

# LIST OF TODOS

Relegate functoriality to a remark . . . . .	2
Define group action to be order preserving and rank preserving, and remove statements of these from the document. . . . .	2
Include a section for generalizations of E and include redfield polya generalization	2
add reference . . . . .	2
Weren't we going to include a question about q-analogs? . . . . .	3
Rephrase so that it has two assertions (in two sentences): first, that $S_n$ acts cover-transitively for all n, and second, that for p prime, $D_{2p}$ and $D_{4p}$ are cover-transitive inside $S_{2p}, S_{4p}$ . . . . .	3
Please note: This paragraph does NOT currently reflect reality. I thought for a while about how we should structure the paper, and this is roughly how I think it should look. . . . .	3
consolidate this section . . . . .	3
consolidate this section . . . . .	3
Obviously come back to this at the end, but since this is the introduction for the REU report, not a journal paper, we should also include descriptions of the sections we don't think will go in the journal paper. . . . .	3
add reference . . . . .	4
I don't think the details of the proof need to be included here, but I've left them here commented out in case we change our minds. Aaron: I agree it is fine to leave them out. . . . .	5
Put all pictures in figures, and make them look nicer . . . . .	7
David, can you fill this in please? . . . . .	10
I think (i.e...) actually makes things more confusing, rather than clearer. Consider removing the clause after i.e.? . . . . .	24
Consider using a word other than special? Maybe you could say a double orbit, or something like that? . . . . .	24
Perhaps consider breaking this proof into a couple lemmas. First I might say that $\lambda_{i+1} - \lambda_i$ is bounded by the number of elements $y$ with tail cycles with respect to some $\sigma_0^r$ so that $y < y'$ with $y' \in \text{Stab}(\sigma_0^r)$ . Then, I would give the bound in the statement of this lemma regarding the size of this set. . . . .	24
I don't understand what it means by an orbit "contributes to the difference." What you really want to say is that an orbit is in a certain set, and the difference is the size of that set. I have edited the previous paragraphs to make this clearer, can you fix the rest of the proof please? . . . . .	25
what does "contribute" mean? Are you are trying to say it lies in a certain set? . . . . .	25
Again, I don't understand . . . . .	25
What does this mean? It sounds like you're thinking of $\lambda_i$ as a variable, which changes size as you look at different $G$ orbits. . . . .	25
How can you choose to ignore a scenario? What does that mean? . . . . .	25
Can you make this sentence more precise? I don't understand what the "x" part is, what a "family" of such special orbits is, etc. . . . .	25
what is this? can you write it precisely as a set? . . . . .	25
what index is this sum over? . . . . .	25
what does it mean to insert a tail cycle, and what is a block of elements? . . . . .	25
with respect to which group element? . . . . .	25
Aaron: Consider removing this sentence, or at the very least inserted after the proof as a remark . . . . .	26
Maybe call this a proposition or theorem instead of corollary? It doesn't really follow immediately from other things, and it's the main result we're showing . . . . .	26
This proof needs more work (self) . . . . .	26

# PECK EDGE POSETS

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## 1. INTRODUCTION

Let  $P$  be a finite graded poset of rank  $n$ . It is natural to study the structure of the edges in the Hasse diagram of  $P$ . To this end, define an endofunctor  $\mathcal{E}^r$  on the category finite graded posets with rank-preserving morphisms as follows

**Definition 1.1.** For  $\mathcal{P}$  the category of graded posets, for each  $r \in \mathbb{N}$ , define the *Functor of Edges*  $\mathcal{E}^r : \mathcal{P} \rightarrow \mathcal{P}$  as follows. The elements of the graded poset  $\mathcal{E}^r(P)$  are  $(x, y)$  where  $x, y \in P$ ,  $x \leq_P y$ , and  $\text{rk}(y) = \text{rk}(x) + r$ . Define the covering relation  $\triangleleft_{\mathcal{E}}$  on  $\mathcal{E}^r(P)$  by  $(x, y) \triangleleft_{\mathcal{E}} (x', y')$  if  $x \triangleleft_P x'$  and  $y \triangleleft_P y'$ . Then define the relation  $\leq_{\mathcal{E}}$  on  $\mathcal{E}^r(P)$  to be the transitive closure of  $\triangleleft_{\mathcal{E}}$ . Let  $Q$  be a finite graded poset of rank  $n$ . Given a morphism  $f : P \rightarrow Q$ , define  $\mathcal{E}^r(f) : \mathcal{E}^r(P) \rightarrow \mathcal{E}^r(Q)$  by  $\mathcal{E}^r(f)(x, y) = (f(x), f(y))$ .

**Example 1.2.** It is important we declare the relation  $\leq_{\mathcal{E}}$  to be the transitive closure of  $\triangleleft_{\mathcal{E}}$ . If instead we defined a relation  $\leq_{\mathcal{E}'}$  on  $\mathcal{E}(P)$  by  $(x, y) \leq_{\mathcal{E}'} (a, b)$  if  $x \leq a, y \leq b$ , then  $\mathcal{E}(P)$  would not necessarily be a graded poset. In Figure 1.2 we give an example of a poset  $P$  for which  $\mathcal{E}(P)$  is not graded with the relation  $\leq_{\mathcal{E}'}$ .

Here, it is clear that  $\mathcal{E}^1(P)$  is a graded poset, with  $\text{rk}(x, y) = \text{rk}(x)$ , but the Hasse diagram on the right represents a poset which does not have a grading.

We will show that  $\mathcal{E}^r$  is well-defined in Section 3. Note that an edge in the Hasse diagram of  $P$  can be written as  $(x, y) \in P \times P$  such that  $x \triangleleft y$ . Such edges are in bijection with elements  $(x, y) \in \mathcal{E}^1(P)$ . We observe that when  $P$  has a nice structure,  $\mathcal{E}^r(P)$  commonly has a nice structure as well. In particular, let the *boolean algebra of rank  $n$* , denoted  $B_n$ , be the poset whose elements are subsets of  $\{1, \dots, n\}$  with the relation given by containment, i.e. for all  $x, y \in B_n$ ,  $x \leq y$  if  $x$  is a subset of  $y$ . It is well-known that if  $G$  is a group of automorphism of  $B_n$ , then  $B_n/G$  is Peck. We conjecture the following.

**Conjecture 1.3.** *Let  $G$  be a group of rank-preserving automorphisms of  $B_n$ . Then  $\mathcal{E}^r(B_n/G)$  is Peck.*

Include a section for generalizations of E and include redfield polya generalization

add reference



## 2. BACKGROUND

In this section we review morphisms of graded posets, Peck posets, and a useful theorem about quotients of Posets.

Throughout the paper we write  $x \leq_P y$  to denote that  $x$  is less than or equal to  $y$  under the relation defined on the poset  $P$ . When the poset is clear we will omit the  $P$  and simply write  $x \leq y$ .

A poset  $P$  is a *graded* poset if there exists a rank function  $\text{rk} : P \rightarrow \mathbb{Z}_{\geq 0}$  satisfying the following conditions.

- (1) If  $x \in P_i$  and  $x < y$ , then  $\text{rk}(x) + 1 = \text{rk}(y)$ ,
- (2) If  $x < y$  then  $\text{rk}(x) < \text{rk}(y)$

Then, define the  $i$ th rank of  $P$  to be  $P_i = \{x \in P \mid \text{rk}(x) = i\}$ . Additionally, if for all  $x \in P$ ,  $0 \leq \text{rk}(x) \leq n$ , and there exists  $x, y$  with  $\text{rk}(x) = 0, \text{rk}(y) = n$ , we say  $P$  is a graded poset of rank  $n$ .

Denote the category of finite graded posets by  $\mathcal{P}$ , and let  $P, Q$  be finite graded posets of rank  $n$ . A map  $f : P \rightarrow Q$  is a *morphism* from  $P$  to  $Q$  if it is rank-preserving and order preserving, i.e. for all  $x, y \in P$ ,  $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$  and  $\text{rk}(x) = \text{rk}(f(x))$ . We say that  $f$  is *injective/surjective/bijective* if it is an *injection/surjection/bijection* from  $P$  to  $Q$  as sets.

*Remark 2.1.* Note that we do not require the implication  $f(x) \leq_Q f(y) \Rightarrow x \leq_P y$  in order for  $f$  to be a morphism. In particular this means a bijective morphism  $f$  need not be an isomorphism, since it will not necessarily have a two-sided inverse.

Write  $p_i = |P_i|$ . If we have

$$p_0 \leq p_1 \leq \dots \leq p_k \geq p_{k+1} \geq \dots \geq p_n$$

for some  $0 \leq k \leq n$ , then  $P$  is *rank-unimodal*, and if  $p_i = p_{n-i}$  for all  $1 \leq i \leq n$ , then  $P$  is *rank-symmetric*. An *antichain* in  $P$  is a set of elements in  $P$  that are pairwise incomparable. If no antichain in  $P$  is larger than the largest rank of  $P$ , then  $P$  is *Sperner*. More generally,  $P$  is *k-Sperner* if no union of  $k$  antichains in  $P$  is larger than the union of the largest  $k$  ranks of  $P$ , and  $P$  is *strongly Sperner* if it is  $k$ -Sperner for all  $1 \leq k \leq n$ . We then make the following definition.

**Definition 2.2.**  $P$  is *Peck* if  $P$  is rank-symmetric, rank-unimodal, and strongly Sperner.

Let  $V(P)$  and  $V(P_i)$  be the complex vector spaces with bases  $\{x : x \in P\}$  and  $\{x : x \in P_i\}$  respectively. Note that we will frequently abuse notation and write  $P$  and  $P_i$  for  $V(P)$  and  $V(P_i)$  when our meaning is clear. In determining whether  $P$  is Peck, it is often useful to consider certain linear transformations on  $V(P)$ .

**Definition 2.3.** A map  $U : V(P) \rightarrow V(P)$  is an *order-raising operator* if  $U(V(P_n)) = 0$  and for all  $0 \leq i \leq n-1$ ,  $x \in P_i$  we have

$$U(x) = \sum_{y \succ x} c_{x,y} y$$

for some constants  $c_{x,y} \in \mathbb{C}$ . We say that  $U$  is the *Lefschetz map* if all  $c_{x,y}$  on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

**Lemma 2.4.**  $P$  is Peck if and only if there exists an order-raising operator  $U$  such that for all  $0 \leq i < \frac{n}{2}$ , the map  $U^{n-2i} : V(P_i) \rightarrow V(P_{n-i})$  is an isomorphism.

**Definition 2.5.** If the Lefschetz map satisfies the condition for  $U$  in Lemma 2.4, then  $P$  is *unitary Peck*.

Let  $G \subseteq \text{Aut}(P)$ , and define the quotient poset  $P/G$  to be the poset whose elements are the orbits of  $G$ , with the relation  $\mathcal{O} \leq \mathcal{O}'$  if there exist  $x \in \mathcal{O}$ ,  $x' \in \mathcal{O}'$  such that  $x \leq x'$ . We will use of the following result later in the paper.

**Theorem 2.6.** *If  $P$  is unitary Peck and  $G \subseteq \text{Aut}(P)$ , then  $P/G$  is Peck.*

### 3. THE FUNCTOR OF EDGES

In Section 3.1 we show that  $\mathcal{E}^r$  as described in Definition 1.1 is a well-defined functor and prove some useful properties of  $\mathcal{E}^r$ . Section 3.2 is devoted to the proof of Theorem 1.5.

**3.1. Definition and Basic Properties.** First we prove Proposition 3.5, which states that  $\mathcal{E}^r$  is a well-defined functor. We break the proof of Proposition 3.5 into Lemmas 3.2, 3.3, and 3.4 which follow below. After showing that  $\mathcal{E}^r$  is well-defined we then define a natural  $G$  action on  $\mathcal{E}^r(P)$  and define a surjection  $\mathcal{E}^r(P)/G \rightarrow \mathcal{E}^r(P/G)$  that will be important for the proof of Theorem 1.5.

**Notation 3.1.** When the poset  $P$  and the value  $r$  are clear, we shall simply use  $\leq_\mathcal{E}$ ,  $\leq_\mathcal{E}$  to refer to  $\leq_{\mathcal{E}^r(P)}$ ,  $\leq_{\mathcal{E}^r(P)}$ . Similarly, in Subsection 3.2, we define an object  $\mathcal{H}^r(B_n)$ , and shall use  $\leq_\mathcal{H}$ ,  $\leq_\mathcal{H}$  in place of  $\leq_{\mathcal{H}^r(P)}$ ,  $\leq_{\mathcal{H}^r(P)}$ .

**Lemma 3.2.** *The relation  $\leq_\mathcal{E}$  defines a partial order on  $\mathcal{E}^r(P)$ .*

*Proof.* We have that  $(x, y) \leq_\mathcal{E} (x, y)$  and that  $\leq_\mathcal{E}$  is transitive by definition. It remains to be shown that  $\leq_\mathcal{E}$  is antisymmetric. Suppose  $(x, y) \leq_\mathcal{E} (x', y')$  and  $(x', y') \leq_\mathcal{E} (x, y)$ . Then  $x \leq_P x' \leq_P x$  and  $y \leq_P y' \leq_P y$ , so  $x = x'$  and  $y = y'$  by antisymmetry of  $\leq_P$ , hence  $(x, y) = (x', y')$ .  $\square$

**Lemma 3.3.** *For  $P$  a graded poset, the object  $\mathcal{E}^r(P)$  is a graded poset.*

*Proof.* To show  $\mathcal{E}^r(P)$  is graded, we must show  $(x, y) <_\mathcal{E} (x', y') \implies \text{rk}(x, y) < \text{rk}(x', y')$  and  $(x, y) <_\mathcal{E} (x', y') \implies \text{rk}(x, y) + 1 = \text{rk}(x', y')$ . Both of these facts follow immediately from the definition of  $\leq_\mathcal{E}$  and the definition  $\text{rk}_\mathcal{E}(x, y) = \text{rk}_P(x)$ .  $\square$

**Lemma 3.4.** *If  $f: P \rightarrow Q$  is morphism of finite ordered posets, then  $\mathcal{E}^r(f): \mathcal{E}^r(P) \rightarrow \mathcal{E}^r(Q)$  is a morphism of finite ordered posets.*

*Proof.* First,  $\mathcal{E}^r(f)$  is rank-preserving, since for all  $(x, y) \in \mathcal{E}^r(P)$  we have

$$\text{rk}_\mathcal{E}(x, y) = \text{rk}_P(x) = \text{rk}_P(f(x)) = \text{rk}_\mathcal{E}(\mathcal{E}^r(f)(x, y)).$$

Suppose  $(x, y) \leq_\mathcal{E} (x', y')$ . Then  $x \leq_P x'$ ,  $y \leq_P y'$ , and since  $f$  is order-preserving, it follows that  $f(x) \leq_P f(x')$ ,  $f(y) \leq_P f(y')$ . Hence  $\mathcal{E}^r(f)(x, y) \leq_\mathcal{E} \mathcal{E}^r(f)(x', y')$ . Thus  $\mathcal{E}^r(f)$  is order preserving and hence a morphism of finite ordered posets.  $\square$

**Proposition 3.5.** *The map  $\mathcal{E}^r(P)$  is an endofunctor on the category of finite graded posets  $\mathcal{P}$ .*

*Proof.* By Lemmas 3.2, 3.3, and 3.4 we have that  $\mathcal{E}^r$  takes objects in  $\mathcal{P}$  to objects in  $\mathcal{P}$  and morphisms of  $\mathcal{P}$  to morphisms of  $\mathcal{P}$ . Furthermore it is clear that  $\mathcal{E}^r(\text{id}_P) = \text{id}_{\mathcal{E}^r(P)}$ , so it remains to be shown that for finite graded posets  $P, Q$ , and  $R$  with morphisms  $f: P \rightarrow Q$  and  $g: Q \rightarrow R$  that  $\mathcal{E}^r(g \circ f) = \mathcal{E}^r(g) \circ \mathcal{E}^r(f)$ . This is clear, however, as for all  $(x, y) \in \mathcal{E}^r(P)$  we have  $\mathcal{E}^r(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}^r(g) \circ \mathcal{E}^r(f))(x, y)$ .  $\square$

Given a group action of  $G$  on  $P$ , we can easily define a natural group action of  $G$  on  $\mathcal{E}^r(P)$  using Proposition 3.5. For all  $g \in G$  we have that multiplication by  $g$  is an automorphism of  $P$ , so it follows that  $\mathcal{E}^r(g)$  is an automorphism of  $\mathcal{E}^r(P)$ . Since  $\mathcal{E}^r$  is a functor this gives a well-defined group action.

**Notation 3.6.** Define a group action of  $G$  on  $\mathcal{E}^r(P)$  by defining  $g \cdot (x, y) = \mathcal{E}^r(g)(x, y) = (g \cdot x, g \cdot y)$ .

I don't think the details of the proof need to be included here, but I've left them here commented out in case we change our minds. Aaron: I agree it is fine to leave them out.

Using Notation 3.6, we have a well-defined quotient poset  $\mathcal{E}^r(P)/G$ . It is then natural to ask whether the operation of quotienting out by  $G$  commutes with  $\mathcal{E}^r$ , that is, whether  $\mathcal{E}^r(P/G) \cong \mathcal{E}^r(P)/G$ . Unfortunately the two posets are rarely isomorphic, but there is always a surjection  $\mathcal{E}^r(P)/G \rightarrow \mathcal{E}^r(P/G)$ , and this surjection is also an injection precisely when the  $G$ -action on  $P$  is CCT.

