). Perhaps explain in words that you are mapping edges to edges. Aaron: I made these edits suggested. I'm not certain whether it is clearer. What do you think?

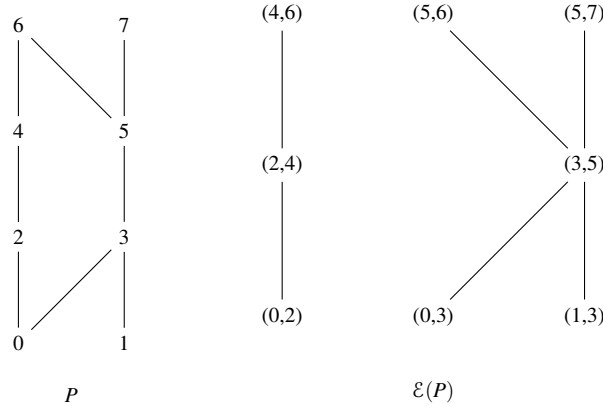
*Proof.* Observe that  $\mathcal{E}(P^{\text{op}})$  is canonically isomorphic to  $\mathcal{E}(P)^{\text{op}}$ , as given by the morphism  $F: \mathcal{E}(P^{\text{op}}) \rightarrow \mathcal{E}(P)^{\text{op}}$ , which sends an edge between two vertices  $x$  and  $y$  of  $P^{\text{op}}$  to the edge between same two vertices  $x$  and  $y$  of  $P$ . The inverse to  $F$  is given by  $G: \mathcal{E}(P)^{\text{op}} \rightarrow \mathcal{E}(P^{\text{op}})$  sending an edge between two vertices  $x$  and  $y$  of  $P$  to the edge between the same two vertices  $x$  and  $y$  of  $P^{\text{op}}$ . These maps are well defined because  $\mathcal{E}(P)^{\text{op}}$  and  $\mathcal{E}(P^{\text{op}})$  are the same as sets, and it follows from the definitions that these are both morphisms of graded posets, and so  $F$  defines an isomorphism.  $\square$

**Proposition 3.10** *If  $P$  is a self-dual poset, then  $\mathcal{E}(P)$  is also self-dual.*

*Proof.* Since  $P$  is self-dual, there is an isomorphism  $f: P \rightarrow P^{\text{op}}$ . By functoriality of  $\mathcal{E}$ , as shown in Lemma 3.3, we obtain that  $\mathcal{E}(f): \mathcal{E}(P) \rightarrow \mathcal{E}(P^{\text{op}})$  is an isomorphism. By Lemma 3.9, there is an isomorphism  $\mathcal{E}(P^{\text{op}}) \cong \mathcal{E}(P)^{\text{op}}$ . Then, letting  $F: \mathcal{E}(P^{\text{op}}) \rightarrow \mathcal{E}(P)^{\text{op}}$ ,  $(x,y) \mapsto (x,y)$  be the same isomorphism defined in the proof of Lemma 3.9, the composition  $F \circ \mathcal{E}(f): \mathcal{E}(P) \rightarrow \mathcal{E}(P)^{\text{op}}$  defines an isomorphism, so  $\mathcal{E}(P)$  is self-dual.  $\square$

Example 3.11: Figure 3.1, not 3.2.  
- As far as I can tell the reference is right, so I'm not sure why we're getting the wrong figure number on compilation. - David Aaron: Me neither. We can just manually reference this figure if we can't figure out how to fix it.

**Example 3.11** While  $\mathcal{E}(P)$  is often Peck for known Peck posets  $P$ ,  $\mathcal{E}(P)$  need not be Peck in general. Furthermore, adding the condition that  $P$  be self-dual does not change this fact. In Figure 3.2 we give an example of a poset  $P$  such that  $P$  is unitary Peck and self-dual, but  $\mathcal{E}(P)$  is not rank-unimodal, hence not Peck.



**Fig. 3.1**  $P$  is self-dual and unitary Peck, but  $\mathcal{E}(P)$  is not Peck

**Remark 3.12** Whenever there is an action  $\psi: G \times [n] \rightarrow [n]$ , we obtain an induced action  $\phi: G \times B_n \rightarrow B_n$  defined by

$$\phi(g, \{x_1, \dots, x_k\}) = \{\psi(g, x_1), \dots, \psi(g, x_k)\}.$$



It is easy to see that any action  $\phi: G \times B_n \rightarrow B_n$  arises in this way. That is, for any action  $\phi$  of  $G$  on  $B_n$  there exists an action  $\psi$  of  $G$  on  $[n]$  such that  $\phi(g, \{x_1, \dots, x_k\}) = \{\psi(g, x_1), \dots, \psi(g, x_k)\}$ . Whenever an action  $\psi$  of  $G$  on  $[n]$  is given, we refer to the action  $\phi$  defined above as the *induced action* on  $B_n$ .

**Corollary 3.13** *For any action  $\phi: G \times B_n \rightarrow B_n$ , we have that*

- $B_n/G$ ,
- $\mathcal{E}(B_n/G)$ , and
- $\mathcal{E}(B_n)/G$

*are self-dual. In particular, they are all rank-symmetric*

*Proof.* By Remark 3.12, any action  $\phi: G \times B_n \rightarrow B_n$  is induced by an action  $\psi: G \times [n] \rightarrow [n]$ . Using this, observe that for any  $\phi$ , the poset  $B_n/G$  is self-dual, as there is an isomorphism  $f: B_n/G \rightarrow (B_n/G)^{\text{op}}$ ,  $G \cdot x \mapsto G \cdot ([n] \setminus x)$ . This map is well defined on  $G$  orbits because every action on  $B_n$  is induced by an action on  $[n]$ . Then, by Proposition 3.10, it follows that  $\mathcal{E}(B_n/G)$  is self-dual.

It only remains to prove that  $\mathcal{E}(B_n)/G$  is self-dual. However, from Proposition 3.10,  $\mathcal{E}(B_n)$  is self-dual, with the isomorphism given by  $\mathcal{E}(f): \mathcal{E}(B_n) \rightarrow \mathcal{E}(B_n^{\text{op}}) \cong \mathcal{E}(B_n)^{\text{op}}$ , that is, the map sending a  $(x, y) \mapsto ([n] \setminus y, [n] \setminus x)$ . Once again, since the action on  $B_n$  is induced by an action on  $[n]$ , this isomorphism descends to an isomorphism  $\mathcal{E}(f)^G: \mathcal{E}(B_n)/G \rightarrow (\mathcal{E}(B_n)/G)^{\text{op}}$ , and so  $\mathcal{E}(B_n)/G$  is self-dual.  $\square$

### 3.3 Equivalent Definitions of CCT Actions

We next give four equivalent definitions of CCT actions. The equivalence of (1) and (2) in the following Lemma 3.14 uses the notion of dual posets, while the equivalence of (1), (3), and (4) use the fact that  $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$ ,  $G(x, y) \mapsto (Gx, Gy)$  is always a surjection.

**Lemma 3.14** *Let  $G$  be a group acting on a graded poset  $P$ . The following are equivalent:*

1. *The action of  $G$  on  $P$  is CCT.*
2. *Whenever  $x < y, x < z$ , and  $y \in Gz$ , there exists some  $g \in \text{Stab}(x)$  with  $gx = z$ .*
3. *The map  $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$  defined by  $q(G(x, y)) = (Gx, Gy)$  is a bijective morphism (but not necessarily an isomorphism).*
4. *For all  $i$  there is an equality  $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$ .*

*Proof.* First, we show (1)  $\Leftrightarrow$  (3). By Proposition 3.6 we know that  $q$  is a bijection exactly when there do not exist distinct orbits  $G(x, y) \neq G(x', y')$  with  $x' \in Gx$ ,  $y' \in Gy$ . Fix  $(x, y), (x', y') \in \mathcal{E}(P)$  such that  $x' \in Gx$  and  $y' \in Gy$ . Pick a  $g \in G$  such that  $g \cdot x' = y$ . Then  $(g \cdot x', y) \in G(x', y')$ , so  $G(x, y) = G(x', y')$  if and only if there exists some  $g' \in G$  such that  $g' \cdot x = g \cdot x'$  and  $g' \cdot y = y$ . Hence  $q$  is a bijection if and only if the  $G$  action is CCT.

