List of Todos

UNIMODALITY PAPER

DAVID HEMMINGER, AARON LANDESMAN, ZIJIAN YAO

David, I did some fancy footwork, which I don't really understand, in order to turn the theorems we repeat into theorems A and B, although I haven't changed proposition to theorem for Theorem B in case you don't like it. This allows us to repeat theorems without a problem though.

1. Introduction

Let P be a finite graded poset of rank n. It is natural to study the structure of the edges in the Hasse diagram of P. To this end, define an endofunctor $\mathcal{E}^r$ on the category finite graded posets with rank-preserving morphisms as follows

Definition 1.1. For $\mathcal{P}$ the category of graded posets, for each $r \in \mathbb{N}$, define the *Functor of Edges* $\mathcal{E}^r : \mathcal{P} \to \mathcal{P}$ as follows. The elements of the graded poset $\mathcal{E}^r(P)$ are (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + r$. Define the covering relation $\lessdot_{\mathcal{E}}$ on $\mathcal{E}^r(P)$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$. Then define the relation $\leq_{\mathcal{E}}$ on $\mathcal{E}^r(P)$ to be the transitive closure of $\lessdot_{\mathcal{E}}$, i.e. $(x, y) \leq_{\mathcal{E}} (x', y')$ if there exists a chain

$$(x, y) \lessdot_{\mathcal{E}} (x_1, y_1) \lessdot_{\mathcal{E}} \ldots \lessdot_{\mathcal{E}} (x_{r-1}, y_{r-1}) \lessdot_{\mathcal{E}} (x', y')$$

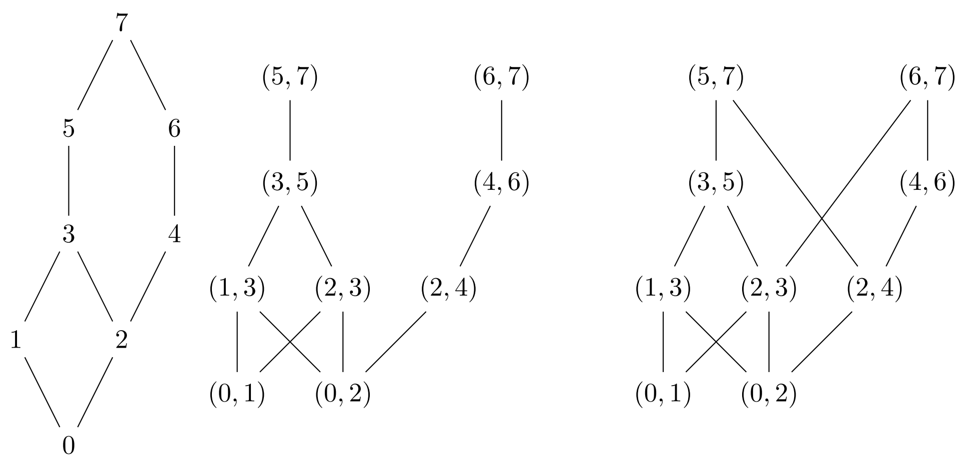
Since people know what the transitive closure is, are we sure we want to use the valuable space in the introduction explicitly defining it?

Let Q be a finited graded poset of rank n. Given a morphism $f : P \to Q$, define $\mathcal{E}^r(f) : \mathcal{E}^r(P) \to \mathcal{E}^r(Q)$ by $\mathcal{E}^r(f)(x, y) = (f(x), f(y))$.

Example 1.2. It is important we declare the relation $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$. If instead we defined a relation $\leq_{\mathcal{E}'}$ on $\mathcal{E}(P)$ by $(x, y) \leq (a, b)$ if $x \leq a, y \leq b$, then $\mathcal{E}(P)$ would not necessarily be a graded poset. In Figure 1.2 we give an example of a poset P for which $\mathcal{E}(P)$ is not graded with the relation $\leq_{\mathcal{E}'}$.

Here, it is clear that $\mathcal{E}^1(P)$ is a graded poset, with $rk(x, y) = rk(x)$, but the Hasse diagram on the right represents a poset which does not have a grading.

We will show that $\mathcal{E}^r$ is well-defined in Section 3. Note that an edge in the Hasse diagram of P can be written as $(x, y) \in P \times P$ such that $x \lessdot y$. Such edges are in bijection with elements $(x, y) \in \mathcal{E}^1(P)$. We observe that when P has a nice structure, $\mathcal{E}^r(P)$ commonly has a nice structure as well. In particular, let the *boolean algebra of rank* n, denoted B_n, be the poset whose elements are subsets of $\{1, \ldots, n\}$ with the relation given by containment,



P $\qquad$ $\mathcal{E}^1(P)$ $\qquad$ $\mathcal{E}^1(P)$ with relation $\leq_{\mathcal{E}'}$ is not graded

i.e. for all $x, y \in B_n$, $x \leq y$ if x is a subset of y. It is well-know than if G is a group of rank-preserving automorphism of B_n, then B_n/G is Peck . We conjecture the following.

Perhaps consider saying either (both order preserving and rank preserving) or (neither of them). I prefer ommiting both, since we later define automorphisms as satisfying both of these properties. The same should be done in 1.3

add reference

Conjecture 1.3. *Let G be a group of rank-preserving automorphisms of B_n. Then $\mathcal{E}^r(B_n/G)$ is Peck.*

We prove this conjecture holds in the case $r = 1$ whenever the group action of G on B_n has the following property.

Weren't we going to include a question about q-analogs?

Definition 1.4. A group action of G on P is *common cover transitive*, (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in Stab(z)$ such that $g \cdot x = y$.

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}^1(B_n/G)$ is Peck.*

A large number of group actions on B_n have the CCT property. In order to construct a family of such actions we first prove that some basic group actions on B_n are CCT, then show that cover transitivity is preserved under semidirect products. Throughout the paper we let a subgroup $G \subseteq S_n$ act on B_n by letting it act on the elements within subsets of $[n] := \{1, \ldots, n\}$, i.e. $g \cdot x = \{g \cdot i : i \in x\}$ for all $g \in G$, $x \in B_n$. We also embed the dihedral group D_{2n} into S_n by letting it act on the vertices of an n-gon.

David, there is an inconsistency in notation here: using ":" for "such that" versus "|" (for "such that". I propose the convention "|" since I think it is slightly more standard, but I'm ok with using ":" though if you feel strongly.

Rephrase so that it has two assertions (in two sentences): first, that S_n acts cover-transitively for all n, and second, that for p prime, $D_{2p} and D_{4p}$ are cover-transitive inside S_{2p}, S_{4p}.

Proposition 1.6. *Let p be prime. The actions of S_n, D_{2p}, and D_{4p} on B_n, B_p and B_{2p}, respectively are CCT.*

Proposition 1.7. *Let $G \subseteq \mathrm{Aut}(P)$, $H \lhd G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

Please note: This paragraph does NOT currently reflect reality. I thought for a while about how we should structure the paper, and this is roughly how I think it should look.

The paper is organized as follows. In Section 2 we cover the necessary background for posets and Peck posets. In Section 3 we show that $\mathcal{E}^r$ is well-defined and prove Theorem 1.7

along with various other nice properties of $\mathcal{E}^r$. Section 5 contains the proofs of Propositions 1.6 and 5.1 as well as some examples of families of group actions shown to be CCT by these propositions. In Section we prove that $\mathcal{E}^r(B_n/G)$ is rank-unimodal for certain group actions that are not CCT. Section discusses a q-analog to Conjecture 1.3, including some partial results.

consolidate this section

consolidate this section

Obviously come back to this at the end, but since this is the introduction for the REU report, not a journal paper, we should also include descriptions of the sections we don't think will go in the journal paper.

2. Background

In this section we review morphisms of graded posets, Peck posets, and a useful theorem about quotients of Posets.

Throughout the paper we write $x \leq_P y$ to denote that x is less than or equal to y under the relation defined on the poset P. When the poset is clear we will omit the P and simply write $x \leq y$.

Let P be a finite *graded* poset of rank n, that is the elements of P are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks* of P, such that if $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$. Define $\text{rk}(x) = k$, where $x \in P_k$.

Denote the category of finite graded posets by $\mathcal{P}$, and let Q be a finite graded poset of rank n. A map $f\colon P \to Q$ is a *morphism* from P to Q if it is rank-preserving and order preserving, i.e. for all $x, y \in P$, $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$ and $\text{rk}(x) = \text{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Remark 2.1. Note that we do not require the implication $f(x) \leq_Q f(y) \Rightarrow x \leq_P y$ in order for f to be a morphism. In particular this means a bijective morphism f need not be an isomorphism, since it will not necessarily have a two-sided inverse.

Write $p_i = |P_i|$. If we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

for some $0 \leq k \leq n$, then P is *rank-unimodal*, and if $p_i = p_{n-i}$ for all $1 \leq i \leq n$, then P is *rank-symmetric*. An *antichain* in P is a set of elements in P that are pairwise incomparable. If no antichain in P is larger than the largest rank of P, then P is *Sperner*. More generally, P is *k-Sperner* if no disjoint union of k antichains in P is larger than the disjoint union of the largest k ranks of P, and P is *strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$. We then make the following definition.

shouldn't the word disjoint be removed here?

Definition 2.2. P is *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x | x \in P\}$ and $\{x | x \in P_i\}$ respectively. Note that we will frequently abuse notation and write P and P_i for $V(P)$ and $V(P_i)$ when our meaning is clear. In determining whether P is Peck, it is often useful to consider certain linear transformations on $V(P)$.

Definition 2.3. A map $U\colon V(P) \to V(P)$ is an *order-raising operator* if $U(V(P_n)) = 0$ and for all $0 \leq i \leq n-1$, $x \in P_i$ we have

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

for some constants $c_{x,y} \in \mathbb{C}$. We say that U is the *Lefschetz map* if all $c_{x,y}$ on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

add reference

Lemma 2.4. *P is Peck if and only if there exists an order-raising operator U such that for all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition 2.5. If the Lefschetz map satisfies the condition for U in Lemma 2.4, then P is *unitary Peck.*

Let $G \subseteq \mathrm{Aut}(P)$, and define the *quotient poset* P/G to be the poset whose elements are the orbits of G, with the relation $\mathcal{O} \leq \mathcal{O}'$ if there exist $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ such that $x \leq x'$. We will use of the following result later in the paper.

Theorem 2.6. *If P is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then P/G is Peck.*

3. The Functor of Edges

In Section 3.1 we show that $\mathcal{E}^r$ as described in Definition 1.1 is a well-defined functor and prove some useful properties of $\mathcal{E}^r$. Section 3.2 is devoted to the proof of Theorem 1.7.

3.1. Definition and Basic Properties.

First we prove Proposition 3.5, which states that $\mathcal{E}^r$ is a well-defined functor. We break the proof of Proposition 3.5 into Lemmas 3.2, 3.3, and 3.4 which follow below. After showing that $\mathcal{E}^r$ is well-defined we then define a natural G action on $\mathcal{E}^r(P)$ and define a surjection $\mathcal{E}^r(P)/G \to \mathcal{E}^r(P/G)$ that will be important for the proof of Theorem 1.7.

Notation 3.1. When the poset P and the value r are clear, we shall simply use $\leq_{\mathcal{E}}, \lessdot_{\mathcal{E}}$ to refer to $\leq_{\mathcal{E}^r(P)}, \lessdot_{\mathcal{E}^r(P)}$. Similarly, in Subsection 3.2, we define an object $\mathcal{H}^r(B_n)$, and shall use $\leq_{\mathcal{H}}, \lessdot_{\mathcal{H}}$ in place of $\leq_{\mathcal{H}^r(P)}, \lessdot_{\mathcal{H}^r(P)}$.

Lemma 3.2. *The relation $\leq_{\mathcal{E}}$ defines a partial order on $\mathcal{E}^r(P)$.*

Proof. We have that $(x, y) \leq_{\mathcal{E}} (x, y)$ and that $\leq_{\mathcal{E}}$ is transitive by definition. It remains to be shown that $\leq_{\mathcal{E}}$ is antisymmetric. Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$ and $(x', y') \leq_{\mathcal{E}} (x, y)$. Then $x \leq_P x' \leq_P x$ and $y \leq_P y' \leq_P y$, so $x = x'$ and $y = y'$ by antisymmetry of $\leq_P$, hence $(x, y) = (x', y')$. □

Lemma 3.3. *For P a graded poset, the object $\mathcal{E}^r(P)$ is a graded poset.*

Proof. To show $\mathcal{E}^r(P)$ is graded, we must show $(x, y) <_{\mathcal{E}} (x', y') \implies rk(x, y) < rk(x', y')$ and $(x, y) \lessdot_{\mathcal{E}} (x', y') \implies rk(x, y) + 1 = rk(x', y')$. Both of these facts follow immediately from the definition of $\lessdot_{\mathcal{E}}$ and the definition $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$. □

I don't think the details of the proof need to be included here, but I've left them here commented out in case we change our minds. Aaron: I agree it is fine to leave them out.

Lemma 3.4. *If $f\colon P \to Q$ is morphism of finite ordered posets, then $\mathcal{E}^r(f)\colon \mathcal{E}^r(P) \to \mathcal{E}^r(Q)$ is a morphism of finite ordered posets.*

Proof. First, $\mathcal{E}^r(f)$ is rank-preserving, since for all $(x, y) \in \mathcal{E}^r(P)$ we have

$$\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x) = \mathrm{rk}_P(f(x)) = \mathrm{rk}_{\mathcal{E}}(\mathcal{E}^r(f)(x, y)).$$

Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$. Then $x \leq_P x'$, $y \leq_P y'$, and since f is order-presevering, it follows that $f(x) \leq_P f(x')$, $f(y) \leq_P f(y')$. Hence $\mathcal{E}^r(f)(x, y) \leq_{\mathcal{E}} \mathcal{E}^r(f)(x', y')$. Thus $\mathcal{E}^r(f)$ is order preserving and hence a morphism of finite ordered posets. □

Proposition 3.5. *The map $\mathcal{E}^r(P)$ is an endofunctor on the category of finite graded posets $\mathcal{P}$.*

Proof. By Lemmas 3.2, 3.3, and 3.4 we have that $\mathcal{E}^r$ takes objects in $\mathcal{P}$ to objects in $\mathcal{P}$ and morphisms of $\mathcal{P}$ to morphisms of $\mathcal{P}$. Furthermore it is clear that $\mathcal{E}^r(\mathrm{id}_P) = \mathrm{id}_{\mathcal{E}^r(P)}$, so it remains to be shown that for finited graded posets P, Q, and R with morphisms $f\colon P \to Q$ and $g\colon Q \to R$ that $\mathcal{E}^r(g \circ f) = \mathcal{E}^r(g) \circ \mathcal{E}^r(f)$. This is clear, however, as for all $(x, y) \in \mathcal{E}^r(P)$ we have $\mathcal{E}^r(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}^r(g) \circ \mathcal{E}^r(f))\,(x, y)$. □

Given a group action of G on P, we can easily define a natural group action of G on $\mathcal{E}^r(P)$ using Proposition 3.5. For all $g \in G$ we have that multiplication by g is an automorphism of P, so it follows that $\mathcal{E}^r(g)$ is an automorphism of $\mathcal{E}^r(P)$. Since $\mathcal{E}^r$ is a functor this gives a well-defined group action.

Notation 3.6. Define a group action of G on $\mathcal{E}^r(P)$ by defining $g \cdot (x, y) = \mathcal{E}^r(g)(x, y) = (g \cdot x\, g \cdot y)$.

Using Notation 3.6, we have a well-defined quotient poset $\mathcal{E}^r(P)/G$. It is then natural to ask whether the operation of quotienting out by G commutes with $\mathcal{E}^r$, that is, whether $\mathcal{E}^r(P/G) \cong \mathcal{E}^r(P)/G$. Unfortunately the two posets are rarely isomorphic, but there is always a surjection $\mathcal{E}^r(P)/G \to \mathcal{E}^r(P/G)$, and this surjection is also an injection precisely when the G-action on P is CCT.

Proposition 3.7. *The map $q\colon \mathcal{E}^r(P)/G \to \mathcal{E}^r(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism.*

Proof. First we show that q is a surjective morphism. Note that q is well defined because if $(x', y') = g(x, y) = (g \cdot x, g \cdot y)$ for some $g \in G$, then $x' \in Gx$ and $y' \in Gy$. Clearly q is rank-preserving and surjective, so it suffices to show that q is order-preserving. Suppose that $G(x, y) \le G(w, z)$. Then there exist some $(x_0, y_0) \in G(x, y)$, $(w_0, z_0) \in G(w, z)$ such that $x_0 \le w_0$ and $y_0 \le z_0$. We then have that $(Gx, Gy) \le (Gw, Gz)$ by definition, hence q is order-preserving. ☐

Lemma 3.8. *Let G be a group acting on a graded poset P. The following are equivalent:*

(1) *The action of G on P is CCT.*
(2) *Whenever $x \lessdot y, x \lessdot z$, and $y \in Gz$, there exists some $g \in Stab(x)$ with $gx = z$.*
(3) *The map $q\colon \mathcal{E}^1(P)/G \to \mathcal{E}^1(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is an bijective morphism (but not necessarily an isomorphism).*

Proof. First, we shall show (1) $\Leftrightarrow$ (3). Observe that q is a bijection exactly when there do not exist distinct orbits $G(x, y) \ne G(x', y')$ with $x' \in Gx$, $y' \in Gy$. Fix $(x, y), (x', y') \in \mathcal{E}^1(P)$ such that $x' \in Gx$ and $y' \in Gy$. Pick a $g \in G$ such that $g \cdot y' = y$. Then $(g \cdot x', y) \in G(x', y')$, so $G(x, y) = G(x', y')$ if and only if there exists some $g' \in G$ such that $g' \cdot x = g \cdot x'$ and $g' \cdot y = y$. Hence q is a bijection if and only if the G action is CCT.

Next, we shall show (2) $\Leftrightarrow$ (3). It is quite analogous to the proof of (1) $\Leftrightarrow$ (3), but we shall include the argument for completeness. First, q is a bijection if and only if there do not exist distinct orbits $G(x, y) \ne G(x', y')$ with $x' \in Gx, y' \in Gy$. Fix $(x, y), (x', y') \in \mathcal{E}^1(P)$ such that $x' \in Gx, y' \in Gy$. Choose $g \in G$ with $gx' = x$. Then, $(x', g \cdot y) \in G(x', y')$. Since $gy' \in Gy$, and q is a bijection, the statement $(x', g \cdot y) \in G(x', y')$ is equivalent to $G(x', y') = G(x, y)$. This, in turn is equivalent to the existence of $g' \in G$ with $g'x = x, g'y = gy'$. So, $g' \in Stab(x), g'y = (gy')$, which is what was claimed in property (2).

