# Peckness of Edge Posets

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August 6, 2014

# Outline of Talk

# Basic Definitions

**Definition**

Let $P$ be a finite graded poset of rank $n$, that is:

- Elements of $P$ are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks*
- If $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$
- Define $\mathrm{rk}(x) = k$, where $x \in P_k$.

**Definition**

A map $f\colon P \to Q$ is a *morphism* from $P$ to $Q$ if $x \leq_P y \implies f(x) \leq_Q f(y)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$. We say that $f$ is *injective/surjective/bijective* if it is an injection/surjection/bijection from $P$ to $Q$ as sets.

# Peck Posets

## Definition

Write $p_i = |P_i|$. P is

- *Rank-symmetric* if $p_i = p_{n-i}$ for all $1 \leq i \leq n$
- *Rank-unimodal* if for some $0 \leq k \leq n$ we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

- *k-Sperner* if no disjoint union of $k$ antichains (sets of pairwise incomparable elements) in $P$ is larger than the disjoint union of the largest $k$ ranks of $P$
- *Strongly Sperner* if it is $k$-Sperner for all $1 \leq k \leq n$.
- *Peck* if $P$ is rank-symmetric, rank-unimodal, and strongly Sperner.

## Definition

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x|x \in P\}$ and $\{x|x \in P_i\}$

## Lemma (Stanley, 19802)

*$P$ is Peck if and only if there exists an linear transformation $U\colon V(P) \to V(P)$ such that*

- *For every basis element $x \in P$,*

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

- *For all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

**Definition**

If the Lefschetz map defined by

$$L(x) = \sum_{y \gtrdot x} y$$

satisfies the second condition in the previous lemma, then $P$ is *unitary Peck*.

# Outline of Talk

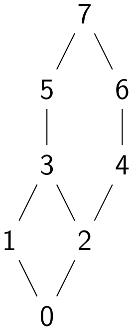
# Definition of the Edge Poset

## Definition

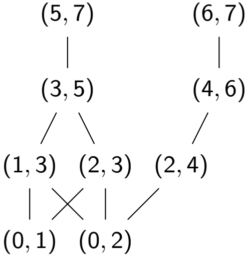
For $P$ a finite graded poset, it's *edge poset* $\mathcal{E}(P)$ is the finite graded poset defined as follows.

- Elements of $\mathcal{E}(P)$ are ordered pairs $(x, y) \in P \times P$ where $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$
- Define $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$
- Define $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$.

# Basic Example




$P$




$\mathcal{E}(P)$

# Conjecture on the Peckness of Edge Posets

**Definition**

The *boolean algebra of rank n* is the poset whose elements are subsets of $[n]$ with order given by containment, i.e. for $x, y \in B_n$, $x \leq y$ if $x \subseteq y$.

**Conjecture (Hemminger, Landesman, and Yao 2014)**

*Let* $G \subseteq \mathrm{Aut}(B_n)$. *Then* $\mathcal{E}(B_n/G)$ *is Peck.*

# Outline of Talk

# Main Result

**Definition**

A group action of $G$ on $P$ is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in \mathrm{Stab}_G(z)$ such that $g \cdot x = y$.

**Theorem (Hemminger, Landesman, and Yao 2014)**

*If a group action of $G$ on $B_n$ is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

## Definition

Given a group action of $G$ on $P$, we define a group action of $G$ on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

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Given a group action of $G$ on $P$, we define a group action of $G$ on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

**Proposition**

*The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism. Furthermore, $q$ is also injective if and only if the action of $G$ on $P$ is CCT.*

**Lemma**

*If $f : P \to Q$ is a bijective morphism and $P$ is Peck then $Q$ is Peck.*

**Theorem (Stanley, 1984)**

*If $P$ is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then $P/G$ is Peck.*

It would then suffice to show that $\mathcal{E}(B_n)$ is unitary Peck, but our proof for this is complicated. Instead we construct a unitary Peck poset $\mathcal{H}(B_n)$ such that there is a bijective morphism $\mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$.

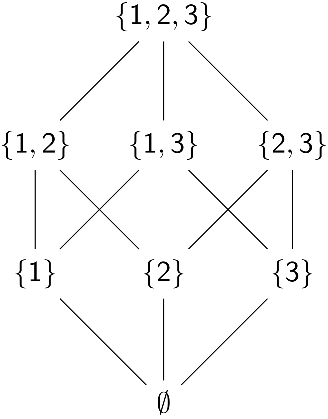
# Definition of $\mathcal{H}(P)$

## Definition

For $P$ a finite graded poset, define the graded poset $\mathcal{H}(P)$ as follows.

- Elements are pairs $(x, y) \in P \times P$ such that $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{H}} (x', y')$ if $x \lessdot_P x', y \lessdot_P y'$ **and** $\mathbf{y \neq x'}$
- Define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$
- Define $rk_{\mathcal{H}}(x, y) = rk_P(x)$.