Second, (2)  $\Leftrightarrow$  (3) by an analogous argument to (1)  $\Leftrightarrow$  (3). Finally, we check (3)  $\Leftrightarrow$  (4). Again using Proposition 3.6, the morphism  $q$  is always surjective. Since a morphism is always rank preserving, it must map  $(\mathcal{E}(P)/G)_i$  surjectively onto  $(\mathcal{E}(P/G))_i$ . However, since the posets are finite, this surjection is a bijection if and only if the sets have the same cardinality.  $\square$

Aaron: Is this sufficient detail, or should we spell it out

**Remark 3.15** While  $q$  is a bijection if and only if the action of  $G$  on  $P$  is CCT, it is *not* true that if the action of  $G$  on  $P$  is CCT, then  $q$  is an isomorphism. For example, take  $G = D_{20} \subset S_{10}$  acting by reflections and rotations on  $\{1, 2, \dots, 10\}$  and consider the induced action on  $B_{10}$ . From Proposition 1.6, this action is CCT. However, consider  $x = \{2, 4\}, y = \{1, 2, 4\}, a = \{2, 4, 7\}, b = \{2, 4, 6, 7\}$ . Then we may observe  $(x, y), (a, b) \in \mathcal{E}(B_{10})$  and  $Gx < Ga, Gy < Gb$ , so  $(Gx, Gy) <_{\mathcal{E}} (Ga, Gb)$ . However, it is not true that  $G(x, y) <_{\mathcal{E}} G(a, b)$ .

### 3.4 Proof of Theorem 3.4

In this section we prove Theorem 3.4, which we recall here:

**Theorem 1.5** *If a group action of  $G$  on  $B_n$  is CCT, then  $\mathcal{E}(B_n/G)$  is Peck.*

The proof is largely based on the following Lemma.

**Lemma 3.16** *Let  $P, Q$  be two graded posets with a morphism  $f: P \rightarrow Q$  that is a bijection (but not necessarily an isomorphism). If  $P$  is Peck, then  $Q$  is Peck.*

*Proof.* Let  $\text{rk}(P) = \text{rk}(Q) = n$ . Since  $P$  is Peck there exists an order-raising operator  $U$  such that  $U^{n-2i}: V(P_i) \rightarrow V(P_{n-i})$  is an isomorphism. Since  $f$  is a poset morphism, it follows that the map  $f \circ U \circ f^{-1}$  is an order-raising operator on  $Q$ . We then have that  $f \circ U^{n-2i} \circ f^{-1} = (f \circ U \circ f^{-1})^{n-2i}: V(Q_i) \rightarrow V(Q_{n-i})$  is an isomorphism since  $U^{n-2i}: V(P_i) \rightarrow V(P_{n-i})$  is an isomorphism and  $f$  is a bijection.  $\square$

By Lemmas 3.14 and 3.16, in order to prove Theorem 3.4 it suffices to prove that  $\mathcal{E}(B_n)/G$  is Peck. One way to do this is to prove that  $\mathcal{E}(B_n)$  is unitary Peck and then apply Theorem 2.6. In fact, this approach generalizes to an arbitrary poset  $P$ .

**Theorem 3.17** *If the action of  $G$  on  $P$  is CCT and  $\mathcal{E}(P)$  is unitary Peck, then  $\mathcal{E}(P/G)$  is Peck.*

*Proof.* Since the  $G$ -action is CCT, there is a bijection  $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$  by Lemma 3.14. Since  $\mathcal{E}(P)$  is unitary Peck we have that  $\mathcal{E}(P)/G$  is Peck by Theorem 2.6, hence  $\mathcal{E}(P/G)$  is Peck by Lemma 3.16.  $\square$

We prove that  $\mathcal{E}(B_n)$  is unitary Peck for  $n > 2$  in Section 8 of our project report<sup>1</sup> for the University of Minnesota at Twin Cities REU, but unfortunately the proof is technical and computational. Note that by Theorem 3.17, this immediately implies Theorem 3.4. Fortunately there is a cleaner – albeit less direct – route to proving Theorem 3.4. In order to avoid showing that  $\mathcal{E}(B_n)$  is unitary Peck, we define a graded peck poset  $\mathcal{H}(B_n)$  which injects into  $\mathcal{E}(B_n)$ .

**Definition 3.18** For  $P$  a graded poset, define the graded poset  $\mathcal{H}(P)$  as follows. Let the elements  $(x, y) \in \mathcal{H}(P)$  be pairs  $(x, y) \in P \times P$  such that  $x \leq y$ . Define  $(x, y) \leq_{\mathcal{H}} (x', y')$  if  $x \leq x', y \leq y'$  and  $x' \neq y$ . Then, define  $\leq_{\mathcal{H}}$  to be the transitive closure of  $\leq_{\mathcal{H}}$ , and define  $\text{rk}_{\mathcal{H}}(x, y) = \text{rk}_P(x)$ .

*Example 3.19* We give an example of the poset  $\mathcal{H}(B_3)$  in Figure 3.2. Observe that  $\mathcal{H}(B_3)$  can be written as a disjoint union of three copies of  $B_2$ . This is a single case of the more general phenomenon proven in Proposition 3.25.

*Remark 3.20* Note that by definition,  $(x, y) \leq_{\mathcal{H}} (x', y')$  precisely when  $(x, y) \leq_{\mathcal{E}} (x', y')$  and  $x' \neq y$ , hence  $(x, y) \leq_{\mathcal{H}} (x', y') \Rightarrow (x, y) \leq_{\mathcal{E}} (x', y')$ . In other words,  $\mathcal{H}(P)$  has the same elements as  $\mathcal{E}(P)$  but with a weaker partial order.

**Lemma 3.21** *For  $P$  a graded poset, the object  $\mathcal{H}(P)$ , as defined in Definition 3.18, is a graded poset.*