☐

Remark 3.9. While q is a bijection if and only if the action of G on P is CCT, it is *not* true that if the action of G on P is CCT, then q is an isomorphism. For example, take $G = D_{20} \subset S_{10}$ acting by reflections and rotations on $[10]$ and hence acting on B_{10}. We shall see in Lemma 11.2 that this action is CCT. However, consider $x = \{2, 4\}, y = \{1, 2, 4\}, a = \{2, 4, 7\}, b = \{2, 4, 6, 7\}$. Then we may observe $(x, y), (a, b) \in \mathcal{E}^1(B_{10})$ and $Gx < Ga, Gy < Gb$, so $(Gx, Gy) <_{\mathcal{E}} (Ga, Gb)$. However, it is not true that $G(x, y) <_{\mathcal{E}} G(a, b)$.

3.2. **Proof of Theorem 1.7.** In this section we prove Theorem 1.7, which we recall here:

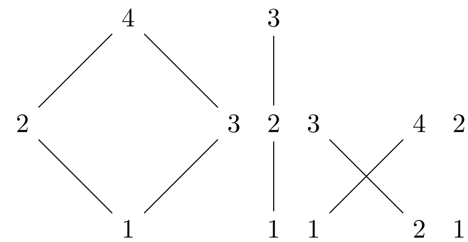
if there's a better way of fixing the numbering for this theorem I'd love to know

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}^1(B_n/G)$ is Peck.*

The proof is largely motivated by the following Lemma.

Lemma 3.10. *If $f : P \to Q$ is a bijection (but not necessarily an isomorphism) and P is Peck then Q is Peck.*

Proof. Let $\mathrm{rk}(P) = \mathrm{rk}(Q) = n$. Since P is Peck there exists an order-raising operator U such that $U^{n-2i}\colon P_i \to P_{n-i}$ is an isomorphism. Since f is a poset morphism, it follows that the map $f \circ U \circ f^{-1}$ is an order-raising operator on Q. We then have that $f \circ U^{n-2i} \circ$



$f^{-1} = \left(f \circ U \circ f^{-1}\right)^{n-2i} : Q_i \to Q_{n-i}$ is an isomorphism since $U^{n-2i}\colon P_i \to P_{n-i}$ is an isomorphism and f is a bijection.

$\square$

By Lemma 3.10 and Proposition 3.7, in order to prove Theorem 1.7 it suffices to prove that $\mathcal{E}^r(B_n)/G$ is Peck. One way to do this is to prove that $\mathcal{E}^r(B_n)$ is unitary Peck and then apply Theorem **??**. While this is the most direct path, we unfortunately do not have a proof that $\mathcal{E}^r(B_n)$ is unitary Peck for $r > 1$. We prove that $\mathcal{E}^1(B_n)$ is unitary Peck in Section 13, but unfortunately the proof is messy and computational.

Fortunately there is a simpler – albeit less direct – route. In order to avoid showing that $\mathcal{E}^r(B_n)$ is unitary Peck, for all r, n, we define a poset $\mathcal{H}^r(B_n)$ such that $\mathcal{H}^r(B_n)$ is easily seen to be unitary Peck. Furthermore, there is always a bijective morphism $f\colon \mathcal{H}^r(B_n)/G \to \mathcal{E}^r(B_n)/G$. By the above discussion, Theorem 1.7 readily follows.

are we obligated to check whether $\mathcal{E}^r(B_n)$ is unitary Peck? Maybe we should at least put it as a question.

Definition 3.11. For P a graded poset, define the graded poset $\mathcal{H}^r(P)$ as follows. Let the elements $(x, y) \in \mathcal{H}^r(P)$ to be pairs $(x, y) \in P \times P$ such that $x < y$ and $rk_P(x)+r = rk_P(y)$,. Define $(x, y) \lessdot_{\mathcal{H}} (x', y')$ if $x \lessdot x', y \lessdot y'$ and $x' \not\leq y$. Then, define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$. and define $rk_{\mathcal{H}}(x, y) = rk_P(x)$.

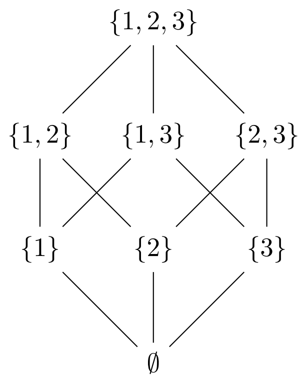
4. The Poset $\mathcal{H}^r(P)$

Remark 4.1. While $\mathcal{E}^r : \mathcal{P} \to \mathcal{P}$ is a functor, $\mathcal{H}^r$ is not a functor. In particular, we have not specified how $\mathcal{H}^r$ acts on morphisms. If, for $f : P \to Q$, we attempted to define $\mathcal{H}^r(f) : \mathcal{P} \to \mathcal{Q}$, we would quickly run into trouble. For example, suppose we took f mapping a diamond P (leftmost below) to a chain of length three Q (second leftmost) Then, $\mathcal{H}^1(P)$ (second rightmost) has four points, with two pairs of edges, whereas $\mathcal{H}^1(Q)$ (rightmost) has two disconnected points. It is clear that there can be no order preserving mapping between these two objects.

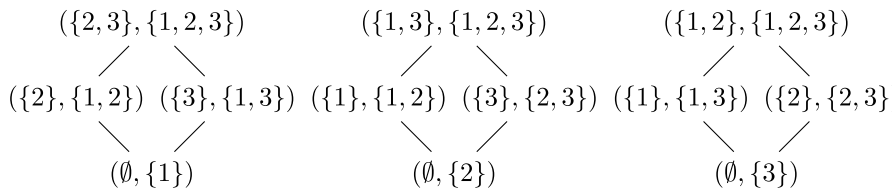
However, it is still true that $\mathcal{H}^r(P)$ can be viewed as a graded poset with $rk_{\mathcal{H}^r(P)}(x, y) = rk_P(x)$.

Put all pictures in figures, and make them look nicer

Example 4.2. Below is an example of the poset $\mathcal{H}^1(B_3)$.



B_3



$\mathcal{H}^1(B_3)$

Lemma 4.3. *For P a graded poset, the object $\mathcal{H}^r(P)$, as defined in Definition 3.11 is a graded poset.*

Proof. This follows immediately from Remark 4.4 and the fact that $\mathcal{E}^r(P)$ is graded. □

Remark 4.4. Note that $(x,y) \lessdot_{\mathcal{H}} (a,b) \Rightarrow (x,y) \lessdot_{\mathcal{E}} (a,b)$.

Similarly to $\mathcal{E}^r(P)$, the action of G on P induces an order-preserving action of G on $\mathcal{H}^r(P)$.

Lemma 4.5. *The function defined by $g \cdot (x,y) = (gx, gy)$ for all $g \in G$, $(x,y) \in \mathcal{H}^r(P)$ is a well-defined group action of G on $\mathcal{H}^r(P)$.*

Proof. Let $g \in G$. Since $\leq_{\mathcal{H}}$ is the transitive closure of $\lessdot_{\mathcal{H}}$ it suffices to show that for all $(x,y),(x',y') \in \mathcal{H}^r(P)$ we have $(x,y) \lessdot_{\mathcal{H}} (x',y') \Leftrightarrow g(x,y) \lessdot_{\mathcal{H}} g(x',y')$. Since g is an automorphism of P, we have $x \leq_P x' \Leftrightarrow gx \leq_P gx'$, $y \leq_P y' \Leftrightarrow gy \leq_P gy'$, and $y \not\leq x' \Leftrightarrow gy \not\leq gx'$, so the result follows from the definition of $\leq_{\mathcal{H}}$. □

Proposition 4.6. *$\mathcal{H}^r(B_n)$ is isomorphic to $\binom{n}{r}$ disjoint copies of B_{n-r}.*

Proof. Suppose we have $(x,y),(x',y') \in \mathcal{H}^r(B_n)$ with $(x,y) \lessdot (x',y')$. Let $j \in [n]$ such that $y' = y \cup \{j\}$, and let $i \in [n]$ such that $x' = x \cup \{i\}$. If $i \neq j$, then $x' \leq y$, contradicting the assumption that $(x,y) \lessdot_{\mathcal{H}} (x',y')$. Thus $x' = x \cup \{i\}$ and $y' = y \cup \{i\}$ for some $i \in [n]$.

Conversely we can easily check that if $i \notin y$, then $(x,y) \lessdot_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$. It follows that for all subsets $w \subset [n]$ such that $|w| = r$, there is an isomorphism from the subposet of elements $\{(x,y)\colon y \setminus x = w\}$ to B_{n-r} defined by $(x,y) \mapsto (x \setminus w, y \setminus w)$. Furthermore if $y \setminus x \neq y' \setminus x'$, then (x,y) and (x',y') are incomparable, so these subposets indexed by w are pairwise disjoint, and $\mathcal{H}^r(B_n)$ is isomorphic to $\binom{n}{r}$ copies of B_{n-r}. □

The primary advantage of Proposition 4.6 is it allows us to easily see that $\mathcal{H}^r(B_n)$ is unitary Peck, which then implies that $\mathcal{H}^r(B_n)/G$ is Peck for any subgroup $G \subseteq S_n$.

Corollary 4.7. *$\mathcal{H}^r(B_n)$ is unitary Peck for all $n \geq r$.*

Proof. This follows immediately from Proposition 4.6 and the fact that B_{n-r} is unitary Peck, as shown, for instance in [?, Theorem 2a] because $B_k = (B_1)^k$ and B_1 is clearly unitary Peck. □

Corollary 4.8. *$\mathcal{H}^r(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. This follows from Corollary 4.7 and Theorem 2.6 □

Lemma 4.9. *The map $f : \mathcal{H}^r(P) \to \mathcal{E}^r(P), G(x, y) \mapsto G(x, y)$ is a bijection (but never an isomorphism for $n - r > 1$.).*

Proof. The elements of $\mathcal{H}^r(P), \mathcal{E}^r(P)$ are precisely the same by definition. Therefore, as long as f is a morphism, it is automatically a bijection. Since f is clearly rank preserving, to show f is a morphism, it suffices to show f is order preserving. This is immediate from Remark 4.4 □

Corollary 4.10. *$\mathcal{E}^r(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. By Corollary 4.8, $\mathcal{H}^r(B_n)/G$ is Peck. By Lemma 4.9, the map $f : \mathcal{H}^r(P) \to \mathcal{E}^r(P), G(x, y) \mapsto G(x, y)$ is a bijection. Then, by Lemma 3.10, it follows that $\mathcal{E}^r(B_n)/G$ is Peck. □

4.1. **A Generalization of $\mathcal{H}^r, \mathcal{E}^r$.** Although for the most part, we shall investigate $\mathcal{E}^1(P/G), \mathcal{E}^1(P)/G$, there is a natural further generalization of $\mathcal{H}^r$, to what we shall call $\mathcal{H}^{\overrightarrow{r}}$, where $\overrightarrow{r} = r_1, r_2, \ldots, r_k$ is an integer valued sequence. The results holding for these generalizations are analogous to those developed above. The purpose of developing these generalizations notion will be to give a more general application to Polya theory than could be given with $\mathcal{H}^r$, for $r \in \mathbb{Z}$.

Definition 4.11. Let $\overrightarrow{r} = r_1, \ldots, r_k$. For a graded poset P, define the graded poset $\mathcal{H}^{r_1,\ldots,r_k}(P)$, also notated $\mathcal{H}^{\overrightarrow{r}}(P)$, whose elements are formal symbols $(x_1, x_2, \cdots, x_{k+1}$ such that $rk(x_i) + r_i = rk(x_{i+1})$, for all $i \in [k]$. Say $(x_1, x_2, \cdots, x_{k+1}) \lessdot_{\mathcal{H}^{\overrightarrow{r}}} (y_1, y_2, \cdots, y_{k+1})$ if $x_i \lessdot_P y_i, y_i \not\leq_P x_{i+1}$ for all $i \in [k + 1]$. Then, define a relation $<_{\mathcal{H}^{\overrightarrow{r}}}$ on $\mathcal{H}^{r_1,\ldots,r_k}(P)$, to be the transitive closure of $\lessdot_{\mathcal{H}^{\overrightarrow{r}}}$. Finally, define $rk_{\mathcal{H}^{\overrightarrow{r}}(P)}(x_1, \ldots, x_{k+1}) = rk_P(x_1)$

Definition 4.12. Let $\overrightarrow{r} = r_1, \ldots, r_k$, with $r_i \in \mathbb{N} \forall i \in [k]$. Let $\mathcal{P}$ be the category of graded posets. Define the functor $\mathcal{E}^{\overrightarrow{r}} : \mathcal{P} \to \mathcal{P}$, also notated $\mathcal{E}^{r_1,\ldots,r_k}$. For a graded poset P, define the graded poset $\mathcal{E}^{r_1,\ldots,r_k}(P)$, also notated whose elements are formal symbols $(x_1, x_2, \cdots, x_{k+1})$ such that $rk(x_i)+r_i = rk(x_{i+1})$, for all $i \in [k]$. Say $(x_1, x_2, \cdots, x_{k+1}) \lessdot_{\mathcal{H}^{\overrightarrow{r}}} (y_1, y_2, \cdots, y_{k+1})$ if $x_i \lessdot_P y_i$ for all $i \in [k + 1]$. Then, define a relation $\leq_{\mathcal{E}^{\overrightarrow{r}}}$ on $\mathcal{E}^{r_1,\ldots,r_k}(P)$, to be the transitive closure of $\lessdot_{\mathcal{E}^{\overrightarrow{r}}}$. Finally, define $rk_{\mathcal{E}^{\overrightarrow{r}}(P)}(x_1, \ldots, x_{k+1}) = rk_P(x_1)$

Remark 4.13. Both $\mathcal{E}^{\overrightarrow{r}}, \mathcal{H}^{\overrightarrow{r}}$ are functors, for the same reasons that $\mathcal{E}^r, \mathcal{H}^r$ are functors. This generalization is essentially taking the nerves of the poset P. See [?] for some related constructions, although their constructions are different in many crucial ways.

In the next Lemmas, we cite analogous results which hold for $\mathcal{H}^{\overrightarrow{r}}$. The proofs are almost identical to those for $\mathcal{H}^r$.

Lemma 4.14. *Given a group action $\phi : G \times P \to P$ there are well defined group actions $\phi_{\mathcal{E}} : G \times \mathcal{E}^{\overrightarrow{r}}(P) \to \mathcal{E}^{\overrightarrow{r}}(P), \phi_{\mathcal{H}} : G \times \mathcal{H}^{\overrightarrow{r}}(P) \to \mathcal{H}^{\overrightarrow{r}}(P)$, both given by $g \cdot (x_1, \cdots, x_{k+1}) = (g \cdot x_1, \cdots, g \cdot x_{k+1})$.*

Proof. The proof is analogous to Lemma 4.5, Lemma 4.5. □

Lemma 4.15. *The poset $\mathcal{H}^{r_1,\ldots,r_k}(B_n)$ is isomorphic to the multinomial coefficient $\binom{n}{r_1, r_2, \ldots, r_k}$ disjoint copies of $B_{n-\sum_{i=1}^k r_i}$. Consequently, $\mathcal{H}^{\overrightarrow{r}}(B_n)$ is unitary Peck and $\mathcal{H}^{\overrightarrow{r}}(B_n)/G$ is Peck.*

Proof. Once again, the proof of the above three statements is analogous to those of 4.6, Corollary 4.7, and Corollary 4.8. The reason there are multinomial coefficients here instead of binomial coefficients, is that an element $(x_1, x_2, \cdots, x_{k+1})$ lies in the copy of $B_{n-\sum_{i=1}^{k} r_i}$ determined by the ordered tuple of subsets $(x_2 \setminus x_1, x_3 \setminus x_2, \ldots, x_{k+1} \setminus x_k)$. The first consists of r_1 elements, the next of r_2 elements, up through the last which consists of r_k elements. Since the total number of ways to choose $r_1, \ldots, r_k$ in k distinct groups is $\binom{n}{r_1, r_2, \ldots, r_k}$, there are exactly this many disjoint copies of $B_{n-\sum_{i=1}^{k} r_i}$. □

5. Common Cover Transitive Actions

In this section, we develop the theory of CCT actions ϕ where G is a group, P is a poset, and $\phi : G \times P \to P$ is an action. Recall Definition 1.4, that ϕ is CCT if whenever $x, y, z \in P, x \lessdot y, y \lessdot z, x \in Gy$ then there exists $g \in Stab(z)$ with $gx = y$. For CCT actions $\phi : G \times P \to P$, we shall show that, $\mathcal{E}^1(P/G)$ is Peck. Additionally, we shall show the CCT property is closed under semidirect products, in the appropriate sense. It is obvious that the action of S_k on B_k is CCT, and we shall also see in Lemma **??** that the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

Example 5.1. Two rather trivial examples of CCT actions are $\phi : S_n \times B_n \to B_n, \psi : G \times B_n \to B_n$ where G is arbitrary, ϕ is the permutation action and ψ is the trivial action. In the former case, $\mathcal{E}^1(B_n/S_l)$ is simply a chain with $n-1$ points, and so is $\mathcal{E}^1(B_n)/S_l$, since all (x, y) are identified under the S_l action. In the latter case, since G acts trivially by ϕ, both $\mathcal{E}^1(B_n/G) \cong \mathcal{E}^1(B_n)$ and $\mathcal{E}^1(B_n)/G \cong \mathcal{E}^1(B_n)$. So again, ψ is CCT.

Developing less trivial examples will take a bit more work, but it will be shown in Lemma **??** that Dihedral groups of order $2p$ and $4p$, for p prime, are CCT.

