List of Todos

1. Notation and Definitions

Let $G \subset \mathfrak{S}_n$, let $x, y \in 2^{[n]}$, and let $1 \le r \le n$. For $r \le i \le n$, let $V_i^{(r)}$ be the $\mathbb{R}$-vector space generated by the basis

$$\{e_{(x,y)}\}_{x \subset y, |y|=i, |x|=i-r}$$

Note that G acts on this space with the action $\sigma e_{(x,y)} = e_{(\sigma x, \sigma y)}$. Let $\left(V_i^{(r)}\right)^G$ be the subspace of $V_i^{(r)}$ that is invariant under this action. Write $q_i^r = \dim\left(\left(V_i^{(r)}\right)^G\right)$. When r is understood to be 1, we will often simply write q_i.

Let p_i be defined as by Pak and Panova, that is

$$p_i = \sum_{Gy, |y|=i} \nu(Gy)$$

where $\nu(Gy)$ is the number of covering relations $Gy \gtrdot Gx$.

2. Current Goals

Our main goal is describe the groups $G \subset \mathfrak{S}_n$ such that for $2^{[n]}/G$ we have $q_i = p_i$. So far we know that this is true for $G = \{e\}$, $G = \mathfrak{S}_n$, and $G = \mathfrak{S}_k \wr \mathfrak{S}_l$ (this is equivalent to Proposition 4.2). Any other groups for which it holds would be good to know about. In fact, if someone wanted to write a program to compute the sequence q_i for all subgroups of $\mathfrak{S}_4$, $\mathfrak{S}_5$, $\mathfrak{S}_6$, $\mathfrak{S}_7$ and such, that would be awesome!

3. Conjectures

4. Results

Proposition 4.1. *For all $G \subset \mathfrak{S}_n$, $1 \le r \le n$, the sequence $q_r^r, q_{r+1}^r, \dots, q_n^r$ is unimodal and symmetric about $\frac{n+r}{2}$.*

There is a natural mapping from the basis of $\left(V_i^{(r)}\right)^G$ to the covering relations $Gy \gtrdot Gx$ given by first identifying the basis elements with the orbits $Ge_{(x,y)}$, and then taking the orbit $Ge_{(x,y)}$ to $Gy \gtrdot Gx$. It is not hard to check that this mapping is both well-defined and surjective. When $G(x,y)$ contains all the pairs (x,y) such that $x \in Gx$, $y \in Gy$, and $x \subset y$, then this mapping is also injective, which implies that the dimension of $\left(V_i^{(r)}\right)^G$ and the number of covering relations $Gy \gtrdot Gx$, $|y| = i$ are equal, meaning that $q_i = p_i$ for all i.

Proposition 4.2. *For $G = \mathfrak{S}_k \wr \mathfrak{S}_l$ and $r = 1$, $q_i = p_i$ for all $1 \le i \le n$.*

Proposition 4.3. *Let p be prime. If G is the embedding of D_p into $\mathfrak{S}_p$ given by letting G act on the vertices of the regular p-gon, then $q_i = p_i$ for all i.*

Proof. By the discussion above, we simply need to check that for any $x, x' \in Gx$, $y, y' \in Gy$ satisfying $x \subset y$ and $x' \subset y'$, there exists some $\pi \in G$ such that $\pi(x,y) = (x', y')$. Pick some $\sigma \in G$ such that $\sigma(y) = y'$, and let $x'' = \sigma^{-1}(x')$. Now if we can find $\tau \in G$ such that $\tau(y) = y$ and $\tau(x) = x''$, then we will have $\sigma\tau(x,y) = (x', y')$, as desired.

We have $x'' \subset y$ because the group operation preserves containment, so let $y = \{a_1, a_2, \dots, a_i\}$, $x = y \smallsetminus \{a_j\}$, and $x'' = y \smallsetminus \{a_k\}$ for some $1 \le j \le i$. Pick some $\rho \in G$ such that $\rho(x) = x''$. If ρ is a reflection of the p-gon, then it is given by a product of disjoint transpositions, one of which must be $(a_j a_k)$, and the rest of which must fix $y \smallsetminus \{a_j, a_k\}$ in order for $\rho(x) = x''$. In this case ρ fixes y, so we're done.

If ρ is not given by a reflection of the p-gon then it must be a rotation. Since p is prime, this means that ρ is given by a single p-cycle $(1, 1+l, 1+2l, \dots, 1+(p-1)l)$, where the entries of the cycle are taken modulo p. Since $\rho(x) = x''$, we must have $x = \{a_k, a_k + l, \dots, a_j - l\}$ and $x'' = \{a_k + l, \dots, a_j - l, a_j\}$. Let τ be the reflection of the p-gon that flips the vertices a_j and a_k. Then τ also flips the pairs of vertices $\{a_k + l, a_j - l\}, \{a_k + 2l, a_j - 2l\}, \dots$, and therefore fixes $y = \{a_k, a_k + l, \dots, a_j\}$, so we're done. $\square$

Lemma 4.4. *$U_i^{(r)}$ is injective for all $i < \frac{n+r}{2}$.*

Lemma 4.5. *For all $\sigma \in G, e_{(x,y)} \in V_i^{(r)}$, we have*

$$U_i^{(r)}(\sigma(e_{(x,y)})) = \sigma(U_i^{(r)}(e_{(x,y)}))$$

Lemma 4.6. *For any group $G \subset \mathfrak{S}_n$ with an action on an $\mathbb{R}$-vector space V with basis $\{v_i\}_{1 \le i \le k}$, the G-invariant subspace V^G of V has basis*

$$\sum_{v_i \in Gv} v_i$$

where the sum is taken over the orbits Gv of the group action.

5. Failed Attempts

Have an idea that failed? Show it off here!