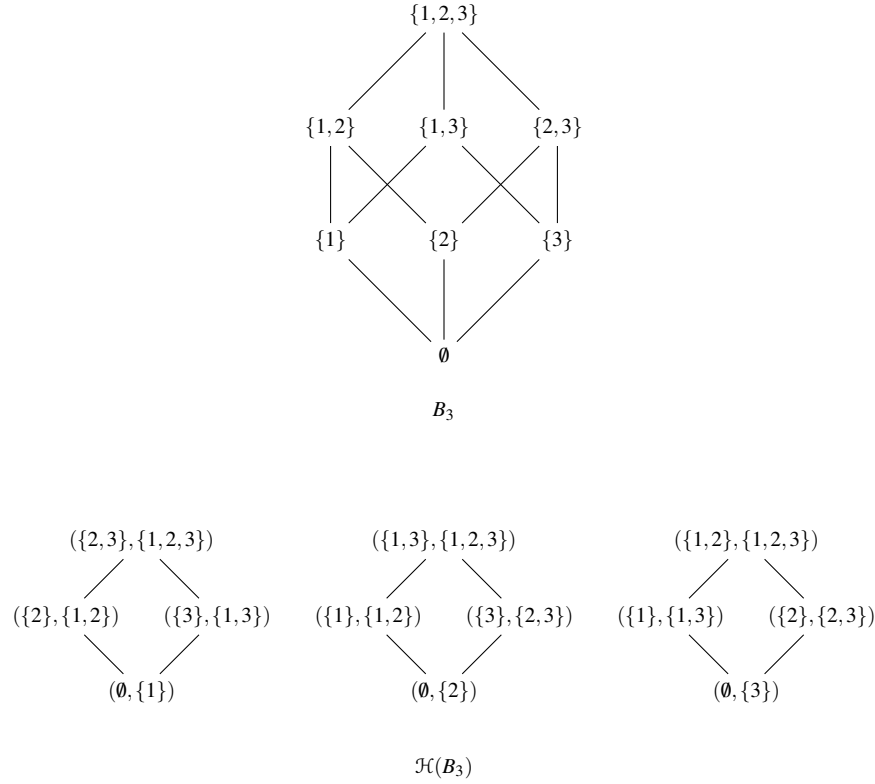
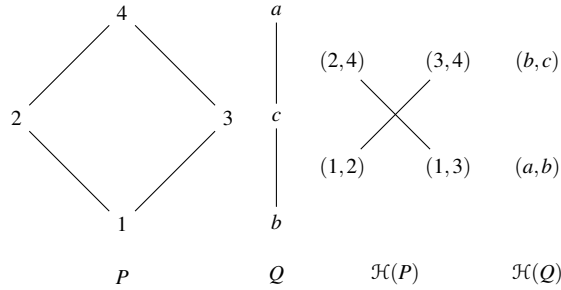
*Proof.* This follows immediately from Remark 3.20 and the fact that  $\mathcal{E}(P)$  is graded.  $\square$

*Remark 3.22* While  $\mathcal{E}: \mathcal{P} \rightarrow \mathcal{P}$  is a functor,  $\mathcal{H}$  is not a functor. In particular, it is not possible to define  $\mathcal{H}(f)$  for  $f$  a morphism. This is illustrated in Figure 3.3. For example, suppose we took  $f: P \rightarrow Q$  defined by  $f(1) = a, f(2) = f(3) = b, f(4) = c$ . Then, there is no possible morphism  $\mathcal{H}(f): \mathcal{H}(P) \rightarrow \mathcal{H}(Q)$  because there are no morphisms whatsoever  $\mathcal{H}(P) \rightarrow \mathcal{H}(Q)$ .

Given an action of a group  $G$  on  $P$ , we define an action of  $G$  on  $\mathcal{H}(P)$  as we did for  $\mathcal{E}(P)$  by again defining  $g \cdot (x, y) = (gx, gy)$  for all  $(x, y) \in P$ . We will then have a well-defined quotient poset  $\mathcal{H}(P)/G$  with the same elements as  $\mathcal{E}(P)/G$ .

**Lemma 3.23** *The automorphism defined by  $g \cdot (x, y) = (gx, gy)$  for all  $g \in G, (x, y) \in \mathcal{H}(P)$  yields a group action of  $G$  on  $\mathcal{H}(P)$ .*

<sup>1</sup> See <http://www.math.umn.edu/reiner/REU/HemmingerLandesmanYao2014.pdf> for the full report. This will be referred to throughout the paper as “the REU report.”

Fig. 3.2  $B_3$  and  $\mathcal{H}(B_3)$ Fig. 3.3 A map of posets  $f: P \rightarrow Q$  with no possible map  $\mathcal{H}(f)$ 

*Proof.* Let  $g \in G$ . Since  $\leq_{\mathcal{H}}$  is the transitive closure of  $\triangleleft_{\mathcal{H}}$  it suffices to show that for all  $(x, y), (x', y') \in \mathcal{H}(P)$  we have  $(x, y) \triangleleft_{\mathcal{H}} (x', y') \Leftrightarrow g(x, y) \triangleleft_{\mathcal{H}} g(x', y')$ . Since  $g$  is an automorphism of  $P$ , we have  $x \leq_P x' \Leftrightarrow gx \leq_P gx'$ ,  $y \leq_P y' \Leftrightarrow gy \leq_P gy'$ , and  $y \neq x' \Leftrightarrow gy \neq gx'$ , so the result follows from the definition of  $\leq_{\mathcal{H}}$ .  $\square$

**Lemma 3.24** *The map  $f: \mathcal{H}(P)/G \rightarrow \mathcal{E}(P)/G$  defined by  $G(x, y) \mapsto G(x, y)$  is a bijective morphism for any group action of  $G$  on  $P$ .*

*Proof.* The elements of  $\mathcal{H}(P)/G, \mathcal{E}(P)/G$  are the same by definition, so it suffices to show that  $f$  is a morphism. Since  $f$  is clearly rank-preserving, it suffices to show  $f$  is order-preserving. This is immediate from Remark 3.20.  $\square$

The remaining step in the proof of Theorem 3.4 is to show that  $\mathcal{H}(B_n)$  is unitary Peck, which we do by generalizing Example 3.19 and showing that  $\mathcal{H}(B_n)$  is isomorphic to the disjoint union of boolean algebras.

**Proposition 3.25**  $\mathcal{H}(B_n)$  is isomorphic to  $n$  disjoint copies of  $B_{n-1}$ .

*Proof.* Let the  $n$  disjoint copies of  $B_{n-1}$  be labeled  $B_{n-1}^{(i)}$ ,  $1 \leq i \leq n$ , with the elements of  $B_{n-1}^{(i)}$  labeled  $x^{(i)}$ ,  $x \subseteq \{1, \dots, n-1\}$ . We will show that the map

$$\begin{aligned} f: \mathcal{H}(B_n) &\longrightarrow \bigcup_{i=1}^n B_{n-1}^{(i)} \\ (x, x \cup i) &\longmapsto x^{(i)} \end{aligned}$$

is an isomorphism. Suppose we have  $(x, y), (x', y') \in \mathcal{H}(B_n)$  with  $(x, y) \leq_{\mathcal{H}} (x', y')$ . Let  $j \in [n]$  such that  $y' = y \cup \{j\}$ , and let  $i \in [n]$  such that  $x' = x \cup \{i\}$ . If  $i \neq j$ , then  $x' = y$ , contradicting the assumption that  $(x, y) \leq_{\mathcal{H}} (x', y')$ . Thus  $x' = x \cup \{i\}$  and  $y' = y \cup \{i\}$  for some  $i \in [n]$ .

Conversely we can easily check that if  $i \notin y$ , then  $(x, y) \leq_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$ . It follows that for all subsets  $w \subset [n]$  such that  $|w| = 1$ , there is an isomorphism from the subposet of elements  $\{(x, y) : y \setminus x = w\}$  to  $B_{n-1}$  defined by  $(x, y) \mapsto (x \setminus w, y \setminus w)$ . Furthermore if  $y \setminus x \neq y' \setminus x'$ , then  $(x, y)$  and  $(x', y')$  are incomparable, so these subposets indexed by  $w$  are pairwise disjoint, and  $\mathcal{H}(B_n)$  is isomorphic to  $n$  copies of  $B_n$ .  $\square$

**Corollary 3.26**  $\mathcal{H}(B_n)$  is unitary Peck for all  $n \geq 0$ .

*Proof.* This follows immediately from Proposition 3.25 and the fact that  $B_{n-1}$  is unitary Peck. Indeed,  $B_n$  is shown to be unitary Peck, in [6, Theorem 2a] by noting that  $B_k = (B_1)^k$  and that  $B_1$  is clearly unitary Peck. Note,  $\mathcal{H}(B_0)$  is the empty poset, and so it is vacuously unitary peck.  $\square$

**Corollary 3.27**  $\mathcal{H}(B_n)/G$  is Peck for any subgroup  $G \subset \text{Aut}(B_n)$ .

*Proof.* This follows from Corollary 3.26 and Theorem 2.6.  $\square$

The next corollary will not be particularly relevant in proving Theorem 3.4, but let us note it as an aside.