**Proposition 3.7.** *The map  $q: \mathcal{E}^r(P)/G \rightarrow \mathcal{E}^r(P/G)$  defined by  $q(G(x, y)) = (Gx, Gy)$  is a surjective morphism.*

*Proof.* First we show that  $q$  is a surjective morphism. Note that  $q$  is well defined because if  $(x', y') = (g \cdot x, g \cdot y) = (g \cdot x, g \cdot y)$  for some  $g \in G$ , then  $x' \in Gx$  and  $y' \in Gy$ . Clearly  $q$  is rank-preserving and surjective, so it suffices to show that  $q$  is order-preserving. Suppose that  $G(x, y) \leq G(w, z)$ . Then there exist some  $(x_0, y_0) \in G(x, y)$ ,  $(w_0, z_0) \in G(w, z)$  such that  $x_0 \leq w_0$  and  $y_0 \leq z_0$ . We then have that  $(Gx, Gy) \leq (Gw, Gz)$  by definition, hence  $q$  is order-preserving.  $\square$

**Lemma 3.8.** *Let  $G$  be a group acting on a graded poset  $P$ . The following are equivalent:*

- (1) *The action of  $G$  on  $P$  is CCT.*
- (2) *Whenever  $x \leq y, x \leq z$ , and  $y \in Gz$ , there exists some  $g \in \text{Stab}(x)$  with  $gx = z$ .*
- (3) *The map  $q: \mathcal{E}^1(P)/G \rightarrow \mathcal{E}^1(P/G)$  defined by  $q(G(x, y)) = (Gx, Gy)$  is an bijective morphism (but not necessarily an isomorphism).*

*Proof.* First, we shall show (1)  $\Leftrightarrow$  (3). Observe that  $q$  is a bijection exactly when there do not exist distinct orbits  $G(x, y) \neq G(x', y')$  with  $x' \in Gx, y' \in Gy$ . Fix  $(x, y), (x', y') \in \mathcal{E}^1(P)$  such that  $x' \in Gx$  and  $y' \in Gy$ . Pick a  $g \in G$  such that  $g \cdot y' = y$ . Then  $(g \cdot x', y) \in G(x', y')$ , so  $G(x, y) = G(x', y')$  if and only if there exists some  $g' \in G$  such that  $g' \cdot x = g \cdot x'$  and  $g' \cdot y = y$ . Hence  $q$  is a bijection if and only if the  $G$  action is CCT.

Next, we shall show (2)  $\Leftrightarrow$  (3). It is quite analogous to the proof of (1)  $\Leftrightarrow$  (3), but we shall include the argument for completeness. First,  $q$  is a bijection if and only if there do not exist distinct orbits  $G(x, y) \neq G(x', y')$  with  $x' \in Gx, y' \in Gy$ . Fix  $(x, y), (x', y') \in \mathcal{E}^1(P)$  such that  $x' \in Gx, y' \in Gy$ . Choose  $g \in G$  with  $gx' = x$ . Then,  $(x', g \cdot y) \in G(x', y')$ . Since  $gy' \in Gy$ , and  $q$  is a bijection, the statement  $(x', g \cdot y) \in G(x', y')$  is equivalent to  $G(x', y') = G(x, y)$ . This, in turn is equivalent to the existence of  $g' \in G$  with  $g'x = x, g'y = gy'$ . So,  $g' \in \text{Stab}(x), g'y = (gy')$ , which is what was claimed in property (2).  $\square$

**Remark 3.9.** While  $q$  is a bijection if and only if the action of  $G$  on  $P$  is CCT, it is *not* true that if the action of  $G$  on  $P$  is CCT, then  $q$  is an isomorphism. For example, take  $G = D_{20} \subset S_{10}$  acting by reflections and rotations on  $[10]$  and hence acting on  $B_{10}$ . We shall see in Lemma 10.2 that this action is CCT. However, consider  $x = \{2, 4\}, y = \{1, 2, 4\}, a = \{2, 4, 7\}, b = \{2, 4, 6, 7\}$ . Then we may observe  $(x, y), (a, b) \in \mathcal{E}^1(B_{10})$  and  $Gx < Ga, Gy < Gb$ , so  $(Gx, Gy) <_{\mathcal{E}} (Ga, Gb)$ . However, it is not true that  $G(x, y) <_{\mathcal{E}} G(a, b)$ .

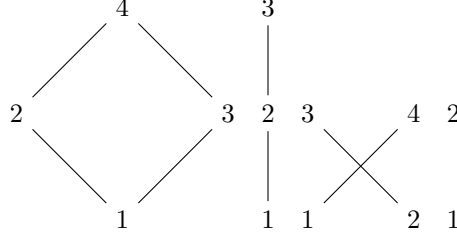
**3.2. Proof of Theorem 1.5.** *In this section we prove Theorem 1.5, which we recall here:*

**Theorem 1.5.** *If a group action of  $G$  on  $B_n$  is CCT, then  $\mathcal{E}^1(B_n/G)$  is Peck.*

*The proof is largely motivated by the following Lemma.*

**Lemma 3.10.** *If  $f: P \rightarrow Q$  is a bijection (but not necessarily an isomorphism) and  $P$  is Peck then  $Q$  is Peck.*

*Proof.* Let  $\text{rk}(P) = \text{rk}(Q) = n$ . Since  $P$  is Peck there exists an order-raising operator  $U$  such that  $U^{n-2i}: P_i \rightarrow P_{n-i}$  is an isomorphism. Since  $f$  is a poset morphism, it follows that the map  $f \circ U \circ f^{-1}$  is an order-raising operator on  $Q$ . We then have that  $f \circ U^{n-2i} \circ f^{-1} = (f \circ U \circ f^{-1})^{n-2i}: Q_i \rightarrow Q_{n-i}$  is an isomorphism since  $U^{n-2i}: P_i \rightarrow P_{n-i}$  is an isomorphism and  $f$  is a bijection.  $\square$



By Lemma 3.10 and Proposition 3.7, in order to prove Theorem 1.5 it suffices to prove that  $\mathcal{E}^r(B_n)/G$  is Peck. One way to do this is to prove that  $\mathcal{E}^r(B_n)$  is unitary Peck and then apply Theorem 2.6. While this is the most direct path, we unfortunately do not have a proof that  $\mathcal{E}^r(B_n)$  is unitary Peck for  $r > 1$ . We prove that  $\mathcal{E}^1(B_n)$  is unitary Peck in Section 12, but unfortunately the proof is messy and computational.

Fortunately there is a simpler – albeit less direct – route. In order to avoid showing that  $\mathcal{E}^r(B_n)$  is unitary Peck, for all  $r, n$ , we define a poset  $\mathcal{H}^r(B_n)$  such that  $\mathcal{H}^r(B_n)$  is easily seen to be unitary Peck. Furthermore, there is always a bijective morphism  $f: \mathcal{H}^r(B_n)/G \rightarrow \mathcal{E}^r(B_n)/G$ . By the above discussion, Theorem 1.5 readily follows.

**Definition 3.11.** For  $P$  a graded poset, define the graded poset  $\mathcal{H}^r(P)$  as follows. Let the elements  $(x, y) \in \mathcal{H}^r(P)$  to be pairs  $(x, y) \in P \times P$  such that  $x < y$  and  $rk_P(x) + r = rk_P(y)$ . Define  $(x, y) \leq_{\mathcal{H}} (x', y')$  if  $x \leq x', y \leq y'$  and  $x' \not\leq y$ . Then, define  $\leq_{\mathcal{H}}$  to be the transitive closure of  $\leq_{\mathcal{H}}$ . and define  $rk_{\mathcal{H}}(x, y) = rk_P(x)$ .

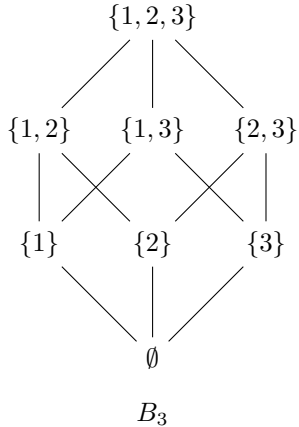
#### 4. THE POSET $\mathcal{H}^r(P)$

*Remark 4.1.* While  $\mathcal{E}^r: \mathcal{P} \rightarrow \mathcal{P}$  is a functor,  $\mathcal{H}^r$  is not a functor. In particular, we have not specified how  $\mathcal{H}^r$  acts on morphisms. If, for  $f: P \rightarrow Q$ , we attempted to define  $\mathcal{H}^r(f): \mathcal{P} \rightarrow \mathcal{Q}$ , we would quickly run into trouble. For example, suppose we took  $f$  mapping a diamond  $P$  (leftmost below) to a chain of length three  $Q$  (second leftmost). Then,  $\mathcal{H}^1(P)$  (second rightmost) has four points, with two pairs of edges, whereas  $\mathcal{H}^1(Q)$  (rightmost) has two disconnected points. It is clear that there can be no order preserving mapping between these two objects.

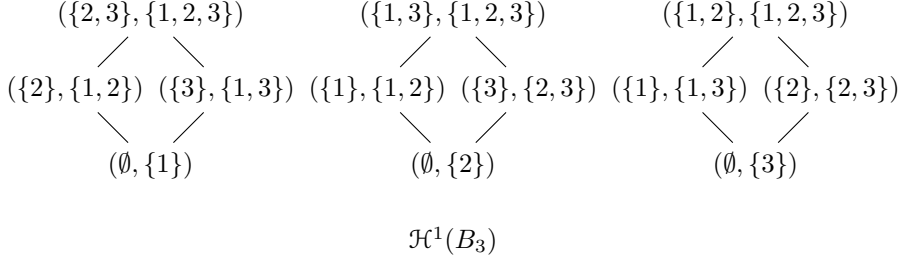
However, it is still true that  $\mathcal{H}^r(P)$  can be viewed as a graded poset with  $rk_{\mathcal{H}^r(P)}(x, y) = rk_P(x)$ .

Put all pictures in figures, and make them look nicer

**Example 4.2.** Below is an example of the poset  $\mathcal{H}^1(B_3)$ .







**Lemma 4.3.** *For  $P$  a graded poset, the object  $\mathcal{H}^r(P)$ , as defined in Definition 3.11 is a graded poset.*

*Proof.* This follows immediately from Remark 4.4 and the fact that  $\mathcal{E}^r(P)$  is graded.  $\square$

*Remark 4.4.* Note that  $(x, y) \leq_{\mathcal{H}} (a, b) \Rightarrow (x, y) \leq_{\mathcal{E}} (a, b)$ .

Similarly to  $\mathcal{E}^r(P)$ , the action of  $G$  on  $P$  induces an order-preserving action of  $G$  on  $\mathcal{H}^r(P)$ .

**Lemma 4.5.** *The function defined by  $g \cdot (x, y) = (gx, gy)$  for all  $g \in G$ ,  $(x, y) \in \mathcal{H}^r(P)$  is a well-defined group action of  $G$  on  $\mathcal{H}^r(P)$ .*

*Proof.* Let  $g \in G$ . Since  $\leq_{\mathcal{H}}$  is the transitive closure of  $\leq_{\mathcal{E}}$  it suffices to show that for all  $(x, y), (x', y') \in \mathcal{H}^r(P)$  we have  $(x, y) \leq_{\mathcal{H}} (x', y') \Leftrightarrow g(x, y) \leq_{\mathcal{H}} g(x', y')$ . Since  $g$  is an automorphism of  $P$ , we have  $x \leq_P x' \Leftrightarrow gx \leq_P gx'$ ,  $y \leq_P y' \Leftrightarrow gy \leq_P gy'$ , and  $y \not\leq x' \Leftrightarrow gy \not\leq gx'$ , so the result follows from the definition of  $\leq_{\mathcal{H}}$ .  $\square$

**Proposition 4.6.**  *$\mathcal{H}^r(B_n)$  is isomorphic to  $\binom{n}{r}$  disjoint copies of  $B_{n-r}$ .*

*Proof.* Suppose we have  $(x, y), (x', y') \in \mathcal{H}^r(B_n)$  with  $(x, y) \leq (x', y')$ . Let  $j \in [n]$  such that  $y' = y \cup \{j\}$ , and let  $i \in [n]$  such that  $x' = x \cup \{i\}$ . If  $i \neq j$ , then  $x' \leq y$ , contradicting the assumption that  $(x, y) \leq_{\mathcal{H}} (x', y')$ . Thus  $x' = x \cup \{i\}$  and  $y' = y \cup \{i\}$  for some  $i \in [n]$ .

Conversely we can easily check that if  $i \notin y$ , then  $(x, y) \leq_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$ . It follows that for all subsets  $w \subset [n]$  such that  $|w| = r$ , there is an isomorphism from the subposet of elements  $\{(x, y) : y \setminus x = w\}$  to  $B_{n-r}$  defined by  $(x, y) \mapsto (x \setminus w, y \setminus w)$ . Furthermore if  $y \setminus x \neq y' \setminus x'$ , then  $(x, y)$  and  $(x', y')$  are incomparable, so these subposets indexed by  $w$  are pairwise disjoint, and  $\mathcal{H}^r(B_n)$  is isomorphic to  $\binom{n}{r}$  copies of  $B_{n-r}$ .  $\square$

The primary advantage of Proposition 4.6 is it allows us to easily see that  $\mathcal{H}^r(B_n)$  is unitary Peck, which then implies that  $\mathcal{H}^r(B_n)/G$  is Peck for any subgroup  $G \subseteq S_n$ .

**Corollary 4.7.**  *$\mathcal{H}^r(B_n)$  is unitary Peck for all  $n \geq r$ .*

*Proof.* This follows immediately from Proposition 4.6 and the fact that  $B_{n-r}$  is unitary Peck, as shown, for instance in [Sta84, Theorem 2a] because  $B_k = (B_1)^k$  and  $B_1$  is clearly unitary Peck.  $\square$

**Corollary 4.8.**  *$\mathcal{H}^r(B_n)/G$  is Peck for any subgroup  $G \subseteq S_n$ .*

*Proof.* This follows from Corollary 4.7 and Theorem 2.6  $\square$

**Lemma 4.9.** *The map  $f : \mathcal{H}^r(P) \rightarrow \mathcal{E}^r(P), G(x, y) \mapsto G(x, y)$  is a bijection (but never an isomorphism for  $n - r > 1$ ).*

*Proof.* The elements of  $\mathcal{H}^r(P), \mathcal{E}^r(P)$  are precisely the same by definition. Therefore, as long as  $f$  is a morphism, it is automatically a bijection. Since  $f$  is clearly rank preserving, to show  $f$  is a morphism, it suffices to show  $f$  is order preserving. This is immediate from Remark 4.4  $\square$

**Corollary 4.10.**  *$\mathcal{E}^r(B_n)/G$  is Peck for any subgroup  $G \subseteq S_n$ .*



*Proof.* By Corollary 4.8,  $\mathcal{H}^r(B_n)/G$  is Peck. By Lemma 4.9, the map  $f : \mathcal{H}^r(P) \rightarrow \mathcal{E}^r(P), G(x, y) \mapsto G(x, y)$  is a bijection. Then, by Lemma 3.10, it follows that  $\mathcal{E}^r(B_n)/G$  is Peck.  $\square$

**4.1. A Generalization of  $\mathcal{H}^r, \mathcal{E}^r$ .** Although for the most part, we shall investigate  $\mathcal{E}^1(P/G), \mathcal{E}^1(P)/G$ , there is a natural further generalization of  $\mathcal{H}^r$ , to what we shall call  $\mathcal{H}^{\vec{r}}$ , where  $\vec{r} = r_1, r_2, \dots, r_k$  is an integer valued sequence. The results holding for these generalizations are analogous to those developed above. The purpose of developing these generalizations notion will be to give a more general application to Polya theory than could be given with  $\mathcal{H}^r$ , for  $r \in \mathbb{Z}$ .

**Definition 4.11.** Let  $\vec{r} = r_1, \dots, r_k$ . For a graded poset  $P$ , define the graded poset  $\mathcal{H}^{\vec{r}_1, \dots, \vec{r}_k}(P)$ , also notated  $\mathcal{H}^{\vec{r}}(P)$ , whose elements are formal symbols  $(x_1, x_2, \dots, x_{k+1})$  such that  $rk(x_i) + r_i = rk(x_{i+1})$ , for all  $i \in [k]$ . Say  $(x_1, x_2, \dots, x_{k+1}) \leq_{\mathcal{H}^{\vec{r}}} (y_1, y_2, \dots, y_{k+1})$  if  $x_i \leq_P y_i, y_i \not\leq_P x_{i+1}$  for all  $i \in [k+1]$ . Then, define a relation  $\leq_{\mathcal{H}^{\vec{r}}}$  on  $\mathcal{H}^{\vec{r}_1, \dots, \vec{r}_k}(P)$ , to be the transitive closure of  $\leq_{\mathcal{H}^{\vec{r}}}$ . Finally, define  $rk_{\mathcal{H}^{\vec{r}}(P)}(x_1, \dots, x_{k+1}) = rk_P(x_1)$

**Definition 4.12.** Let  $\vec{r} = r_1, \dots, r_k$ , with  $r_i \in \mathbb{N} \forall i \in [k]$ . Let  $\mathcal{P}$  be the category of graded posets. Define the functor  $\mathcal{E}^{\vec{r}} : \mathcal{P} \rightarrow \mathcal{P}$ , also notated  $\mathcal{E}^{r_1, \dots, r_k}$ . For a graded poset  $P$ , define the graded poset  $\mathcal{E}^{r_1, \dots, r_k}(P)$ , also notated whose elements are formal symbols  $(x_1, x_2, \dots, x_{k+1})$  such that  $rk(x_i) + r_i = rk(x_{i+1})$ , for all  $i \in [k]$ . Say  $(x_1, x_2, \dots, x_{k+1}) \leq_{\mathcal{E}^{\vec{r}}} (y_1, y_2, \dots, y_{k+1})$  if  $x_i \leq_P y_i$  for all  $i \in [k+1]$ . Then, define a relation  $\leq_{\mathcal{E}^{\vec{r}}}$  on  $\mathcal{E}^{r_1, \dots, r_k}(P)$ , to be the transitive closure of  $\leq_{\mathcal{E}^{\vec{r}}}$ . Finally, define  $rk_{\mathcal{E}^{\vec{r}}(P)}(x_1, \dots, x_{k+1}) = rk_P(x_1)$

*Remark 4.13.* Both  $\mathcal{E}^{\vec{r}}, \mathcal{H}^{\vec{r}}$  are functors, for the same reasons that  $\mathcal{E}^r, \mathcal{H}^r$  are functors. This generalization is essentially taking the nerves of the poset  $P$ . See [EB05] for some related constructions, although their constructions are different in many crucial ways.