Theorem 5.2. *If an action $\phi : G \times P \to P$ is CCT, and $\mathcal{E}^1(P)$ is Peck, then $\mathcal{E}^1(P/G)$ is Peck.*

Proof. By Proposition 3.7, there is a surjection $q : \mathcal{E}^1(P)/G \to \mathcal{E}^1(P/G)$. Additionally, by Lemma 3.7, if ϕ is CCT, q is a bijection. Finally, by Lemma 3.10, since $q : \mathcal{E}^1(P)/G \to \mathcal{E}^1(P/G)$ is a bijection and $\mathcal{E}^1(P)/G$ is Peck, it follows that $\mathcal{E}^1(P/G)$ is Peck. □

5.1. **Preservation Under Semidirect Products.**

Recall Proposition 5.1, as stated in the introduction. This Proposition essentially says the CCT property is preserved under semidirect products. We shall use this to obtain simple proofs that CCT actions are also preserved under direct products and wreath products.

Recall 5.1:

Proposition 1.7. *Let $G \subsetneq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

David, can you fill this in please?

Proof.

□

Theorem 5.3. *For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.*

Proof. First, by definition of CCT, if either G or H acts trivially, it is clear the action of $G \times H$ is CCT. To see this, suppose G acts trivially, the case of H acting trivially is similar. Then, if $(x, y) \lessdot (a, b), (r, s) \lessdot (a, b), (x, y) \in G \times H(r, s)$, then $x \in Gr$ so $x = r$. If $b = y = s$, then $(e, e) \in Stab(x, y)$ and $(e, e)(x, y) = (r, s)$. Otherwise, it must be that $x = r = a$, and then $y \lessdot b, s \lessdot b, y \in Hs$, and so by cover transitivity of G, there exists $g \in G$ so that $(e, g)(x, y) = (r, s)$ and $(e, g) \in Stab(a, b)$.

Next, Observe that $G\times H$ can be viewed as the semidirect product of $(G\times\{e\})\rtimes(\{e\}\times H)$. Since the action of G on P is CCT, by the above paragraph, the action of $G\times\{e\}$ on $P\times Q$ is CCT. Also, since the action of H on Q is CCT, it follows that the action of $\{e\}\times H$ on $(P/G)\times Q$ is CCT. Therefore, the actions of $(G\times\{e\})\rtimes(\{e\}\times H)$ satisfies the conditions of Proposition 5.1 and so the action of $G\times H$ is CCT. □

Next, we use Proposition 5.1 to prove in Theorem 5.8 that the CCT property is preserved under wreath products with the symmetric group. First, we need some definitions of wreath product.

Definition 5.4. For G, H groups, with $H\subset S_l$, the *wreath product*, notated $G\wr H$, is the group whose elements are pairs $(g,h)\in G^l\times H$ with multiplication defined by

$$((g_1',\dots,g_l'),h')\cdot((g_1,\dots,g_l),h)=((g'_{h'(1)}g_1,\dots,g'_{h'(l)}g_l),hh')$$

where $h\in H$ acts on $[l]$ by the restriction of the permutation action of S_l to H.

In other words, $G\wr H$ can be viewed as a certain semidirect product of $G^l\rtimes H$.

Notation 5.5. For any group G with a given action $\psi: G\times P\to P$, we obtain an induced action of $G\wr H$, $\phi: G\wr H\times P^l\to P^l$ defined by

$$((g_1,\dots,g_l),h)(a_1,\dots,a_l)=(g_{h^{-1}(1)}\cdot a_{h^{-1}(1)},\dots,g_{h^{-1}(l)}\cdot a_{h^{-1}(l)}).$$

Remark 5.6. Heuristically, one may think of the above action as obtained by first having G act separately on the l distinct copies of P, and then letting H act by permuting the copies.

Lemma 5.7. *For P a poset, the action $\phi: S_l\times P^l\to P^l, \sigma\cdot(x_1,\dots,x_l)=(x_{\sigma(1)},\dots,x_{\sigma(l)})$ is CCT.*

Proof. For $a\in P^l$ notate $a=(a_1,\dots,a_l)$. Suppose $x,y,z\in P^l$ with $x\lessdot z, y\lessdot z$, and $x\in S_l y$. This means there is a unique i such that $x_i\lessdot z_i, x_k=z_k$ for $k\neq i$. Additionally, there is a unique j for which $y_j\lessdot z_j, y_k=z_k$ for $k\neq j$. Since $x\in S_l y$, we obtain the equality of multisets $\{x_1,\dots,x_l\}=\{y_1,\dots,y_l\}$. But for $k\neq i,j, x_k=z_k=y_k$, we also obtain equality of sets $\{x_i,x_j\}=\{y_i,y_j\}$. Since $rk(y_j)\lessdot rk(x_j)$, we obtain $y_j=x_i, y_i=x_j$. Then, taking the transposition. $\sigma\in S_l=(ij)$, it follows $\sigma\in Stab(z)$ and $\sigma\cdot x=y$. □

Theorem 5.8. *If $\psi: G\times P\to P$ is CCT, then $\phi: G\wr S_l\times P^l\to P^l$ where ϕ is the induced action defined in Notation 5.5 is also CCT.*

Proof. First, the wreath product $G\wr S_l$ can be viewed as the semidirect product $G^l\rtimes S_l$. Observe that since the action of G on P is CCT, using Theorem 5.3, we obtain the action of G^l on P^l is CCT, by the action $(g_1,\dots,g_l)(x_1,\dots,x_l)=(g_1\cdot x_1,\dots,g_l\cdot x_l)$. Additionally, since $P^l/G^l\cong(P/G)^l$. So, for $\sigma\in S_l, x_i\in P/G$, the action of $S_l\times(P/G)^l\to(P/G)^l, (\sigma,(x_1,\dots,x_l))\mapsto(x_{\sigma(1)},\dots,x_{\sigma(l)})$ is CCT by Lemma 5.7. So, the action ϕ, satisfies the conditions of Proposition 5.1 and so ϕ is CCT. □

remove commented out old proofs of wreath products and direct products? The old proof of direct product might actually be simpler than this one though, what do you think?

5.2. **An application to rooted trees.** In this subsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Theorem 5.8 and Theorem 5.3, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT.

Definition 5.9. A poset P is a *rooted tree* if P is a graded poset, there is a unique element $x\in P$ of maximal rank, called the *root*, and for all $x\in P$, other than the root, there exists a unique $y\in P$ with $y\gtrdot x$.

Definition 5.10. For P a rooted tree, an element $x\in P$ is a *leaf* if there is no $z\in P$ with $x>z$. Denote the set of all leaves of P by $L(P)$.

Lemma 5.11. *Let P be a rooted tree and let $L(P)$ be the set of leaves of P. Then, the action of $Aut(P)$ on P induces an action of $Aut(P)$ on $L(P)$. Furthermore, there is also an induced action of $Aut(P)$ on B_n where $n = |L(P)|$.*

Proof. First, we must show that $Aut(P)$ induces an action on $L(P)$. To show this, it suffices to show that for any $g \in Aut(P), x \in L(P)$, then $gx \in L(P)$. This fact, however, is easy to see, because if $gx \notin L(P)$, then there exists $y < gx$. However, then $g^{-1}y < x$, contradicting the assumption that x is a leaf. We then obtain the induced action $Aut(P) \times L(P) \to L(P), (g, x) \mapsto gx$.

To complete the proof, we just have to give the induced action of $Aut(P)$ on B_n. To do this, identify $L(P) \cong [n]$ as sets, where $|L(P)| = n$ by assumption. Then, for $g \in Aut(P), \{l_1, \ldots, l_k\} \in B_n$, the induced action of $Aut(P)$ on B_n is given by $g\{l_1, \ldots, l_k\} = \{g \cdot l_1, \ldots, g \cdot l_k\}$. □

Convention 5.12. For the rest of this section only, fix a rooted tree P and denote by G the group of automorphisms $Aut(P)$. Let G act on B_n, where $n = |L(P)|$ by the induced action $\phi : G \times P \to P$, defined in the proof of Claim 5.11.

Notation 5.13. For $x \in P$, denote $D(x) = \{y \in P | y \leq x\}$.

Proposition 5.14. *Let P be a rooted tree with root vertex labeled r. Then, if let $\{A_1, \ldots, A_m\}$ denote the set of isomorphism classes of $\{D(x)|x \lessdot r\}$. For $T \in A_j$, denote $G_j = Aut(T)$. Then,*

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \tag{5.1}$$

In particular, $Aut(P)$ can be expressed as a sequence of direct products and wreath products of symmetric groups.

Proof. It is clear that if P is rank 1, with $|P_0| = n$, then the automorphism group is S_n. So, let us proceed by induction. That is, label the vertices of P by $\{0, 1, \ldots, s\}$ such that the root is labeled 0 and the vertices just below the root are labeled $1, \ldots, k$. Let $A_1, \ldots, A_m$ denote the distinct isomorphism classes of trees in the set $\{D(1), \ldots, D(k)\}$. For $T \in A_j$, denote $G_j = Aut(T)$. Let $T_j = \{t | t \lessdot 0, t \in A_j\}$. Then, letting Q_j be the subtree of P whose elements lie in the set $0 \cup (\cup_{t \in T_j} D(t))$, it follows $Aut(Q_j) \cong G_j \wr S_{i_j}$, because after choosing a permutation of the elements of T_j, we are free to choose any element of G_j to permute each $D(t), t \in T_j$. If $t_1 \lessdot 0, t_2 \lessdot 0, g \cdot t_1 = t_2$, then it must be that $D(t_1) = D(t_2)$. Therefore, $Aut(P)$ must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \tag{5.2}$$

Since each G_j is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (5.2) that so is $Aut(P)$. □

Corollary 5.15. *For P the automorphism group of a rooted tree, $Aut(P)$ is CCT.*

Proof. First, the permutation action $\phi : S_n \times B_n \to B_n$, is clearly CCT, as was noted in Example 5.1. By Theorem 5.8, wreath products with symmetric groups preserve the CCT property, and by Theorem 5.3 the direct product of two CCT groups is again CCT. Therefore, by Proposition 5.14, all groups of the form $Aut(P)$ are built up from these operations, and so $Aut(P)$ is also CCT. □

5.3. **Further Cover Transitive Actions.**

5.3.1. *Automorphisms of Polytopes.* We describe several linear automorphism groups of polytopes whose actions are CCT. In particular, we prove that the linear automorphism group of simplices, octahedrons are common cover transitive, although both of these are fairly straightforward.

Definition 5.16. Let M be a polytope, with a particular embedding in R^n. The *group of linear automorphisms of* M is the subgroup of GL_n whose elements are $\{g \in GL_n | g \cdot M = M\}$.

Proposition 5.17. *Let G be the group linear automorphisms on the $n-1$ simplex, Δ, whose vertices lie at the standard basis vectors in $\mathbb{R}^n$. The action of G on the n-simplex induces an action on $[n]$, given by identifying $[n]$ with the n vertices of the $n-1$ simplex. Hence, it induces an action on B_n. This induced action on B_n is CCT.*

Proof. The group of linear automorphisms in this case induces the permutation action of S_n on B_n, because any permutation matrix defines a linear map on $\mathbb{R}^n$. However, we know the action of S_n on B_n is CCT from Example 5.1 $\square$

Proposition 5.18. *Let G be the group of linear automorphisms of the n-octahedron, embedded inside $\mathbb{R}^n$, whose vertices are located at $\pm e_i$, where $e_1, \dots e_n$ are the standard basis vectors of $\mathbb{R}^n$. Then, the action of G on the octahedron induces an action of G on the $2n$ vertices of the octahdron, and hence on B_{2n}. This induced action on B_{2n} is common cover transitive.*

Proof. It is well know that the group of linear automorphisms of the n-octahedron is the hyperoctahedral group. (The hyperoctahedral group is commonly notated B_n, since it is the type B coexeter group, but we do not use this here to avoid confusing with the boolean algebra.) It is well know that they hyperoctahedral group can be written as $S_2 \wr S_n$. This can be seen directly by letting the i^{th} copy of S_2 in the wreath product act by the matrix sending e_i to $-e_i$, and fixing all other basis vectors, and then letting S_n act by permuting coordinates. Then, by Theorem 5.8, it follows that $S_2 \wr S_n$ is CCT. $\square$

Do we need to cite this theory on the hyperoctahedral group?

Remark 5.19. Later in Lemma 11.2, we shall also see that the action of the dihedral group on a regular n-gon, for $n = p, 2p$, is cover transitive. This is the group of linear automorphisms of the regular n-gon. Hence, gives yet another example of automorphisms of the linear automorphism group of polytopes being CCT.

5.3.2. *Regular Cover Transitive Actions of $\mathbb{Z}_2^k$.* In this subsubsection, we show that any embedding of $\mathbb{Z}_2^k$ into S_n defines an action on B_n is CCT. In particular, this implies that the regular action of $\mathbb{Z}_2^k$, that is the action of $\mathbb{Z}_2^k \times \mathbb{Z}_2^k \to \mathbb{Z}_2^k, (g,h) \mapsto g \cdot h$ is a CCT action. However, it turns out that this is the only class of groups for which the regular action is CCT.

Lemma 5.20. *Let n, k be arbitrary. Let $G = \mathbb{Z}_2^k$ and pick any order preserving, rank preserving action $\phi : G \times [n] \to [n]$. Then the induced action $\phi : G \times B_n \to B_n$ is CCT.*

Proof. First, observe that all elements $g \in G$ have order 2. Suppose $x \lessdot z, y \lessdot z, x = gy$. We will show $g \in Stab(z)$. By definition, this would imply that ϕ is common cover transitive.

Let a be the unique element such that $a \in y, a \notin x$. In order to show $g \in Stab(z)$, it suffices to show that $ga \in x$. Let $b = g^{-1}a$. We know $b \in x$. However, since g has order 2, it must be that $ga = g^2 b = b \in x$. Hence, $g \in Stab(z)$, and ϕ is CCT. $\square$

Corollary 5.21. *Let $\mathbb{Z}_2^k$ act on itself by the regular action $\psi : \mathbb{Z}_2^k \times \mathbb{Z}_2^k \to \mathbb{Z}_2^k, (g,h) \mapsto g \cdot h$. By identifying $\mathbb{Z}_2^k \cong [2^k]$ as sets, we obtain an induced action $\phi : \mathbb{Z}_2^k \times B_{2^k} \to B_{2^k}$ which is cover transitive.*

Proof. By Lemma 5.20, any embedding of $\mathbb{Z}_2^k \to S_n$ defines a CCT action on B_n. So, this particular embedding defines a CCT action. $\square$

Proposition 5.22. *Let G be a finite group, acting on itself by the left regular action, $\psi : G \times G \to G, (g,h) \mapsto g \cdot h$, with $|G| = n$. By identifying $G \cong [n]$ as sets, ψ this induces an action $\phi : G \times B_n \to B_n$. Then, ϕ is cover transitive if and only if $G \cong \mathbb{Z}_2^k$ for some k.*

Proof. First, if $G \cong \mathbb{Z}_2^k$, then by Corollary 5.21, we know the action $\phi : G \times B_{2^k} \to B_{2^k}$ is indeed CCT.

Note that $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$. The forward implication is obvious.

To see the converse, first note that if $\forall g \in G, g^2 = e$, then G is abelian because $aba^{-1}b^{-1} = abab = (ab)^2 = e$. Then, if G is an abelian group, all of whose elements have order two, the structure theorem of finite abelian groups tells us $G \cong \mathbb{Z}_2^k$.

So, Suppose $G \ncong \mathbb{Z}_2^k$. Then, there exists $g \in G, g^2 \neq e$. Clearly $\{e\} \lessdot \{e \cup g\}, \{g\} \lessdot \{e \cup g\}, \{g\} \in G\{e\}$. So, to show the induced action $\phi : G \times B_n \to B_n$ is not cover transitive, it suffices to show there is no $h \in G, h \in Stab(\{e \cup g\}), h \cdot \{e\} = \{g\}$. If $h \in Stab(\{e \cup g\})$ then $h \cdot e = e$ or $h \cdot e = g$. Since the regular action is simply transitive, either $h = e$ or $h = g$. If $h = e$, then we would have $h\{e\} = g$, which would imply $g = e$, contradicting the assumption that $g^2 \neq e$. On the other hand, if $h = g$, since $h \in Stab(\{e \cup g\})$ it must be that $h \cdot g = e$. Then, $g^2 = g \cdot g = h \cdot g = e$, contradicting the assumption $g^2 \neq e$. Therefore, there does not exist such an h and the regular action is not CCT. □

Remark 5.23. One nice way of restating the above results is that for G an abstract group, there exists some n and some embedding $G \subset S_n$ such that the induced action of G on B_n is not CCT if and only if $G \ncong Z_2^k$ for some k. If $G \cong Z_2^k$, we have seen this holds in Lemma 5.20. If $G \ncong Z_2^k$, then by Proposition 5.22, the regular action of G on $B_{|G|}$ defines an action which is not CCT.

6. Wreath Product of two Symmetric Groups

In this section,we prove a result similar to that of [**?**, Theorem 1.1]. We shall construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. It shall give an alternate proof of Theorem 1.1 in the case that $r = 1$.

Notation 6.1. We shall now show For this section, fix l, m with $n = l \cdot m$ and fix $G = S_m \wr S_l$. Let S_m act on B_m by the permutation representation, and then let G act on $B_m^l \cong B_{m \cdot l}$ by the action defined in Notation 5.5.

6.1. Recalling Pak and Panova's Result.

We first review the necessary definitions and then state [**?**, Theorem 1.1]:

A *partition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *parition of* n, notated $\lambda \vdash n$. A *composition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$. That is, it is a partition where order matters. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *composition of* n. Let $P_n(l, m)$ denote the set of partitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, such that $\lambda_1 \leq m, k \leq l$. That is, $P_n(l, m)$ is the partitions which fit inside an $l \times m$ rectangle.

Notation 6.2. [**?**, Section 1] For λ a partition, let $\nu(\lambda)$ be the number of distinct nonzero part sizes of λ.

Notation 6.3. [**?**, Section 1] Let $p_k(l, m, r) = \sum_{\lambda \in P_k(l,m)} \binom{\nu(\lambda)}{r}$.

Theorem 6.4. [**?**, Theorem 1.1] *The sequence* $p_r(l, m, r), p_{r+1}(l, m, r), \dots, p_{l \cdot m}(l, m, r)$ *is unimodal and symmetric.*

6.2. The Poset $\mathcal{E}^r(B_n)/G$.

Now that we have managed to state Pak and Panova's Theorem, we will begin the trek toward showing how the $p_1(l, m, 1)$ are actually ranks of $\mathcal{E}^1(B_n/G)$. The first step is to describe the Peck poset $\mathcal{E}^1(B_n)/G$. In particular, we shall obtain an explicit formula for the sizes of its ranks in Theorem 6.19. To do this, we will first describe representatives for the vertices of B_n/G and then analogous representatives for vertices of $\mathcal{E}^1(B_n)/G$.