**Corollary 3.28** Both  $\mathcal{E}(B_n)$  and  $\mathcal{H}(B_n)$  have symmetric chain decompositions (SCD).

*Proof.*  $\mathcal{H}(B_n)$  has an SCD by Proposition 3.25 and the fact that  $B_{n-1}$  has an SCD, as shown in [2]. By Lemma 3.24 there is a bijective morphism  $f: \mathcal{H}(B_n) \rightarrow \mathcal{E}(B_n)$ , and since a bijective morphism takes an SCD to an SCD it follows that  $\mathcal{E}(B_n)$  has an SCD.  $\square$

**Corollary 3.29**  $\mathcal{E}(B_n)/G$  is Peck for any subgroup  $G \subset \text{Aut}(B_n)$ .

*Proof.* By Corollary 3.27,  $\mathcal{H}(B_n)/G$  is Peck. By Lemma 3.24, the map  $f: \mathcal{H}(B_n)/G \rightarrow \mathcal{E}(B_n)/G, G(x, y) \mapsto G(x, y)$  is a bijective morphism. Then, by Lemma 3.16, it follows that  $\mathcal{E}(B_n)/G$  is Peck.  $\square$

We now deduce Theorem 3.4.

*Proof of Theorem 3.4.* By Corollary 3.29,  $\mathcal{E}(B_n)/G$  is Peck for any group action of  $G$  on  $B_n$ . Since the  $G$ -action is CCT, there is a bijective morphism from  $\mathcal{E}(B_n)/G$  to  $\mathcal{E}(B_n/G)$  by Lemma 3.14. Hence  $\mathcal{E}(B_n/G)$  is Peck by Lemma 3.16.  $\square$

Note that we have also developed several generalizations of  $\mathcal{E}$ , for which many similar results hold. For more information, see Subsection 3.3 of the REU report.

#### 4 Common Cover Transitive Actions

In this section, we develop the theory of CCT actions  $\phi$  where  $G$  is a group,  $P$  is a poset, and  $\phi : G \times P \rightarrow P$  is an action. Recall Definition 1.4, that  $\phi$  is CCT if whenever  $x, y, z \in P, x \leq z, y \leq z, x \in G y$  then there exists  $g \in \text{Stab}(z)$  with  $gx = y$ . We show that the CCT property is closed under semidirect products, in the appropriate sense. From Proposition 1.6, which will be proven in Subsubsection 4.2.4, the action of  $S_n$  on  $B_n$  and the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

*Example 4.1* Two rather trivial examples of CCT actions are  $\phi : S_n \times B_n \rightarrow B_n$  and  $\psi : G \times B_n \rightarrow B_n$  where  $G$  is arbitrary,  $\phi$  is the action induced by  $S_n$  permuting the elements of  $[n]$ , and  $\psi$  is the trivial action. In the former case,  $\mathcal{E}(B_n/S_n)$  is simply a chain with  $n$  points, and so is  $\mathcal{E}(B_n)/S_n$ , since all  $(x, y)$  are identified under the  $S_n$  action. In the latter case, since  $G$  acts trivially by  $\phi$ , both  $\mathcal{E}(B_n/G) \cong \mathcal{E}(B_n)$  and  $\mathcal{E}(B_n)/G \cong \mathcal{E}(B_n)$ . So again,  $\psi$  is CCT.

##### 4.1 Preservation Under Semidirect Products

**Lemma 4.2** *Let  $G \subseteq \text{Aut}(P)$ ,  $H \triangleleft G$ ,  $K \subset G$  such that  $G = H \rtimes K$ . We then have a well defined group action  $K \times P/H \rightarrow P/H, (k, Hx) \mapsto H(k \cdot x)$ .*

*Proof.* Note that if  $x, x' \in Hx$ , we have  $x' = h \cdot x$  for some  $h \in H$ . Since  $H$  is normal in  $G$ , we have that for all  $k \in G$  there exists  $h' \in H$  so that  $khk^{-1} = h'$ . So

$$k \cdot x' = kh \cdot x = k(k^{-1}h'k) \cdot x = h' \cdot (k \cdot x)$$

Hence  $k \cdot x$  and  $k \cdot x'$  are in the same  $H$ -orbit, so we have a well-defined group action of  $K$  on  $P/H$  defined by  $k \cdot Hx = H(k \cdot x)$ .  $\square$

Recall Proposition 1.7, as stated in the introduction, which says that the CCT property is preserved under semidirect products. We will use Proposition 1.7 to construct more examples of CCT group actions, in particular using it to give a simple proofs that CCT actions are preserved under direct products and wreath products.

**Proposition 4.3** *Let  $G \subseteq \text{Aut}(P)$ ,  $H \triangleleft G$ ,  $K \subset G$  such that  $G = H \rtimes K$ . Suppose that the action of  $H$  is on  $P$  is CCT and the action of  $K$  on  $P/H$  is CCT. Then the action of  $G$  on  $P$  is CCT.*

*Proof.* Since  $G = H \rtimes K$ , every element  $g \in G$  can be written uniquely as a product  $hk$  for some  $h \in H, k \in K$ . Let  $x, y, z \in P$  such that  $x \leq z, y \leq z$  and such that there exists some  $h_0 k_0 \in G$  such that  $h_0 k_0 \cdot x = y$ . It suffices to show that there exists some  $g \in \text{Stab}_G(z)$  such that  $g \cdot x = y$ .

The orbits  $Hx, Hy, Hz \in P/H$  satisfy  $Hx \leq Hz, Hy \leq Hz$  such that  $k_0 \cdot Hx = Hy$ . Thus, since the action of  $K$  on  $P/H$  is CCT, there exists some  $k_1 \in K$  such that  $k_1 \in \text{Stab}_K(Hz)$  and  $k_1 \cdot Hx = Hy$ . It follows that there exists some  $h_1 \in H$  such that  $h_1 k_1 h_0 \in \text{Stab}_G(z)$  and  $h_1 k_1 h_0 \cdot x \in Hy$ .

Write  $x' = h_1 k_1 h_0 \cdot x$ . Since the group action of  $G$  must be order-preserving by definition, we have that  $x' \leq z$ . We already had that  $y \leq z$  and  $x' \in Hy$ , hence there exists some  $h_2 \in \text{Stab}_H(z)$  such that  $h_2 \cdot x' = y$  by the fact that the action of  $H$  on  $P$  is CCT. Then we have that  $h_2 h_1 k_1 h_0 \cdot x = h_2 \cdot x' = y$  and  $h_2 h_1 k_1 h_0 \cdot z = h_2 \cdot z = z$ , as desired.  $\square$

**Proposition 4.4** *For  $\phi : G \times P \rightarrow P, \psi : H \times Q \rightarrow Q$  two CCT actions, then the direct product*

$$\phi \times \psi : (G \times H) \times (P \times Q) \rightarrow (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$$

*is also CCT.*

*Proof.* First note that if either  $G$  or  $H$  acts trivially, then it can be easily checked that the action of  $G \times H$  is CCT. Next, observe that  $G \times H$  can be viewed as the semidirect product of  $(G \times \{e\}) \rtimes (\{e\} \times H)$ . Since the action of  $G$  on  $P$  is CCT, the action of  $G \times \{e\}$  on  $P \times Q$  is CCT. Also, since the action of  $H$  on  $Q$  is CCT, it follows that the action of  $\{e\} \times H$  on  $P \times Q/(G \times \{e\})$  is CCT. Therefore, the action of  $(G \times \{e\}) \rtimes (\{e\} \times H)$  satisfies the conditions of Proposition 1.7 and so the action of  $G \times H$  on  $P \times Q$  is CCT.  $\square$

Next, we use Proposition 1.7 to prove in Proposition 4.9 that the CCT property is preserved under wreath products with the symmetric group. First, we need the definition of the wreath product.

