1 Progress on the problem of the Boolean Algebra problem

First we introduce/recall notations. B_n is the Boolean algebra on n elements $\{1, ..., n\}$, $G < S_n$ is a subgroup of the symmetric group S_n. Let $V_i^{(r)}$ be generated by basis elements $\{(x, y)\}$ such that $y \in P_i$ and $x \in P_{i-r}$ where $x < y$, for convenience we denote the set of the basis elements by A_i, so

$$A_i = \{(x, y)|x \in P_{i-1}, y \in P_i,\ x < y\}, \quad \text{and} \quad V_i^1 = \mathbb{R}A_i.$$

For each i we define a relation $\sim$ on A_i as follows: we say that

$$(x, y) \sim (x', y') \text{ iff there exists} \sigma, \tau \in G \text{ such that } \sigma x = x', \tau y = y'.$$

I want to emphasize the fact that since $(x', y') \in A_i$, the relation necessarily requires $\sigma x < \tau y$, i.e., $(\tau^{-1}\sigma)x < y$. One can easily verify that $\sim$ is a an equivalent relation on A_i. Let us denote the number of such classes on A_i by p_i, i.e.,

$$p_i = \# \text{ of equivalent classes formed by } \sim.$$

In the Hasse diagram of the graded poset B_n/G, the number p_i precisely counts the number of edges doing down from rank level i to rank level $(i-1)$. (And it is consistent with the notation Vic gave for the case $r = 1$).

Now we define another equivalence relation $\equiv$ on A_i by the following group action

$$G \times A_i \to A_i \quad \text{where } \sigma(x, y) \mapsto (\sigma x, \sigma y).$$

and define q_i to be the number of G-orbits for each i such that $1 \leq i \leq n$. Note that the integer q_i is the dimension of $(V_i^{(1)})^G$, the invariant subspace of $V_i^{(1)}$. (This is step-by-step analogous to Stanley's proof).

Next I shall prove several facts about q_i and p_i and propose some possible directions. I shall also include in the end some methods that I have tried and have not yet succeeded in concluding anything.

Lemma 1.1. For any $G < S_n$ and all $1 \leq i \leq n$, we have $q_i \geq p_i$.

Proof. This is almost trivial, since each G-orbit lives in some $\sim$ equivalent class. $\square$

Note that we are particularly interested in the case for G where $q_i = p_i$, since this would imply that the sequence p_i are unimodal, as a corollary of results 4.4, 4.5 and the exercise Vic gave, which says $U_i^r : V_i^{(r)} \to V_{i+1}^{(r)}$ is injective. For details, see **??**.

Proposition 1.2. Let $G = S_m \wr S_l$ where $n = ml$, and p_i, q_i as defined above. Then $p_i = q_i$.

Remark 1.3. This gives a different proof that for group $S_m \wr S_l$, the statistics p_i are unimodal, which is the case $r = 1$ in the paper by Pak and Panova.

Proof. To prove the proposition, we need to show that each $\sim$ equivalent class is precisely a G-orbit (i.e., a $\equiv$ equivalent class).

We formulate in terms of the following statement: for any pairs $(x, y), (x', y') \in A_i$ with $\sigma, \tau \in G$ satisfying $\sigma x = x', \tau y = y'$ and $\tau^{-1}\sigma x < y$, then there exists $g \in G$ such that $gx = x'$ and $gy = y'$.

Note that it suffices to prove the statement when $\tau = 1$, i.e., if there is sigma satisfying $\sigma x < y$, then there exists $g \in G$ such that $gx = \sigma x$ and $gy = y$, since we can replace σ and g by $\tau^{-1}\sigma$ and $\tau^{-1}g$ respectively. We can further reduce the proof to the case where y is has the shape of a young tableau for the following reason: let δ be the wreath transformation that takes y to y', where y' is of the shape of a YT (i.e., size of the rows weakly decreases from top to bottom). Since $\sigma x < y$, then $\delta\sigma x < \delta y = y'$. Hence, if we know that there is some g' such that $g'x = \delta\sigma x$ and $g'y = y'$, then letting $g = \delta^{-1}g'$ gives us $gx = \delta^{-1}g'x = \sigma x$ and $gy = y$.