*In the next Lemmas, we cite analogous results which hold for  $\mathcal{H}^{\vec{r}}$ . The proofs are almost identical to those for  $\mathcal{H}^r$ .*

**Lemma 4.14.** Given a group action  $\phi : G \times P \rightarrow P$  there are well defined group actions  $\phi_{\mathcal{E}} : G \times \mathcal{E}^{\vec{r}}(P) \rightarrow \mathcal{E}^{\vec{r}}(P), \phi_{\mathcal{H}} : G \times \mathcal{H}^{\vec{r}}(P) \rightarrow \mathcal{H}^{\vec{r}}(P)$ , both given by  $g \cdot (x_1, \dots, x_{k+1}) = (g \cdot x_1, \dots, g \cdot x_{k+1})$ .

*Proof.* The proof is analogous to Lemma 4.5, Lemma 4.5.  $\square$

**Lemma 4.15.** The poset  $\mathcal{H}^{r_1, \dots, r_k}(B_n)$  is isomorphic to the multinomial coefficient  $\binom{n}{r_1, r_2, \dots, r_k}$  disjoint copies of  $B_{n - \sum_{i=1}^k r_i}$ . Consequently,  $\mathcal{H}^{\vec{r}}(B_n)$  is unitary Peck and  $\mathcal{H}^{\vec{r}}(B_n)/G$  is Peck.

*Proof.* Once again, the proof of the above three statements is analogous to those of 4.6, Corollary 4.7, and Corollary 4.8. The reason there are multinomial coefficients here instead of binomial coefficients, is that an element  $(x_1, x_2, \dots, x_{k+1})$  lies in the copy of  $B_{n - \sum_{i=1}^k r_i}$  determined by the ordered tuple of subsets  $(x_2 \setminus x_1, x_3 \setminus x_2, \dots, x_{k+1} \setminus x_k)$ . The first consists of  $r_1$  elements, the next of  $r_2$  elements, up through the last which consists of  $r_k$  elements. Since the total number of ways to choose  $r_1, \dots, r_k$  in  $k$  distinct groups is  $\binom{n}{r_1, r_2, \dots, r_k}$ , there are exactly this many disjoint copies of  $B_{n - \sum_{i=1}^k r_i}$ .  $\square$

## 5. COMMON COVER TRANSITIVE ACTIONS

*In this section, we develop the theory of CCT actions  $\phi$  where  $G$  is a group,  $P$  is a poset, and  $\phi : G \times P \rightarrow P$  is an action. Recall Definition 1.4, that  $\phi$  is CCT if whenever  $x, y, z \in P, x \leq y, y \leq z, x \in Gy$  then there exists  $g \in \text{Stab}(z)$  with  $gx = y$ . For CCT actions  $\phi : G \times P \rightarrow P$ , we shall show that,  $\mathcal{E}^1(P/G)$  is Peck. Additionally, we shall show the CCT property is closed under semidirect products, in the appropriate sense. It is obvious that the action of  $S_k$  on  $B_k$  is CCT, and we shall also see in Lemma 10.2, that the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT*

groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

*Remark 5.1.* Before continuing with the description of cover transitive, it is worth noting that the notion of cover transitivity generalizes to  $\mathcal{E}^r$  and even  $\mathcal{E}^{\vec{r}}$ . Letting  $\vec{r} = r_1, \dots, r_k$ , one would call a group action  $\phi : G \times P \rightarrow P$ ,  $\vec{r}$  common cover transitive if for any chains  $(x_0, \dots, x_k), (y_0, \dots, y_k)$ , such that there exists  $g_i \in G$  with  $x_i = g_i y_i$ , for all  $i, 0 \leq i \leq k$ , then there exists a single  $g \in G$  with  $x_i = g y_i$ . It is trivial to see that 1 common cover transitive agrees with our definition of common cover transitive. Furthermore, it is also simple to see that an action is  $\vec{r}$  common cover transitive if and only if the natural surjection  $q : \mathcal{E}^{\vec{r}}(P)/G \rightarrow \mathcal{E}^{\vec{r}}(P/G)$  is a bijection. Additionally, very many of the properties developed in this section will also hold for this generalized notion of  $\vec{r}$  common cover transitive.

**Example 5.2.** Two rather trivial examples of CCT actions are  $\phi : S_n \times B_n \rightarrow B_n, \psi : G \times B_n \rightarrow B_n$  where  $G$  is arbitrary,  $\phi$  is the action induced by  $S_n$  permuting the elements of  $[n]$ , and  $\psi$  is the trivial action. In the former case,  $\mathcal{E}^1(B_n/S_l)$  is simply a chain with  $n-1$  points, and so is  $\mathcal{E}^1(B_n)/S_l$ , since all  $(x, y)$  are identified under the  $S_l$  action. In the latter case, since  $G$  acts trivially by  $\phi$ , both  $\mathcal{E}^1(B_n/G) \cong \mathcal{E}^1(B_n)$  and  $\mathcal{E}^1(B_n)/G \cong \mathcal{E}^1(B_n)$ . So again,  $\psi$  is CCT.

**Theorem 5.3.** *If an action  $\phi : G \times P \rightarrow P$  is CCT, and  $\mathcal{E}^1(P)$  is Peck, then  $\mathcal{E}^1(P/G)$  is Peck.*

*Proof.* By Proposition 3.7, there is a surjection  $q : \mathcal{E}^1(P)/G \rightarrow \mathcal{E}^1(P/G)$ . Additionally, by Lemma 3.7, if  $\phi$  is CCT,  $q$  is a bijection. Finally, by Lemma 3.10, since  $q : \mathcal{E}^1(P)/G \rightarrow \mathcal{E}^1(P/G)$  is a bijection and  $\mathcal{E}^1(P)/G$  is Peck, it follows that  $\mathcal{E}^1(P/G)$  is Peck.  $\square$

**5.1. Preservation Under Semidirect Products.** Recall Proposition 1.7, as stated in the introduction. This Proposition essentially says the CCT property is preserved under semidirect products. We shall use this to obtain simple proofs that CCT actions are also preserved under direct products and wreath products.

**Proposition 1.7.** *Let  $G \subseteq \text{Aut}(P)$ ,  $H \triangleleft G$ ,  $K \subset G$  such that  $G = H \rtimes K$ . Suppose that the action of  $H$  on  $P$  is CCT and the action of  $K$  on  $P/H$  is CCT. Then the action of  $G$  on  $P$  is CCT.*

*Proof.*

$\square$

**Proposition 5.4.** *For  $\phi : G \times P \rightarrow P, \psi : H \times Q \rightarrow Q$  two CCT actions, then the direct product  $\phi \times \psi : (G \times H) \times (P \times Q) \rightarrow (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$  is also CCT.*

*Proof.* First, by definition of CCT, if either  $G$  or  $H$  acts trivially, it is clear the action of  $G \times H$  is CCT. To see this, suppose  $G$  acts trivially, the case of  $H$  acting trivially is similar. Then, if  $(x, y) \leq (a, b), (r, s) \leq (a, b), (x, y) \in (G \times H)(r, s)$ , then  $x \in Gr$  so  $x = r$ . There are now two cases, either  $b = y = s$  or  $x = r = a$ . First, if  $b = y = s$ , then  $(e, e) \in \text{Stab}(x, y)$  and  $(e, e)(x, y) = (r, s)$ . Second, if  $x = r = a$ , and then  $y \leq b, s \leq b, y \in Hs$ , and so by cover transitivity of  $G$ , there exists  $g \in G$  so that  $(e, g)(x, y) = (r, s)$  and  $(e, g) \in \text{Stab}(a, b)$ .

Next, Observe that  $G \times H$  can be viewed as the semidirect product of  $(G \times \{e\}) \rtimes (\{e\} \times H)$ . Since the action of  $G$  on  $P$  is CCT, by the above paragraph, the action of  $G \times \{e\}$  on  $P \times Q$  is CCT. Also, since the action of  $H$  on  $Q$  is CCT, it follows that the action of  $\{e\} \times H$  on  $(P/G) \times Q$  is CCT. Therefore, the action of  $(G \times \{e\}) \rtimes (\{e\} \times H)$  satisfies the conditions of Proposition 1.7 and so the action of  $G \times H$  is CCT.  $\square$

Next, we use Proposition 1.7 to prove in Proposition 5.9 that the CCT property is preserved under wreath products with the symmetric group. First, we need some definitions of wreath product.

David, can you fill this in please?

**Definition 5.5.** For  $G, H$  groups, with  $H \subset S_l$ , the *wreath product*, notated  $G \wr H$ , is the group whose elements are pairs  $(g, h) \in G^l \times H$  with multiplication defined by

$$((g'_1, \dots, g'_l), h') \cdot ((g_1, \dots, g_l), h) = ((g'_{h^{-1}(1)}g_1, \dots, g'_{h^{-1}(l)}g_l), hh')$$

where  $h \in H$  acts on  $[l]$  by the restricting of the action of  $S_l$  on  $[l]$  by all permutations to  $H$ .

*In other words,  $G \wr H$  can be viewed as a certain semidirect product of  $G^l \rtimes H$ .*

**Notation 5.6.** For any group  $G$  with a given action  $\psi : G \times P \rightarrow P$ , we obtain an induced action of  $G \wr H$ ,  $\phi : G \wr H \times P^l \rightarrow P^l$  defined by

$$((g_1, \dots, g_l), h)(a_1, \dots, a_l) = (g_{h^{-1}(1)} \cdot a_{h^{-1}(1)}, \dots, g_{h^{-1}(l)} \cdot a_{h^{-1}(l)}).$$

*Remark 5.7.* Heuristically, one may think of the above action as obtained by first having  $G$  act separately on the  $l$  distinct copies of  $P$ , and then letting  $H$  act by permuting the copies.

**Lemma 5.8.** For  $P$  a poset, the action  $\phi : S_l \times P^l \rightarrow P^l$ ,  $\sigma \cdot (x_1, \dots, x_l) = (x_{\sigma(1)}, \dots, x_{\sigma(l)})$  is CCT.

*Proof.* For  $a \in P^l$  notate  $a = (a_1, \dots, a_l)$ . Suppose  $x, y, z \in P^l$  with  $x \leq z, y \leq z$ , and  $x \in S_l y$ . This means there is a unique  $i$  such that  $x_i \leq z_i, x_k = z_k$  for  $k \neq i$ . Additionally, there is a unique  $j$  for which  $y_j \leq z_j, y_k = z_k$  for  $k \neq j$ . Since  $x \in S_l y$ , we obtain the equality of multisets  $\{x_1, \dots, x_l\} = \{y_1, \dots, y_l\}$ . But for  $k \neq i, j, x_k = z_k = y_k$ , we also obtain equality of sets  $\{x_i, x_j\} = \{y_i, y_j\}$ . Since  $rk(y_j) < rk(x_j)$ , we obtain  $y_j = x_i, y_i = x_j$ . Then, taking the transposition  $\sigma = (ij) \in S_l$ , it follows that  $\sigma \in \text{Stab}(z)$  and  $\sigma \cdot x = y$ .  $\square$

**Proposition 5.9.** If  $\psi : G \times P \rightarrow P$  is CCT, then  $\phi : G \wr S_l \times P^l \rightarrow P^l$  where  $\phi$  is the induced action defined in Notation 5.6 is also CCT.

*Proof.* First, the wreath product  $G \wr S_l$  can be viewed as the semidirect product  $G^l \rtimes S_l$ . Observe that since the action of  $G$  on  $P$  is CCT, using Proposition 5.4, we obtain the action of  $G^l$  on  $P^l$  is CCT. Explicitly, this action is  $(g_1, \dots, g_l)(x_1, \dots, x_l) = (g_1 \cdot x_1, \dots, g_l \cdot x_l)$ . Additionally, since  $P^l/G^l \cong (P/G)^l$ . So, for  $\sigma \in S_l, x_i \in P/G$ , the action  $S_l \times (P/G)^l \rightarrow (P/G)^l$ ,  $(\sigma, (x_1, \dots, x_l)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(l)})$  is CCT by Lemma 5.8. So, the action  $\phi$ , satisfies the conditions of Proposition 1.7 and so  $\phi$  is CCT.  $\square$

**5.2. Examples of Cover Transitive Actions.** *In this subsection, we describe several classes of cover transitive actions. First, we show that the automorphism group of any rooted tree is cover transitive. Second, we show linear automorphisms of simplices and octahedra are cover transitive. Third, we show that the left multiplication action is cover transitive if and only if the group is  $\mathbb{Z}_2^k$ , and that any action of  $\mathbb{Z}_2^k$  on  $[n]$  induces a cover transitive action on  $B_n$ .*

**5.2.1. An application to rooted trees.** *In this subsubsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Proposition 5.9 and Proposition 5.4, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT.*

**Definition 5.10.** A graded poset  $P$  is a *rooted tree* if there is a unique element  $x \in P$  of maximal rank, called the *root*, and for all  $x \in P$ , other than the root, there exists a unique  $y \in P$  with  $y > x$ .

**Definition 5.11.** For  $P$  a rooted tree, an element  $x \in P$  is a *leaf* if there is no  $z \in P$  with  $x > z$ . Denote the set of all leaves of  $P$  by  $L(P)$ .

**Lemma 5.12.** Let  $P$  be a rooted tree. Then, the action of  $\text{Aut}(P)$  on  $P$  induces an action of  $\text{Aut}(P)$  on  $L(P)$ . Furthermore, there is also an induced action of  $\text{Aut}(P)$  on  $B_n$  where  $n = |L(P)|$ .

*Proof.* First, we must show that  $\text{Aut}(P)$  induces an action on  $L(P)$ . That is, we must show that for any  $g \in \text{Aut}(P)$ ,  $x \in L(P)$ , then  $gx \in L(P)$ . If  $x \in L(P)$  but  $gx \notin L(P)$ , then there exists  $y < gx$ . However, then  $g^{-1}y < x$ , contradicting the assumption that  $x$  is a leaf. We then obtain the induced action  $\text{Aut}(P) \times L(P) \rightarrow L(P)$ ,  $(g, x) \mapsto gx$ .

Finally, to obtain the induced action of  $\text{Aut}(P)$  on  $B_n$  identify  $L(P) \cong [n]$  as sets, where  $|L(P)| = n$ , and from this we get of action of  $\text{Aut}(P)$  on  $B_n$ .  $\square$

**Convention 5.13.** For the rest of this section only, fix a rooted tree  $P$  and denote by  $G$  the group of automorphisms  $\text{Aut}(P)$ . Let  $G$  act on  $B_n$ , where  $n = |L(P)|$  by the induced action  $\phi : G \times P \rightarrow P$ , given by restricting  $\text{Aut}(P)$  to  $\text{Aut}(L(P))$ , as described in the proof of Claim 5.12.

**Notation 5.14.** For  $x \in P$ , denote  $D(x) = \{y \in P : y \leq x\}$ .

**Proposition 5.15.** *Let  $P$  be a rooted tree with root vertex labeled  $r$ . Let  $\{A_1, \dots, A_m\}$  denote the set of isomorphism classes of  $\{D(x) : x \leq r\}$ . For  $T \in A_j$ , denote  $G_j = \text{Aut}(T)$ . Then,*

$$(5.1) \quad \text{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

*In particular,  $\text{Aut}(P)$  can be expressed as a sequence of direct products and wreath products of symmetric groups.*

*Proof.* It is clear that if  $P$  is rank 0, then  $\text{Aut}(P)$  is trivial. So, let us proceed by induction. That is, label the vertices of  $P$  by  $\{0, 1, \dots, s\}$  such that the root is labeled 0 and the vertices just below the root are labeled  $1, \dots, k$ . Let  $A_1, \dots, A_m$  denote the distinct isomorphism classes of trees in the set  $\{D(1), \dots, D(k)\}$ . For  $T \in A_j$ , denote  $G_j = \text{Aut}(T)$ . Let  $T_j = \{t : t \leq 0, D(t) \cong A_j\}$ . Then, letting  $Q_j$  be the subtree of  $P$  whose elements lie in the set  $0 \cup (\cup_{t \in T_j} D(t))$ , it follows  $\text{Aut}(Q_j) \cong G_j \wr S_{i_j}$ , because after choosing a permutation of the elements of  $T_j$ , we are free to choose any element of  $G_j$  to permute each  $D(t)$ ,  $t \in T_j$ . If  $t_1 \leq 0, t_2 \leq 0, g \cdot t_1 = t_2$ , then it must be that  $D(t_1) = D(t_2)$ . Therefore,  $\text{Aut}(P)$  must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$(5.2) \quad \text{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

Since each  $G_j$  is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (5.2) that so is  $\text{Aut}(P)$ .  $\square$

**Corollary 5.16.** *For  $P$  the automorphism group of a rooted tree,  $\text{Aut}(P)$  is CCT.*

*Proof.* By Proposition 5.9, wreath products with symmetric groups preserve the CCT property, and by Proposition 5.4 the direct product of two CCT groups is again CCT. Therefore, by Proposition 5.15, all groups of the form  $\text{Aut}(P)$  are built up from these operations, and so  $\text{Aut}(P)$  is also CCT.  $\square$

**5.2.2. Automorphisms of Polytopes.** *As another class of CCT actions, we describe several linear automorphism groups of polytopes whose actions are CCT. In particular, we prove that the linear automorphism group of simplices, and octahedra are common cover transitive, although both of these are fairly straightforward. Later in Lemma 10.2, we shall also see that the action of the dihedral group on a regular  $n$ -gon, for  $n = p, 2p$ , is cover transitive. This is the group of linear automorphisms of the regular  $n$ -gon. This action gives yet another example of automorphisms of the linear automorphism group of polytopes being CCT.*

**Definition 5.17.** Let  $M$  be a polytope, with a particular embedding in  $\mathbb{R}^n$ . The group of linear automorphisms of  $M$  is the subgroup of  $GL_n$  whose elements are  $\{g \in GL_n : g \cdot M = M\}$ .