**Definition 4.5** For  $G, H$  groups, with  $H \subset S_\ell$ , the *wreath product*, denoted by  $G \wr H$ , is the group whose elements are pairs  $(g, h) \in G^\ell \times H$  with multiplication defined by

$$((g'_1, \dots, g'_\ell), h') \cdot ((g_1, \dots, g_\ell), h) = ((g'_{h'(1)}g_1, \dots, g'_{h'(\ell)}g_\ell), hh')$$

where  $H$  acts on  $[\ell]$  via the embedding of  $H$  into  $S_\ell$ .

In other words,  $G \wr H$  can be viewed as a certain semidirect product of  $G^\ell \rtimes H$ .

**Definition 4.6** For any group  $G$  with a given action  $\psi: G \times P \rightarrow P$ , we obtain an induced action of  $G \wr H$ ,  $\phi: G \wr H \times P^\ell \rightarrow P^\ell$  defined by

$$((g_1, \dots, g_\ell), h)(a_1, \dots, a_\ell) = (g_{h^{-1}(1)} \cdot a_{h^{-1}(1)}, \dots, g_{h^{-1}(\ell)} \cdot a_{h^{-1}(\ell)}).$$

*Remark 4.7* Heuristically, one may think of the above action as obtained by first having  $G$  act separately on the  $\ell$  distinct copies of  $P$ , and then letting  $H$  act by permuting the copies.

**Lemma 4.8** For  $P$  a graded poset, the action

$$\phi: S_\ell \times P^\ell \rightarrow P^\ell, \sigma \cdot (x_1, \dots, x_\ell) = (x_{\sigma(1)}, \dots, x_{\sigma(\ell)})$$

is CCT.

*Proof.* For  $a \in P^\ell$  notate  $a = (a_1, \dots, a_\ell)$ . Suppose  $x, y, z \in P^\ell$  with  $x \leq z, y \leq z$ , and  $x \in S_\ell y$ , but  $x \neq y$ . This means there is a unique  $i$  such that  $x_i \leq z_i$  and  $x_k = z_k$  for  $k \neq i$ . Additionally, there is a unique  $j$  for which  $y_j \leq z_j$  and  $y_k = z_k$  for  $k \neq j$ . Since  $x \in S_\ell y$ , we obtain the equality of multisets  $\{x_1, \dots, x_\ell\} = \{y_1, \dots, y_\ell\}$ . But for  $k \neq i, j$  we have  $x_k = z_k = y_k$ , so we also obtain equality of sets  $\{x_i, x_j\} = \{y_i, y_j\}$ . Since  $y_j \leq x_j$ , we obtain  $y_j = x_i, y_i = x_j$ . Then, taking the transposition  $\sigma = (ij) \in S_\ell$ , it follows that  $\sigma \in \text{Stab}(z)$  and  $\sigma \cdot x = y$ .  $\square$

**Proposition 4.9** If  $\psi: G \times P \rightarrow P$  is CCT, let  $\phi: G \wr S_\ell \times P^\ell \rightarrow P^\ell$  where  $\phi$  is the induced action defined in Definition 4.6. Then  $\phi$  is also CCT.

*Proof.* Note that the wreath product  $G \wr S_\ell$  can be viewed as a semidirect product  $G^\ell \rtimes S_\ell$ . Since the action of  $G$  on  $P$  is CCT we obtain that the action of  $G^\ell$  on  $P^\ell$  is CCT by Proposition 4.4. Furthermore the action  $S_\ell \times (P/G)^\ell \rightarrow (P/G)^\ell$  defined by  $(\sigma, (x_1, \dots, x_\ell)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(\ell)})$  for  $\sigma \in S_\ell$  and  $x_i \in P/G$  is CCT by Lemma 4.8. Since  $P^\ell/G^\ell \cong (P/G)^\ell$ , it follows that the action  $\phi$  satisfies the conditions of Proposition 1.7, so  $\phi$  is CCT.  $\square$

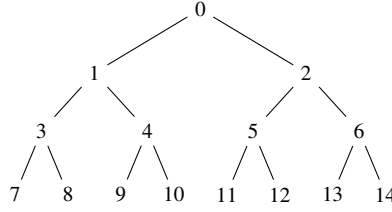
## 4.2 Examples of CCT Actions

In this subsection, we describe several classes of CCT actions. First, we show that the automorphism group of any rooted tree is CCT. Second, we show linear automorphisms of simplices and octahedra are CCT. Third, we show that the left multiplication action is CCT if and only if the group is  $\mathbb{Z}_2^k$ , and that any action of  $\mathbb{Z}_2^k$  on  $[n]$  induces a CCT action on  $B_n$ . In the end of this subsection, we prove Proposition 1.6, which shows certain symmetric group and dihedral group actions are CCT.

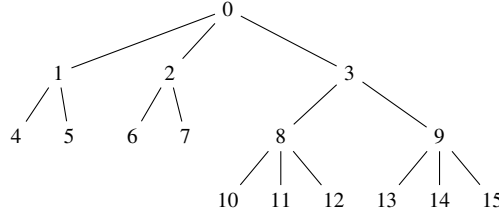
### 4.2.1 An application to rooted trees

In this subsubsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Proposition 4.9 and Proposition 4.4, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT.

**Definition 4.10** A graded poset  $P$  is a *rooted tree* if there is a unique element  $x \in P$  of maximal rank, called the *root*, and for all  $x \in P$ , other than the root, there exists a unique  $y \in P$  with  $y \succ x$ .



**Fig. 4.1** An example of a rooted tree with 8 leaves



**Fig. 4.2** An example of a rooted tree with 10 leaves

*Example 4.11* In Figure 4.1 and Figure 4.2, we give two examples of rooted trees.

**Definition 4.12** For  $P$  a rooted tree, an element  $x \in P$  is a *leaf* if there is no  $z \in P$  with  $x > z$ . Denote the set of all leaves of  $P$  by  $L(P)$ .

We next recall a useful, elementary lemma, whose proof we omit.

**Lemma 4.13** Let  $P$  be a rooted tree. Then, the action of  $\text{Aut}(P)$  on  $P$  induces an action of  $\text{Aut}(P)$  on  $L(P)$ . Furthermore, there is also an induced action of  $\text{Aut}(P)$  on  $B_n$  where  $n = |L(P)|$ .

For the rest of this section only, fix a rooted tree  $P$  and denote by  $G$  the group of automorphisms  $\text{Aut}(P)$ . Let  $G$  act on  $B_n$ , where  $n = |L(P)|$ , by the induced action  $\phi: G \times L(P) \rightarrow L(P)$  described in the proof of Lemma 4.13. For  $x \in P$ , denote  $D(x) = \{y \in P: y \leq x\}$ , so  $D(x)$  is the maximal subposet of  $P$  with maximum element  $x$ .

**Proposition 4.14** Let  $P$  be a rooted tree with root vertex labeled 0. Let  $\{A_1, \dots, A_m\}$  denote the set of isomorphism classes of  $\{D(x): x \leq 0\}$ , and let  $i_k$  denote the number of subtrees in  $\{D(x): x \leq 0\}$  in the isomorphism class  $A_k$ . For  $A_k \in \{A_1, \dots, A_m\}$ , denote  $G_k = \text{Aut}(A_k)$ . Then,

$$\text{Aut}(P) \cong (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}) \quad (4.1)$$

In particular,  $\text{Aut}(P)$  can be expressed as a sequence of direct products and wreath products of symmetric groups.