Definition 6.5. For $x \subset [l \cdot m]$ we shall say x is *left justified* if whenever $a \in x$, such that $a \neq 1 \pmod{l}$, then $a - 1 \in x$. In other words, when we pick out the boxes of the $l \times m$ rectangle which lie in x the resulting diagram is left justified.

Definition 6.6. For $x \subset [l \cdot m]$, define the *composition of* x, denoted $comp(x) = \lambda_1, \dots, \lambda_l$ where $\lambda_i = |\{a \in x | i \cdot (m-1) < a \leq i \cdot m\}|$. That is, if we pick out the boxes of the $l \times m$ rectangle which lie in x, the composition is just the sequence of the number of boxes in each row. Similarly, define the *partition of* x, notated $part(x)$ as follows. Let $\pi \in S_l$ be a permutation such that $\pi(i) \geq \pi(i+1)$ for all $i, 1 \leq i \leq l-1$. Then $part(x) = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(l)})$. That is, $part(x)$ is the $comp(x)$, written in decreasing order.

Lemma 6.7. *For any $g \in G, x \in B_n$, it follows that $part(x) = part(gx)$.*

Proof. It suffices to show this holds for any generator g. Then. G is generated by elements which permute rows, and elements that swap rows. It is clear that if g only permutes elements in a single row, then $comp(gx) = comp(x)$, so in particular, $part(gx) = part(x)$. If g swaps two rows, then $comp(gx)$ is simply a reordering of the parts of $comp(x)$, and so again $part(gx) = part(x)$. □

Definition 6.8. For $x \subset [l \cdot m]$, say x is a *Young Diagram* if x is left justified and $comp(x)$ is a partition.

Lemma 6.9. *For each $x \in B_{l \cdot m}$ there exists a unique representative $z \in Gx$ such that z is a Young Diagram.*

Proof. Uniqueness is clear, because for any $g \in G$, by Claim 6.7, $part(gx) = part(x)$. So, to we only have to show there is some $g \in G$ for which gx is a Young Diagram. Indeed, first, choose $h_1 \in G$ so that $h_1 x$ is left justified. This can be done because every permutation of a single row in the $l \times m$ rectangle lies in the wreath product, so, we can take h_1 to be the product of the elements that left justify each individual row. Then, let h_2 be the element that swaps the rows of $h_1 x$ so that they increase going down. Finally, taking $g = h_2 h_1$, it follows that gx is a Young diagram. □

Notation 6.10. For $x \in B_n$ denote by $\overline{x}$ the unique element in Gx such that $\overline{x}$ is a Young Diagram.

Now that representatives for each G orbit in B_n have been described, we shall move on to describing representatives for each G orbit in $\mathcal{E}(B_n)$.

Notation 6.11. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition, introduce the alternate notation $\lambda = a_1^{b_1} \cdots a_s^{b_s}$ if the first b_1 parts of λ are equal to a_1, the next b_2 parts of λ are equal to b_2, and in general, for $1 \leq h \leq b_j$, the parts $\lambda_{h + \sum_{i=1}^{j-1} b_i}$ are all equal to a_j. Furthermore, all a_j must be distinct.

Lemma 6.12. *If $(x, y), (w, y) \in \mathcal{E}^r(B_n)_i$ then $(x, y) \in G(w, y)$ if and only if for all $g \in G$ such that gy is a Young diagram, we have $(gx, gy) \in G(gw, gy)$. In particular, if $gy = \overline{y}$ then $(x, y) \in G(w, y)$ if and only if $(gx, \overline{y}) \in (gw, \overline{y})$.*

Proof. Clearly $g(x, y) \in G(g(w, y))$ if and only if $(x, y) \in G(w, y)$. □

The point of the proceeding Lemma is that in order to determine when two elements are identified, we may assume that y is a young diagram.

Notation 6.13. For y a Young Diagram with $part(y) = comp(y) = a_1^{b_1} \cdots a_s^{b_s}$, and $x \subset y$, for all $i, 1 \leq i \leq s$, define y_i to be the $a_i \times b_i$ rectangle consisting of the rows $1 + \sum_{j=1}^{i-1} b_j, 2 + \sum_{j=1}^{i-1} b_j, \ldots, \sum_{j=1}^{i} b_j$. That is, y_i is just the rectangle of all parts of y which are of length a_i.

Proposition 6.14. *If y is a Young Diagram so that $part(y) = comp(y) = a_1^{b_1} \cdots a_s^{b_s}$, and $(x, y), (w, y) \in \mathcal{E}^r(B_n)_j$ then $(x, y) \in G(w, y)$ if and only if for all $i, 1 \leq i \leq s$, it holds that $part(x \cap y_i) = part(w \cap y_i)$.*

Proof. First, suppose for all $i, 1 \leq i \leq s$, that $part(x \cap y_i) = part(w \cap y_i)$. We know both $x \subset y, w \subset y$. Further, since y_i is a rectangle, there exists some $g \in \mathrm{Stab}(y_i)$ so that $gx = w$. Then, for each i, there exists some $g_{1,i} \in \mathrm{Stab}(y_i)$ so $g_{1,i}(x \cap y_i) = \overline{x \cap y_i}$, which only interchange elements in y_i. Similarly, there exists $g_{2,i} \in \mathrm{Stab}(y_i), g_{2,i}(w \cap y_i) = \overline{w \cap y_i}$. However, the assumption $part(x \cap y_i) = part(w \cap y_i)$ precisely means $\overline{x \cap y_i} = \overline{w \cap y_i}$. Therefore, $g_{2,i}^{-1} g_{1,i} \in \mathrm{Stab}(y)$ and $g_{2,i}^{-1} g_{2,i}(x \cap y_i) = (w \cap y_i)$. Applying this same procedure for all i and multiplying the corresponding group elements together gives an element $g \in \mathrm{Stab}(y)$ with $gx = w$. Therefore, $g(x,y) = (gx, gy) = (gx, y) = (w, y)$, so $(x,y) \in G(w,y)$.

Conversely, note that $(x,y) \in G(w,y)$ is equivalent to the existence of a $g \in \mathrm{Stab}(y)$ with $gx = w$. However, any $g \in \mathrm{Stab}(y)$ can only interchange rows of the same length. Therefore, we obtain the stronger result that we can write $g = h_1 \cdots h_s$ where $h_i \in \mathrm{Stab}(y_i)$, and $h_i(p) = p$ for all $p \notin y_i$. Then, it follows that $h_i(x \cap y_i) = w \cap y_i$ and so $part(x \cap y_i) = part(w \cap y_i)$, as claimed. □

Remark 6.15. So, one way of viewing G orbits of an element $(x,y) \in \mathcal{E}^r(B_n)$ is as "outer" Young Diagrams, made up of a sequence of rectangles stacked on top of one another, each one wider than the next, which form the Young diagram $\bar{y}$. Then, to each such rectangle we associate an "inner" Young Diagram. The inner young diagram corresponds to the elements in that rectangle in y but not in x. Two elements are in the same G orbit if and only if their "outer" Yound Diagram and "inner" Young Diagrams are all the same.

The next step is to give explicitly formulas for the ranks of $\mathcal{E}^r(B_n)$.

Notation 6.16. For $p(x) = \sum_{i=0}^{N} c_i x^i$, a polynomial, define the notation $[p(x)]_r = c_r$.

Now, we briefly introduce notation for q binomial coefficients, so that we can state the next propositions. Let $q \in \mathbb{R}$ and let $[n]_q = \sum_{i=0}^{n-1} q^i$. Then, denote $[n]_q! = \prod_{i=1}^{n} [i]_q$. Finally, let $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.

Notation 6.17. For $\lambda = (a_1^{b_1} \cdots a_s^{b_s})$ a partition, denote $\eta(\lambda, r) = \left[\prod_{i=1}^{s} \binom{a_i+b_i}{a_i}_q\right]_r$.

Proposition 6.18. [?, Proposition 1.3.19] *There is an equality* $[\binom{a+b}{b}_q]_j = |P_j(l,m)|$.

Theorem 6.19. *The sizes of the ranks of $\mathcal{E}^r(B_n)/G$ can be written as*
$$|(\mathcal{E}^r(B_n)/G)_i| = \sum_{\lambda \in P_i(l,m)} \eta(\lambda, r).$$

Proof. By Proposition 6.14 each orbit (x,y) has a representative such that y is a Young Diagram, with $part(y) = \prod_{i=1}^{s} a_i^{b_i}$, and each $x \cap y_i$ is a Young Diagram. Therefore, for a fixed y, the number of orbits (x,y) with $|x| + r = |y|$ is exactly determined by the Young Diagrams of $\overline{x \cap y_i}$, for $1 \leq i \leq s$. Equivalently, it is determined by the Young Diagrams $[y_i \setminus x]$, for $1 \leq i \leq s$. So, we wish to calculate the number of tuples of Young Diagrams of the form $(\overline{y_1 \setminus x}, \overline{y_2 \setminus x}, \ldots, \overline{y_s \setminus x})$ so that $\sum_{i=1}^{s} |y_i \setminus x| = r$. Now, define j_i by $j_i = |y_i \setminus x|$, still of course with, $\sum_{i=1}^{s} j_i = r$. Then, by Proposition 6.18, the number of such partitions is $[\binom{a_i+b_i}{b_i}_q]_{j_i}$. Therefore, the number of elements $G(x,y)$ with $j_i = |y_i \setminus x|$, is simply the product $\prod_{i=1}^{s} [\binom{a_i+b_i}{b_i}_q]_{j_i}$. Therefore, since j_i were chosen arbitrarily, only subject to the constraint that $\sum_{i=1}^{s} j_i = r$, It follows that the total number of elements $G(x,y)$, for y fixed, is equal to
$$\sum_{\substack{(j_1,\ldots,j_s),\\ \sum_{i=1}^{s} j_i = r}} \prod_{i=1}^{s} \left[\binom{a_i+b_i}{b_i}_q\right]_{j_i} = \left[\prod_{i=1}^{s} \binom{a_i+b_i}{b_i}_q\right]_r = \eta(\lambda, r).$$

Then, summing this over all Young Diagrams y, gives that

$$|\mathcal{E}^r(B_n)/G|_i = \sum_{\lambda \in P_i(l,m)} \left[\prod_{i=1}^{s} \binom{a_i+b_i}{b_i}_q\right]_r = \sum_{\lambda \in P_i(l,m)} \eta(\lambda, r).$$

□

the following commented out section was based on the old definition of $\mathcal{E}^r$. Should we remove it?

6.3. **Ensuing Conclusions.** We now reap the fruits of our labor in order to draw some nice results.

Corollary 6.20. *The sequence* $\sum_{\lambda \in P_r(l,m)} \eta(\lambda, r), \ldots, \sum_{\lambda \in P_n(l,m)} \eta(\lambda, r)$, *is the sequence of ranks of the Peck poset* $\mathcal{E}^r(B_{l\cdot m})/G$. *In particular, the sequence is symmetric and unimodal.*

Proof. By Theorem 6.19, $\sum_{\lambda \in P_i(l,m)} \eta(\lambda, r)$ is the size of the $i - r$th rank of the Peck poset $\mathcal{E}^r(B_{l\cdot m})/G$. Since Peck posets are unimodal and symmetric, this sequence is as well. □

We now obtain an easy proof of Theorem 6.4 for the case $r = 1$. Using the fact that $S_m \wr S_l$ is CCT. Namely,

Corollary 6.21. *The sequence* $p_1(l, m, 1), p_{r+1}(l, m, 1), \ldots, p_{l\cdot m}(l, m, 1)$ *is the rank generating function of* $\mathcal{E}^1(B_{l\cdot m})/G$. *In particular, the sequence is unimodal and symmetric.*

Proof. First, we know $F^1(B_{l\cdot m})/G$ is Peck, since it is the quotient of a unitary Peck poset by a group action. However, it is easy to see that $\eta(\lambda, 1) = \nu(\lambda)$, since both count the number Young Diagrams of size $|\lambda| - 1$ fitting inside the Young Diagram corresponding to λ. Therefore, $\sum_{\lambda \in P_i(l,m)} \eta(\lambda, r) = \sum_{\lambda \in P_i(l,m)} \nu(\lambda) = p_i(l, m, 1)$. However, by Corollary 6.20, $\sum_{\lambda \in P_r(l,m)} \eta(\lambda, r), \ldots, \sum_{\lambda \in P_n(l,m)} \eta(\lambda, r)$, are the ranks of $\mathcal{E}^1(B_{l\cdot m})/G$. Therefore, $p_i(l, m, 1)$ are also the ranks of $\mathcal{E}^1(B_{l\cdot m})/G$, hence symmetric and unimodal. □

7. An Application to Polya Theory

In this section, we use the generalized notion of $\mathcal{H}^{\overrightarrow{r}}(B_n)$ to obtain the unimodality of the coefficients a certain class of polynomials from Polya Theory, which will turn out to be the rank generating function of $\mathcal{H}^{\overrightarrow{r}}(B_n)/G$. The result we prove is a generalization of [**?**, Corollary 7.16].

7.1. **A Brief Review of Polya Theory.** We shall follow the treatment from [**?**, Chapter 7]. First, we build up some definitions to state Polya's Theorem.

Definition 7.1. Let $G \subset S_n$ act on $[n]$ by the restriction of the permutation action. For $\pi \in G$, the action of π on $[n]$ can be written in cycle notation so that there are c_i cycles of length i. Define the *cycle indicator* of π to be the monomial $Z_\pi(z_1, \ldots, z_n) = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}$.

Definition 7.2. The *cycle indicator* for a group $G \subset S_n$, is the polynomial

$$Z_G(z_1, \ldots, z_k) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi(z_1, \ldots, z_k).$$

Definition 7.3. A *coloring* of a set S by the colors $R = \{r_1, \ldots, r_k\}$ is a map $S \to R$. Heuristically, a coloring of S can be thought of as an assignment of a "color" from the set R to each of the elements of S.

Notation 7.4. For A, B sets, let $A^B = Hom_{\text{sets}}(B, A)$. Let $G \subset S_n$ act on B_n. Then, G acts on $R^{B_n} = Hom_{\text{sets}}(B_n, R)$ by $g \cdot f(S) = f(g \cdot S)$ where $g \in G, S \in B_n, f \in R^{B_n}$. Then, let R^{B_n}/G denote the quotient of R^{B_n} by the action of G defined above.

For $f \in R^S$, say $color(f) = (i_1, \ldots, i_k)$ if $|\{s \in S | f(s) = j\}| = i_j$ for $1 \leq j \leq k$. Observe that if $f \in Gh$ then $color(f) = color(h)$, so the map *color* descends to a map on G orbits.

Define $\kappa(i_1, \ldots, i_k) = |\{Gf \in R^{B_n}/G \text{ such that } color(f) = (i_1, \ldots, i_k)\}|$.

Finally, let $F_G(r_1, \ldots, r_k) = \sum_{i_1,\ldots,i_k} \kappa(i_1, \ldots, i_k) r_1^{i_1} \cdots r_k^{i_k}$.

Remark 7.5. The above notation may seem extremely cumbersome. It is simply a formal way of saying that $\kappa(i_1, \ldots, i_k)$ is the number of inequivalent colorings of subsets of B_n, under the G action. Once again, see [**?**, Chapter 7] for a more lengthy exposition.

Theorem 7.6. *(Polya's Theorem) Let* $G \subset S_n$ *act on* B_n. *With* F_G, Z_G *as defined above, the following equality holds.*

$$F_G(r_1, \ldots, r_k) = Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n).$$

7.2. A Rank Generating Function.

Notation 7.7. Let $[f(x_1,\dots,x_k)]_{x_{j_1}^{i_1}\cdots x_{j_l}^{i_l}}$ denote the coefficient of $x_{j_1}^{i_1}\cdots x_{j_l}^{i_l}$ in $f(x_1,\dots,x_k)$, which may itself be a polynomial.

Lemma 7.8. *The number $\kappa(i_1,\dots,i_k)$ is equal to $|(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|$. Consequently,*

$$[Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)]_{r_1^{i_1}\cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|.$$

Proof. Define a map $m : (\mathcal{E}^{r_2,\dots,r_{k-1}}(B_n))_{i_1} \to \{f \in R^{B_n} | color(f) = (i_1,\dots,i_k)\}$ as follows. For $(x_2,\dots,x_k) \in \mathcal{E}^{r_2,\dots,r_{k-1}}(B_n)$, let $m(x_2,\dots,x_k) = f$, where

$$f(t) = \begin{cases} r_1 & \text{if } t \in x_2 \\ r_i & \text{if } t \in x_{i+1} \setminus x_i \\ r_k & \text{if } t \notin x_k \end{cases}.$$

Observe that m is infact a bijection, as we can easily define an inverse map by sending a coloring f to $(x_2,\dots,x_k)$, where x_j is the set of all elements $t \in [n]$ for which $f(t) = r_l, l < j$, for all $j \in [k-1]$. Next, the two group actions were precisely defined so that so that,

$$g(x_1,\dots,x_{k-1}) = g(y_2,\dots,y_{k-1}) \Leftrightarrow g \cdot m(x_1,\dots,x_{k-1})) = g \cdot m(y_2,\cdots,y_{k-1})$$

Therefore, m descends to a bijection $m^G : (\mathcal{E}^{r_2,\dots,r_{k-1}}(B_n))_{i_1}/G \to R^{B_n}/G$, This implies $\kappa(i_1,\dots,i_k)$ is equal to $|(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|$.

Then, by Polya's Theorem 7.6,

$$[Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)]_{r_1^{i_1}\cdots r_k^{i_k}} = \kappa(i_1,\dots,i_k) = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|.$$

□

Now we arrive at our main result of the section. The following result provides a rank generating function for $\mathcal{E}^{\overrightarrow{r}}(B_n)/G$

Proposition 7.9. *Let $z_j = \sum_{l=1}^k r_l^j$. Let*

$$Z_G^{i_2,\dots,i_{k-1}}(r_1) = [Z_G(z_1,\dots,z_{k-1},1)]_{r_2^{i_2}\cdots r_{k-1}^{i_{k-1}}},$$

be a polynomial in r_1. Then, the coefficients of $Z_G^{i_2,\dots,i_{k-1}}(r_1)$ are the ranks of the peck poset $\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G$. In particular, they form a symmetric, unimodal sequence.

