

# Peckness of Edge Posets

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# Basic Definitions

## Definition

Let  $P$  be a finite graded poset of rank  $n$ , that is:

- Elements of  $P$  are a disjoint union of  $P_0, P_1, \dots, P_n$ , called the *ranks*
- If  $x \in P_i$  and  $x \leq y$ , then  $y \in P_{i+1}$
- Define  $\text{rk}(x) = k$ , where  $x \in P_k$ .

## Definition

A map  $f: P \rightarrow Q$  is a *morphism* from  $P$  to  $Q$  if

$x \leq_P y \implies f(x) \leq_Q f(y)$  and  $\text{rk}(x) = \text{rk}(f(x))$ . We say that  $f$  is *injective/surjective/bijective* if it is an injection/surjection/bijection from  $P$  to  $Q$  as sets.

# Peck Posets

## Definition

Write  $p_i = |P_i|$ .  $P$  is

- *Rank-symmetric* if  $p_i = p_{n-i}$  for all  $1 \leq i \leq n$
- *Rank-unimodal* if for some  $0 \leq k \leq n$  we have

$$p_0 \leq p_1 \leq \dots \leq p_k \geq p_{k+1} \geq \dots \geq p_n$$

- *k-Sperner* if no disjoint union of  $k$  antichains (sets of pairwise incomparable elements) in  $P$  is larger than the disjoint union of the largest  $k$  ranks of  $P$
- *Strongly Sperner* if it is  $k$ -Sperner for all  $1 \leq k \leq n$ .
- *Peck* if  $P$  is rank-symmetric, rank-unimodal, and strongly Sperner.

## Definition

Let  $V(P)$  and  $V(P_i)$  be the complex vector spaces with bases  $\{x|x \in P\}$  and  $\{x|x \in P_i\}$

## Lemma (Stanley, 1980)

$P$  is Peck if and only if there exists an linear transformation  $U: V(P) \rightarrow V(P)$  such that

- For every basis element  $x \in P$ ,

$$U(x) = \sum_{y \succ x} c_{x,y} y$$

- For all  $0 \leq i < \frac{n}{2}$ , the map  $U^{n-2i}: V(P_i) \rightarrow V(P_{n-i})$  is an isomorphism.

## Definition

If the Lefschetz map defined by

$$L(x) = \sum_{y \succ x} y$$

satisfies the second condition in the previous lemma, then  $P$  is *unitary Peck*.

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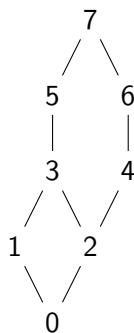
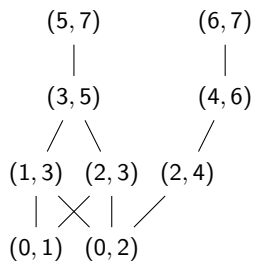
# Definition of the Edge Poset

## Definition

For  $P$  a finite graded poset, its *edge poset*  $\mathcal{E}(P)$  is the finite graded poset defined as follows.

- Elements of  $\mathcal{E}(P)$  are ordered pairs  $(x, y) \in P \times P$  where  $x \lessdot y$
- Define  $(x, y) \lessdot_{\mathcal{E}} (x', y')$  if  $x \lessdot_P x'$  and  $y \lessdot_P y'$
- Define  $\leq_{\mathcal{E}}$  to be the transitive closure of  $\lessdot_{\mathcal{E}}$
- Define  $\text{rk}_{\mathcal{E}}(x, y) = \text{rk}_P(x)$ .



 $P$  $\mathcal{E}(P)$

# A Conjecture on the Peckness of Edge Posets

## Definition

The *boolean algebra of rank  $n$*  is the poset whose elements are subsets of  $[n]$  with order given by containment, i.e. for  $x, y \in B_n$ ,  $x \leq y$  if  $x \subseteq y$ .

## Conjecture (Hemminger, Landesman, and Yao 2014)

Let  $G \subseteq \text{Aut}(B_n)$ . Then  $\mathcal{E}(B_n/G)$  is Peck.

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# Main Result

## Definition

A group action of  $G$  on  $P$  is *cover transitive* if whenever  $x, y, z \in P$  such that  $x \triangleleft z$ ,  $y \triangleleft z$ , and  $y \in Gx$ , there exists some  $g \in \text{Stab}_G(z)$  such that  $g \cdot x = y$ .