**Proposition 5.18.** *Let  $G$  be the group linear automorphisms of the  $n - 1$  simplex, whose vertices lie at the standard basis vectors in  $\mathbb{R}^n$ . The action of  $G$  on the  $n - 1$ -simplex induces an action on  $[n]$ , given by identifying  $[n]$  with the  $n$  vertices of the  $n - 1$  simplex. Hence, it induces an action on  $B_n$ . This induced action on  $B_n$  is CCT.*

*Proof.* The group of linear automorphisms in this case induces the usual action of  $S_n$  on  $B_n$ , because any permutation matrix defines a linear map on  $\mathbb{R}^n$ . However, we know the action of  $S_n$  on  $B_n$  is CCT from Example 5.2  $\square$

**Proposition 5.19.** *Let  $G$  be the group of linear automorphisms of the  $n$ -octahedron, embedded inside  $\mathbb{R}^n$ , whose vertices are located at  $\pm e_i$ , where  $e_1, \dots, e_n$  are the standard basis vectors of  $\mathbb{R}^n$ . Then, the action of  $G$  on the octahedron induces an action of  $G$  on the  $2n$  vertices of the octahedron, and hence on  $B_{2n}$ . This induced action on  $B_{2n}$  is common cover transitive.*

*Proof.* It is simple to see that the group of linear automorphisms of the  $n$ -octahedron with vertices at  $\pm e_i$ , where  $e_1, \dots, e_n$  is the hyperoctahedral group, since it is generated by the permutation matrices, together with the matrix  $A$  with  $A_{1,1} = -1, A_{i,i} = 1, A_{j,k} = 0$  where  $i \neq 1, j \neq k$ . (The hyperoctahedral group is commonly notated  $B_n$ , since it is the type  $B$  Coxeter group, but we do not use this here to avoid confusing with the boolean algebra.) It is well known that the hyperoctahedral group can be written as  $S_2 \wr S_n$ . Then, by Proposition 5.9, it follows that  $S_2 \wr S_n$  is CCT.  $\square$

5.2.3. *Cover Transitive Actions of  $\mathbb{Z}_2^k$ .* In this subsection, we show that any embedding of  $\mathbb{Z}_2^k$  into  $S_n$  defines an action on  $B_n$  which is CCT. In particular, this implies that the left multiplication action of  $\mathbb{Z}_2^k$  is CCT. However, it turns out that this is the only class of groups for which the left multiplication action is CCT.

**Lemma 5.20.** *Let  $n, k$  be arbitrary. Let  $G = \mathbb{Z}_2^k$  and pick any order preserving, rank preserving action  $\phi : G \times [n] \rightarrow [n]$ . Then the induced action  $\phi : G \times B_n \rightarrow B_n$  is CCT.*

*Proof.* First, observe that all elements  $g \in G$  have order 2. Suppose  $x \prec z, y \prec z, x = gy$ . We will show  $g \in \text{Stab}(z)$ . By definition, this would imply that  $\phi$  is common cover transitive.

Let  $a$  be the unique element such that  $a \in y, a \notin x$ . Certainly  $g(y \setminus a) \subset gx = y \subset z$ . So, to show  $g \in \text{Stab}(z)$ , it suffices to show that  $ga \in x$ . Let  $b = g^{-1}a$ . We know  $b \in x$ . However, since  $g$  has order 2, it must be that  $ga = g^2b = b \in x$ . Hence,  $g \in \text{Stab}(z)$ , and  $\phi$  is CCT.  $\square$

**Corollary 5.21.** *Let  $\mathbb{Z}_2^k$  act on itself by left multiplication  $\psi : \mathbb{Z}_2^k \times \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2^k, (g, h) \mapsto g \cdot h$ . By identifying  $\mathbb{Z}_2^k \cong [2^k]$  as sets, we obtain an induced action  $\phi : \mathbb{Z}_2^k \times B_{2^k} \rightarrow B_{2^k}$  which is cover transitive.*

*Proof.* By Lemma 5.20, any embedding of  $\mathbb{Z}_2^k \rightarrow S_n$  defines a CCT action on  $B_n$ . So, this particular embedding defines a CCT action.  $\square$

**Proposition 5.22.** *Let  $G$  be a finite group, acting on itself by left multiplication,  $\psi : G \times G \rightarrow G, (g, h) \mapsto g \cdot h$ , with  $|G| = n$ . By identifying  $G \cong [n]$  as sets,  $\psi$  induces an action  $\phi : G \times B_n \rightarrow B_n$ . Then,  $\phi$  is cover transitive if and only if  $G \cong \mathbb{Z}_2^k$  for some  $k$ .*

*Proof.* If  $G \cong \mathbb{Z}_2^k$ , then by Corollary 5.21, we know the action  $\phi : G \times B_{2^k} \rightarrow B_{2^k}$  is indeed CCT.

Next, we prove the converse. First, let us show  $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$ . The forward implication is obvious. To see the converse, first note that if  $\forall g \in G, g^2 = e$ , then  $G$  is abelian because  $aba^{-1}b^{-1} = abab = (ab)^2 = e$ . Then,  $G$  is an abelian group, all of whose elements have order two, the structure theorem of finite abelian groups tells us  $G \cong \mathbb{Z}_2^k$ .

So, Suppose  $G \not\cong \mathbb{Z}_2^k$ . Then, there exists  $g \in G, g^2 \neq e$ . Clearly  $\{e\} \prec \{e \cup g\}, \{g\} \prec \{e \cup g\}, \{g\} \in G\{e\}$ . So, to show the induced action  $\phi : G \times B_n \rightarrow B_n$  is not cover transitive, it suffices to show there is no  $h \in G, h \in \text{Stab}(\{e \cup g\}), h \cdot \{e\} = \{g\}$ . If  $h \in \text{Stab}(\{e \cup g\})$  then  $h \cdot e = e$  or  $h \cdot e = g$ . In both cases, it is simple to see that  $g^2 = e$ . Therefore, there does not exist such an  $h$  and left multiplication is not CCT.  $\square$

*Remark 5.23.* One nice way of restating the above results is that for  $G$  an abstract group, there exists some  $n$  and some embedding  $G \subset S_n$  such that the induced action of  $G$  on  $B_n$  is not CCT if and only if  $G \not\cong \mathbb{Z}_2^k$  for some  $k$ . If  $G \cong \mathbb{Z}_2^k$ , we have seen this holds in



Lemma 5.20. If  $G \not\cong Z_2^k$ , then by Proposition 5.22, the left multiplication action of  $G$  on  $B_{|G|}$  defines an action which is not CCT.

## 6. WREATH PRODUCT OF TWO SYMMETRIC GROUPS

In this section, we prove a result similar to that of [IP13, Theorem 1.1]. We shall construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. It shall give an alternate proof of Theorem 1.1 in the case that  $r = 1$ .

**Notation 6.1.** We shall now show For this section, fix  $l, m$  with  $n = l \cdot m$  and fix  $G = S_m \wr S_l$ . Let  $S_m$  act on  $B_m$  by the permutation representation, and then let  $G$  act on  $B_m^l \cong B_{m \cdot l}$  by the action defined in Notation 5.6.

**6.1. Recalling Pak and Panova's Result.** We first review the necessary definitions and then state [IP13, Theorem 1.1]:

A partition is a sequence of numbers  $\lambda = (\lambda_1, \dots, \lambda_k)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ . If  $\sum_{i=1}^k \lambda_i = n$  then  $\lambda$  is a partition of  $n$ , notated  $\lambda \vdash n$ . A composition is a sequence of numbers  $\lambda = (\lambda_1, \dots, \lambda_k)$ . That is, it is a partition where order matters. If  $\sum_{i=1}^k \lambda_i = n$  then  $\lambda$  is a composition of  $n$ . Let  $P_n(l, m)$  denote the set of partitions  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ , such that  $\lambda_1 \leq m, k \leq l$ . That is,  $P_n(l, m)$  is the partitions which fit inside an  $l \times m$  rectangle.

**Notation 6.2.** [IP13, Section 1] For  $\lambda$  a partition, let  $\nu(\lambda)$  be the number of distinct nonzero part sizes of  $\lambda$ .

**Notation 6.3.** [IP13, Section 1] Let  $p_k(l, m, r) = \sum_{\lambda \in P_k(l, m)} \binom{\nu(\lambda)}{r}$ .

**Theorem 6.4.** [IP13, Theorem 1.1] The sequence  $p_r(l, m, r), p_{r+1}(l, m, r), \dots, p_{l \cdot m}(l, m, r)$  is unimodal and symmetric.

**6.2. The Poset  $\mathcal{E}^r(B_n)/G$ .** Now that we have managed to state Pak and Panova's Theorem, we will begin the trek toward showing how the  $p_1(l, m, 1)$  are actually ranks of  $\mathcal{E}^1(B_n)/G$ . The first step is to describe the Peck poset  $\mathcal{E}^1(B_n)/G$ . In particular, we shall obtain an explicit formula for the sizes of its ranks in Theorem 6.19. To do this, we will first describe representatives for the vertices of  $B_n/G$  and then analogous representatives for vertices of  $\mathcal{E}^1(B_n)/G$ .

**Definition 6.5.** For  $x \subset [l \cdot m]$  we shall say  $x$  is *left justified* if whenever  $a \in x$ , such that  $a \not\equiv 1 \pmod{l}$ , then  $a - 1 \in x$ . In other words, when we pick out the boxes of the  $l \times m$  rectangle which lie in  $x$  the resulting diagram is left justified.

**Definition 6.6.** For  $x \subset [l \cdot m]$ , define the *composition* of  $x$ , denoted  $\text{comp}(x) = \lambda_1, \dots, \lambda_l$  where  $\lambda_i = |\{a \in x : i \cdot (m - 1) < a \leq i \cdot m\}|$ . That is, if we pick out the boxes of the  $l \times m$  rectangle which lie in  $x$ , the composition is just the sequence of the number of boxes in each row. Similarly, define the *partition* of  $x$ , notated  $\text{part}(x)$  as follows. Let  $\pi \in S_l$  be a permutation such that  $\pi(i) \geq \pi(i+1)$  for all  $i, 1 \leq i \leq l-1$ . Then  $\text{part}(x) = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(l)})$ . That is,  $\text{part}(x)$  is the  $\text{comp}(x)$ , written in decreasing order.

**Lemma 6.7.** For any  $g \in G, x \in B_n$ , it follows that  $\text{part}(x) = \text{part}(gx)$ .

*Proof.* It suffices to show this holds for any generator  $g$ . Then  $G$  is generated by elements which permute rows, and elements that swap rows. It is clear that if  $g$  only permutes elements in a single row, then  $\text{comp}(gx) = \text{comp}(x)$ , so in particular,  $\text{part}(gx) = \text{part}(x)$ . If  $g$  swaps two rows, then  $\text{comp}(gx)$  is simply a reordering of the parts of  $\text{comp}(x)$ , and so again  $\text{part}(gx) = \text{part}(x)$ .  $\square$

**Definition 6.8.** For  $x \subset [l \cdot m]$ , say  $x$  is a *Young Diagram* if  $x$  is left justified and  $\text{comp}(x)$  is a partition.

**Lemma 6.9.** For each  $x \in B_{l \cdot m}$  there exists a unique representative  $z \in Gx$  such that  $z$  is a Young Diagram.

*Proof.* Uniqueness is clear, because for any  $g \in G$ , by Claim 6.7,  $\text{part}(gx) = \text{part}(x)$ . So, to we only have to show there is some  $g \in G$  for which  $gx$  is a Young Diagram. Indeed, first, choose  $h_1 \in G$  so that  $h_1x$  is left justified. This can be done because every permutation of a single row in the  $l \times m$  rectangle lies in the wreath product, so, we can take  $h_1$  to be the product of the elements that left justify each individual row. Then, let  $h_2$  be the element that swaps the rows of  $h_1x$  so that they increase going down. Finally, taking  $g = h_2h_1$ , it follows that  $gx$  is a Young diagram.  $\square$

**Notation 6.10.** For  $x \in B_n$  denote by  $\bar{x}$  the unique element in  $Gx$  such that  $\bar{x}$  is a Young Diagram.

Now that representatives for each  $G$  orbit in  $B_n$  have been described, we shall move on to describing representatives for each  $G$  orbit in  $\mathcal{E}(B_n)$ .

**Notation 6.11.** For  $\lambda = (\lambda_1, \dots, \lambda_k)$  a partition, introduce the alternate notation  $\lambda = a_1^{b_1} \dots a_s^{b_s}$  if the first  $b_1$  parts of  $\lambda$  are equal to  $a_1$ , the next  $b_2$  parts of  $\lambda$  are equal to  $a_2$ , and in general, for  $1 \leq h \leq b_j$ , the parts  $\lambda_{h+\sum_{i=1}^{j-1} b_i}$  are all equal to  $a_j$ . Furthermore, all  $a_j$  must be distinct.

**Lemma 6.12.** If  $(x, y), (w, y) \in \mathcal{E}^r(B_n)_i$  then  $(x, y) \in G(w, y)$  if and only if for all  $g \in G$  such that  $gy$  is a Young diagram, we have  $(gx, gy) \in G(gw, gy)$ . In particular, if  $gy = \bar{y}$  then  $(x, y) \in G(w, y)$  if and only if  $(gx, \bar{y}) \in (gw, \bar{y})$ .

*Proof.* Clearly  $g(x, y) \in G(g(w, y))$  if and only if  $(x, y) \in G(w, y)$ .  $\square$

The point of the preceding Lemma is that in order to determine when two elements are identified, we may assume that  $y$  is a young diagram.

**Notation 6.13.** For  $y$  a Young Diagram with  $\text{part}(y) = \text{comp}(y) = a_1^{b_1} \dots a_s^{b_s}$ , and  $x \subset y$ , for all  $i, 1 \leq i \leq s$ , define  $y_i$  to be the  $a_i \times b_i$  rectangle consisting of the rows  $1 + \sum_{j=1}^{i-1} b_j, 2 + \sum_{j=1}^{i-1} b_j, \dots, \sum_{j=1}^i b_j$ . That is,  $y_i$  is just the rectangle of all parts of  $y$  which are of length  $a_i$ .

**Proposition 6.14.** If  $y$  is a Young Diagram so that  $\text{part}(y) = \text{comp}(y) = a_1^{b_1} \dots a_s^{b_s}$ , and  $(x, y), (w, y) \in \mathcal{E}^r(B_n)_j$  then  $(x, y) \in G(w, y)$  if and only if for all  $i, 1 \leq i \leq s$ , it holds that  $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$ .

*Proof.* First, suppose for all  $i, 1 \leq i \leq s$ , that  $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$ . We know both  $x \subset y, w \subset y$ . Further, since  $y_i$  is a rectangle, there exists some  $g \in \text{Stab}(y_i)$  so that  $gx = w$ . Then, for each  $i$ , there exists some  $g_{1,i} \in \text{Stab}(y_i)$  so  $g_{1,i}(x \cap y_i) = \overline{x \cap y_i}$ , which only interchange elements in  $y_i$ . Similarly, there exists  $g_{2,i} \in \text{Stab}(y_i)$ ,  $g_{2,i}(w \cap y_i) = \overline{w \cap y_i}$ . However, the assumption  $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$  precisely means  $\overline{x \cap y_i} = \overline{w \cap y_i}$ . Therefore,  $g_{2,i}^{-1}g_{1,i} \in \text{Stab}(y)$  and  $g_{2,i}^{-1}g_{1,i}(x \cap y_i) = (w \cap y_i)$ . Applying this same procedure for all  $i$  and multiplying the corresponding group elements together gives an element  $g \in \text{Stab}(y)$  with  $gx = w$ . Therefore,  $g(x, y) = (gx, y) = (w, y)$ , so  $(x, y) \in G(w, y)$ .

Conversely, note that  $(x, y) \in G(w, y)$  is equivalent to the existence of a  $g \in \text{Stab}(y)$  with  $gx = w$ . However, any  $g \in \text{Stab}(y)$  can only interchange rows of the same length. Therefore, we obtain the stronger result that we can write  $g = h_1 \dots h_s$  where  $h_i \in \text{Stab}(y_i)$ , and  $h_i(p) = p$  for all  $p \notin y_i$ . Then, it follows that  $h_i(x \cap y_i) = w \cap y_i$  and so  $\text{part}(x \cap y_i) = \text{part}(w \cap y_i)$ , as claimed.  $\square$

**Remark 6.15.** So, one way of viewing  $G$  orbits of an element  $(x, y) \in \mathcal{E}^r(B_n)$  is as “outer” Young Diagrams, made up of a sequence of rectangles stacked on top of one another, each one wider than the next, which form the Young diagram  $\bar{y}$ . Then, to each such rectangle we associate an “inner” Young Diagram. The inner young diagram corresponds to the elements in that rectangle in  $y$  but not in  $x$ . Two elements are in the same  $G$  orbit if and only if their “outer” Young Diagram and “inner” Young Diagrams are all the same.

The next step is to give explicitly formulas for the ranks of  $\mathcal{E}^r(B_n)$ .



**Notation 6.16.** For  $p(x) = \sum_{i=0}^N c_i x^i$ , a polynomial, define the notation  $[p(x)]_r = c_r$ .

Now, we briefly introduce notation for  $q$  binomial coefficients, so that we can state the next propositions. Let  $q \in \mathbb{R}$  and let  $[n]_q = \sum_{i=0}^{n-1} q^i$ . Then, denote  $[n]_q! = \prod_{i=1}^n [i]_q$ . Finally, let  $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$ .