Proof. In Lemma 7.8, we saw

$$[Z_G(z_1,\dots,z_k)]_{r_1^{i_1}\cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|.$$

However, since $\sum_{j=1}^k i_j = n$, it follows that i_k is determined by the numbers $i_1,\dots,i_{k-1}$, and so

$$[Z_G(z_1,\dots,1)]_{r_1^{i_1}\cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|.$$

Next, write $Z_G^{i_2,\dots,i_{k-1}}(r_1)$, defined in the statement of the Theorem, as $\sum_t c_t(r_1)^t$. We have just shown that $c_t = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|$. Since $\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G$ is a Peck poset by Lemma 4.15, its rank sizes form a symmetric and unimodal sequence. Therefore, the coefficients of $Z_G^{i_2,\dots,i_{k-1}}(r_1)$ form a symmetric, unimodal sequence. □

Remark 7.10. Note that in the case $\overrightarrow{r} = 0$, the above result precisely becomes [**?**, Corollary 7.16].

8. Further Identities for the Wreath Product

In this section, we draw on the methods developed earlier in this section to write down some interesting generating functions for the case that $G = S_m \wr S_l$. First, we shall use Proposition 7.9 to obtain an explicit generating function for $p_i(l,m,1)$, and then we shall relate the sum $\sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t}$ to the tth Bell number, using $\mathcal{E}^{\overrightarrow{r}}(B_n)/G$, where $\overrightarrow{r}$ is of the form $\overrightarrow{r} = 1, \ldots, 1$.

Notation 8.1. For this section only, firx $m, l \in \mathbb{N}$ and fix $G = S_m \wr S_l$. Additionally, fix $n = m \cdot l$.

8.1. A Generating Function for $p_i(l,m,1)$..

Proposition 8.2. *Let c_i be the number of i cycles in π and define*

$$W_\pi(z_1, \ldots, z_n) = \begin{cases} \frac{Z_\pi(z_1,\ldots,z_n)}{z_1} & \text{if } c_1 > 0 \\ 0 & \text{if } c_1 = 0 \end{cases}.$$

Then, there is an equality

$$\sum_{i=0}^{n} p_i(l,m,1) r_1^i r_2^{n-i-1} = \sum_{\pi \in G} |\mathrm{Fix}(g)| W_\pi(r_1 + r_2, r_1^2 + r_2^2, \ldots, r_1^n + r_2^n).$$

Proof. Recall from Corollary 6.21 that $p_i(l,m,1)$ was the rank generating function of $\mathcal{E}^1(B_{l\cdot m})/G$.

As was seen in Proposition 7.9, it is also the case that

$$[Z_G(r_1 + r_2 + r_3, \ldots, (r_1)^n + (r_2)^n + (r_3)^n)]_{(r_3)^1}$$

is also the rank generating function of $|(\mathcal{E}^1(B_n)/G)_{i_1}|$. Therefore,

$$p_i(l,m,1) = [Z_G(r_1 + r_2 + r_3, \ldots, r_1^n + r_2^n + r_3^n)]_{r_3}.$$

So, to complete the proof, it suffices to show

$$[Z_G(r_1 + r_2 + r_3, \ldots, r_1^n + r_2^n + r_3^n)]_{r_3} = \sum_{\pi \in G} |\mathrm{Fix}(g)| W_\pi(r_1 + r_2, r_1^2 + r_2^2, \ldots, r_1^n + r_2^n).$$

To show this, it further suffices to show that for all $\pi \in G$,

$$Z_\pi(r_1 + r_2 + r_3, \ldots, r_1^n + r_2^n + r_3^n)]_{r_3} = |\mathrm{Fix}(\pi)| W_\pi(r_1 + r_2, r_1^2 + r_2^2, \ldots, r_1^n + r_2^n).$$

Indeed, this is easy to see, because for r_3 to have a nonzero coefficient in the expansion of r_3 in Z_π, first, π must have some 1-cycle, as otherwise, no variable could appear in the expansion of Z_π raised only to the first power. Second, if π has some 1-cycle, then the coefficient of r_3 in

$$Z_\pi(r_1 + r_2 + r_3, r_1^2 + r_2^2 + r_3^2, \ldots, r_1^n + r_2^n + r_3^n) = \prod_{i=1}^{n} (r_1^i + r_2^i + r_3^i)^{c_i}$$

is precisely

$$c_1 \cdot W_\pi(r_1 + r_2, r_1^2 + r_2^2, \ldots, r_1^n + r_2^n) = |\mathrm{Fix}(\pi)| W_\pi(r_1 + r_2, r_1^2 + r_2^2, \ldots, r_1^n + r_2^n)$$

because c_1 is the number of 1-cycles in π, which is by definition the number of fixed points of π. This is exactly what we wanted to show. □

8.2. Bounded Partition Sizes.

This section is an application of Polya theory, and is moderately tangential to the rest of the paper. However, it relates to the functor $\mathcal{E}^{\overrightarrow{r}}$, in that it counts the $|(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0| = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$, as described in Remark 8.6.

Notation 8.3. Let $P_{[t]}[l,m]$ denote the set of partitions of the set $[t]$ into at most l sets, such that each set in the partition has size at most m. Let $P[l,m] = \cup_{t \in \mathbb{N}} P_{[t]}[l,m]$ be the set of all partitions into at most l sets such that each set in the partition has size at most m. Denote $p_{[t]}[l,m] = |P_{[t]}[l,m]|$, and $p[l,m] = |P[l,m]|$.

Remark 8.4. It is fairly simple to see that $p_{[t]}[l,m], p[l,m]$ satisfy the recursive formulas

$$p_{[t]}[l,m] = \sum_{k=1}^{l} p_{[t-k]}[l-1,m].$$

$$p[l,m] = \sum_{k=1}^{l} p[l-1,m].$$

These can be reached by considering the number of elements in the first row. It is also straightforward to relate these statistics to the number set partitions of $[m \cdot l]$ that fit inside a m times l box, but just multiplying by an appropriate binomial coefficient.

make this more precise

Proposition 8.5. *There is an equality* $p_{[t]}[l,m] = \frac{t!}{l!(m!)^l} \sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t}$.

Proof. Using Polya's Theorem 7.6, we know

$$\mathcal{E}_G(r_1, \ldots, r_k) = \frac{1}{|G|} Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n).$$

In particular,

$$[\mathcal{E}_G(r_1, \ldots, r_k)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}} = [\frac{1}{|G|} Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}}.$$

Of course, by definition of F_G, we know

$$[mathcalE_G(r_1, \ldots, r_k)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}} = \kappa(1, 1, \ldots, 1, m \cdot l - t, 0, \ldots, 0)$$

where there are t 1's in the above expression. By definition κ is just the number of inequivalent ways to distinctly color t numbers in an $l \times m$ rectangle, up to the action of $G = S_m \wr S_l$. For any such coloring, let square a_i be colored with color r_i. Each G orbit of colorings has a unique representative where along each row the colors are sorted in increasing order, and additionallly, going down the first column, the colors are sorted in increasing order. Here, we are basically ignoring color $t+1$, just viewing it as a placeholder to color the remaining squares of the grid. Then, the colorings defined above are in bijection with partitions of $[t]$ so that this partition has at most l sets, and each set in this partition has at most m elements. The set of such partitions is exactly $P_{[t]}[l,m]$. Therefore, the number of such partitions is $p_{[t]}[l,m]$. Then, it follows that

$$p_{[t]}[l,m] = \kappa(1, 1, \ldots, 1, m \cdot l - t, 0, \ldots, 0) = [\frac{1}{|G|} Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}}.$$

Of course, $\frac{1}{|G|} = \frac{1}{l!(m!)^l}$ when $G = S_m \wr S_l$. So, to complete the proof, we just need to show that

$$[Z_G(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}} = t! \sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t}.$$

This follows from the definition of Z_G. That is, it is the sum over all the cycle indicators $\sum_{\pi \in G} Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n)$. So, to show the above holds, it suffices to show that

$$[Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \ldots, \sum_{i=1}^k r_i^n)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}} = t! \binom{\mathrm{Fix}(\pi)}{t}$$

This is now apparent, because $Z_\pi = \prod_{i < m \cdot l} (\sum_j r_j^i)^{c_i}$, where c_i is the number of i cycles in π. The only way we can obtain a monomial of the form $r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}$ is if $c_1 > t$. Then, the coefficient of such a term will exactly be the number of ways to choose an ordered set of t elements from c_1 terms. That is, it is precisely $t! \binom{c_1}{t}$. However, c_1 is the number of 1-cycles, that is $c_1 = \mathrm{Fix}(\pi)$. Hence, $[Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)]_{r_1 r_2 \cdots r_t r_{t+1}^{m \cdot l - t}} = t!\binom{\mathrm{Fix}(\pi)}{t}$ as desired. ∎

Remark 8.6. Thanks to Proposition 7.9, an equivalent way to state the above result is that $\sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t} = |(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0|$, if $\overrightarrow{r}$ is the vector consisting of t ones. Also, letting $\overrightarrow{s}$ be the vector with $t-1$ ones, we may note $|(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0| = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$, since one may think of elements of $\mathcal{E}^{\overrightarrow{s}}(B_n)/G$ as tuples of $t-1$ elements, each contained in the next, whose lowest element is of rank 1, and one may think of elements of $(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0$ as tuples of t elements, each contained in the next, whose lowest element is of rank 0. There is an obvious bijection between these two sets, given by adding or removing the element $\emptyset$ or rank 0 in B_n. Therefore, we also obtain $\sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t} = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$.

We can also get a formula for the size of the whole set $p[l, m]$:

Proposition 8.7. *Define the function* $f : \mathbb{N} \cup 0 \to \mathbb{N}$, *by*

$$f(x) = \begin{cases} \lfloor e \cdot n! \rfloor & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

Then,

$$p[l, m] = \frac{1}{l!(m!)^l} \sum_{\pi \in G} f(|\mathrm{Fix}(\pi)|).$$

Proof. First, note that $\sum_{i=1}^k i!\binom{k}{i} = f(k)$. This can be seen fairly easily, because $\sum_{i=1}^k i!\binom{k}{i} = k!\sum_{i=0}^k \frac{1}{i!}$, while $k! \cdot e = k!\sum_{i=0}^\infty \frac{1}{i!}$, and for $k > 1$, it is easy to bound the difference $\sum_{i=k+1}^\infty \frac{1}{i!} < \frac{1}{k!}$, which implies $\sum_{i=1}^k i!\binom{k}{i} = \lfloor k! \cdot e \rfloor = f(k)$ for $k > 1$.

By definition, $p[l, m] = \sum_{t=0}^{l \cdot m} p_{[t]}(l, m)$. Therefore, by Proposition 8.5,

$$\begin{aligned} p[l, m] &= \sum_{t=0}^{l \cdot m} p_{[t]}(l, m) \\ &= \sum_{t=0}^{l \cdot m} \frac{t!}{l!(m!)^l} \sum_{\pi \in G} \binom{\mathrm{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{t=0}^{l \cdot m} \sum_{\pi \in G} t!\binom{\mathrm{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} \sum_{t=0}^{l \cdot m} t!\binom{\mathrm{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} \sum_{t=0}^{l \cdot m} f(|\mathrm{Fix}(\pi)|). \end{aligned}$$

∎

Perhaps also put in the recursive relations which the $P[l, m]$ satisfy

9. Groups of order n.

In this section, we shall show that for any group G such that $|G| = n$, with an order-preserving, rank preserving, transitive action on B_n, then $|\mathcal{E}^1(B_n)/G)_i| = \binom{n-1}{i-1}$ Any such action can be described by identifying $[n] \cong G$, and then having G act on itself by the left multiplication. We can then use a simple result in group theory to calculate the ranks of $\mathcal{E}^1(B_n)/G$ for all transitive actions $\phi : G \times P \to P$, where G is abelian.

For the rest of this section, fix a group G with $|G| = n$, and a transitive action of G on $[n]$, which induces an action of G on B_n.

Lemma 9.1. *For G a group acting transitively on $[n]$, it follows that* $\operatorname{Stab}(x) = \{e\}$.

Proof. Since G acts transitively on n we have the number of orbits $|[n]/G| = 1$, Therefore, using Burnside's Lemma, together with the observation that $e \in \operatorname{Stab}(x)$,

$$|G| = \sum_{x \in X} |\operatorname{Stab}(x)| \geq \sum_{x \in X} 1 = |X| = |G|.$$

Then, the above string of inequalities tells us that we must have $|\operatorname{Stab}(x)| = 1$ for all x, or in other words, $\operatorname{Stab}(x) = \{e\}$. ☐

Proposition 9.2. *For any element $(x, y) \in \mathcal{E}^1(B_n)$, then* $\operatorname{Stab}(x, y) = e$.

Proof. From Lemma 9.1 for any $x \in B_n$ with $|x| = 1$, we have $\operatorname{Stab}(x) = \{e\}$. Then, by definition $\operatorname{Stab}(x, y) = \operatorname{Stab}(x) \cap \operatorname{Stab}(y) = \operatorname{Stab}(x) \cap \operatorname{Stab}(y \setminus x)$. However, $\operatorname{Stab}(y \setminus x)$ is by assumption a 1 element set, because $|y| = |x| + 1, x \subset y$. This implies that $\operatorname{Stab}(x, y) = \operatorname{Stab}(x) \cap \operatorname{Stab}(y \setminus x) = \operatorname{Stab}(x) \cap \{e\} = \{e\}$, for all $(x, y) \in \mathcal{E}^1(B_n)$, as claimed. ☐

Lemma 9.3. *The number $|(\mathcal{E}^1(B_n)/G)_i| = \binom{n-1}{i}$.*

Proof. Since there are $\binom{n}{i}$ elements $x \in (B_n)_i$, each with $n - i$ elements $y \in (B_n)_{i+1}$ with $x \subset y$, there are a total of $\binom{n}{i} \cdot (n - i)$ elements $(x, y) \in \mathcal{E}^r(B_n)_i$. By Proposition 9.2, we have $|\operatorname{Stab}(x, y)| = 1$. Therefore, by the Orbit Stabilizer Theorem, it follows $|G(x, y)| = n$. So, all orbits have the same size. From this,

$$|\mathcal{E}^1(B_n)_i/G| = |\mathcal{E}^1(B_n)_i|/|G| = \frac{\binom{n}{i} \cdot (n-i)}{n} = \binom{n-1}{i}.$$

☐

Lemma 9.4. *For G a group let $H \subset G$ be the subgroup such that $hx = x$ for all $x \in X, h \in H$. Then, H is a normal subgroup of G and if $|G/H| = n$, then $\mathcal{E}^1(B_n)/G = \mathcal{E}^1(B_n)/(G/H)$.*

Proof. First, we check H is normal. For any $g \in G, x \in x$ it follows $ghg^{-1}x = gh(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = x$, and so $ghg^{-1} \in H$. Hence, G/H is a group.

Then, observe $Gx = (G/H)x$ because H acts trivially. Therefore, $\mathcal{E}^1(B_n)/G \cong \mathcal{E}^1(B_n)/(G/H)$, since $G, G/H$ have exactly the same orbits. ☐

Lemma 9.5. *All faithful transitive abelian groups A acting on the set $[n]$ have $|A| = n$.*

Proof. This is a well known, easy to prove, result. See for instance [?] ☐

Corollary 9.6. *Any abelian group A which acts transitively on $[n]$ has $|\mathcal{E}(B_n)/A|_i = \binom{n-1}{i}$.*

Proof. By Lemma 9.4, we can assume A acts faithfully as well. By Lemma 9.5, $|A| = n$. Then, by Lemma 9.3, $|\mathcal{E}(B_n)/A|_i = \binom{n-1}{i}$. ☐

10. Quotient by the Cyclic Group

In this section, we show that for $G = C_n$ the cyclic group of order n, the poset $\mathcal{E}^1(B_n/G)$ is rank symmetric and rank unimodal. The rank symmetric part is obvious. From Corollary 9.6, we know that for $G = C_n$, the size of the i^{th} rank of the poset $\mathcal{E}^1(B_n)/C_n$ is

$$|\mathcal{E}^1(B_n)/C_n|_i = \binom{n-1}{i}.$$

We will bound $|\mathcal{E}^1(B_n/C_n)|_i$ by $|\mathcal{E}^1(B_n)/C_n|_i$ to obtain the unimodality.

Notation 10.1. Throughout the section, we fix an arbitrary n to start with, and set

$$q_i = |\mathcal{E}^1(B_n)/C_n|_i, \quad p_i = |\mathcal{E}^1(B_n/C_n)|_i.$$

$|\mathcal{E}^1(B_n/C_n)|_i$ should probably be notated as $|\mathcal{E}^1(B_n/C_n)_i|$, since it is the size of the ith rank of the poset. You should check to make sure you did not do this anywhere else (it is also done in the q-analog case, currently in 12.1, for instance)

It might be clearer

The quotient B_n/C_n is the well-studied necklace poset. The elements in the poset are represented by n filled or empty beads ordered cyclically. Label the positions of the beads in the necklace by $1, 2, ..., n$. More specifically, since each element $x \in B_n$ is represented by a sequence of empty or filled beads, an element $G(x) \in B_n/C_n$ is represented by such a sequence up to rotational equivalence. Similarly, an element $(Gx, Gy) \in [\mathcal{E}^1(B_n)/C_n]_i$ where $x \lessdot y$ can be regarded as a necklace where $i+1$ beads out of n are filled (which represent y), and 1 of the filled bead is distinct from all others (so the other i beads represent x).

Perhaps consider writing $(P)_i$ instead of $[P]_i$ for the ith rank of P. Above I used the notation $[f(x,y)]^a_x$ for the coefficient of x^a in $f(x,y)$, and we have also been using parentheses above for this. Again, this is something that appears several times in this and the next section, and if you agree, you should make sure to change it everywhere.

Notation 10.2. We fix $\sigma_0 = (1\,2\,...\,n)$, and call σ_0 the 1-*click rotation,* which generates the group C_n.