# The Boolean Algebra $B_3$




$B_3$

# $\mathcal{H}(B_3)$ is unitary Peck

$$(\{2,3\},\{1,2,3\}) \qquad (\{1,3\},\{1,2,3\}) \qquad (\{1,2\},\{1,2,3\})$$

$$(\{2\},\{1,2\}) \; (\{3\},\{1,3\}) \; (\{1\},\{1,2\}) \; (\{3\},\{2,3\}) \; (\{1\},\{1,3\}) \; (\{2\},\{2,3\})$$

$$(\emptyset,\{1\}) \qquad (\emptyset,\{2\}) \qquad (\emptyset,\{3\})$$

$\mathcal{H}(B_3)$

**Definition**

As before, for $G$ acting on $P$, define $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

**Remark**

Since $\mathcal{E}(P)$ and $\mathcal{H}(P)$ have the same elements and $(x, y) \leq_{\mathcal{H}} (x', y') \implies (x, y) \leq_{\mathcal{E}} (x', y')$, there is a natural bijective morphism $\mathcal{H}(P)/G \to \mathcal{E}(P)/G$.

**Proof of Main Result.**

$\mathcal{H}(B_n)$ unitary Peck $\implies \mathcal{H}(B_n)/G$ Peck $\implies \mathcal{E}(B_n)/G$ Peck $\implies \mathcal{E}(B_n/G)$ Peck. $\square$

# Outline of Talk

# CCT actions

## Lemma

*Let $G$ be a group acting on a graded poset $P$. The following are equivalent:*

1. *The action of $G$ on $P$ is CCT.*
2. *Whenever $w \lessdot x, w \lessdot y$, and $x \in Gy$, there exists some $g \in Stab(w)$ with $gx = y$.*
3. *The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x,z)) = (Gx, Gz)$ is a bijective morphism (but not necessarily an isomorphism).*
4. *For all $i$ there is an equality $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$*

# Some examples of CCT actions

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1. The trivial group;
2. The group $S_n$ acting on $B_n$;
3. The group $D_{2n}$ acting on $B_n$ when $n = p$ or $n = 2p$, and $p$ is a prime;
4. The elementary 2-group $(\mathbb{Z}/2\mathbb{Z})^k$ with any action on $P$.

# The Dihedral group $D_{2p}$ and $D_{4p}$

Add pictures.

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Add pictures.

# The direct product

**Lemma**

*For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.*

# The semi-direct product

**Proposition**

*Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that $H$ acts CC transitively on $P$ and $K$ acts CC transitively on $P/H$. Then $G$ acts CC transitively on $P$.*

# The wreath product

**Corollary**

*If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where $\phi$ is the induced action is also CCT.*

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**Corollary**

*The action* $S_m \wr S_l \times B_n \to B_n$ *is CCT, where* $n = ml$.

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**Corollary**

*The action $S_m \wr S_l \times B_n \to B_n$ is CCT, where $n = ml$.*

**Remark**

This recovers a special case of a theorem obtained by Pak & Panova:
The poset $\mathcal{E}(B_n/S_m \wr S_l)$ is rank symmetric and rank unimodal.
(Furthermore, it is Peck!)

# The automorphism of rooted trees

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# Automorphism of rooted trees

## Proposition

*Let $P$ be a rooted tree with root vertex labeled $r$. Then, if let $\{A_1, \ldots, A_m\}$ denote the set of isomorphism classes of $\{D(x) | x \lessdot r\}$. For $T \in A_j$, denote $G_j = Aut(T)$. Then,*

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

*In particular, $Aut(P)$ can be expressed as a sequence of direct products and wreath products of symmetric groups.*

# Outline of Talk

# Unimodality of ranks of certain edge posets

**Lemma**

*Let $C_n$ be the cyclic group which acts naturally on $B_n$, then size of the $i^{th}$ rank of the poset $\mathcal{E}(B_n)/C_n$ is*

$$|\mathcal{E}(B_n)/C_n|_i = \binom{n-1}{i}.$$

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**Lemma**

*Let $C_p$ be the cyclic group with prime order $p$, then*

$$|\mathcal{E}(B_n)/C_n|_i - |\mathcal{E}(B_n/C_n)|_i = \frac{p-1}{2}.$$

# Unimodality of ranks of certain edge posets

**Lemma**

*Let $D_{2n}$ be the dihedral group of size $2n$ which acts naturally on $B_n$, then size of the $i^{th}$ rank of the poset $\mathcal{E}(B_n)/D_{2n}$ is*

$$|\mathcal{E}(B_n)/D_{2n}|_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+2)}+1]\cdot\binom{\lceil n/2\rceil - 1}{\lceil (i+1)/2\rceil - 1}\right)$$

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*The poset $\mathcal{E}(B_n/D_{2n})$ is rank symmetric and rank unimodal.*

# The $q$ analog of the problem

# Possible Generalizations of $\mathcal{E}$

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# Acknowledgements


1. Thanks to Dr. Vic Reiner and Elise DelMas for mentoring and TAing this project.
2. The work for this project took place at the Minnesota at Twin Cities REU. Thanks to Dr. Gregg Musiker and the University of Minnesota School of Mathematics for coordinating and hosting the REU.
3. This research was supported by the RTG grant NSF/DMS-1148634.