**Notation 6.17.** For  $\lambda = (a_1^{b_1} \cdots a_s^{b_s})$  a partition, denote  $\eta(\lambda, r) = \left[ \prod_{i=1}^s \binom{a_i + b_i}{a_i}_q \right]_r$ .

**Proposition 6.18.** [Sta11, Proposition 1.3.19] *There is an equality  $[(\frac{a+b}{b})_q]_j = |P_j(l, m)|$ .*

**Theorem 6.19.** *The sizes of the ranks of  $\mathcal{E}^r(B_n)/G$  can be written as*

$$|(\mathcal{E}^r(B_n)/G)_i| = \sum_{\lambda \in P_i(l, m)} \eta(\lambda, r).$$

*Proof.* By Proposition 6.14 each orbit  $(x, y)$  has a representative such that  $y$  is a Young Diagram, with  $\text{part}(y) = \prod_{i=1}^s a_i^{b_i}$ , and each  $x \cap y_i$  is a Young Diagram. Therefore, for a fixed  $y$ , the number of orbits  $(x, y)$  with  $|x| + r = |y|$  is exactly determined by the Young Diagrams of  $\overline{x \cap y_i}$ , for  $1 \leq i \leq s$ . Equivalently, it is determined by the Young Diagrams  $[y_i \setminus x]$ , for  $1 \leq i \leq s$ . So, we wish to calculate the number of tuples of Young Diagrams of the form  $(y_1 \setminus x, y_2 \setminus x, \dots, y_s \setminus x)$  so that  $\sum_{i=1}^s |y_i \setminus x| = r$ . Now, define  $j_i$  by  $j_i = |y_i \setminus x|$ , still of course with,  $\sum_{i=1}^s j_i = r$ . Then, by Proposition 6.18, the number of such partitions is  $[(\frac{a_i + b_i}{b_i})_q]_{j_i}$ . Therefore, the number of elements  $G(x, y)$  with  $j_i = |y_i \setminus x|$ , is simply the product  $\prod_{i=1}^s [(\frac{a_i + b_i}{b_i})_q]_{j_i}$ . Therefore, since  $j_i$  were chosen arbitrarily, only subject to the constraint that  $\sum_{i=1}^s j_i = r$ , It follows that the total number of elements  $G(x, y)$ , for  $y$  fixed, is equal to

$$\sum_{\substack{(j_1, \dots, j_s), \\ \sum_{i=1}^s j_i = r}} \prod_{i=1}^s \left[ \binom{a_i + b_i}{b_i}_q \right]_{j_i} = \left[ \prod_{i=1}^s \binom{a_i + b_i}{b_i}_q \right]_r = \eta(\lambda, r).$$

Then, summing this over all Young Diagrams  $y$ , gives that

$$|\mathcal{E}^r(B_n)/G|_i = \sum_{\lambda \in P_i(l, m)} \left[ \prod_{i=1}^s \binom{a_i + b_i}{b_i}_q \right]_r = \sum_{\lambda \in P_i(l, m)} \eta(\lambda, r).$$

□

**6.3. Ensuing Conclusions.** *We now reap the fruits of our labor in order to draw some nice results.*

**Corollary 6.20.** *The sequence  $\sum_{\lambda \in P_r(l, m)} \eta(\lambda, r), \dots, \sum_{\lambda \in P_n(l, m)} \eta(\lambda, r)$ , is the sequence of ranks of the Peck poset  $\mathcal{E}^r(B_{l \cdot m})/G$ . In particular, the sequence is symmetric and unimodal.*

*Proof.* By Theorem 6.19,  $\sum_{\lambda \in P_i(l, m)} \eta(\lambda, r)$  is the size of the  $i - r$ th rank of the Peck poset  $\mathcal{E}^r(B_{l \cdot m})/G$ . Since Peck posets are unimodal and symmetric, this sequence is as well. □

*We now obtain an easy proof of Theorem 6.4 for the case  $r = 1$ . Using the fact that  $S_m \wr S_l$  is CCT. Namely,*

**Corollary 6.21.** *The sequence  $p_1(l, m, 1), p_{r+1}(l, m, 1), \dots, p_{l \cdot m}(l, m, 1)$  is the rank generating function of  $\mathcal{E}^1(B_{l \cdot m})/G$ . In particular, the sequence is unimodal and symmetric.*

*Proof.* First, we know  $\mathcal{E}^1(B_{l \cdot m})/G$  is Peck, since it is the quotient of a unitary Peck poset by a group action. However, it is easy to see that  $\eta(\lambda, 1) = \nu(\lambda)$ , since both count the number Young Diagrams of size  $|\lambda| - 1$  fitting inside the Young Diagram corresponding to  $\lambda$ . Therefore,  $\sum_{\lambda \in P_i(l, m)} \eta(\lambda, r) = \sum_{\lambda \in P_i(l, m)} \nu(\lambda) = p_i(l, m, 1)$ . However, by Corollary 6.20,  $\sum_{\lambda \in P_r(l, m)} \eta(\lambda, r), \dots, \sum_{\lambda \in P_n(l, m)} \eta(\lambda, r)$ , are the ranks of  $\mathcal{E}^1(B_{l \cdot m})/G$ . Therefore,  $p_i(l, m, 1)$  are also the ranks of  $\mathcal{E}^1(B_{l \cdot m})/G$ , hence symmetric and unimodal. □

## 7. RELATIONS TO POLYA THEORY

In this section, we obtain summation formulas for the ranks of  $\mathcal{E}^{\vec{\tau}}(B_n)/G$ . These formulas are given by certain polynomials in Polya theory. In particular, we obtain polynomials from Polya theory which are the rank generating functions of  $\mathcal{E}^{\vec{\tau}}(B_n)/G$ , which implies that these polynomials have symmetric, unimodal coefficients. The result we prove is a generalization of [Sta13, Corollary 7.16].

**7.1. A Brief Review of Polya Theory.** We shall follow the treatment from [Sta13, Chapter 7]. First, we build up some definitions to state Polya's Theorem.

**Definition 7.1.** Let  $G \subset S_n$  act on  $[n]$  by the restriction of the action of  $S_n$  on  $[n]$  by all permutations. For  $\pi \in G$ , the action of  $\pi$  on  $[n]$  can be written in cycle notation so that there are  $c_i$  cycles of length  $i$ . Define the *cycle indicator* of  $\pi$  to be the monomial  $Z_\pi(z_1, \dots, z_n) = z_1^{c_1} z_2^{c_2} \dots z_n^{c_n}$ .

**Definition 7.2.** The *cycle indicator* for a group  $G \subset S_n$ , is the polynomial

$$Z_G(z_1, \dots, z_k) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi(z_1, \dots, z_k).$$

**Definition 7.3.** A *coloring* of a set  $S$  by the colors  $R = \{r_1, \dots, r_k\}$  is a map  $S \rightarrow R$ . Heuristically, a coloring of  $S$  can be thought of as an assignment of a “color” from the set  $R$  to each of the elements of  $S$ .

**Notation 7.4.** For  $A, B$  sets, let  $A^B = \text{Hom}_{\text{sets}}(B, A)$ . Let  $G \subset S_n$  act on  $B_n$ . Then,  $G$  acts on  $R^{B_n} = \text{Hom}_{\text{sets}}(B_n, R)$  by  $g \cdot f(S) = f(g \cdot S)$  where  $g \in G, S \in B_n, f \in R^{B_n}$ . Then, let  $R^{B_n}/G$  denote the quotient of  $R^{B_n}$  by the action of  $G$  defined above.

Suppose  $R = \{r_1, \dots, r_k\}$ , and define the map  $c : R^S \rightarrow \mathbb{Z}^k$ , by sending  $f \in R^S$  to  $c(f) = (i_1, \dots, i_k)$  if  $|\{s \in S : f(s) = r_j\}| = i_j$  for  $1 \leq j \leq k$ . Observe that if  $f \in Gh$  then  $c(f) = c(h)$ , so the map *color* descends to a map on  $G$  orbits.

Define

$$\kappa(i_1, \dots, i_k) = |\{Gf \in R^{B_n}/G \text{ such that } c(f) = (i_1, \dots, i_k)\}|.$$

Finally, let

$$F_G(r_1, \dots, r_k) = \sum_{i_1, \dots, i_k} \kappa(i_1, \dots, i_k) r_1^{i_1} \dots r_k^{i_k}.$$

*Remark 7.5.* The above notation may seem extremely cumbersome. It is simply a formal way of saying that  $\kappa(i_1, \dots, i_k)$  is the number of inequivalent colorings of subsets of  $B_n$ , under the  $G$  action. Once again, see [Sta13, Chapter 7] for a more lengthy exposition.

**Theorem 7.6.** (*Polya's Theorem*) Let  $G \subset S_n$  act on  $B_n$ . With  $F_G, Z_G$  as defined above, the following equality holds.

$$F_G(r_1, \dots, r_k) = Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right).$$

**7.2. A Rank Generating Function.** Next, we produce rank generating functions for  $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$ . The reason for starting at  $i_2$  and ending at  $i_{k-1}$  is that we shall include additional numbers  $i_1$  and  $i_k$  to be colored by the first color and the  $k$ th color. Given an element  $x_2, \dots, x_k \in \mathcal{E}^{i_2, \dots, i_{k-1}}$ , the first color will be used to color the elements in  $x_2$  and the  $k$ th color will be used to color the elements in  $[n] \setminus x_k$ .

**Notation 7.7.** Let  $[f(x_1, \dots, x_k)]_{x_{j_1}^{i_1} \dots x_{j_l}^{i_l}}$  denote the coefficient of  $x_{j_1}^{i_1} \dots x_{j_l}^{i_l}$  in  $f(x_1, \dots, x_k)$ , which may itself be a polynomial.

**Lemma 7.8.** The number  $\kappa(i_1, \dots, i_k)$  is equal to  $|\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G|_{i_1}$ . Consequently,

$$\left[ Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

*Proof.* Define a map  $m : (\mathcal{E}^{r_2, \dots, r_{k-1}}(B_n))_{i_1} \rightarrow \{f \in R^{B_n} : c(f) = (i_1, \dots, i_k)\}$  as follows. For  $(x_2, \dots, x_k) \in \mathcal{E}^{r_2, \dots, r_{k-1}}(B_n)$ , let  $m(x_2, \dots, x_k) = f$ , where

$$f(t) = \begin{cases} r_1 & \text{if } t \in x_2 \\ r_j & \text{if } t \in x_{j+1} \setminus x_j \\ r_k & \text{if } t \notin x_k \end{cases}$$

Observe that  $m$  is in fact a bijection, as we can easily define an inverse map by sending a coloring  $f$  to  $(x_2, \dots, x_k)$ , where  $x_j$  is the set of all elements  $t \in [n]$  for which there exists  $l < j$  with  $f(t) = r_l$ . Next, the two group actions were defined so that so that,

$$g(x_1, \dots, x_{k-1}) = (y_2, \dots, y_{k-1}) \Leftrightarrow g \cdot m(x_1, \dots, x_{k-1}) = m(y_2, \dots, y_{k-1})$$

Therefore,  $m$  descends to a bijection  $m^G : (\mathcal{E}^{r_2, \dots, r_{k-1}}(B_n))_{i_1}/G \rightarrow R^{B_n}/G$ . This implies  $\kappa(i_1, \dots, i_k)$  is equal to  $|(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|$ .

Then, by Polya's Theorem 7.6,

$$\left[ Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{r_1^{i_1} \dots r_k^{i_k}} = \kappa(i_1, \dots, i_k) = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

□

Now we arrive at our main result of the section. The following result provides a rank generating function for  $\mathcal{E}^{\vec{r}}(B_n)/G$ .

**Proposition 7.9.** *Let  $z_j = \sum_{j=1}^k r_j^j$ . Let*

$$Z_G^{i_2, \dots, i_{k-1}}(r_1) = [Z_G(z_1, \dots, z_{k-1}, 1)]_{r_2^{i_2} \dots r_{k-1}^{i_{k-1}}} = \sum_t c_t r_1^t,$$

*be a polynomial in  $r_1$ . Then, the coefficients  $c_t$  are the ranks of the Peck poset  $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$ . In particular, the coefficients form a symmetric, unimodal sequence.*

*Proof.* In Lemma 7.8, we saw

$$[Z_G(z_1, \dots, z_k)]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

However, since  $\sum_{j=1}^k i_j = n$ , it follows that  $i_k$  is determined by the numbers  $i_1, \dots, i_{k-1}$ , and so

$$[Z_G(z_1, \dots, 1)]_{r_1^{i_1} \dots r_k^{i_k}} = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|.$$

Next, write  $Z_G^{i_2, \dots, i_{k-1}}(r_1) = \sum_t c_t (r_1)^t$ . We have just shown that  $c_t = |(\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G)_{i_1}|$ . Since  $\mathcal{E}^{i_2, \dots, i_{k-1}}(B_n)/G$  is a Peck poset by Lemma 4.15, its rank sizes form a symmetric and unimodal sequence. Therefore, the coefficients of  $Z_G^{i_2, \dots, i_{k-1}}(r_1)$  form a symmetric, unimodal sequence. □

*Remark 7.10.* Note that in the case  $\vec{r} = 0$ , the above result precisely becomes [Sta13, Corollary 7.16].

**7.3. Further Identities for the Wreath Product.** *In this section, we draw on the methods developed earlier in this section to write down some interesting generating functions for the case that  $G = S_m \wr S_l$ . First, we shall use Proposition 7.9 to obtain an explicit generating function for  $p_i(l, m, 1)$ , and then we shall relate the sum  $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}$  to counting certain types of set partitions, which also count  $|\mathcal{E}^{\vec{r}}(B_n)/G|_0$ , where  $\vec{r}$  is of the form  $\vec{r} = 1, \dots, 1$ .*

**Notation 7.11.** For this section only, fix  $m, l \in \mathbb{N}$  and fix  $G = S_m \wr S_l$ . Additionally, fix  $n = m \cdot l$ .

7.3.1. *A Generating Function for  $p_i(l, m, 1)$ .*

**Proposition 7.12.** *Let  $c_i$  be the number of  $i$  cycles in  $\pi$  and define*

$$W_\pi(z_1, \dots, z_n) = \begin{cases} \frac{Z_\pi(z_1, \dots, z_n)}{z_1} & \text{if } c_1 > 0 \\ 0 & \text{if } c_1 = 0 \end{cases}.$$

*Then, there is an equality*

$$\sum_{i=0}^n p_i(l, m, 1) r_1^i r_3^{n-i-1} = \sum_{\pi \in G} |\text{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

*Proof.* Recall from Corollary 6.21 that  $p_i(l, m, 1)$  is the rank generating function of  $\mathcal{E}^1(B_{l \cdot m})/G$ .

As was seen in Proposition 7.9, it is also the case that

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}$$

is the rank generating function of  $|\mathcal{E}^1(B_n)/G|$ . Therefore,

$$p_i(l, m, 1) = [Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}.$$

So, to complete the proof, it suffices to show

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2} = \sum_{\pi \in G} |\text{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To show this, it further suffices to show that for all  $\pi \in G$ ,

$$Z_\pi(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)_{r_2} = |\text{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To see, we may note that  $r_2$  must have a nonzero coefficient in the expansion of  $r_2$  in  $Z_\pi$ . First, if  $r_2$  has a nonzero coefficient,  $\pi$  must have some 1-cycle, as otherwise, no variable could appear in the expansion of  $Z_\pi$  raised only to the first power. Second, if  $\pi$  has some 1-cycle, then the coefficient of  $r_2$  in

$$Z_\pi(r_1 + r_2 + r_3, r_1^2 + r_2^2 + r_3^2, \dots, r_1^n + r_2^n + r_3^n) = \prod_{i=1}^n (r_1^i + r_2^i + r_3^i)^{c_i}$$

is precisely

$$c_1 \cdot W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n) = |\text{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n)$$

because  $c_1$  is the number of 1-cycles in  $\pi$ , which is by definition the number of fixed points of  $\pi$ . This is exactly what we wanted to show.  $\square$

**7.3.2. Bounded Partition Sizes.** *This subsubsection is an application of Polya theory, and is moderately tangential to the rest of the paper. However, it relates to the functor  $\mathcal{E}^{\vec{r}}$ , in that it counts the  $|\mathcal{E}^{\vec{r}}(B_n)/G|_0 = |\mathcal{E}^{\vec{s}}(B_n)/G|_1$ , as described in Remark 7.16.*

**Notation 7.13.** Let  $P_{[t]}[l, m]$  denote the set of partitions of the set  $[t]$  into at most  $l$  blocks, such that each block has size at most  $m$ . Let  $P[l, m] = \cup_{t \in \mathbb{N}} P_{[t]}[l, m]$ . Denote  $p_{[t]}[l, m] = |P_{[t]}[l, m]|$ , and  $p[l, m] = |P[l, m]|$ .

*Remark 7.14.* The numbers  $p_{[t]}[l, m]$  satisfy the recursive formula

$$p_{[t]}[l, m] = \sum_{k=1}^l \binom{t-1}{k-1} p_{[t-k]}[l-1, m].$$

This formula can be shown by splitting up the set  $P_{[t]}[l, m]$  based on how many elements appear in the first row. The reason for including  $t-1$  instead of  $t$  is that 1 must always appear in the first row.