## Theorem (Hemminger, Landesman, and Yao 2014)

If a group action of  $G$  on  $B_n$  is cover transitive, then  $\mathcal{E}(B_n/G)$  is Peck.

## Definition

Given a group action of  $G$  on  $P$ , we define a group action of  $G$  on  $\mathcal{E}(P)$  by letting  $g \cdot (x, y) = (g \cdot x, g \cdot y)$  for all  $g \in G$ .

## Definition

Given a group action of  $G$  on  $P$ , we define a group action of  $G$  on  $\mathcal{E}(P)$  by letting  $g \cdot (x, y) = (g \cdot x, g \cdot y)$  for all  $g \in G$ .

## Proposition

*The map  $q: \mathcal{E}(P)/G \rightarrow \mathcal{E}(P/G)$  defined by  $q(G(x, y)) = (Gx, Gy)$  is a surjective morphism. Furthermore,  $q$  is also injective if and only if the action of  $G$  on  $P$  is cover transitive.*

## Lemma

*If  $f: P \rightarrow Q$  is a bijective morphism and  $P$  is Peck then  $Q$  is Peck.*

## Theorem (Stanley, 1984)

*If  $P$  is unitary Peck and  $G \subseteq \text{Aut}(P)$ , then  $P/G$  is Peck.*

It then suffices to show that  $\mathcal{E}(B_n)$  is unitary Peck. Unfortunately while this is true, the only proof that we know is long and computational. Instead we outline a nicer –albeit less direct– proof.

# Definition of $\mathcal{H}(P)$

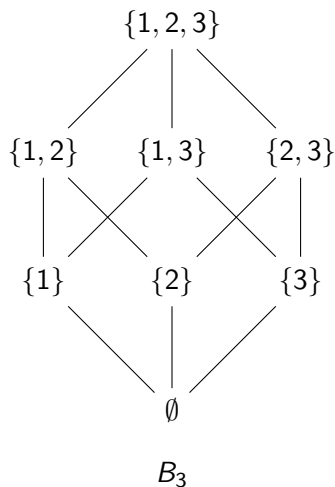
## Definition

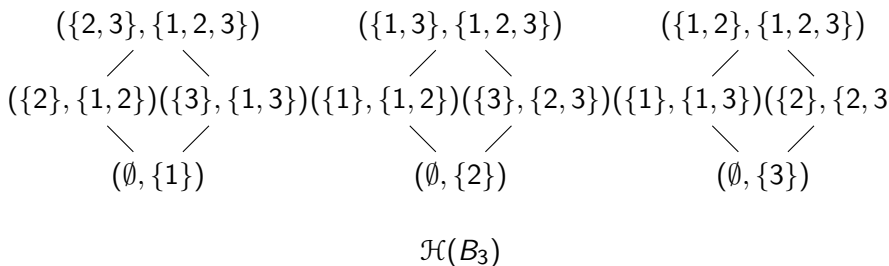
For  $P$  a finite graded poset, define the graded poset  $\mathcal{H}(P)$  as follows.

- Elements are pairs  $(x, y) \in P \times P$  such that  $x \leq y$
- Define  $(x, y) \leq_{\mathcal{H}} (x', y')$  if  $x \leq_P x', y \leq_P y'$  **and**  $y \neq x'$
- Define  $\leq_{\mathcal{H}}$  to be the transitive closure of  $\leq_{\mathcal{H}}$
- Define  $rk_{\mathcal{H}}(x, y) = rk_P(x)$ .



# The Boolean Algebra $B_3$



$\mathcal{H}(B_3)$  is unitary Peck

## Definition

As before, for  $G$  acting on  $P$ , define  $g \cdot (x, y) = (g \cdot x, g \cdot y)$ .

## Remark

Since  $\mathcal{E}(P)$  and  $\mathcal{H}(P)$  have the same elements and  $(x, y) \leq_{\mathcal{H}} (x', y') \implies (x, y) \leq_{\mathcal{E}} (x', y')$ , there is a natural bijective morphism  $\mathcal{H}(P)/G \rightarrow \mathcal{E}(P)/G$ .

## Proof of Main Result.

$\mathcal{H}(B_n)$  unitary Peck  $\implies \mathcal{H}(B_n)/G$  Peck  $\implies \mathcal{E}(B_n)/G$  Peck  
 $\implies \mathcal{E}(B_n/G)$  Peck. □