Perhaps the word "1-click" isn't really necessary, (although I agree it's important to fix σ_0 since you only use "1-click" rotation one or two times, and it might be clearer to say something about $\sigma_0^d y$ rather than something about the d-click rotation of y. Additionally, you don't define d-click rotation anywhere, but again, I don't think you should use this work "click" at all, even though I know you like thinking in these terms

Notation 10.3. We shall abuse notation by writing (Gx, Gy) in place of $(Gx, Gy) \cap \mathcal{E}^1(B_n)$ as sets, and hence, viewing $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}^1(B_n) \subset \mathcal{E}^1(B_n)$.

maybe we should make different notation for this, something like $(G \times G)(x,y)$, instead of abusing notation. I put this as a notation because it is used later in lemma 11.0.14 as well

Lemma 10.4. *Let $G = C_n$ where n is a prime, then $q_i - p_i = (n-1)/2$ for $1 \le i \le n-2$.*

Proof. Let $G = C_n$ where n is a prime. We consider the action of G on B_n. Note that n being a prime guarantees that the action of any nontrivial $\sigma \in G$ has no fixed points, since σ is always an n-cycle in its cycle decomposition. Now suppose $(x, y) \in A_i$ is an element such that $\sigma x = y$ for some $\sigma \neq e$. Then, there is no $g \in C_n$ such that $gx = \sigma x$ and $gy = y$, since $gx = \sigma x \Rightarrow g = \sigma$ and $gy = y \Rightarrow g = e$. Let (Gx, Gy) be an element in $[\mathcal{E}^1(B_n/G)]_i$, i.e., an edge class in the necklace poset, with representative (x, y). We abuse notation by writing (Gx, Gy) in place of $(Gx, Gy) \cap \mathcal{E}^1(B_n) \subset \mathcal{E}^1(B_n)$. We view $G(x, y)$ as a subset of $\mathcal{E}^1(B_n)$, namely $G(x, y) = \{(gx, gy) : g \in G\}$. Clearly $G(x, y) \subset (Gx, Gy)$ as subsets of $\mathcal{E}^1(B_n)$. It is also clear that the number of distinct $\sigma \in G$ such that $\sigma x < y$ is precisely the number of G-orbits in (Gx, Gy). In particular, if the only σ such that $\sigma x < y$ is identity, then $(Gx, Gy) = G(x, y)$ as subsets of $\mathcal{E}^1(B_n)$.

Elise: very minor note, but if you are setting $G = C_n$, try to be consistent and only use G throughout the proof. It's clearer

what is A_i. I think you may mean $\mathcal{E}^1(B_n)_i$?

again, consider changing brackets to parentheses, although I'll stop pointing this out

Note that q_i is the number of the disjoint subsets $G(x, y)$ in $\mathcal{E}^1(B_n)$, so the problem of counting $q_i - p_i$ is the same as that of counting the number of distinct G-orbits $G(x, y)$ with non-identity $\sigma \in G$ such that $\sigma x < y$, and we claim that for $i \neq 1, n-1$, this number is $(n-1)/2$.

Probably it would be best to put the lemma later saying there are at most two $G(x,y)$ in the same (Gx, Gy). If you don't want to do this, you might have to say that there are at most 2 G orbits in the same $G \times G$ orbit (which is nonobvious, and very special to the cyclic group.) I think you really need to put this paragraph after the next paragraph.

First consider $\sigma = \sigma_0$ and (x, y) such that $x \lessdot y$ and $\sigma x \lessdot y$, i.e., we take out the distinct filled bead from y in the necklace, and rotate y by a 1-click rotation, then the remaining i filled beads remain in σy. It is clear that the only possibility for this to occur is if we have $i+1$ consecutive filled beads, i.e., $y = \{1, 2, ..., i+1\}$ and $x = \{1, 2, ..., i\}$ (up to rotational equivalence by actions of G).

I think (i.e...) actually makes things more confusing, rather than clearer. Consider removing the clause after i.e.?

In general, for any $\sigma = \sigma_0^j$, $j < n/2$, there is a precisely one orbit $G(x, y)$ such that $x \lessdot y$ and $\sigma x < y$. Notice that the necklace $\{1, j+1, 2j+1, ...\}$ and $\{1, (n-j)+1, 2(n-j)+1, ...\}$ are the same necklaces, so it suffices to count the number of (x, y) with $x \lessdot y, \sigma_0^j x \lessdot y$, with $j < n/2$. It is also easy to show that, for all distinct $j < n/2$, the necklaces representated by $\{1, j+1, 2j+1, ..., i \cdot j+1\}$ are all distinct, since n is a prime number. $\square$

Corollary 10.5. *Let $G = C_n$ where n is a prime, and p_i as defined above. Then the sequence p_i is unimodal.*

Proof. By the Proposition and the symmetry of p_i, we only need to show $p_0 \le p_1$, but $p_0 = 1$, so the claim holds. $\square$

A similar method can be used to bound $q_i - p_i$ for n not necessarily prime. Our next goal in the section is to show that:

It seems like there is never a proof of this lemma, although presumably you mean to put it right above 10.13. It might also be a little confusing to state this lemma in its precise form here, and why it implies unimodality, since it seems

Lemma 10.6. *Let $G = C_n$, and p_i, q_i be defined as above. Let $\lambda_i = q_i - p_i$, then for sufficiently large n, the difference $\lambda_{i+1} - \lambda_i$ is bounded above by $q_{i+1} - q_i$.*

Note that this suffices to show that the p_i are unimodal, since the Lemma implies that

$$p_{i+1} - p_i = (q_{i+1} - \lambda_{i+1}) - (q_i - \lambda_i) = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0.$$

Notation 10.7. Let B_m/C_m be the necklace quotient poset, and let $q(i,m) = |\mathcal{E}^1(B_m)/C_m|_i$ and similarly let $p(i,m) = |\mathcal{E}^1(B_m/C_m)|_i$.

Perhaps define this together with p_i above, since p_i really depend on n as well? See the comment on your definition of p_i, q_i.

Definition 10.8. Let $\sigma \in G$. we call each subset $s \subset x \in B_n$ that is fixed by σ a *full cycle* (under the action of σ), and $x \setminus (\cup_{\text{full cycles } s} s)$ is called a *tail cycle* (under the action of σ).

Perhaps put this lemma above 10.4 because then you don't have to say there are at most two special orbits in the same orbit in that proof.

Lemma 10.9. *Let $G = C_n$, n not necessarily prime. Assume that $\sigma_0^r x \lessdot y$ for some (x, y), where $x \lessdot y$ and $r < n/2$. Also assume that there is no $g \in G$ such that $gx = \sigma_0^r(x)$ and $gy = y$. Then r is the only integer $1 \leq r < n/2$ such that $\sigma_0^r(x) \lessdot y$.*

Proof. By assumption, x has a tail cycle under the action of σ_0^r, for otherwise x is fixed by σ_0^r, because then $g = id$ satisfies $gx = \sigma_0^r x, gy = y$. Additionally, y must have some tail cycle under the action of σ_0^r, for otherwise y is fixed under σ_0^r and we can take $g = \sigma_0^r$. Call γ the tail cycle of y, so γ consists of some filled beads in the necklace, one of which is the element $z = y \backslash x$. Let σ be any rotation such that $\sigma x \lessdot y$. then σ has to take a full cycle of x (under σ_0^r) to another full cycle (under σ_0^r), so $\sigma(\gamma \backslash z) \lessdot \gamma$. Recall that $\sigma_0^r(\gamma \backslash z) \lessdot \gamma$ by assumption, so the elements in γ are r-positions away from each other, namely $\gamma = \{z, z-r, z-2r, ...z-lr\}$, but $z+r \notin \gamma$, since γ a tail cycle. Since $\sigma(\gamma \backslash z) \lessdot \gamma$, there is some m for which $\sigma(z - mr) = z$. But then, $z - (m-1)r \in x$, and $\sigma(z - (m-1)r) = z + r \notin y$, a contradiction. $\square$

I don't think you need to say let q_i, p_i be defined as usual

Lemma 10.10. *Let $G = C_n$, q_i and p_i be defined as usual, then*

$$q_i - p_i = \#\{G(x,y) : \exists\ \sigma_0^r x \lessdot y \text{ where } r < n/2, \text{ but } \nexists\, g \text{ s.t. } gx = \sigma_0^r x, gy = y\}.$$

inconsistent notation for size of a set, change # to $|*|$.

Proof. The difference of $q_i - p_i$ is counting the difference of the number of G-orbits $G(x, y)$ and the number of $(Gx, Gy) \in \mathcal{E}^1(B_n/C_n)$. It is clear that the G-orbits correspond to the (rotational equivalence classes of) fillings of the necklaces, where we fill $i+1$ elements with 1 of them being marked as the distinct element. Lemma 10.9 tells us that each subset $(Gx, Gy) \subset \mathcal{E}^1(B_n)$, contains at most two G-orbits (also regarded as subsets of $\mathcal{E}^1(B_n)$). The number of (Gx, Gy) containing exactly two G orbits is precisely the number of pairs of G-orbits $(G(x,y), G(\sigma x, y))$ (where $x \lessdot y$ and $\sigma x \lessdot y$) with $G(x,y) \neq G(\sigma x, y)$. The number of such pairs is then the number of $G(x,y)$ such that $\exists r, r < \frac{n}{2}, G(\sigma_0^r x, y) \neq G(x,y), \sigma_0^r x \lessdot y$. We restrict $r < n/2$ to avoid double counting pairs. This proves the Lemma. $\square$

Definition 10.11. Fix $r < n/2$ to be an integer and let $G(x,y)$ be an G-orbit of edges. We call $G(x,y)$ a *special orbit* if $\sigma_0^r x \lessdot y$, but there does not exist $g \in C_n$ such that $gx = \sigma_0^r x$, and $gy = y$.

Consider using a word other than special? Maybe you could say a double orbit, or something like that?

Perhaps consider breaking this proof into a couple lemmas. First I might say that $\lambda_{i+1} - \lambda_i$ is bounded by the number of elements y with tail cycles with respect to some σ_0^r so that $y \lessdot y'$ with $y' \in Stab(\sigma_0^r)$. Then, I would give the bound in the statement of this lemma regardin the size of this set.

Lemma 10.12. *Let $G = C_n$ and let λ_i, $q(i,m)$ be as defined above, then*

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n,i+1) \\ 3 \leq k}} q\left(\frac{(i+1)}{k}, \frac{n}{k}\right)$$

Proof. Recall from Lemma 10.10 that

$$q_i - p_i = |\{G(x,y) : \exists\ \sigma_0^r x \lessdot y \text{ where } r < n/2,\ \text{ but } \nexists\, g \text{ s.t. } gx = \sigma_0^r x, gy = y\}|.$$

Hence, to count $q_i - p_i = \lambda_i$, we only need to count the number of distinct subset $G(x,y) \subset \mathcal{E}^1(B_n)$ for which there exists $r < \frac{n}{2}$ so that $G(x,y)$ is special with respect to σ_0^r.

Note that, for each $r < n/2$ such that $(r,n) = 1$, there is precisely one such G-orbit, since in this case σ_0^r is a full n-cycle [See proof of Lemma 10.4]. This holds regardless of i.

Now consider $r < n/2$ such that $(r,n) > 1$. In this case, σ_0^r fixes elements of the form $\{s, s+r, s+2r, ...,\}$ for any starting point s, which are proper subsets of $[n]$. For $y \in B_n$, one may write y as the union of full cycles and at most one tail cycle under the action of σ_0^r. Now, if $y \in B_n$ has no tail cycles under the action of σ_0^r then $(\sigma_0^r x, \sigma_0^r y) = (\sigma_0^r x, y) \in G(x,y)$. This means that if $G(x,y) \in q_i - p_i = |\{G(x,y) : \exists\ \sigma_0^t x \lessdot y \text{ where } t < n/2,\ \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^t x, gy = y\}|$, then it must be $\sigma_0^s x \lessdot y$ with $s \neq r, s < n/2$.

So, $p_i - q_i$ is equal to the number of $G(x,y)$ such that there exists r with $x \lessdot y, \sigma_0^r x \lessdot y$ and y has a tail cycle under the action of σ_0^r. Now let us focus on the difference of $\lambda_{i+1} - \lambda_i$. Note that the difference only occurs when we count $r \leq n/2$ and $(r,n) > 1$. Let $G(x,y)$ be a G-orbit where $x \lessdot y$ such that $|y| = i+1$, and let $\sigma_0^r(x) \lessdot y$. Suppose that there is no g such that $gx = \sigma_0^r(x)$ and $gy = y$. From the discussion above, we know that there a tail cycle of y, written $w := \{s, s+r, s+2r, \ldots, s+lr\}$ for some l, and $x = y \backslash \{l\}$. Suppose that $y \cup \{s-r\}$ (mod n whenever necessary) still has a tail cycle $w \cup \{(s-r)\}$, with respect to σ_0^r. Then, if we let $y' = y \cup \{(s-r)\}$ and $x' = x \cup \{(s-r)\}$, the orbit $G(x,y')$ where $|y'| = i+2$ is also a special orbit, thus does contribute 1 when we count λ_{i+1}. In this case, our counting process for λ_{i+1} and λ_i both increase by 1, so such special orbit does not contribute to $\lambda_{i+1} - \lambda_i$. Now suppose that adding $s - r$ completes the sequence z into a tail cycle, namely $\{s, s+r, \ldots, s-r, s\}$. Now y' will not be a special orbit anymore, since σ_0^r fixes y'. In this situation, we increase 1 for λ_i only. If we choose to ignore this scenario, we will get an upper bound for $\lambda_{i+1} - \lambda_i$.

I don't understand what it means by an orbit "contributes to the difference." What you really want to say is that an orbit is in a certain set, and the difference is the size of that set. I have edited the previous paragraphs to make this clearer, can you fix the rest of the proof please?

Elise: does this mean $\lambda_{i+1} - \lambda_i \neq 0$?

what does "contribute" mean? Are you are trying to say it lies in a certain

Again, I don't understand

How can you choose to ignore a scenario? What does that mean?

Now return to the situation where $G(x,y)$ is not a special orbit, but there exists $s \in [n]$ so that $G(y, y \cup s)$ is a special orbit, where $y' = y \cup \{s'\}$ for any $s' \notin y$. We need to count how many times this happens. First we note that this happens when $d|n$, and $(n/d)|(i+1)$ where $d = (r,n)$. Note that, for different r such that (r,n) remains the same, there is only family of such special orbits, since the x part will be d-rotationally symmetric. We claim that, this is at most counting, over all $d|n$, where $1 < d < n$ and $(n/d)|i+1$, the sum $\sum q\left(\frac{(i+1)d}{n}, d\right)$. This is because we have at most $q(\frac{(i+1)d}{n}, d)$ ways to insert the tail cycle (with precisely 1 distinct bead) in the block of d elements.

What does this mean? It sounds like you're thinking of λ_i as a variable, which changes size as you

Can you make this sentence more precise? I don't understand what the "x" part is, what a "family" of such special orbits is, etc.

what is this? can you write it precisely as a set?

what index is this sum

what does it mean to insert a tail cycle, and what is a block of elements?

We can introduce $k = n/d$ and rewrite the sum as

$$\sum_{\substack{k|(i+1,n),\\ 1<k<n}} q\left(\frac{i+1}{k}, \frac{n}{k}\right).$$

Note that in our sum, k can never be 2, since otherwise the tail cycle of x has precisely 1

with repect to which group element?

element, and then y is fixed (consisting of full cycles). So this proves the assertion that

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n,i+1)\\ 3\leq k}} q(\frac{i+1}{k}, \frac{n}{k}).$$

Note that, from the process of calculating $\lambda_{i+1} - \lambda_i$, we can potentially obtain a better bound if we need to. □

Elise: of what? Be precise

Aaron: Consider removing this sentence, or at the very least inserted after the proof as a remark

Maybe call this a proposition or theorem instead of corollary? It doesn't really follow immediately from other things, and it's the main result we're showing

Corollary 10.13. *The poset $\mathcal{E}^1(B_n/C_n)$ is rank symmetric and rank unimodal.*

Proof. We need to show that for $G = C_n$, the p_i are unimodal. So, we only need to bound $\sum_{\substack{k|(n,i+1)\\ 3\leq k<n/2}} q(\frac{i+1}{k}, \frac{n}{k})$. It is clear that for sufficiently large n, say for $n \geq 9$,

$$q((i+1)/k, n/k) = \binom{n/k - 1}{(i+1)/k - 1} \leq \binom{\lceil n/3 - 1\rceil}{\lceil (i+1)/3 - 1\rceil}.$$

Since $k \leq i$, and $i + 1 \geq n/2$, we coarsely bound the sum by

$$\frac{n}{2} \cdot \binom{\lceil n/3 - 1\rceil}{\lceil (i+1)/3 - 1\rceil}.$$

We want to show that this is smaller than the difference of

$$q_i - q_{i-1} = \frac{(n-1)!}{(i)!(n-i)!} - \frac{(n-1)!}{(i-1)!(n-i+1)!} = \binom{n}{i}\frac{n-2i}{n} \geq \binom{n}{i}\frac{2}{n}.$$

Namely, we want to show that, for sufficiently large n,

$$\left(\frac{n}{2}\right)^2 \cdot \binom{\lceil n/3 - 1\rceil}{\lceil (i+1)/3 - 1\rceil} \leq \binom{n}{i}.$$

This bound works for $n \geq 12$. For smaller n the claim can be easily checked.

Now, by Lemma 10.12, we know that

$$p_{i+1} - p_i = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0,$$

which proves that the p_i are increasing for $i < n/2$. By symmetry, the p_i are unimodal. □

11. Quotient by the Dihedral Group

In this section, we show a similar result for the Dihedral groups that act naturally on B_n. Namely, for $G = D_{2n}$ the dihedral group of order $2n$, the poset $\mathcal{E}^1(B_n/G)$ is rank symmetric and rank unimodal.