**Proposition 7.15.** *There is an equality  $p_{[t]}[l, m] = \frac{t!}{l!(m!)^t} \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}$ .*

*Proof.* Using Polya's Theorem 7.6, we know

$$F_G(r_1, \dots, r_k) = \frac{1}{|G|} Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right).$$

In particular,

$$[F_G(r_1, \dots, r_k)]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = \left[ \frac{1}{|G|} Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}}.$$

Of course, by definition of  $F_G$ , we know

$$[F_G(r_1, \dots, r_k)]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = \kappa(1, 1, \dots, 1, m \cdot l - t, 0, \dots, 0)$$

where there are  $t$  1's in the above expression. By definition,  $\kappa$  is just the number of inequivalent ways to distinctly color  $t$  numbers in an  $l \times m$  rectangle, up to the action of  $G = S_m \wr S_l$ . For any such coloring, let square  $a_i$  be colored with color  $r_i$ . Each  $G$  orbit of colorings has a unique representative such that the colors are sorted in increasing order along each row, and in increasing order down the first column. Then, the colorings defined above are in bijection with partitions of  $[t]$  so that this partition has at most  $l$  blocks, and each block has at most  $m$  elements. The set of such partitions is exactly  $P_{[t]}[l, m]$ . Therefore, the number of such partitions is  $p_{[t]}[l, m]$ . Then, it follows that

$$p_{[t]}[l, m] = \kappa(1, 1, \dots, 1, m \cdot l - t, 0, \dots, 0) = \left[ \frac{1}{|G|} Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}}.$$

Of course,  $\frac{1}{|G|} = \frac{1}{l!(m!)^l}$  when  $G = S_m \wr S_l$ . So, to complete the proof, we just need to show that

$$(7.1) \quad \left[ Z_G \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = t! \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t}.$$

We reduce this problem to showing  $Z_G$  it is the sum over all the cycle indicators  $\sum_{\pi \in G} Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)$ . So, to show Equation (7.1) holds, it suffices to show that

$$(7.2) \quad \left[ Z_\pi \left( \sum_{j=1}^k r_j, \sum_{j=1}^k r_j^2, \dots, \sum_{j=1}^k r_j^n \right) \right]_{i_1 i_2 \dots i_t i_{t+1}^{m \cdot l - t}} = t! \binom{\text{Fix}(\pi)}{t}$$

This is now apparent, because  $Z_\pi = \prod_{i < m \cdot l} (\sum_j r_j^i)^{c_i}$ , where  $c_i$  is the number of  $i$  cycles in  $\pi$ . The only way we can obtain a monomial of the form  $r_1 r_2 \dots r_t r_{t+1}^{m \cdot l - t}$  is if  $c_1 \geq t$ . Then, the coefficient of such a term will exactly be the number of ways to choose an ordered set of  $t$  elements from  $c_1$  terms. That is, it is precisely  $t! \binom{c_1}{t}$ . However,  $c_1$  is the number of 1-cycles, that is  $c_1 = |\text{Fix}(\pi)|$ . Hence, (7.2) holds.  $\square$

*Remark 7.16.* Thanks to Proposition 7.9, an equivalent way to state the above Proposition 7.15 is that  $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} = |(\mathcal{E}^{\vec{r}}(B_n)/G)_0|$ , if  $\vec{r}$  is the vector consisting of  $t$  ones. Also, letting  $\vec{s}$  be the vector with  $t-1$  ones, we may note  $|(\mathcal{E}^{\vec{r}}(B_n)/G)_0| = |(\mathcal{E}^{\vec{s}}(B_n)/G)_1|$ , since one may think of elements of  $\mathcal{E}^{\vec{s}}(B_n)/G$  as tuples of  $t-1$  elements, each contained in the next, whose lowest element is of rank 1, and one may think of elements of  $(\mathcal{E}^{\vec{r}}(B_n)/G)_0$  as tuples of  $t$  elements, each contained in the next, whose lowest element is of rank 0. There is an obvious bijection between these two sets, given by adding or removing the element  $\emptyset$  or rank 0 in  $B_n$ . Therefore, we also obtain  $\sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} = |(\mathcal{E}^{\vec{s}}(B_n)/G)_1|$ .

We can also obtain a formula for the size of the whole set  $p[l, m]$ , by simply summing over all possible values of  $t$ .

**Proposition 7.17.** Define the function  $f : \mathbb{N} \cup 0 \rightarrow \mathbb{N}$ , by

$$f(x) = \begin{cases} \lfloor e \cdot n! \rfloor & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

Then,

$$p[l, m] = \frac{1}{l!(m!)^l} \sum_{\pi \in G} f(|\text{Fix}(\pi)|).$$

*Proof.* First, note that  $\sum_{i=1}^k i! \binom{k}{i} = f(k)$ . This can be seen fairly easily, because  $\sum_{i=1}^k i! \binom{k}{i} = k! \sum_{i=0}^k \frac{1}{i!}$ , while  $k! \cdot e = k! \sum_{i=0}^{\infty} \frac{1}{i!}$ . For  $k > 1$ , it is easy to bound the difference  $\sum_{i=k+1}^{\infty} \frac{1}{i!} < \frac{1}{k!}$ , which implies  $\sum_{i=1}^k i! \binom{k}{i} = \lfloor k! \cdot e \rfloor = f(k)$  for  $k > 1$ .

By definition,  $p[l, m] = \sum_{t=0}^{l \cdot m} p_{[t]}(l, m)$ . Therefore, by Proposition 7.15,

$$\begin{aligned} p[l, m] &= \sum_{t=0}^{l \cdot m} p_{[t]}(l, m) \\ &= \sum_{t=0}^{l \cdot m} \frac{t!}{l!(m!)^l} \sum_{\pi \in G} \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{t=0}^{l \cdot m} \sum_{\pi \in G} t! \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} \sum_{t=0}^{l \cdot m} t! \binom{\text{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l} \sum_{\pi \in G} f(|\text{Fix}(\pi)|). \end{aligned}$$

□

## 8. GROUPS OF ORDER $n$ .

In this section, we shall show that for any group  $G$  such that  $|G| = n$ , with an order-preserving, rank preserving, transitive action on  $B_n$ , then

$$|(\mathcal{E}^1(B_n)/G)_i| = \binom{n-1}{i-1}.$$

The most important example of such an action is the left multiplication action

$$\phi : G \times G \rightarrow G, (g, h) \mapsto g \cdot h.$$

By identifying  $G \cong [n]$  as sets, this induces an action of  $G$  on  $B_n$ . We can then use a simple result in group theory to calculate the ranks of  $\mathcal{E}^1(B_n)/G$  for all transitive actions  $\phi : G \times P \rightarrow P$ , where  $G$  is abelian.

For the rest of this section, fix a group  $G$  with  $|G| = n$ , and a transitive action of  $G$  on  $[n]$ , which induces an action of  $G$  on  $B_n$ .

**Lemma 8.1.** *For  $G$  a group acting transitively on  $[n]$ , it follows that  $\text{Stab}(x) = \{e\}$ .*

*Proof.* Since  $G$  acts transitively on  $n$  we have the number of orbits  $|[n]/G| = 1$ . Therefore, using Burnside's Lemma, together with the observation that  $e \in \text{Stab}(x)$ ,

$$|G| = \sum_{x \in X} |\text{Stab}(x)| \geq \sum_{x \in X} 1 = |X| = |G|.$$

Then, the above string of inequalities tells us that we must have  $|\text{Stab}(x)| = 1$  for all  $x$ , or in other words,  $\text{Stab}(x) = \{e\}$ .  $\square$

**Proposition 8.2.** *For any element  $(x, y) \in \mathcal{E}^1(B_n)$ , then  $\text{Stab}(x, y) = e$ .*

*Proof.* From Lemma 8.1 for any  $x \in B_n$  with  $|x| = 1$ , we have  $\text{Stab}(x) = \{e\}$ . Then, by definition  $\text{Stab}(x, y) = \text{Stab}(x) \cap \text{Stab}(y) = \text{Stab}(x) \cap \text{Stab}(y \setminus x)$ . However,  $\text{Stab}(y \setminus x)$  is by assumption a 1 element set, because  $|y| = |x| + 1, x \subset y$ . This implies that  $\text{Stab}(x, y) = \text{Stab}(x) \cap \text{Stab}(y \setminus x) = \text{Stab}(x) \cap \{e\} = \{e\}$ , for all  $(x, y) \in \mathcal{E}^1(B_n)$ , as claimed.  $\square$

**Lemma 8.3.** *The number  $|(\mathcal{E}^1(B_n)/G)_i| = \binom{n-1}{i}$ .*

*Proof.* Since there are  $\binom{n}{i}$  elements  $x \in (B_n)_i$ , each with  $n - i$  elements  $y \in (B_n)_{i+1}$  with  $x \subset y$ , there are a total of  $\binom{n}{i} \cdot (n - i)$  elements  $(x, y) \in \mathcal{E}^1(B_n)_i$ . By Proposition 8.2, we have  $|\text{Stab}(x, y)| = 1$ . Therefore, by the Orbit Stabilizer Theorem, it follows  $|G(x, y)| = n$ . So, all orbits have the same size. From this,

$$|\mathcal{E}^1(B_n)_i/G| = |\mathcal{E}^1(B_n)_i|/|G| = \frac{\binom{n}{i} \cdot (n - i)}{n} = \binom{n-1}{i}.$$

$\square$

**Lemma 8.4.** *For  $G$  a group let  $H \subset G$  be the subgroup such that  $hx = x$  for all  $x \in X, h \in H$ . Then,  $H$  is a normal subgroup of  $G$  and if  $|G/H| = n$ , then  $\mathcal{E}^1(B_n)/G = \mathcal{E}^1(B_n)/(G/H)$ .*

*Proof.* First, we check  $H$  is normal. For any  $g \in G, x \in x$  it follows  $ghg^{-1}x = gh(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = x$ , and so  $ghg^{-1} \in H$ . Hence,  $G/H$  is a group.

Then, observe  $Gx = (G/H)x$  because  $H$  acts trivially. Therefore,  $\mathcal{E}^1(B_n)/G \cong \mathcal{E}^1(B_n)/(G/H)$ , since  $G, G/H$  have exactly the same orbits.  $\square$

**Lemma 8.5.** *All faithful transitive abelian groups  $A$  acting on the set  $[n]$  have  $|A| = n$ .*

*Proof.* This is a well known, easy to prove, result. See for instance [?]  $\square$

**Corollary 8.6.** *Any abelian group  $A$  which acts transitively on  $[n]$  has*

$$|(\mathcal{E}(B_n)/A)_i| = \binom{n-1}{i}.$$

*Proof.* By Lemma 8.4, we can assume  $A$  acts faithfully as well. By Lemma 8.5,  $|A| = n$ . Then, by Lemma 8.3,  $|(\mathcal{E}(B_n)/A)_i| = \binom{n-1}{i}$ .  $\square$

## 9. QUOTIENT BY THE CYCLIC GROUP

In this section, we show that for  $G = C_n$  the cyclic group of order  $n$ , the poset  $\mathcal{E}^1(B_n/G)$  is rank symmetric and rank unimodal. The rank symmetric part is obvious. From Corollary 8.6, we know that for  $G = C_n$ , the size of the  $i^{\text{th}}$  rank of the poset  $\mathcal{E}^1(B_n)/C_n$  is

$$|(\mathcal{E}^1(B_n)/C_n)_i| = \binom{n-1}{i}.$$

We will bound  $|(\mathcal{E}^1(B_n/C_n))_i|$  by  $|(\mathcal{E}^1(B_n)/C_n)_i|$  to obtain the unimodality.



**Notation 9.1.** We fix an arbitrary  $n$  to start with, set

$$Q(n) = \mathcal{E}^1(B_n)/C_n, \quad P(n) = \mathcal{E}^1(B_n/C_n)$$

and let

$$q(i, n) = |Q(n)_i|, \quad p(i, n) = |P(n)_i|.$$

When there is no ambiguity of which  $n$  we are referring to, we denote  $Q(n), P(n)$  respectively by  $Q, P$ , and write  $q_i = q(i, n), p_i = p(i, n)$ .

*The quotient  $B_n/C_n$  is the well-studied necklace poset. The elements in the poset are represented by  $n$  filled or empty beads ordered cyclically. Label the positions of the beads in the necklace by  $1, 2, \dots, n$ . More specifically, since each element  $x \in B_n$  is represented by a sequence of empty or filled beads, an element  $G(x) \in B_n/C_n$  is represented by such a sequence up to rotational equivalence. Similarly, an element  $(Gx, Gy) \in (\mathcal{E}^1(B_n)/C_n)_i$  where  $x \leq y$  can be regarded as a necklace where  $i+1$  beads out of  $n$  are filled (which represent  $y$ ), and 1 of the filled bead is distinct from all others (so the other  $i$  beads represent  $x$ ).*

**Notation 9.2.** We fix  $\sigma_0 = (1\ 2\ \dots\ n)$ , the generator of the group  $C_n$ .

**Notation 9.3.** We shall abuse notation by writing  $(Gx, Gy)$  in place of  $(Gx, Gy) \cap \mathcal{E}^1(B_n)$  as sets, and hence, viewing  $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}^1(B_n) \subset \mathcal{E}^1(B_n)$ . Namely,

$$(Gx, Gy) = \{(\sigma x, \tau y) \in \mathcal{E}^1(B_n) : \sigma, \tau \in G; \sigma x \leq \tau y\}.$$

Similarly, we view

$$G(x, y) = \{(gx, gy) \in \mathcal{E}^1(B_n) : g \in G\}$$

**Definition 9.4.** Let  $\sigma \in G$ . We call each subset  $s \subset x \in B_n$  that is fixed by  $\sigma$  a *full cycle* (under the action of  $\sigma$ ). Let  $\mathcal{S}$  be the set of all full cycles in  $x$ , we call  $x \setminus (\cup_{s \in \mathcal{S}} s)$  a *tail cycle* (under the action of  $\sigma$ ).

**Lemma 9.5.** Suppose that  $\sigma_0^r x \leq y$  for some  $(x, y)$ , where  $x \leq y$  and  $r < n/2$ . Also suppose that there is no  $g \in G$  such that  $gx = \sigma_0^r(x)$  and  $gy = y$ . Then  $r$  is the only integer  $1 \leq r < n/2$  such that  $\sigma_0^r(x) \leq y$ .

*Proof.* By assumption,  $x$  has a tail cycle under the action of  $\sigma_0^r$ , for otherwise  $x$  is fixed by  $\sigma_0^r$ , because then  $g = id$  satisfies  $gx = \sigma_0^r x, gy = y$ . Additionally,  $y$  must have some tail cycle under the action of  $\sigma_0^r$ , for otherwise  $y$  is fixed under  $\sigma_0^r$  and we can take  $g = \sigma_0^r$ . Call  $\gamma$  the tail cycle of  $y$ , so  $\gamma$  consists of some filled beads in the necklace, one of which is the element  $z = y \setminus x$ . Let  $\sigma$  be any rotation such that  $\sigma x \leq y$ . then  $\sigma$  has to take a full cycle of  $x$  (under  $\sigma_0^r$ ) to another full cycle (under  $\sigma_0^r$ ), so  $\sigma(\gamma \setminus z) \leq \gamma$ . Recall that  $\sigma_0^r(\gamma \setminus z) \leq \gamma$  by assumption, so the elements in  $\gamma$  are  $r$ -positions away from each other, namely  $\gamma = \{z, z-r, z-2r, \dots, z-lr\}$ , but  $z+r \notin \gamma$ , since  $\gamma$  a tail cycle. Since  $\sigma(\gamma \setminus z) \leq \gamma$ , there is some  $m$  for which  $\sigma(z-mr) = z$ . But then,  $z - (m-1)r \in x$ , and  $\sigma(z - (m-1)r) = z + r \notin y$ , a contradiction.  $\square$

**Lemma 9.6.** Let  $G = C_n$ , and  $q_i = q(i, n), p_i = p(i, n)$  as usual, then

$$q_i - p_i = |\{G(x, y) : \exists \sigma_0^r x \leq y \text{ where } r < n/2, \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^r x, gy = y\}|.$$

*Proof.* From definition,  $q_i$  counts number of the disjoint subsets  $G(x, y)$  in  $\mathcal{E}^1(B_n)$  and  $p_i$  counts the number of disjoint subsets  $(Gx, Gy)$ . Hence difference of  $q_i - p_i$  is counting the difference of the number of  $G$ -orbits  $G(x, y)$  and the number of  $(Gx, Gy) \in \mathcal{E}^1(B_n/C_n)$ . Note that as subsets of  $\mathcal{E}^1(B_n)$ ,  $G(x, y) \subset (Gx, Gy)$ , so the problem of counting  $q_i - p_i$  is the same as that of counting the number of distinct  $G$ -orbits  $G(x, y)$  with non-identity  $\sigma \in G$  such that  $\sigma x < y$ .

It is clear that the  $G$ -orbits correspond to the (rotational equivalence classes of) fillings of the necklaces, where we fill  $i+1$  elements with 1 of them being marked as the distinct element. Lemma 9.5 tells us that each subset  $(Gx, Gy) \subset \mathcal{E}^1(B_n)$ , contains at most two  $G$ -orbits (also regarded as subsets of  $\mathcal{E}^1(B_n)$ ). The number of  $(Gx, Gy)$  containing exactly two  $G$  orbits is precisely the number of pairs of  $G$ -orbits  $(G(x, y), G(\sigma x, y))$  (where  $x \leq y$  and  $\sigma x < y$ ) with  $G(x, y) \neq G(\sigma x, y)$ . The number of such pairs is then the number of  $G(x, y)$

such that  $\exists r, r < \frac{n}{2}, G(\sigma_0^r x, y) \neq G(x, y), \sigma_0^r x \leq y$ . We restrict  $r < n/2$  to avoid double counting pairs. This proves the Lemma.  $\square$

*First we consider a special case where  $n = p$  is a prime, for which we have a simple formula for  $p_i$  based on the formula for  $q_i$ .*

**Lemma 9.7.** *Let  $G = C_p$  where  $p$  is a prime, then  $q_i - p_i = (p - 1)/2$  for  $1 \leq i \leq p - 2$ .*

*Proof.* We consider the action of  $G$  on  $B_p$ . Note that  $p$  being a prime guarantees that the action of any nontrivial  $\sigma \in G$  has no fixed points, since  $\sigma$  is always an  $p$ -cycle in its cycle decomposition. Now suppose  $(x, y) \in \mathcal{E}^1(B_p)_i$  is an element such that  $\sigma x = y$  for some  $\sigma \neq e$ . Then, there is no  $g \in C_p$  such that  $gx = \sigma x$  and  $gy = y$ , since  $gx = \sigma x \Rightarrow g = \sigma$  and  $gy = y \Rightarrow g = e$ . Let  $(Gx, Gy)$  be an element in  $(\mathcal{E}^1(B_p/G))_i$ , in other words, an edge class in the necklace poset with representative  $(x, y)$ . Note that we can also view  $G(x, y)$  as a subset of  $\mathcal{E}^1(B_p)$ , namely  $G(x, y) = \{(gx, gy) : g \in G\}$ . Clearly  $G(x, y) \subset (Gx, Gy)$  as subsets of  $\mathcal{E}^1(B_p)$ . It is clear that the number of distinct  $\sigma \in G$  such that  $\sigma x < y$  is precisely the number of  $G$ -orbits in  $(Gx, Gy)$ . In particular, if the only  $\sigma$  such that  $\sigma x < y$  is identity, then  $(Gx, Gy) = G(x, y)$  as subsets of  $\mathcal{E}^1(B_p)$ .