*Proof.* We proceed by induction on the rank of  $P$ . It is clear that if  $P$  is rank 0, then  $\text{Aut}(P)$  is trivial. If the rank of  $P$  is greater than 0, label the vertices of  $P$  by  $\{0, 1, \dots, s\}$  such that the root is labeled 0 and the vertices just below the root are labeled  $1, \dots, k$ . Let  $A_1, \dots, A_m$  denote the distinct isomorphism classes of trees in the set  $\{D(1), \dots, D(k)\}$ . For  $A_k \in \{A_1, \dots, A_m\}$  denote  $G_k = \text{Aut}(A_k)$ . Let  $T_j = \{t: t \leq 0, D(t) \cong A_j\}$ . Then, letting  $Q_j$  be the subtree of  $P$  whose elements lie in the set  $\{0\} \cup (\cup_{t \in T_j} D(t))$ , we have that  $\text{Aut}(Q_j) \cong G_j \wr S_{i_j}$ , because after choosing a permutation of the elements of  $T_j$ , we are free to choose any element of  $G_j$  to permute each  $D(t), t \in T_j$ . If  $t_1 \leq 0, t_2 \leq 0, g \cdot t_1 = t_2$ , then it must be that  $g \cdot D(t_1) = D(t_2)$ . Therefore,  $\text{Aut}(P)$  must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$\text{Aut}(P) \cong (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \quad (4.2)$$

Since each  $G_j$  is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (4.1) that so is  $\text{Aut}(P)$ .  $\square$

*Example 4.15* Let  $P_1$  be the rooted tree in Figure 4.1 and  $P_2$  be the rooted tree in Figure 4.2. Proposition 4.14 says that  $\text{Aut}(P_1) \cong (S_2 \wr S_2) \wr S_2$ ; and  $\text{Aut}(P_2) \cong (S_2 \wr S_2) \times (S_3 \wr S_2)$ .

**Corollary 4.16** *For  $P$  a rooted tree,  $\text{Aut}(P)$  is CCT.*

*Proof.* Let the rank of  $P$  be  $n$ . By Proposition 4.9, wreath products with symmetric groups preserve the CCT property, and by Proposition 4.4 the direct product of two CCT groups is again CCT. Therefore, by the proof of Proposition 4.14, the group  $\text{Aut}(P)$  can be built up by repeating this pair of observations  $n - 1$  times.  $\square$

#### 4.2.2 Automorphisms of Polytopes

As another class of CCT actions, we describe several linear automorphism groups of polytopes whose induced actions on  $B_n$  are CCT. In particular, we prove that the linear automorphism groups of simplices and octahedra are CCT. Once we prove Proposition 1.6 in Subsubsection 4.2.4 we will also see that the action of the dihedral group on a regular  $n$ -gon is CCT for  $n = p, 2p$ . Since the dihedral group is the group of all linear automorphisms of the regular  $n$ -gon, this action gives another example of the linear automorphism group of a polytope being CCT.

**Definition 4.17** Let  $M$  be a polytope, with a particular embedding in  $\mathbb{R}^n$ . The group of linear automorphisms of  $M$  is the subgroup of  $\text{GL}_n$  whose elements are  $\{g \in \text{GL}_n : g \cdot M = M\}$ .

First, we look at linear automorphisms of simplices. Let  $G$  be the group of linear automorphisms of the  $(n - 1)$ -simplex, whose vertices lie at the standard basis vectors in  $\mathbb{R}^n$ . The action of  $G$  on the  $(n - 1)$ -simplex induces an action on  $[n]$ , given by identifying  $[n]$  with the  $n$  vertices of the  $(n - 1)$ -simplex. Hence, it induces an action on  $B_n$ .

*Example 4.18* The action of the group of linear automorphisms of the  $(n - 1)$ -simplex, acting on  $B_n$ , as defined above, is CCT. To see this, observe that the group of linear automorphisms in this case induces the usual action of  $S_n$  on  $B_n$ , because any permutation matrix defines a linear map on  $\mathbb{R}^n$ . However, we know the action of  $S_n$  on  $B_n$  is CCT from Example 4.1.

Next, we look at linear automorphisms of octahedra. Let  $G$  be the group of linear automorphisms of the  $n$ -octahedron, embedded inside  $\mathbb{R}^n$ , whose vertices are located at  $\pm e_i$ , where  $e_1, \dots, e_n$  are the standard basis vectors of  $\mathbb{R}^n$ . Then, the action of  $G$  on the octahedron induces an action of  $G$  on the  $2n$  vertices of the octahedron, and hence on  $B_{2n}$ .

**Proposition 4.19** *The induced action of the group of linear automorphisms of the  $n$ -octahedron on  $B_{2n}$  is CCT.*

*Proof.* It is simple to see that the group of linear automorphisms of the  $n$ -octahedron is the hyperoctahedral group, since it is generated by the permutation matrices, together with the matrix  $A$  with  $A_{1,1} = -1, A_{i,i} = 1, A_{j,k} = 0$  where  $i \neq 1, j \neq k$ .<sup>2</sup> It is well known that the hyperoctahedral group can be written as  $S_2 \wr S_n$ . Then, by Proposition 4.9, it follows that  $S_2 \wr S_n$  is CCT.  $\square$

*Remark 4.20* Let us give a brief recap of which linear automorphisms of polytopes are known to induce actions on  $B_n$  which are CCT. First, by the above lemmas, for octahedra and simplices, the induced action is CCT. By Proposition 1.6 and Remark 4.23 the linear automorphism group of an  $n$ -gon induces a CCT action on  $B_n$  if and only if one of  $n \in \{1, p, 2p\}$  for  $p$  a prime. Additionally, using python code, we have verified that automorphisms of the 3-cube, with vertices at  $(\pm 1, \pm 1, \pm 1)$  induces a CCT action. It is still unknown whether the linear automorphisms of  $n$ -cubes is CCT for  $n > 3$ , and also whether the remaining five exceptional regular polytopes (namely the dodecahedron and icosahedron in  $\mathbb{R}^3$  as well as the 24-cell, 120-cell, and 600-cell polytopes in  $\mathbb{R}^4$ ) induce CCT actions. These questions are repeated in Question 6.6 and Question 6.7.

<sup>2</sup> The hyperoctahedral group is commonly denoted by  $B_n$ , since it is the type  $B$  Coxeter group. We do not use this notation here to avoid confusing it with the boolean algebra.



### 4.2.3 CCT Actions of $\mathbb{Z}_2^k$

In this subsection, we show that any embedding of  $\mathbb{Z}_2^k$  into  $S_n$  defines an action on  $B_n$  which is CCT. In particular, this implies that every action of  $\mathbb{Z}_2^k$  is CCT. However, it turns out that this is the only class of groups for which every action is CCT.

**Proposition 4.21** *Recall that  $G$  is an elementary abelian 2-group if  $G \cong (\mathbb{Z}/2\mathbb{Z})^k$  for some  $k \in \mathbb{N}$ .*

1. *For any  $n \in \mathbb{N}$ , and  $G$  an elementary abelian 2-group, every  $G$ -action  $\phi : G \times B_n \rightarrow B_n$  is CCT.*
2. *For every finite group  $G$  which is not an elementary abelian 2-group, there exists at least one  $G$  action which is not CCT, namely the action of  $G$  on  $B_n$  induced by the left-regular action of  $G$  on itself, where  $n = |G|$ .*

*Proof.* First, we show 1 holds. Let  $x, y, z \in B_n$  such that  $x \leq z, y \leq z$ , and  $x = gy$  for some  $g \in G$ . Since  $x \neq y$  we have  $z = x \cup y$ . Furthermore, since every element in  $\mathbb{Z}_2^k$  has order 2 we have that  $gy = g^2x = x$  and thus  $gz = gx \cup gy = y \cup x = z$ . Hence  $g \in \text{Stab}(z)$  and thus  $\phi$  is CCT.