Let us recap what we want to prove: $(x, y) \in A_i$, with y in the shape of a Young tableau, and $\sigma \in G$ such that $\sigma x < y$. First consider the case when σ only permute some rows but do not swap them. Let x be formed by taking out a square Q from some row R, note that $x \backslash R = y \backslash R$, so σ fixes y on all other rows except for R. If $\sigma y = y$, then we are done by taking $g = \sigma$, otherwise, since $\sigma x < y$, we know that $\sigma(x \cap R \backslash Q) < (x \cap R)$ and $\sigma(Q) = Q'$ is a square out of y in the row R. Let σ' be the transformation that takes Q' to Q and fixes everything else, then it is clear that $\sigma'\sigma x = \sigma x$ and $\sigma'\sigma y = y$, so we found such $g = \sigma'\sigma$. Now without loss of generality we can assume that x is formed by taking out a corner square Q from y in row R

and σ swap the rows. If σ does not affect Q then again take $g = \sigma$, otherwise, since $\sigma x < y$, we know that σ can only swap row R with some other row below it, say R', where $x \cap R'$ has the same size as $x \cap R$, but in this case, swapping rows do not affect the shape of x, so we can simply take $g = 1$ the identity. This concludes the proposition. $\square$

Next step, generalize for $r \geq 2$.

Now what about the Necklace poset where $G = C_n = \langle (1\ 2\ ...\ n) \rangle$?

For this situation we claim the following:

Proposition 1.4. Let $G = C_n$, then for i such that $2 \leq i \leq n-1$, we have that $q_i > p_i$. In addition, we can calculate q_i fairly easily in this particular case:

$$q_i = \binom{n-1}{i-1}.$$

Note that this gives an upper bound for p_i.

Proof. To show that $q_i > p_i$ when $2 \leq i \leq n-1$, we consider a pair of element (x_0, y_0) where $x_0 = \{1, 2, ..., i-1\}$ and $y_0 = \{1, 2, ..., i\}$. Take σ being the generating permutation $(1\ 2\ ...\ n)$ and τ the identity. Then clearly $\sigma x = \{2, ..., i\} < y$. Now consider $g = \sigma^d$ for some integer power d, if $gx = \sigma x$, then necessarily $g = \sigma$. Since $i \leq n-1$, $\sigma y = \{2, ..., i+1\} \neq y$. This shows that the $\equiv$ equivalent class of (x, y) contains more than 1 G-orbit, therefore $q_i > p_i$.

Now consider the second part of the statement, first we show that the action $G \times A_i \to A_i$ is faithful, namely for any $(x, y) \in A_i$, $\tau(x, y) = (x, y)$ if and only if $\tau = 1$, the identity element of G. Note that, if $\tau x = x$ and $\tau y = y$, then $\tau(y \backslash x) = y \backslash x$, where $y \backslash x \in P_1$ is a one element set, since $\tau = (1\ 2\ ...\ n)^d$ for some power d (which is a rotation), τ fixes the one element set iff $\tau = 1$. Now for any $(x, y) \in A_i$, the stabilizer $\mathrm{Stab}_{(x,y)}$ is trivial. By the orbit-stabilizer lemma, we know the orbit $G_{(x,y)}$ of (x, y) always contains $|G| = n$ elements. It is easy to show that $|A_i| = \binom{n}{i} \times i$, so $q_i = |A_i| / |G_{(x,y)}| = \binom{n}{i} \times \frac{i}{n} = \binom{n-1}{i-1}$.

$\square$

Now we have (at least) two jobs to do, first to determine for which groups G one does have $p_i = q_i$, secondly, to show that p_i is unimodal for $G = C_n$ and other groups.

For the second part with $G = C_n$, I compared the list of $(p_1, ..., p_n)$ with $(q_1, ..., q_n)$ for a few cases (this could lead to a proof by realizing how q_i differs from p_i, i.e., how G-orbits group to form the $\equiv$ classes).

Here are the experimental data: i starting from 1 to n.