, and we claim that for  $i \neq 1, p - 1$ , this number is  $(p - 1)/2$ .

First consider  $\sigma = \sigma_0$  and  $(x, y)$  such that  $x < y$  and  $\sigma x < y$ , i.e., we take out the distinct filled bead from  $y$  in the necklace, and rotate  $y$  by a 1-click rotation, then the remaining  $i$  filled beads remain in  $\sigma y$ . It is clear that the only possibility for this to occur is if we have  $i + 1$  consecutive filled beads, i.e.,  $y = \{1, 2, \dots, i + 1\}$  and  $x = \{1, 2, \dots, i\}$  (up to rotational equivalence by actions of  $G$ ).

In general, for any  $\sigma = \sigma_0^j, j < n/2$ , there is a precisely one orbit  $G(x, y)$  such that  $x < y$  and  $\sigma x < y$ . Notice that the necklace  $\{1, j + 1, 2j + 1, \dots\}$  and  $\{1, (n - j) + 1, 2(n - j) + 1, \dots\}$  are the same necklaces, so it suffices to count the number of  $(x, y)$  with  $x < y, \sigma_0^j x < y$ , with  $j < n/2$ . It is also easy to show that, for all distinct  $j < n/2$ , the necklaces represented by  $\{1, j + 1, 2j + 1, \dots, i \cdot j + 1\}$  are all distinct, since  $n$  is a prime number.  $\square$

**Corollary 9.8.** *Let  $G = C_n$  where  $n$  is a prime, and  $p_i$  as defined above. Then the sequence  $p_i$  is unimodal.*

*Proof.* By the Proposition and the symmetry of  $p_i$ , we only need to show  $p_0 \leq p_1$ , but  $p_0 = 1$ , so the claim holds.  $\square$

*A similar method can be used to bound  $q_i - p_i$  for  $n$  not necessarily prime. Let  $\lambda_i = q_i - p_i$ , our next goal in the section is to show that, for sufficiently large  $n$ , the difference  $\lambda_{i+1} - \lambda_i$  is bounded above by  $q_{i+1} - q_i$ .*

*Note that this suffices to show that the  $p_i$  are unimodal, since*

$$p_{i+1} - p_i = (q_{i+1} - \lambda_{i+1}) - (q_i - \lambda_i) = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0.$$

**Definition 9.9.** Fix  $r < n/2$  to be an integer and let  $G(x, y)$  be an  $G$ -orbit of edges. We call  $G(x, y)$  a *special orbit* if  $\sigma_0^r x \leq y$ , but there does not exist  $g \in C_n$  such that  $gx = \sigma_0^r x$ , and  $gy = y$ .

**Lemma 9.10.** *Let  $G = C_n$  and let  $\lambda_i, q(i, m)$  be as defined above, then*

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n, i+1) \\ 3 \leq k}} q\left(\frac{(i+1)}{k}, \frac{n}{k}\right)$$

*Proof.* Recall from Lemma 9.6 that

$$q_i - p_i = |\{G(x, y) : \exists \sigma_0^r x \leq y \text{ where } r < n/2, \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^r x, gy = y\}|.$$

I think (i.e...) actually makes things more confusing, rather than clearer. Consider removing the clause after i.e.?

Consider using a word other than special? Maybe you could say a double orbit, or something like that?

Perhaps consider breaking this proof into a couple lemmas. First I might say that  $\lambda_{i+1} - \lambda_i$  is bounded by the number of elements  $y$  with tail cycles with respect to some  $\sigma_0^r$  so that  $y \leq y'$  with  $y' \in \text{Stab}(\sigma_0^r)$ . Then, I would give the bound in the statement of this lemma regarding the size of this set.

Hence, to count  $q_i - p_i = \lambda_i$ , we only need to count the number of distinct subset  $G(x, y) \subset \mathcal{E}^1(B_n)$  for which there exists  $r < \frac{n}{2}$  so that  $G(x, y)$  is special with respect to  $\sigma_0^r$ .

Note that, for each  $r < n/2$  such that  $(r, n) = 1$ , there is precisely one such  $G$ -orbit, since in this case  $\sigma_0^r$  is a full  $n$ -cycle [See proof of Lemma 9.7]. This holds regardless of  $i$ .

Now consider  $r < n/2$  such that  $(r, n) > 1$ . In this case,  $\sigma_0^r$  fixes elements of the form  $\{s, s+r, s+2r, \dots\}$  for any starting point  $s$ , which are proper subsets of  $[n]$ . For  $y \in B_n$ , one may write  $y$  as the union of full cycles and at most one tail cycle under the action of  $\sigma_0^r$ . Now, if  $y \in B_n$  has no tail cycles under the action of  $\sigma_0^r$  then  $(\sigma_0^r x, \sigma_0^r y) = (\sigma_0^r x, y) \in G(x, y)$ . This means that if  $G(x, y) \in q_i - p_i = |\{G(x, y) : \exists \sigma_0^t x < y \text{ where } t < n/2, \text{ but } \nexists g \text{ s.t. } gx = \sigma_0^t x, gy = y\}|$ , then it must be  $\sigma_0^s x < y$  with  $s \neq r, s < n/2$ .

So,  $p_i - q_i$  is equal to the number of  $G(x, y)$  such that there exists  $r$  with  $x < y, \sigma_0^r x < y$  and  $y$  has a tail cycle under the action of  $\sigma_0^r$ . Now let us focus on the difference of  $\lambda_{i+1} - \lambda_i$ . Note that the difference only occurs when we count  $r \leq n/2$  and  $(r, n) > 1$ . Let  $G(x, y)$  be a  $G$ -orbit where  $x < y$  such that  $|y| = i+1$ , and let  $\sigma_0^r(x) < y$ . Suppose that there is no  $g$  such that  $gx = \sigma_0^r(x)$  and  $gy = y$ . From the discussion above, we know that there is a tail cycle of  $y$ , written  $w := \{s, s+r, s+2r, \dots, s+lr\}$  for some  $l$ , and  $x = y \setminus \{l\}$ . Suppose that  $y \cup \{s-r\}$  (mod  $n$  whenever necessary) still has a tail cycle  $w \cup \{(s-r)\}$ , with respect to  $\sigma_0^r$ . Then, if we let  $y' = y \cup \{(s-r)\}$  and  $x' = x \cup \{(s-r)\}$ , the orbit  $G(x, y')$  where  $|y'| = i+2$  is also a special orbit, thus does contribute 1 when we count  $\lambda_{i+1}$ . In this case, our counting process for  $\lambda_{i+1}$  and  $\lambda_i$  both increase by 1, so such special orbit does not contribute to  $\lambda_{i+1} - \lambda_i$ . Now suppose that adding  $s-r$  completes the sequence  $z$  into a tail cycle, namely  $\{s, s+r, \dots, s-r, s\}$ . Now  $y'$  will not be a special orbit anymore, since  $\sigma_0^r$  fixes  $y'$ . In this situation, we increase 1 for  $\lambda_i$  only. If we choose to ignore this scenario, we will get an upper bound for  $\lambda_{i+1} - \lambda_i$ .

Now return to the situation where  $G(x, y)$  is not a special orbit, but there exists  $s \in [n]$  so that  $G(y, y \cup s)$  is a special orbit, where  $y' = y \cup \{s'\}$  for any  $s' \notin y$ . We need to count how many times this happens. First we note that this happens when  $d|n$ , and  $(n/d)|(i+1)$  where  $d = (r, n)$ . Note that, for different  $r$  such that  $(r, n)$  remains the same, there is only family of such special orbits, since the  $x$  part will be  $d$ -rotationally symmetric. We claim that, this is at most counting, over all  $d|n$ , where  $1 < d < n$  and  $(n/d)|(i+1)$ , the sum  $\sum q\left(\frac{(i+1)d}{n}, d\right)$ . This is because we have at most  $q\left(\frac{(i+1)d}{n}, d\right)$  ways to insert the tail cycle (with precisely 1 distinct bead) in the block of  $d$  elements.

We can introduce  $k = n/d$  and rewrite the sum as

$$\sum_{\substack{k|(i+1, n), \\ 1 < k < n}} q\left(\frac{i+1}{k}, \frac{n}{k}\right).$$

Note that in our sum,  $k$  can never be 2, since otherwise the tail cycle of  $x$  has precisely 1 element, and then  $y$  is fixed (consisting of full cycles). So this proves the assertion that

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n, i+1) \\ 3 \leq k}} q\left(\frac{i+1}{k}, \frac{n}{k}\right).$$

I don't understand what it means by an orbit "contributes to the difference." What you really want to say is that an orbit is in a certain set, and the difference is the size of that set. I have edited the previous paragraphs to make this clearer, can you fix the rest of the proof please?

what does "contribute" mean? Are you are trying to say it lies in a certain set?

Again, I don't understand

What does this mean? It sounds like you're thinking of  $\lambda_i$  as a variable, which changes size as you look at different  $G$ -orbits.

How can you choose to ignore a scenario? What does that mean?

Can you make this sentence more precise? I don't understand what the "x" part is, what a "family" of such special orbits is, etc.

what is this? can you write it precisely as a set?

what index is this sum over?

what does it mean to insert a tail cycle, and what is a block of elements?

with respect to which group element?

Note that, from the process of calculating  $\lambda_{i+1} - \lambda_i$ , we can potentially obtain a better bound if we need to.  $\square$

Aaron: Consider removing this sentence, or at the very least inserted after the proof as a remark

**Corollary 9.11.** *The poset  $\mathcal{E}^1(B_n/C_n)$  is rank symmetric and rank unimodal.*

*Proof.* We need to show that for  $G = C_n$ , the  $p_i$  are unimodal. So, we only need to bound

$$\sum_{\substack{k|(n,i+1) \\ 3 \leq k < n/2}} q\left(\frac{i+1}{k}, \frac{n}{k}\right). \text{ It is clear that for sufficiently large } n, \text{ say for } n \geq 9,$$

$$q((i+1)/k, n/k) = \binom{n/k - 1}{(i+1)/k - 1} \leq \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil}.$$

Since  $k \leq i$ , and  $i+1 \geq n/2$ , we coarsely bound the sum by

$$\frac{n}{2} \cdot \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil}.$$

We want to show that this is smaller than the difference of

$$q_i - q_{i-1} = \frac{(n-1)!}{(i)!(n-i)!} - \frac{(n-1)!}{(i-1)!(n-i+1)!} = \binom{n}{i} \frac{n-2i}{n} \geq \binom{n}{i} \frac{2}{n}.$$

Namely, we want to show that, for sufficiently large  $n$ ,

$$\left(\frac{n}{2}\right)^2 \cdot \binom{\lceil n/3 - 1 \rceil}{\lceil (i+1)/3 - 1 \rceil} \leq \binom{n}{i}.$$

This bound works for  $n \geq 12$ . For smaller  $n$  the claim can be easily checked.

Now, by Lemma 9.10, we know that

$$p_{i+1} - p_i = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0,$$

which proves that the  $p_i$  are increasing for  $i < n/2$ . By symmetry, the  $p_i$  are unimodal.  $\square$

## 10. QUOTIENT BY THE DIHEDRAL GROUP

In this section, we show a similar result for the Dihedral groups that act naturally on  $B_n$ . Namely, for  $G = D_{2n}$  the dihedral group of order  $2n$ , the poset  $\mathcal{E}^1(B_n/G)$  is rank symmetric and rank unimodal.

**Notation 10.1.** In this section, we fix an arbitrary  $n$  again, set

$$Q = \mathcal{E}^1(B_n)/D_{2n}, \quad P = \mathcal{E}^1(B_n/D_{2n})$$

and let

$$q_i = |(\mathcal{E}^1(B_n)/D_{2n})_i|, \quad p_i = |(\mathcal{E}^1(B_n/D_{2n}))_i|.$$

**Lemma 10.2.** *Let  $G = D_{2p}$  or  $G = D_{4p}$  where  $p$  is prime, and  $p_i, q_i$  as defined above. Then  $p_i = q_i$ . In other words, the action of  $D_{2p}$  (respectively  $D_{4p}$ ) on  $B_p$  (respectively  $B_{2p}$ ) is CCT.*

*Proof.* The proof of the Lemma is similar to the proof of Lemma 9.7. For example, the group  $G = D_{2n}$  contains  $C_n$  as a subgroup and is generated by  $C_n$  with arbitrary reflections. The additional reflections guarantee that there are no special orbits  $G(x, y)$  for any  $r < n/2$ , which implies  $p_i = q_i$ . Similarly,  $D_{4n}$  is generated by the rotational subgroup  $C_{2n}$  and reflections, and it is not hard to see that for any pair of elements  $(x, z), (y, z) \in \mathcal{E}^1(B_n)$  such that  $x$  and  $y$  differ by a rotation, which does not fix  $z$ , then there always exists a reflection  $\tau$  such that  $\tau x = y$  and  $\tau$  fixes  $z$ . Following the proof of 9.7, we know that the action of  $D_{4n}$  on  $B_n$  is CCT.  $\square$

Maybe call this a proposition or theorem instead of corollary? It doesn't really follow immediately from other things, and it's the main result we're showing

This proof needs more work (self)

*Remark 10.3.* Let  $G$  be as in Lemma 10.2. We remark that, in addition to unimodality, Corollary ?? implies that the poset  $\mathcal{E}^1(B_n/G)$  is Peck.

**Lemma 10.4.** *We have an explicit formula for  $q_i = |(\mathcal{E}^1(B_n)/D_{2n})_i|$ :*

$$q_i = \frac{1}{2} \left( \binom{n-1}{i} + \frac{1}{2} [(-1)^{n(i+1)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1} \right)$$

*Proof.* Consider an element  $(x, y) \in \mathcal{E}^1(B_n)$  where  $x \triangleleft y$ . Note that any element  $\tau$  that fixes both  $x$  and  $y$  has to fix the difference  $y \setminus x$ , which is a one-element set. Since  $G = D_{2n}$ , we know that  $1 \leq |\text{Stab}(x, y)| \leq 2$ . Let  $\mu_1$  be the number of  $G$ -orbits with the trivial stabilizer and  $\mu_2$  be the number of  $G$ -orbits with the stabilizer of size 2, which contains the identity and a reflection. By the Orbit-Stabilizer Theorem, each orbit with the trivial stabilizer is of size  $|D_{2n}|/1 = 2n$ , while all other orbit is of size  $|D_{2n}|/2 = n$ . Therefore

$$\mu_1 \cdot 2n + \mu_2 \cdot n = |\{(x, y)\}| = \binom{n}{i+1} \cdot \binom{i+1}{1} = n \binom{n-1}{i}.$$

Our goal is to calculate  $q_i = \mu_1 + \mu_2$ , so it remains to calculate  $\mu_2$ , which counts the number of  $G$ -orbits such that the reflection also fixes  $x$ . Without loss of generality, we may assume that  $y \setminus x = \{1\}$ . We break into cases:

- (1) If  $n$  is odd and  $i+1$  is even, then it is clear that no reflections fix  $x$  for any  $x$ , so  $\mu_2 = 0$ .
- (2) For all other cases, there are precisely  $\lceil n/2 \rceil - 1$  places to insert  $\lceil (i+1)/2 \rceil - 1$  elements of  $[n]$  to form  $x$ .

This gives the desired formula  $\mu_2 = \frac{1}{2} [(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1}$ . Therefore,

$$q_i = \frac{1}{2} \left( \binom{n-1}{i} + \frac{1}{2} [(-1)^{n(i+2)} + 1] \cdot \binom{\lceil n/2 \rceil - 1}{\lceil (i+1)/2 \rceil - 1} \right)$$

□

*Remark 10.5.* An alternate proof of Lemma 10.4 follows directly from Polya's Theorem 7.6, by counting the number of ways to color  $n$  elements with three colors, so that there are  $i$  of the first color, 1 of the second color, and  $n - i - 1$  of the third color.