Next, we show 2 holds. First, let us show  $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$ . The forward implication is obvious. To see the converse, first note that if  $\forall g \in G, g^2 = e$ , then  $G$  is abelian because  $aba^{-1}b^{-1} = abba = a^2 = e$ . Then,  $G$  is an abelian group, all of whose elements have order two. The structure theorem of finite abelian groups tells us  $G \cong \mathbb{Z}_2^k$ .

So, Suppose  $G \not\cong \mathbb{Z}_2^k$ . Then, there exists  $g \in G, g^2 \neq e$ . Clearly  $\{e\} \leq \{e \cup g\}, \{g\} \leq \{e \cup g\}, \{g\} \in G\{e\}$ . So, to show the induced action  $\phi : G \times B_n \rightarrow B_n$  is not CCT, it suffices to show there is no  $h \in G, h \in \text{Stab}(\{e \cup g\}), h \cdot \{e\} = \{g\}$ . If  $h \in \text{Stab}(\{e \cup g\})$  then  $h \cdot e = e$  or  $h \cdot e = g$ . In both cases, it is simple to see that  $g^2 = e$ . Therefore, there does not exist such an  $h$  and left multiplication is not CCT.  $\square$

### 4.2.4 The proof of 1.6

Let  $x, y \in B_n$  and let  $G$  act on  $B_n$ . For the remainder of this section, rather than using  $(Gx, Gy)$  to properly notate  $\{(a, b) : a \in Gx, b \in Gy\}$ , we abuse notation by writing  $(Gx, Gy)$  in place of  $(Gx, Gy) \cap \mathcal{E}(B_n)$ . Hence, we view  $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}(B_n) \subset \mathcal{E}(B_n)$ . Namely,

$$(Gx, Gy) = \{(\sigma x, \tau y) \in \mathcal{E}(B_n) : \sigma, \tau \in G; \sigma x \leq \tau y\}.$$

Similarly, we notate

$$G(x, y) = \{(gx, gy) \in \mathcal{E}(B_n) : g \in G\}.$$

**Proposition 4.22** *Let  $n$  be a positive integer and  $p$  be prime. The following actions are CCT.*

1. *The action of  $S_n$  on  $B_n$ ,*
2. *The action of  $D_{2p}$  on  $B_p$ ,*
3. *The action of  $D_{4p}$  on  $B_{2p}$ ,*

*Proof.* We have already seen in Example 4.1 that (1) holds trivially. We prove part (2). The proof of part (3) is similar.

Let  $x, y$  be elements of  $B_n$  such that  $x$  is covered by  $y$ . We wish to prove that given  $G(x, y) \in \mathcal{E}(B_p)$  such that  $\sigma x \leq \tau y$  where  $\sigma \in D_{2p}$ , there exists some  $\tau \in D_{2p}$  such that  $\tau x = \sigma x$  and  $\tau y = y$ .

The action of  $D_{2p}$  on  $B_p$  is induced by the action of  $D_{2p}$  on  $[p]$  where  $[p]$  is identified with vertices of the regular  $p$ -gon. Note that any element in  $D_{2p}$  is either some reflection  $r$  by one of the lines of symmetry of the polygon, or some rotation  $\sigma_0^d$  where  $\sigma_0$  is the generator  $\sigma_0 = (12 \cdots p)$  and  $d$  is some integer. Hence we only need to show the claim when  $\sigma = r$  or  $\sigma = \sigma_0^d$ . It is clear that the claim holds for  $\sigma = r$ : if  $x \leq y$  and  $r \cdot x \leq y$ , then  $r \cdot y = y$ , since  $r$  is of order 2. Now suppose  $\sigma_0^d \cdot x \leq y$  for some  $G(x, y) \in \mathcal{E}(B_p)_i$ . It is fairly straightforward to see that  $(x, y)$  is of form  $x = \{s, s+d, \dots, s+(i-1) \cdot d\}$  for some starting point  $s \in [n]$  and  $y = \{s, s+d, \dots, s+(i-1)d, s+i \cdot d\}$ . Now let  $r_0$  be the reflection that sends  $x \mapsto (2s+i \cdot d) - x$  for all  $x \in [n]$ , reducing mod  $n$  whenever necessary. Then  $r_0 x = \sigma_0^d x$  and  $r_0$  fixes  $y$  by construction. Therefore, the action  $D_{2p}$  on  $B_p$  is CCT.  $\square$

*Remark 4.23* It is easy to see that if  $n \neq p, n \neq 2p, n > 8$  for any prime  $p$ , then the action of  $D_{2n}$  on  $B_n$  is not CCT. To see this, we give an example of a non-CCT pair. Assume  $n \neq p, 2p$ , then  $n = mk$  for some  $m \geq k \geq 3$ . Let us consider elements  $x, y, z$  where  $z = \{1, m+1, 2m+1, \dots, (k-1)m+1, 2, m+2\}$ . Let  $x = z \setminus \{m+2\}$  and  $y = z \setminus \{2\}$ . We immediately have that  $x, y \prec z$ , and  $x \in D_{2n}y$  since  $x$  is sent to  $y$  by the permutation  $(12 \cdots n)^m \in D_{2n}$ . It is also clear that there is no  $g \in D_{2n}$  translating  $x$  to  $y$  while fixing  $z$ , from the asymmetry of the element  $z$ . Therefore, the action of  $D_{2n}$  on  $B_n$  as described is CCT if and only if  $n = p$  or  $n = 2p$  for some prime  $p$ .

Hence, a complete list of  $n$  for which  $D_{2n}$  is CCT is given by  $n = p, n = 2p, n = 1, n = 8$  where  $p$  varies over all primes.

*Remark 4.24* There are several other results related to  $\mathcal{E}(C_n), \mathcal{E}(D_{2n})$ , with  $C_n = \mathbb{Z}/n\mathbb{Z}$ , which are proven in Section 7 of the REU report. Notably,

1. For  $G$  any group of order  $n$ , acting transitively on  $[n]$ , the induced action of  $G$  on  $B_n$  defines a quotient poset with  $|\mathcal{E}(B_n)/G|_i = \binom{n-1}{i}$ .
2. For all  $n$ ,  $\mathcal{E}(B_n/C_n)$  is symmetric and unimodal.
3. For all  $n$ ,  $\mathcal{E}(B_n/D_{2n})$  is symmetric and unimodal.

## 5 A Unimodality Result

Using Corollary 4.16, we prove a result related to that of Pak and Panova in [3, Theorem 1.1]. We construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. This construction gives an alternate proof of [3, Theorem 1.1] in the case that  $r = 1$ .

For this section, fix  $\ell, m$  with  $n = \ell \cdot m$  and fix  $G = S_m \wr S_\ell$ . Let  $S_m$  act on  $B_m$  by the permutation representation, and then let  $G$  act on  $B_m^\ell \cong B_{m \cdot \ell}$  by the action defined in Definition 4.6.

### 5.1 Restatement of the Unimodality Result

We first review the necessary definitions and then state [3, Theorem 1.1]:

A *partition*  $\lambda$  of  $n$  is a sequence of numbers  $\lambda = (\lambda_1, \dots, \lambda_k)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$  and that  $\sum_{i=1}^k \lambda_i = n$ , denoted by  $\lambda \vdash n$ . Let  $P_n(\ell, m)$  denote the set of partitions  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ , such that  $\lambda_1 \leq m, k \leq \ell$ . That is,  $P_n(\ell, m)$  is the set of partitions which fit inside an  $\ell \times m$  rectangle.

For  $\lambda$  a partition, let  $v(\lambda)$  be the number of distinct nonzero part sizes of  $\lambda$ . Let  $p_k(\ell, m, r) = \sum_{\lambda \in P_k(\ell, m)} \binom{v(\lambda)}{r}$ , as defined in [3, Section 1].