Notation 11.1. In this section, we fix an arbitrary n again, and set

$$q_i = |\mathcal{E}^1(B_n)/D_{2n}|_i, \quad p_i = |\mathcal{E}^1(B_n/D_{2n})|_i.$$

Once again consider first calling these posets P,Q

Lemma 11.2. *Let $G = D_{2n}$ or $G = D_{4n}$ where n is prime , and p_i, q_i as defined above. Then $p_i = q_i$. In other words, the action of D_{2n} (respectively D_{4n}) on B_n (respectively B_{2n} is CCT.*

Proof. The proof of the Lemma is similar to the proof of Lemma 10.4. For example, the group $G = D_{2n}$ contains C_n as a subgroup and is generated by C_n with arbitrary reflections. The additional reflections guarantee that there are no special orbits $G(x, y)$ for any $r < n/2$, which implies $p_i = q_i$. Similarly, D_{4n} is generated by the rotational subgroup C_{2n} and reflections, and it is not hard to see that for any pair of elements $(x, z), (y, z) \in \mathcal{E}^1(B_n)$ such that x and y differ by a rotation, which does not fix z, then there always exists a reflection τ such that $\tau x = y$ and τ fixes z. Following the proof of 10.4, we know that the action of D_{4n} on B_n is CCT. □

here, you just repeated the definition of CCT. You should either say it is not hard to see it is CCT, but it seems like that's really not proving anything. You really need to comment on the 2 cycles and p-cycles being irrelevant.

Remark 11.3. Let G be as in Lemma 11.2. We remark that, in addition to unimodality, Corollary **??** implies that the poset $\mathcal{E}^1(B_n/G)$ is Peck.

Lemma 11.4. *We have an explicit formula for* $q_i = [\mathcal{E}^1(B_n)/D_{2n}]_i$

Elise: earlier you were using $|*|$ for rank size

$$q_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+1)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1}\right)$$

Proof. Consider an element $(x, y(\in \mathcal{E}^1(B_n)$ where $x \lessdot y$. Note that any element τ that fixes both x and y has to fix the difference $y \backslash x$, which is a one-element set. Since $G = D_{2n}$, we know that $1 \leq |\operatorname{Stab}(x, y)| \leq 2$. Let μ_1 be the number of G-orbits with the trivial stabilizer and μ_2 be the number of G-orbits with the stabilizer of size 2, which contains the identity and a reflection. By the Orbit-Stabilizer Theorem, each orbit with the trivial stabilizer is of size $|D_{2n}|/1 = 2n$, while all other orbit is of size $|D_{2n}|/2 = n$. Therefore

$$\mu_1 \cdot 2n + \mu_2 \cdot n = |\{(x, y)\}| = \binom{n}{i+1} \cdot \binom{i+1}{1} = n\binom{n-1}{i}.$$

Our goal is to calculate $q_i = \mu_1 + \mu_2$, so it remains to calculate μ_2, which counts the number of G-orbits such that the reflection also fixes x. Without loss of generality, we may assume that $y \backslash x = \{1\}$. We break into cases:

(1) If n is odd and $i + 1$ is even, then it is clear that no reflections fix x for any x, so $\mu_2 = 0$.
(2) For all other cases, there are precisely $\lceil n/2 - 1 \rceil$ places to insert $\lceil (i+1)/2 - 1 \rceil$ elements of $[n]$ to form x.

This gives the desired formula $\mu_2 = \frac{1}{2}[(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1}$. Therefore,

$$q_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1}\right)$$

□

Remark 11.5. An alternate proof of Lemma 11.4 follows directly from Polya's Theorem 7.6, by counting the number of ways to color n elements with three colors, so that there are i of the first color, 1 of the second color, and $n - i - 1$ of the third color.

Corollary 11.6. *The poset* $\mathcal{E}^1(B_n/D_{2n})$ *is rank symmetric and rank unimodal.*

Proof. The proof is similar to the proof of Lemma 10.12 and Corollary 10.13. In fact, we can bound the difference of $\lambda_{i+1} - \lambda_i$ by precisely the same bound used in Lemma 10.12, since the difference we get here (where $G = D_{2n}$) is smaller than the previous difference (where $G = C_n$). Then, a similar proof shows that the difference of $q_{i+1} - q_i$ obtained in Lemma 11.2 is significantly larger than the upper bound. □

12. A q-analog

In this section we consider a q-analog of the Boolean algebra and prove Lemma 12.4, which is a q-analog to Lemma 9.3.

Let q be a prime and $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, graded naturally by dimensions.

Why do you introduce $V_n(q)$ in place of $\mathbb{F}_q^n$? It seems like they're equally long. Would you be ok with just using $\mathbb{F}_q^n$ everywhere? Is the reason you introduce this because you don't like that $\mathbb{F}_q^n$ has a basis?

Let $C_n(q)$ be the multiplicative subgroup of the finite field $\mathbb{F}_{q^n}$, so $C_n(q)$ acts $\mathbb{F}_q$-linearly on $\mathbb{F}_{q^n}$, which is isomorphic to $V_n(q)$ as an $\mathbb{F}_q$ vector space. The action of $C_n(q)$ on $V_n(q)$ induces an order preserving and rank preserving action on the poset $B_n(q)$. The poset $B_n(q)/C_n(q)$ is the q-analog of the necklace poset.

Would you be able to more clearly explain the action of $\mathbb{F}_{q^n}$? I would copy what Vic said in the email he sent us almost verbatim

The following Lemma is a standard result:

Cite a reference

Lemma 12.1. *The number $|B_n(q)_i|$, that is to say, the number of i dimensional subspace in $V_n(q)$, is $\binom{n}{i}_q$*

Notation 12.2. Define the graded poset $T = \mathcal{E}^1\big(B_n(q)\big)/C_n(q)$ and let $t_i = |T_i|$.

Lemma 12.3. *For any $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$, so that $(V_x, V_y) \in \mathcal{E}^1\big(B_n(q)\big)/C_n(q)$, then $Stab_{C_n(q)}(V_x, V_y) \cong \mathbb{F}_q^\times$. In particular, $|Stab_{C_n(q)}(V_x, V_y)| = q-1$.*

Proof. We claim that if $\tau \in C_n(q)$ such that $\tau V_x = V_x$ and $\tau V_y = V_y$, then $\tau \in \mathbb{F}_q^\times$. First, it is clear that any scalar $c \in \mathbb{F}_q^\times$ fixes any $V \in B_n(q)$. Now assume $V_x \subset V_y$ and $\tau V_x = V_x, \tau V_y = V_y$ as in the claim. Then, in particular τ permutes all the elements in V_x, i.e., $\tau : V_x \to V_x$ is an isomorphism of $\mathbb{F}_q$-vector spaces. Pick a nonzero element $a \in V_x$, it is clear that $\tau^{q^i-1}a = a$. This equation implies that $\tau^{q^i-1} - 1 = 0$, so τ satisfies the polynomial $X^{q^i-1} - 1 \in \mathbb{F}_q[X]$. Similarly, since $\tau V_y = V_y$, τ is a root of the polynomial $X^{q^{i+1}-1} - 1 \in \mathbb{F}_q[X]$. Let $f(X)$ be a monic irreducible polynomial of τ over $\mathbb{F}_q$. Letting $ord(\tau)$ be the order of τ, we obtain that $ord(\tau)|q^{i+1}-1, ord(\tau)|q^i - 1$. Therefore, $ord(\tau)|(q^{i+1}-1)-(q^i-1) = q^i(q-1)$. However, we know $q \not| ord(\tau)$ because $\tau \in C_n(q)$ and so $ord(\tau)|q^n - 1$. This implies $ord(\tau)|q-1$. This implies $\tau \in \mathbb{F}_q^\times$. ∎

Lemma 12.4. *Let q be a prime, then*

$$|\mathcal{E}^1\big(B_n(q)\big)/C_n(q)|_i = \binom{n-1}{i}_q.$$

Proof. Let $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$ so that $(V_x, V_y) \in F^1(B_n(q))$. By Lemma 12.3, for any $(V_x, V_y) \in F^1(B_n(q))_i, |Stab_{C_n(q)}(V_x, V_y)| = q-1$. By the Orbit-Stabilizer Theorem, we know that the size of each orbit in $\mathcal{E}^1((B_n(q))_i$ under the action of $C_n(q)$ is

$$|C_n(q)|/(q-1) = \frac{q^n - 1}{q-1} = (n)_q.$$

Now we can calculate the number q_i, which is the total number of elements in $(B_n(q))_i$ divided by the size of each orbit:

$$q_i = \frac{\binom{n}{i+1}_q \binom{i+1}{1}_q}{(n)_q} = \binom{n-1}{i}_q.$$

∎

In general, for $G \in Gl_n(\mathbb{F}_q)$ acting on $B_n(q)$, we want to ask the following questions:

(1) Is $\mathcal{E}^1\big(B_n(q)\big)$ unitary Peck?
(2) Is $\mathcal{E}^1\big(B_n(q)/G\big)$ Peck? or more weakly, is it rank unimodal?

13. The object $\mathcal{E}^1(B_n)$.

In this section, we explicitly compute the raising operators corresponding to both $\mathcal{H}^1(B_n), \mathcal{E}^1(B_n)$ and explictly show that $\mathcal{E}^1(B_n)$ is unitary Peck by showing certain raising maps are invertible.

Notation 13.1. For this section, we will use $M^{n-2i-1} = \mathcal{E}^1(U)^{n-2i-1}$, where $\mathcal{E}^1(U)^{n-2i-1} : \mathcal{E}^1(B_n)_i \to \mathcal{E}^1(B_n)_{n-i-1}$ is the Lefschetz map, as defined in **??**.

Notation 13.2. For the remainder of this section, we shall take $|a| = n-i-1, |b| = n-i, |x| = i, |y| = i+1$. Let $k = n-2i-1$, that is, $k = |a| - |x|$. Additionally, whenever we write an expression of the form (x,y) or (a,b), it is assumed that $x \lessdot y, a \lessdot b$.

Theorem 13.3. *Defining L to be the Lefschetz map $\mathcal{H}^1(B_n) \to \mathcal{H}^1(B_n)$, we have $L^{n-2i-1} : \mathcal{H}^1(B_n)_i \to \mathcal{H}^1(B_n)_{n-i-1}$, and explicitly*

$$L^{n-2i-1}(x,y) = k! \sum_{\substack{y \not\subset a,\\ x \subset a,\\ y \subset b}} (a,b).$$

Proof. Note that the conditions $y \not\subset a, x \subset a, y \subset b$ are equivalent to $(a,b) >_{\mathcal{H}^1(B_n)} (x,y)$. Clearly, if $(a,b) \not>_{\mathcal{H}^1(B_n)} (x,y)$, then the coefficient of (a,b) in $L^{n-2i-1}(x,y)$ is 0. So, to complete the proof, it suffices to show that if $(a,b) >_{\mathcal{H}^1(B_n)} (x,y)$ then the coefficient of (a,b) in $L^{n-2i-1}(x,y)$ is $k!$. However, this coefficient is precisely the number of sequences $(x,y) = (x_0,y_0) \lessdot_{\mathcal{H}^1(B_n)} (x_1,y_1) \lessdot_{\mathcal{H}^1(B_n)} \cdots \lessdot_{\mathcal{H}^1(B_n)} (x_k,y_k) = (a,b)$. By definition of $\mathcal{H}^1$, we must have that $y_k \setminus x_k = y \setminus x$ for all k. Therefore, the number of such sequences is equal to the number of sequences $x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$, since choosing the x_i determine y_i because $y_i = x_i \cup (y \setminus x)$. Finally, the number of such sequences $x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$, is equivalent to the number of ways to add the elements in $a \setminus x$ to x. This is because each sequence $x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$ is determined uniquely by the singletons $x_{i+1} \setminus x_i, 1 \le i \le k$. Since in total we are adding k elements to x in order to obtain a, there are $k!$ ways to do this. Therefore, coefficient of (a,b) in $L^{n-2i-1}(x,y)$ is $k!$ $\square$

Theorem 13.4. *The map M satisfies*

$$M^{n-2i-1}(x,y) = (2^k-1)(k-1)! \sum_{y \subset a} (a,b) + k! \sum_{\substack{y \not\subset a,\\ x \subset a,\\ y \subset b}} (a,b)$$

Proof. For a particular $(x,y),(a,b)$, if either $y \subset a$ or $y \not\subset a, x \subset a, y \subset b$, then $(a,b) \not>_{\mathcal{H}^1(B_n)} (x,y)$, and so the coefficient of (a,b) in $M^{n-2i-1}(x,y)$ is 0.

Clearly, we cannot have both $y \subset a$ and $y \not\subset a, x \subset a, y \subset b$, hold at the same time. So, suppose $y \not\subset a, x \subset a, y \subset b$. This implies that $b \setminus a = y \setminus x$. Then, the coefficient of (a,b) in $M^{n-2i-1}(x,y)$ is precisely the number of sequences $(x,y) = (x_0,y_0) \lessdot_{\mathcal{E}^1(B_n)} (x_1,y_1) \lessdot_{\mathcal{E}^1(B_n)} \cdots \lessdot_{\mathcal{E}^1(B_n)} (x_k,y_k) = (a,b)$. However, since $y \setminus x = b \setminus a$, it must be that $y_k \setminus x_k = y \setminus x$ as well. Therefore, the number of such sequences is equal to $k!$, as was shown in the proof of Theorem 13.3.

To complete the proof, we need to show that if $y \subset a$ then the coefficient of (a,b) in $M^{n-2i-1}(x,y)$ is $(2^k-1)(k-1)!$. Equivalently, we need to show that the number of sequences

$$(x,y) = (x_0,y_0) \lessdot_{\mathcal{E}^1(B_n)} (x_1,y_1) \lessdot_{\mathcal{E}^1(B_n)} \cdots \lessdot_{\mathcal{E}^1(B_n)} (x_k,y_k) = (a,b) \tag{13.1}$$

is $(2^k-1)(k-1)!$.

For the moment, fix j and consider the set of all sequences of the form in Equation (13.1) such that $(a \setminus b) \cup y_{j-1} = y_j$ First, let us show the number of such with this j fixed is $(k-1)!2^j$. To do this, start by considering the number of sequence $y_0, \ldots, y_k$ such that $y = y_0 \lessdot_{B_n} y_1 \lessdot_{B_n} \cdots \lessdot_{B_n} y_k = b$. Since we enforce $y_j = y_{j-1} \cup (a \setminus b)$, the number of such sequences is precisely the number of ways to order the elements $a \setminus y$. Since $|y| - |a| = k - 1$, there are $(k-1)!$ such ways. Additionally, for $l < j$ we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$ or $x_{l+1} = x_l \cup (y_l \setminus x_l)$. Either of these is possible at every step. Additionally, for $l \geq j$, we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$. So for any fixed sequence of y_k with so that j is minimal with $(a \setminus b) \in y_j$, there are precisely 2^{j-1} possible sequences $x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$ so that $(x, y) = (x_0, y_0) \lessdot_{\mathcal{E}^1(B_n)} (x_1, y_1) \lessdot_{\mathcal{E}^1(B_n)} \cdots \lessdot_{\mathcal{E}^1(B_n)} (x_k, y_k) = (a, b)$.

So, in total, there are $2^j(k-1)!$ sequences of the form in (13.1) with $(a \setminus b) \cup y_{j-1} = y_j$. Now, there clearly must be some such $j, 1 \leq j \leq k$. Therefore, the coefficient we are looking for is $\sum_{j=1}^{k} 2^{j-1}(k-1)! = (2^k - 1)(k-1)!$, as claimed.

□

13.1. **Proof that $\mathcal{E}(B_n)$ is unitary Peck.** In this subsection, we will show the rows of M form a basis by showing we can make a change of basis to a map which is takes M^{n-2i-1} to L^{n-2i-1}. Since we know L^{n-2i-1} is an isomorphism, it will follow that M^{n-2i-1} is as well.

Notation 13.5. Let $\beta = \frac{2^k - 1}{k}$. Denote

$$v_{(a,b)} = \beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y).$$

Note that $v_{(a,b)}$ are simply the rows of M^{n-2i-1}, each divided by the constant $k!$.

Notation 13.6. For any set s of size at least $n - i$,

$$z_s = \frac{1}{\beta\binom{|s|-i-1}{n-2i-2}(|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}} \sum_{b \subset s, a \subset b} v_{(a,b)}$$

Lemma 13.7. *For any set s of size at least $n - i$, we have $z_s = \sum_{y \subset s}(x, y)$. In particular, $\sum_{y \subset s}(x, y)$ lies in the span of $v_{(a,b)}$*

Proof. We have

$$\begin{aligned}
&\left(\beta\binom{|s| - i - 1}{n - 2i - 2}(|s| - n + i + 1) + \binom{|s| - i - 1}{n - 2i - 1}\right) z_s \\
&= \sum_{b \subset s, a \subset b} v_{(a,b)} \\
&= \sum_{b \subset s, a \subset b} \left(\beta \sum_{y \subset a}(x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y)\right) \\
&= \sum_{b \subset s, a \subset b} \beta \sum_{y \subset a}(x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\
&= \beta\binom{|s| - i - 1}{n - 2i - 2}(|s| - n + i + 1) \sum_{y \subset s}(x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\
&= \beta\binom{|s| - i - 1}{n - 2i - 2}(|s| - n + i + 1) \sum_{y \subset s}(x, y) + \binom{|s| - i - 1}{n - 2i - 1} \sum_{y \subset s}(x, y) \\
&= \left(\frac{1}{\beta\binom{|s|-i-1}{n-2i-2}(|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}}\right) \sum_{y \subset s}(x, y).
\end{aligned}$$

In going between the fourth line and the fifth line, one needs to count the number of y satisfying $y \subset a \subset b \subset s$. If we fix s, a there are $(|s| - n + i + 1)$ choices for the element b, since $b = a \cup \{s\}$ for $s \notin a$. Then, we need to count the number of a with $y \subset a \subset s$. This is exactly $\binom{|s|-i-1}{n-2i-2}$. Hence, the total of number of such y is the product $\binom{|s|-i-1}{n-2i-2}(|s| - n + i + 1)$.

In going from the fifth line to the sixth line, for y, s fixed, we count the number of b with $y \subset b \subset s$. This is exactly $\binom{|s|-i-1}{n-2i-1}$. □

Notation 13.8. For any set s of size at least $n-i$, let $w_s = \sum_{a\subset s, t\notin s} v_{(a,a\cup\{t\})}$.