* n =3, $\{p_i\} = \{1, 1, 1\}; \{q_i\} = \{1, 2, 1\}$
* n =4, $\{p_i\} = \{1, 2, 2, 1\}; \{q_i\} = \{1, 3, 3, 1\}$
* n =5, $\{p_i\} = \{1, 2, 4, 2, 1\}; \{q_i\} = \{1, 4, 6, 4, 1\}$
* n =6, $\{p_i\} = \{1, 3, 9, 9, 3, 1\}; \{q_i\} = \{1, 5, 10, 10, 5, 1\}$
* n =7, $\{p_i\} = \{1, 3, 12, 17, 12, 3, 1\}; \{q_i\} = \{1, 6, 15, 20, 15, 6, 1\}$

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I was hoping that this comparison would point to some directions, and it does, at least to some extend. Notice that for $n = 3, 5, 7$, we have that $q_i - p_i = (n-1)/2$ for $2 \le i \le n-1$. So I think that this is true for all $n =$ primes.

Proposition 1.5. Let $G = C_n$ where n is a prime, then $q_i - p_i = (n-1)/2$ for $2 \le i \le n-1$.

Proof. My idea is to group the G-orbits to form $\sim$ classes, in fact, we count the number of G-orbits in each $\sim$ classes. Note that n being a prime guarantees that the action of any nontrivial $\sigma \in C_n$ has no fixed points, since σ is always an n-cycle in its cycle decomposition. Now suppose $(x, y) \in A_i$ is a pair such that $\sigma x \in y$ for some no trivial σ, then there is no $g \in C_n$ such that $gx = \sigma x$ and $gy = y$, since $gx = \sigma x \Rightarrow g = \sigma$ and $gy = y \Rightarrow g = 1$. Let $\mathcal{O}$ be a $\sim$ class with representative (x, y), then by the argument above, the number of distinct $\sigma \in G$ such that $\sigma x < y$ is precisely the number of G-orbits in $\mathcal{O}$, in particular, if the only σ such that $\sigma x < y$ is identity, then $\mathcal{O}$ is a G-orbit.

Therefore, the problem of counting $q_i - p_i$ is to count the number of distinct G-orbits $G_{(x,y)}$ with distinct $\sigma \in G$ such that $\sigma x < y$, and we claim that regardless of i, this number is $(n-1)/2$.

We prove the claim using a combinatorial method, we consider rotations of the necklace pattern of the G-orbits of y, where $y \in P_i$. We label the "pearls" - empty cells - of the necklace by $1, 2, ..., n$. For example, to represent $\{1, 2, 3\} \equiv \{2, 3, 4\}...$, we simply fill 3 consecutive pearls to be solid, others would empty. Let $\sigma_0 = (1\,2\,...\,n)$ be the generator of C_n, which is a 1-click rotation.

First consider $\sigma = \sigma_0$ and $\sigma x < y$, i.e., we take out a filled pearl from y, and rotate y by a 1-click rotation, the remaining filled pearls fits in y. It is clear that the only possibility for this to occur is we have n consecutive filled cells, i.e., $y = \{1, 2, ..., i\}$ and $x = \{1, 2, ..., i - 1\}$ (up to actions by G).

Now consider $\sigma = \sigma_0^2$ and $\sigma x < y$, i.e., we take out a filled pearl from y, and rotate y by a 2-click rotation, the remaining filled pearls fits in y. Now we again trace back from the "last" filled pearl in y (any filled pearl could be the last one since we are in a circle), say at position P, then at position $P-2, P-4, ...$ (reduce mod n whenever necessary) there need to be a filled pearl. In this case, there is again only one possibility, namely G-orbits of $\{1, 3, ...2i - 1\}$ and $x = \{1, 3, ..., 2i - 3\}$.

Similarly, for any $\sigma = \sigma_0^j$, $j < n/2$, there is a precisely one possibility for $\sigma x < y$. Notice that for $j > n/2$, the necklace $\{1, j + 1, 2j + 1, ...\}$ and $\{1, (n - j) + 1, 2(n - j) + 1, ...\}$ are the same necklaces, so we do not count them again. It is also easy to show that, for all distinct $j < n/2$, the necklaces $\{1, j+1, 2j+1, ..., (i-1)j+1\}$ are all distinct, since n is a prime number.

This proves the claim. □

As a corollary we have the following theorem:

Theorem 1.6. Let $G = C_n$ where n is a prime, and p_i as defined above. Then the sequence p_i is unimodal.

Proof. By the proposition and the symmetry of p_i, we only need to show $p_1 \leq p_2$, but $p_1 = 1$, so we are done. □