A *Young diagram* is a finite collection of boxes arranged so that their rows are left-justified and their row lengths are weakly increasing from top to bottom.

**Theorem 5.1** [3, Theorem 1.1] *The sequence  $p_r(\ell, m, r), p_{r+1}(\ell, m, r), \dots, p_{\ell-m}(\ell, m, r)$  is unimodal and symmetric.*

### 5.2 A Proof of Theorem 5.1 for $r = 1$

Now that we have stated Pak and Panova's Theorem, we give an alternative proof of Theorem 5.1 in the case of  $r = 1$ . In fact, we do better, by realizing  $p_i(\ell, m, 1)$  as ranks of a Peck poset.

**Proposition 5.2** *There is an equality  $|\mathcal{E}(B_n)/G|_i = |\mathcal{E}(B_n/G)|_i = p_{1+i}(\ell, m, 1)$ . In particular, Theorem 5.1 holds in the case  $r = 1$ .*

*Proof.* First, observe that  $S_m \wr S_\ell$  can be described as the automorphism group of a rooted tree of rank 2 with  $\ell$  elements at rank 1 and  $m \cdot \ell$  elements at rank 2, such that each element at rank 1 is above  $m$  elements at rank 2. Then, by Corollary 4.16, it follows that the action of  $G$  on  $B_{m \cdot \ell}$  is CCT and so  $\mathcal{E}(B_n/G)$  is Peck.

Next, note that each equivalence class in  $B_n/G$  has a unique representative which is a Young diagram. The correspondence is given by sending an equivalence class to the representative which is left-justified and bottom-justified. Here, we think of  $B_n = B_{\ell \cdot m}$  as subsets of the  $\ell \cdot m$  boxes in an  $\ell \times m$  rectangle. For a complete proof, see, for example, Lemma 5.11 of the REU report.

Now, let  $Gx, Gy$  be two  $G$  orbits with  $\bar{x}$  the Young diagram corresponding to  $x$ , and  $\bar{y}$  the Young diagram corresponding to  $y$ . Suppose  $Gx \triangleleft Gy$ . Then,  $\bar{x}$  must be contained in  $\bar{y}$ , but with a single box removed. Since  $\bar{x}, \bar{y}$  are both Young diagram, the removed box must be one of the corners of  $\bar{y}$ . Observe that the number of corners of a partition is precisely the number of distinct part sizes, and so  $|\{Gx : Gx \triangleleft Gy\}| = v(\bar{y})$ . Therefore,

$$\begin{aligned} |\mathcal{E}(B_n/G)_i| &= \sum_{(Gx, Gy) \in \mathcal{E}(B_n/G)_i} 1 \\ &= \sum_{Gy \in (B_n/G)_{i+1}} \left( \sum_{Gx \triangleleft Gy} 1 \right) \\ &= \sum_{Gy \in (B_n/G)_{i+1}} v(\bar{y}) \\ &= \sum_{\lambda \in P_{i+1}(\ell, m)} v(\lambda). \end{aligned}$$

Therefore,  $|(\mathcal{E}(B_n)/G)_i| = |\mathcal{E}(B_n/G)_i| = p_{1+i}(\ell, m, 1)$ . Since  $\mathcal{E}(B_n)/G$  is Peck,  $p_1(\ell, m, 1), p_2(\ell, m, 1), \dots, p_{\ell-m}(\ell, m, 1)$  is unimodal and symmetric, and hence Theorem 5.1 holds in the case  $r = 1$ .  $\square$

## 6 Final Remarks

In this section, we discuss several related results and list further questions.

**Remark 6.1** Observe that Theorem 3.4 trivially implies that  $\mathcal{E}(B_n)$  is Peck. In fact, it is true that  $\mathcal{E}(B_n)$  is unitary Peck. For a proof, see Section 8 of the REU report.

**Definition 6.2** Let  $B_n(q)$ , the  $q$ -analogue of the boolean algebra, be the graded poset whose elements are linear subspaces  $V \subset \mathbb{F}_q^n$  with  $V \leq W$  if  $V \subset W$ .

A natural extension of Conjecture 1.3 would be an analogous result for  $q$ -analogs. We suspect the method used in Section 8 of the REU report to prove  $\mathcal{E}(B_n)$  is unitary peck may solve Question 6.3.

**Question 6.3** Is  $\mathcal{E}(B_n(q))$  unitary peck?

In particular, if the answer to the previous Question 6.3 is affirmative, it would immediately hold that  $\mathcal{E}(B_n(q))/G$  is Peck, and if  $G$  is CCT then  $\mathcal{E}(B_n(q)/G)$  is Peck. Hence, we pose the following question:

**Question 6.4** For  $G$  a group with a CCT action on  $B_n(q)$  is  $\mathcal{E}(B_n(q)/G)$  Peck?

Even more generally, we wonder if the  $q$ -analogue of Conjecture 1.3 holds.

**Question 6.5** For  $G$  a group acting on  $B_n(q)$ , is  $\mathcal{E}(B_n(q)/G)$  Peck? If not, is  $\mathcal{E}(B_n(q)/G)$  rank-unimodal?

We found several additional interesting examples of CCT actions. Once such action is the linear automorphism of the  $n$ -cube. Using python code which we wrote, we found that for  $n \leq 3$ , the linear automorphisms of the  $n$ -cube induces a CCT action on  $B_{2^n}$ . We wonder if this generalizes.

**Question 6.6** Does the group of linear automorphism of an  $n$ -cube in  $\mathbb{R}^n$ , whose vertices lie at  $(\pm 1, \dots, \pm 1)$  induce a CCT action on  $B^{2^n}$ ?

There is also the question of which regular polytopes induce CCT actions. We have shown it holds for the octahedron and the simplex (tetrahedron) in Subsubsection 4.2.2. We also checked using python code which we wrote that it holds for the  $n$ -cube for  $n \leq 3$ . We wonder whether it holds for the five exceptional regular polytopes, which are the only regular polytopes other than simplices, octahedra, and cubes.

Remark 6.1 should be converted to a corollary and should be moved to the introduction, to immediately follow Theorem 1.5. Aaron: I'm fine with moving this to the introduction, but I don't think we should call it a corollary. Should we perhaps make this a separate theorem or proposition, and just refer people to the REU report?

**Question 6.7** Do the groups of linear automorphisms of the five exceptional regular polytopes (namely the dodecahedron and icosahedron in  $\mathbb{R}^3$  as well as the 24-cell, 120-cell, and 600-cell polytopes in  $\mathbb{R}^4$ ) induce CCT actions?

We found using python code we wrote that the group of invertible linear maps on  $\mathbb{F}_2^3$  acting on the seven nonzero points of  $\mathbb{F}_2^3$  induces an action on  $B_7$  which is CCT. We wonder if this generalizes to other groups of invertible linear maps on finite fields.

**Question 6.8** Is the action of  $\mathrm{GL}_n(\mathbb{F}_q)$  on  $B_{q^n-1}$  (induced by the action of  $\mathrm{GL}_n(\mathbb{F}_q)$  on  $(\mathbb{F}_q^n)^\times$ ) CCT? What about the action of  $\mathrm{PGL}_n(\mathbb{F}_q)$  on  $B_n(q)$ ? If not, what about the action of  $\mathrm{PGL}_n(\mathbb{F}_2)$  on  $B_n(2)$ ?

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References - check if the Pak and Panova article has appeared. Also, several words should be capitalized in the bibliography - Sperner (twice), Lefschetz, Volume. - I fixed everything except for capitalizing "Volume", which is up to BibTex's bibliography style. I think this style is the one. Order asks for, so we shouldn't change anything? - David Aaron: I think it's fine not to change anything else, just mention what we don't change in the additional document