Lemma 13.9. *We have*

$$w_s = \sum_{t\notin s}\left(\beta\binom{|s|-i-1}{n-2i-2}\sum_{y\subset s}(x, x\cup\{t\}) + \binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x, x\cup\{t\})\right).$$

Proof.

$$\begin{aligned}
w_s &= \sum_{a\subset s, t\notin s} v_{(a,a\cup\{t\})}\\
&= \sum_{a\subset s, t\notin s}\left(\beta\sum_{y\subset a}(x,y) + \sum_{y\subset b, x\subset a, y\not\subset a\cup\{t\}}(x,y)\right)\\
&= \sum_{t\notin s}\left(\sum_{a\subset s}\beta\sum_{y\subset a}(x,y) + \sum_{a\subset s}\sum_{y\subset b, x\subset a, y\not\subset a\cup\{t\}}(x,y)\right)\\
&= \sum_{t\notin s}\left(\beta\binom{|s|-i-1}{n-2i-2}\sum_{y\subset s}(x,y) + \sum_{a\subset s}\sum_{y\subset b, x\subset a, y\not\subset a\cup\{t\}}(x,y)\right)\\
&= \sum_{t\notin s}\left(\beta\sum_{y\subset s}(x,y) + \binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x, x\cup\{t\})\right)
\end{aligned}$$

The equalities between lines three and four, and four and five hold for similar reasons as the equalities between lines four and five, and five and six in 13.7 □

Notation 13.10. Let $u_s = \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2}z_s}{\binom{|s|-i}{n-2i-1}} + z_s$.

Lemma 13.11. *We have* $u_s = \sum_{x\subset s, y\supset x}(x,y)$. *In particular,* $\sum_{x\subset s, y\supset x}(x,y)$ *lies in the span of* $v_{(a,b)}$.

Proof. By 13.9 and 13.7 we have

$$\begin{aligned}
u_s &= \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2}z_s}{\binom{|s|-i}{n-2i-1}} + z_s\\
&= \frac{\sum_{t\notin s}\left(\beta\binom{|s|-i-1}{n-2i-2}\sum_{y\subset s}(x,y) + \binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x,x\cup\{t\})\right) - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2}\sum_{y\subset s}(x,y)}{\binom{|s|-i}{n-2i-1}}\\
&\qquad + \sum_{y\subset s}(x,y)\\
&= \frac{\sum_{t\notin s}\left(\binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x,x\cup\{t\})\right)}{\binom{|s|-i}{n-2i-1}} + \sum_{y\subset s}(x,y)\\
&= \sum_{t\notin s}\left(\binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x,x\cup\{t\})\right) + \sum_{y\subset s}(x,y)\\
&= \sum_{x\subset s, y\supset x}(x,y)
\end{aligned}$$

The penultimate line is equal to the ultimate line because any element (x,y) with $x\subset s$ must either have $y\subset s$ or else $y\subset s\cup\{t\}$ for $t\notin s$. The two terms on the penultimate line cover precisely these two cases. □

Notation 13.12. For any $s\subset[n]$, with $|s|\le n-i$, define $h_s = z_{[n]} - u_s$.

Lemma 13.13. *With h_s as defined above $h_s = \sum_{x \not\subset s}(x,y)$*

Proof. Using 13.7 and 13.11

$$\begin{aligned} h_s &= z_{[n]} - u_s \\ &= \sum_{x \subset y}(x,y) - \sum_{x \subset s, x \subset y}(x,y) \\ &= \sum_{x \not\subset s}(x,y) \end{aligned}$$

□

Notation 13.14. For $I \subset [n]$, let $I^c = [n] \setminus I$, the complement of I in $[n]$.

Lemma 13.15. *Let I be fixed. If l_J lies in the span of $v_{(a,b)}$ for $J \subset I$, then so does l_I.*

Proof. Let $I = \{i_1, \ldots, i_s\}$. Let $A_k = \{x | i_k \in x\}$. Then, $\cap_{k=1}^s A_k = \{x | I \subset x\}$. Using the principle of inclusion exclusion, we can write $\sum_{x \in \cap_{k=1}^s A_k}(x,y)$ as a sum of terms $\pm \sum_{x \in \cap_{k \in J} A_k}(x,y)$, for $J \subset I$. Since we are assuming $\pm \sum_{x \in \cap_{k \in J} A_k}(x,y)$, lie in the span of $v_{(a,b)}$ it follows that $l_I = \sum_{x \in \cap_{k=1}^s A_k}(x,y)$ does as well.

□

Lemma 13.16. *For all $J \subset [n]$, with $|J| \leq i$, we have l_J lies in the span of $v_{(a,b)}$.*

Proof. We shall prove this by induction on the size of I. First, we know $h_{[n]}$ lies in the span of $v_{(a,b)}$ and so have completed the base case $|I| = 0$ since $h_{[n]} = l_\emptyset$. So, to complete the proof, it suffices to show that if we know this Lemma holds for all $|s| > j$ then it holds for $|s| = j$. This is exactly what 13.15 proves. □

Lemma 13.17. *For $|x| = i$, we have $l_x = \sum_{y \supset x}(y,x)$.*

Proof. By definition, $l_{\bar{x}} = \sum_{\bar{x} \subset x, \bar{x} \cap x = \bar{x}, x \subset y}(x,y) = \sum_{\bar{x} \subset y} \bar{(}x,y)$. Replacing $\bar{x}$ by x gives the result. □

Notation 13.18. Define $m_a = \sum_{x \subset a} l_x$.

Lemma 13.19. *With m_a as defined above, we have $m_a = \sum_{x \subset a}(x,y)$.*

Proof. By 13.17, we obtain

$$m_a = \sum_{x \subset a} l_x = \sum_{x \subset a} \sum_{y \supset x}(y,x) = \sum_{x \subset a}(x,y)$$

□

Notation 13.20. Denote $r_a = \sum_{b, b \subset a} v_{(a,b)}$.

Lemma 13.21. *With r_a as defined above, we have*

$$r_a = ((i+1)\beta - 1)\sum_{y \subset a}(x,y) + \sum_{x \subset a}(x,y).$$

Proof.

$$\begin{aligned}
r_a &= \sum_{b\supset a} v_{(a,b)} \\
&= \sum_{b\supset a}\left(\beta\sum_{y\subset a}(x,y) + \sum_{\substack{y\subset b,\\ x\subset a,\\ y\not\subset a}}(x,y).\right) \\
&= \sum_{b\supset a}\beta\sum_{y\subset a}(x,y) + \sum_{b\supset a}\sum_{\substack{y\subset b,\\ x\subset a,\\ y\not\subset a}}(x,y). \\
&= (i+1)\beta\sum_{y\subset a}(x,y) + \sum_{b\supset a}\sum_{\substack{y\subset b,\\ x\subset a,\\ y\not\subset a}}(x,y). \\
&= (i+1)\beta\sum_{y\subset a}(x,y) + \sum_{x\subset a, y\not\subset a}(x,y). \\
&= ((i+1)\beta-1)\sum_{y\subset a}(x,y) + \left(\sum_{x\subset a, y\not\subset a}(x,y) + \sum_{y\subset a}(x,y)\right). \\
&= ((i+1)\beta-1)\sum_{y\subset a}(x,y) + \sum_{x\subset a}(x,y).
\end{aligned}$$

□

Notation 13.22. Assuming we do not have $(i+1)\beta = 1$, define $t_a = \frac{r_a - m_a}{(i+1)\beta - 1}$

Lemma 13.23. *With t_a as defined above, $t_a = \sum_{y\subset a}(x,y)$*

Proof. By 13.21 and 13.19, we have

$$\begin{aligned}
t_a &= \frac{r_a - m_a}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta-1)\sum_{y\subset a}(x,y) + \sum_{x\subset a}(x,y) - \sum_{x\subset a}(x,y)}{(i+1)\beta-1} \\
&= \frac{((i+1)\beta-1)\sum_{y\subset a}(x,y)}{(i+1)\beta-1} \\
&= \sum_{y\subset a}(x,y)
\end{aligned}$$

□

Notation 13.24. Assuming $\beta(i+1)\neq 1$, let $q_{a,b} = v_{(a,b)} - \beta t_a$.

Lemma 13.25. *We have $q_{a,b} = \sum_{\substack{y\subset b,\\ x\subset a,\\ y\not\subset a}}(x,y)$.*

Proof. Using 13.23

$$\begin{aligned} q_{a,b} &= v_{(a,b)} - \beta t_a \\ &= \beta \sum_{y \subset a} (x,y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y) - \beta \sum_{y \subset a} (x,y) \\ &= \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y) \end{aligned}$$

□

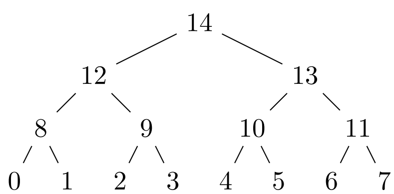
Theorem 13.26. *For $n > 2$, the matrix M^{n-2i-1} is invertible.*

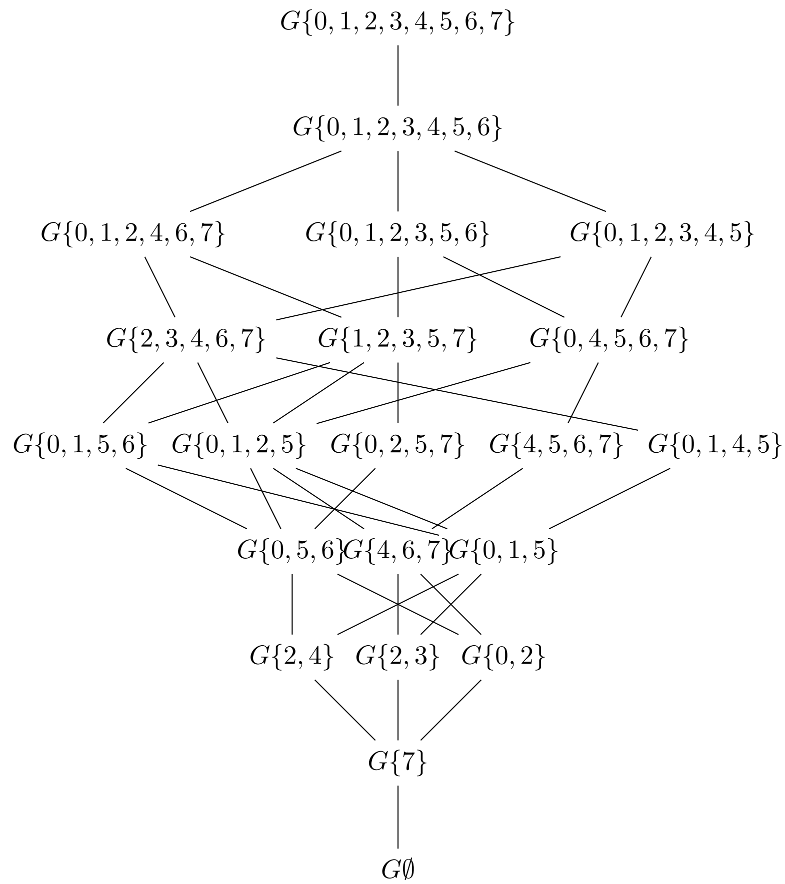
Proof. We saw that if $\beta(i+1) \neq 1$, we have that $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y)$ lie in the span of $v_{(a,b)}$. It is always the case that $\beta \geq 1, i \geq 0$. The only time in which $\beta = 1, i = 0$ is when $n = 2$. Therefore, $q_{a,b}$ are always defined for $n > 2$. However, $q_{a,b}$ are exactly the rows of L^{n-2i-1} as defined in Theorem 13.3. Using Theorem **??**, we know $\mathcal{H}^1(B_n)$ is unitary Peck, and therefore the rows of L^{n-2i-1} are independent. Therefore, the rows of M^{n-2i-1} span an independent set in a vector space of the same dimension. So, the rows of M^{n-2i-1} are independent. Hence, M^{n-2i-1} is indeed an isomorphism, when $n > 2$. □

14. Odds, Ends, and Failed Attempts

In this section, we include various remarks, as well as failed attempts, which don't particularly belong in other sections.

14.1. **Quotients by Automorphism Groups of Rooted Trees.** In Subsection 5.2, we saw that the full automorphism groups of rooted trees are CCT. It is well known that the quotient $B_{ml}/(S_m \wr S_l)$ is a distributive lattice. Hence, we were curious whether B_{ml}/G is a distributive lattice as well, where G is the automorphism group of a rooted tree. The answer is a definitive no, as can be seen in the case $G = (S_2 \wr S_2) \wr S_2$, where G acts on a the binary rooted tree P with $|L(P)| = 8$, whose vertices are labeled $0, \ldots, 7$ from left to right. In this case, it is easy to see that $G\{0,5,6\} \lessdot G\{0,1,5,6\}, G\{0,5,6\} \lessdot G\{0,1,2,5\}, G\{0,1,5\} \lessdot G\{0,1,5,6\}, G\{0,1,5\} \lessdot G\{0,1,2,5\}$. Hence, $G\{0,5,6\}, G\{0,1,5\}$ do not have a well defined join, and so B_{ml}/G is not a lattice.





14.2. **A Failed Attempt for Showing $\mathcal{E}^1(B_n/G)$ is unimodal.** In this section, we describe one path we were pursuing in order to show $\mathcal{E}^1(B_n/G)$ is unimodal. We were attempting to do this by trying to show there were injective order raising maps $U_i : \mathcal{E}^1(B_n/G)_i \to \mathcal{E}^1(B_n/G)_{i+1}$.

We shall now define several maps, so that we can draw a certain commuting diagram in Remark 14.2

Notation 14.1. Let $U_i : P_i \to P_{i+1}$ be the raising operator for the poset P. Then, we obtain an induced map

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \to P_{i+1} \otimes P_{i+1}, (x, y) \mapsto U(x) \otimes U(y).$$

We also have the natural inclusions

$$\begin{aligned} k_i : \mathcal{E}^1(P)_i &\to P_i \otimes P_{i+1}, \\ (x, y) &\mapsto (x, y) \\ k_i^{G\times G} : \mathcal{E}^1(P/G)_i &\to (P/G)_i \otimes (P/G)_{i+1}, \\ (Gx, Gy) &\mapsto (Gx, Gy), \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Next, we define the map

$$\begin{aligned} j_i : (P/G)_i \otimes (P/G)_{i+1} &\to P_i \otimes P_{i+1}, \\ (Gx, Gy) &\mapsto \left(\frac{1}{|G|} \sum_{g\in G} gx, \frac{1}{|G|} \sum_{h\in G} hy). \right) \end{aligned}$$

where x is an arbitrary representative of Gx and y is an arbitrary representative of Gy

Then, define the map

$$\begin{aligned} p_i : P_i \otimes P_{i+1} &\to (P/G)_i \otimes (P/G)_{i+1}, \\ (x, y) &\mapsto (Gx, Gy). \end{aligned}$$

Further, define the map

$$\begin{aligned} (U_i \otimes U_{i+1})^{G\times G} : P_i \otimes P_{i+1} &\to P_{i+1} \otimes P_{i+1}, \\ (Gx, Gy) &\mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx, Gy). \end{aligned}$$

We also have the projections inclusions

$$\begin{aligned} \pi_i : P_i \otimes P_{i+1} &\to \mathcal{E}^1(P)_i, \\ (x, y) &\mapsto \begin{cases} (x, y), & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases} \\ \pi_i^{G\times G} : (P/G)_i \otimes (P/G)_{i+1} &\to \mathcal{E}^1(P/G)_i, \\ (Gx, Gy) &\mapsto \begin{cases} (Gx, Gy), & \text{if } Gx \lessdot Gy \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Finally, denote

$$\begin{aligned} \mathcal{E}^1(U)_i : \mathcal{E}^1(P)_i &\to \mathcal{E}^1(P)_{i+1} \\ (x, y) &\mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x, y) \\ \mathcal{E}^1(U)_i^{G\times G} : \mathcal{E}^1(P/G)_i &\to \mathcal{E}^1(P/G)_{i+1} \\ (Gx, Gy) &\mapsto k_i^{G\times G} \circ (U \otimes U)^{G\times G} \circ \pi_{i+1}^{G\times G}(Gx, Gy) \end{aligned}$$

where it is defined above on a basis and we extend to the whole space by linearity.

Remark 14.2. For $i < \frac{n}{2}$ we obtain the following (almost commuting, but $j_{i+1} \circ p_{i+1} \neq \text{id}$.) diagram

$$\begin{tikzcd}
\mathcal{E}^1(P)_i \arrow[r, hook, "k_i"] \arrow[rrr, bend left, "\mathcal{E}^1(U)_i"] & P_i \otimes P_{i+1} \arrow[r, hook, "U_i \otimes U_{i+1}"] & P_{i+1} \otimes P_{i+2} \arrow[r, two heads, "\pi_{i+1}"] \arrow[d, two heads, shift right, "p_{i+1}"'] & \mathcal{E}^1(P)_{i+1} \\
\mathcal{E}^1(P/G)_i \arrow[r, hook, "k_i^{G\times G}"] \arrow[rrr, bend right, "\mathcal{E}^1(U)_i^{G\times G}"'] & (P/G)_i \otimes (P/G)_{i+1} \arrow[r, hook, "(U_i \otimes U_{i+1})^{G\times G}"] \arrow[u, hook, "j_i"'] & (P/G)_{i+1} \otimes (P/G)_{i+2} \arrow[r, two heads, "\pi_{i+1}^{G\times G}"] \arrow[u, hook, shift right, "j_{i+1}"'] & \mathcal{E}^1(P/G)_{i+1}
\end{tikzcd}$$

Unfortunately, in general, with the above definitions of the maps,

$$\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \not\subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}).$$

However, if we did have

$$\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}),$$

using the fact that $\mathcal{E}^1(U)_i, U_i, U_{i+1}$ are all injective, a fairly simple diagram chase would reveal $\mathcal{E}^1(U)_i^{G\times G}$ is injective. This, in turn, would imply $\mathcal{E}^1(B_n/G)_i$ is symmetric, unimodal, and sperner. We have tried several variations on these exact maps, but were never quite able to obtain the desired $\ker\left(\pi_{i+1}^{G\times G}\circ p_{i+1}\right)\cap \operatorname{Im}((U\otimes U)\circ j_i\circ k_i^{G\times G})\subset \ker \pi_{i+1}\cap \operatorname{Im}((U\otimes U)\circ j_i\circ k_i^{G\times G})$.

David Hemminger, Duke University
E-mail address: david.hemminger@duke.edu

Aaron Landesman, Harvard University
E-mail address: aaronlandesman@gmail.com

Zijian Yao, Department of Mathematics, Brown University
E-mail address: zijian_yao@brown.edu