**Corollary 10.6.** *The poset  $\mathcal{E}^1(B_n/D_{2n})$  is rank symmetric and rank unimodal.*

*Proof.* The proof is similar to the proof of Lemma 9.10 and Corollary 9.11. In fact, we can bound the difference of  $\lambda_{i+1} - \lambda_i$  by precisely the same bound used in Lemma 9.10, since the difference we get here (where  $G = D_{2n}$ ) is smaller than the previous difference (where  $G = C_n$ ). Then, a similar proof shows that the difference of  $q_{i+1} - q_i$  obtained in Lemma 10.2 is significantly larger than the upper bound. □

## 11. A $q$ -ANALOG

*In this section we consider a  $q$ -analog of the Boolean algebra and prove Lemma 11.4, which is a  $q$ -analog to Lemma 8.3.*

*Let  $q$  be a prime and  $B_n(q)$  be the poset of all  $\mathbb{F}_q$ -subspaces of  $V_n(q) := (\mathbb{F}_q)^n$ , graded naturally by dimensions.*

*Let  $C_n(q)$  be the multiplicative subgroup of the finite field  $\mathbb{F}_{q^n}$ , so  $C_n(q)$  acts  $\mathbb{F}_q$ -linearly on  $\mathbb{F}_{q^n}$  by multiplication, which is isomorphic to  $V_n(q)$  as an  $\mathbb{F}_q$  vector space. The action of  $C_n(q)$  on  $V_n(q)$  induces an order preserving and rank preserving action on the poset  $B_n(q)$ . The poset  $B_n(q)/C_n(q)$  is the  $q$ -analog of the necklace poset.*

*The following Lemma is a standard result. For a proof of the lemma, we refer the reader to, for example, [Stall].*

**Lemma 11.1.** *The number  $|B_n(q)_i|$ , that is to say, the number of  $i$  dimensional subspace in  $V_n(q)$ , is  $\binom{n}{i}_q$*

**Notation 11.2.** Define the graded poset  $T = \mathcal{E}^1(B_n(q))/C_n(q)$  and let  $t_i = |T_i|$ .

**Lemma 11.3.** *For any  $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$ , so that  $(V_x, V_y) \in \mathcal{E}^1(B_n(q))/C_n(q)$ , then  $\text{Stab}_{C_n(q)}(V_x, V_y) \cong \mathbb{F}_q^\times$ . In particular,  $|\text{Stab}_{C_n(q)}(V_x, V_y)| = q - 1$ .*

*Proof.* We claim that if  $\tau \in C_n(q)$  such that  $\tau V_x = V_x$  and  $\tau V_y = V_y$ , then  $\tau \in \mathbb{F}_q^\times$ . First, it is clear that any scalar  $c \in \mathbb{F}_q^\times$  fixes any  $V \in B_n(q)$ . Now assume  $V_x \subset V_y$  and  $\tau V_x = V_x, \tau V_y = V_y$  as in the claim. Then, in particular  $\tau$  permutes all the elements in  $V_x$ , i.e.,  $\tau : V_x \rightarrow V_x$  is an isomorphism of  $\mathbb{F}_q$ -vector spaces. Pick a nonzero element  $a \in V_x$ , it is clear that

$$\tau^{q^i}(a) = \tau^{q^i}a = \tau a.$$

The latter equation implies that  $\tau^{q^i} - \tau = 0$ , so  $\tau$  satisfies the polynomial  $X^{q^i} - X \in \mathbb{F}_q[X]$ . Similarly, since  $\tau V_y = V_y$ ,  $\tau$  is a root of the polynomial  $X^{q^{i+1}} - X \in \mathbb{F}_q[X]$ . Let  $f(X)$  be a monic irreducible polynomial of  $\tau$  over  $\mathbb{F}_q$ , then  $\deg f(X) \mid i$  and  $\deg f(X) \mid (i+1)$ . This shows that  $\deg f(X) = 1$ , in another word,  $\tau$  is a  $\mathbb{F}_q$ -scalar.  $\square$

**Lemma 11.4.** *Let  $q$  be a prime, then*

$$|(\mathcal{E}^1(B_n(q))/C_n(q))_i| = \binom{n-1}{i}_q.$$

*Proof.* Let  $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$  so that  $(V_x, V_y) \in \mathcal{E}^1(B_n(q))$ . By Lemma 11.3, for any  $(V_x, V_y) \in \mathcal{E}^1(B_n(q))_i$ ,  $|\text{Stab}_{C_n(q)}(V_x, V_y)| = q - 1$ . By the Orbit-Stabilizer Theorem, we know that the size of each orbit in  $\mathcal{E}^1((B_n(q))_i)$  under the action of  $C_n(q)$  is

$$|C_n(q)|/(q-1) = \frac{q^n - 1}{q - 1} = (n)_q.$$

Now we can calculate the number  $q_i$ , which is the total number of elements in  $(B_n(q))_i$  divided by the size of each orbit:

$$q_i = \frac{\binom{n}{i+1}_q \binom{i+1}{1}_q}{(n)_q} = \binom{n-1}{i}_q.$$

$\square$

## 12. THE OBJECT $\mathcal{E}^1(B_n)$ .

*In this section, we explicitly compute the raising operators corresponding to both  $\mathcal{H}^1(B_n), \mathcal{E}^1(B_n)$  and explicitly show that  $\mathcal{E}^1(B_n)$  is unitary Peck by showing certain raising maps are invertible.*

**Notation 12.1.** For this section, we will use  $M^{n-2i-1} = \mathcal{E}^1(U)^{n-2i-1}$ , where  $\mathcal{E}^1(U)^{n-2i-1} : \mathcal{E}^1(B_n)_i \rightarrow \mathcal{E}^1(B_n)_{n-i-1}$  is the Lefschetz map, as defined in ??.

**Notation 12.2.** For the remainder of this section, we shall take  $|a| = n - i - 1, |b| = n - i, |x| = i, |y| = i + 1$ . Let  $k = n - 2i - 1$ , that is,  $k = |a| - |x|$ . Additionally, whenever we write an expression of the form  $(x, y)$  or  $(a, b)$ , it is assumed that  $x \triangleleft y, a \triangleleft b$ .

**Theorem 12.3.** *Defining  $L$  to be the Lefschetz map  $\mathcal{H}^1(B_n) \rightarrow \mathcal{H}^1(B_n)$ , we have  $L^{n-2i-1} : \mathcal{H}^1(B_n)_i \rightarrow \mathcal{H}^1(B_n)_{n-i-1}$ , and explicitly*

$$L^{n-2i-1}(x, y) = k! \sum_{\substack{y \not\subset a, \\ x \subset a, \\ y \subset b}} (a, b).$$

*Proof.* Note that the conditions  $y \not\subset a, x \subset a, y \subset b$  are equivalent to  $(a, b) >_{\mathcal{H}^1(B_n)} (x, y)$ . Clearly, if  $(a, b) \not>_{\mathcal{H}^1(B_n)} (x, y)$ , then the coefficient of  $(a, b)$  in  $L^{n-2i-1}(x, y)$  is 0. So, to complete the proof, it suffices to show that if  $(a, b) >_{\mathcal{H}^1(B_n)} (x, y)$  then the coefficient of  $(a, b)$  in  $L^{n-2i-1}(x, y)$  is  $k!$ . However, this coefficient is precisely the number of sequences  $(x, y) = (x_0, y_0) \triangleleft_{\mathcal{H}^1(B_n)} (x_1, y_1) \triangleleft_{\mathcal{H}^1(B_n)} \cdots \triangleleft_{\mathcal{H}^1(B_n)} (x_k, y_k) = (a, b)$ . By definition of



$\mathcal{H}^1$ , we must have that  $y_k \setminus x_k = y \setminus x$  for all  $k$ . Therefore, the number of such sequences is equal to the number of sequences  $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$ , since choosing the  $x_i$  determine  $y_i$  because  $y_i = x_i \cup (y \setminus x)$ . Finally, the number of such sequences  $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$ , is equivalent to the number of ways to add the elements in  $a \setminus x$  to  $x$ . This is because each sequence  $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$  is determined uniquely by the singletons  $x_{i+1} \setminus x_i$ ,  $1 \leq i \leq k$ . Since in total we are adding  $k$  elements to  $x$  in order to obtain  $a$ , there are  $k!$  ways to do this. Therefore, coefficient of  $(a, b)$  in  $L^{n-2i-1}(x, y)$  is  $k!$   $\square$

**Theorem 12.4.** *The map  $M$  satisfies*

$$M^{n-2i-1}(x, y) = (2^k - 1)(k - 1)! \sum_{y \subset a} (a, b) + k! \sum_{\substack{y \not\subset a, \\ x \subset a, \\ y \subset b}} (a, b)$$

*Proof.* For a particular  $(x, y), (a, b)$ , if either  $y \subset a$  or  $y \not\subset a, x \subset a, y \subset b$ , then  $(a, b) \not\prec_{\mathcal{H}^1(B_n)} (x, y)$ , and so the coefficient of  $(a, b)$  in  $M^{n-2i-1}(x, y)$  is 0.

Clearly, we cannot have both  $y \subset a$  and  $y \not\subset a, x \subset a, y \subset b$ , hold at the same time. So, suppose  $y \not\subset a, x \subset a, y \subset b$ . This implies that  $b \setminus a = y \setminus x$ . Then, the coefficient of  $(a, b)$  in  $M^{n-2i-1}(x, y)$  is precisely the number of sequences  $(x, y) = (x_0, y_0) \leq_{\mathcal{E}^1(B_n)} (x_1, y_1) \leq_{\mathcal{E}^1(B_n)} \cdots \leq_{\mathcal{E}^1(B_n)} (x_k, y_k) = (a, b)$ . However, since  $y \setminus x = b \setminus a$ , it must be that  $y_k \setminus x_k = y \setminus x$  as well. Therefore, the number of such sequences is equal to  $k!$ , as was shown in the proof of Theorem 12.3.

To complete the proof, we need to show that if  $y \subset a$  then the coefficient of  $(a, b)$  in  $M^{n-2i-1}(x, y)$  is  $(2^k - 1)(k - 1)!$ . Equivalently, we need to show that the number of sequences

$$(12.1) \quad (x, y) = (x_0, y_0) \leq_{\mathcal{E}^1(B_n)} (x_1, y_1) \leq_{\mathcal{E}^1(B_n)} \cdots \leq_{\mathcal{E}^1(B_n)} (x_k, y_k) = (a, b)$$

is  $(2^k - 1)(k - 1)!$ .

For the moment, fix  $j$  and consider the set of all sequences of the form in Equation (12.1) such that  $(a \setminus b) \cup y_{j-1} = y_j$ . First, let us show the number of such with this  $j$  fixed is  $(k - 1)!2^j$ . To do this, start by considering the number of sequence  $y_0, \dots, y_k$  such that  $y = y_0 \leq_{B_n} y_1 \leq_{B_n} \cdots \leq_{B_n} y_k = b$ . Since we enforce  $y_j = y_{j-1} \cup (a \setminus b)$ , the number of such sequences is precisely the number of ways to order the elements  $a \setminus y$ . Since  $|y| - |a| = k - 1$ , there are  $(k - 1)!$  such ways. Additionally, for  $l < j$  we must have  $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$  or  $x_{l+1} = x_l \cup (y_l \setminus x_l)$ . Either of these is possible at every step. Additionally, for  $l \geq j$ , we must have  $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$ . So for any fixed sequence of  $y_k$  with so that  $j$  is minimal with  $(a \setminus b) \in y_j$ , there are precisely  $2^{j-1}$  possible sequences  $x = x_0 \leq_{B_n} x_1 \leq_{B_n} \cdots \leq_{B_n} x_k = a$  so that  $(x, y) = (x_0, y_0) \leq_{\mathcal{E}^1(B_n)} (x_1, y_1) \leq_{\mathcal{E}^1(B_n)} \cdots \leq_{\mathcal{E}^1(B_n)} (x_k, y_k) = (a, b)$ .

So, in total, there are  $2^j(k - 1)!$  sequences of the form in (12.1) with  $(a \setminus b) \cup y_{j-1} = y_j$ . Now, there clearly must be some such  $j$ ,  $1 \leq j \leq k$ . Therefore, the coefficient we are looking for is  $\sum_{j=1}^k 2^{j-1}(k - 1)! = (2^k - 1)(k - 1)!$ , as claimed.  $\square$

**12.1. Proof that  $\mathcal{E}(B_n)$  is unitary Peck.** *In this subsection, we will show the rows of  $M$  form a basis by showing we can make a change of basis to a map which takes  $M^{n-2i-1}$  to  $L^{n-2i-1}$ . Since we know  $L^{n-2i-1}$  is an isomorphism, it will follow that  $M^{n-2i-1}$  is as well.*

**Notation 12.5.** Let  $\beta = \frac{2^k - 1}{k}$ . Denote

$$v_{(a,b)} = \beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y).$$

Note that  $v_{(a,b)}$  are simply the rows of  $M^{n-2i-1}$ , each divided by the constant  $k!$ .



**Notation 12.6.** For any set  $s$  of size at least  $n - i$ ,

$$z_s = \frac{1}{\beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}} \sum_{b \subset s, a \subset b} v_{(a,b)}$$

**Lemma 12.7.** For any set  $s$  of size at least  $n - i$ , we have  $z_s = \sum_{y \subset s} (x, y)$ . In particular,  $\sum_{y \subset s} (x, y)$  lies in the span of  $v_{(a,b)}$

*Proof.* We have

$$\begin{aligned} & \left( \beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1} \right) z_s \\ &= \sum_{b \subset s, a \subset b} v_{(a,b)} \\ &= \sum_{b \subset s, a \subset b} \left( \beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a} (x, y) \right) \\ &= \sum_{b \subset s, a \subset b} \beta \sum_{y \subset a} (x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\ &= \beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) \sum_{y \subset s} (x, y) + \sum_{b \subset s, a \subset b} \sum_{y \subset b, x \subset a, y \not\subset a} (x, y). \\ &= \beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) \sum_{y \subset s} (x, y) + \binom{|s|-i-1}{n-2i-1} \sum_{y \subset s} (x, y) \\ &= \left( \frac{1}{\beta \binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1) + \binom{|s|-i-1}{n-2i-1}} \right) \sum_{y \subset s} (x, y). \end{aligned}$$

In going between the fourth line and the fifth line, one needs to count the number of  $y$  satisfying  $y \subset a \subset b \subset s$ . If we fix  $s, a$  there are  $(|s| - n + i + 1)$  choices for the element  $b$ , since  $b = a \cup \{s\}$  for  $s \notin a$ . Then, we need to count the number of  $a$  with  $y \subset a \subset s$ . This is exactly  $\binom{|s|-i-1}{n-2i-2}$ . Hence, the total of number of such  $y$  is the product  $\binom{|s|-i-1}{n-2i-2} (|s| - n + i + 1)$ .

In going from the fifth line to the sixth line, for  $y, s$  fixed, we count the number of  $b$  with  $y \subset b \subset s$ . This is exactly  $\binom{|s|-i-1}{n-2i-1}$ .  $\square$

**Notation 12.8.** For any set  $s$  of size at least  $n - i$ , let  $w_s = \sum_{a \subset s, t \notin s} v_{(a, a \cup \{t\})}$ .

**Lemma 12.9.** We have

$$w_s = \sum_{t \notin s} \left( \beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} (x, x \cup \{t\}) + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} (x, x \cup \{t\}) \right).$$

*Proof.*

$$\begin{aligned}
w_s &= \sum_{a \subset s, t \notin s} v_{(a, a \cup \{t\})} \\
&= \sum_{a \subset s, t \notin s} \left( \beta \sum_{y \subset a} (x, y) + \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\
&= \sum_{t \notin s} \left( \sum_{a \subset s} \beta \sum_{y \subset a} (x, y) + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\
&= \sum_{t \notin s} \left( \beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y) + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} (x, y) \right) \\
&= \sum_{t \notin s} \left( \beta \sum_{y \subset s} (x, y) + \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right)
\end{aligned}$$

The equalities between lines three and four, and four and five hold for similar reasons as the equalities between lines four and five, and five and six in [12.7](#)  $\square$

**Notation 12.10.** Let  $u_s = \frac{w_s - (n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} z_s}{\binom{|s| - i}{n - 2i - 1}} + z_s$ .

**Lemma 12.11.** We have  $u_s = \sum_{x \subset s, y \supset x} (x, y)$ . In particular,  $\sum_{x \subset s, y \supset x} (x, y)$  lies in the span of  $v_{(a, b)}$ .

*Proof.* By [12.9](#) and [12.7](#) we have

$$\begin{aligned}
u_s &= \frac{w_s - (n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} z_s}{\binom{|s| - i}{n - 2i - 1}} + z_s \\
&= \frac{\sum_{t \notin s} \left( \beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y) + \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right)}{\binom{|s| - i}{n - 2i - 1}} \\
&\quad - \frac{(n - |s|) \beta \binom{|s| - i - 1}{n - 2i - 2} \sum_{y \subset s} (x, y)}{\binom{|s| - i}{n - 2i - 1}} + \sum_{y \subset s} (x, y) \\
&= \frac{\sum_{t \notin s} \left( \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right)}{\binom{|s| - i}{n - 2i - 1}} + \sum_{y \subset s} (x, y) \\
&= \sum_{t \notin s} \left( \binom{|s| - i}{n - 2i - 1} \sum_{x \subset s} (x, x \cup \{t\}) \right) + \sum_{y \subset s} (x, y) \\
&= \sum_{x \subset s, y \supset x} (x, y)
\end{aligned}$$

The penultimate line is equal to the ultimate line because any element  $(x, y)$  with  $x \subset s$  must either have  $y \subset s$  or else  $y \subset s \cup \{t\}$  for  $t \notin s$ . The two terms on the penultimate line cover precisely these two cases.  $\square$

**Notation 12.12.** For any  $s \subset [n]$ , with  $|s| \leq n - i$ , define  $h_s = z_{[n]} - u_s$ .

**Lemma 12.13.** With  $h_s$  as defined above  $h_s = \sum_{x \not\subset s} (x, y)$

*Proof.* Using 12.7 and 12.11

$$\begin{aligned}
 h_s &= z_{[n]} - u_s \\
 &= \sum_{x \subset y} (x, y) - \sum_{x \subset s, x \subset y} (x, y) \\
 &= \sum_{x \not\subset s} (x, y)
 \end{aligned}$$

□

**Notation 12.14.** For  $I \subset [n]$ , let  $I^c = [n] \setminus I$ , the complement of  $I$  in  $[n]$ .

**Lemma 12.15.** *Let  $I$  be fixed. If  $l_J$  lies in the span of  $v_{(a,b)}$  for  $J \subset I$ , then so does  $l_I$ .*

*Proof.* Let  $I = \{i_1, \dots, i_s\}$ . Let  $A_k = \{x | i_k \in x\}$ . Then,  $\cap_{k=1}^s A_k = \{x | I \subset x\}$ . Using the principle of inclusion exclusion, we can write  $\sum_{x \in \cap_{k=1}^s A_k} (x, y)$  as a sum of terms  $\pm \sum_{x \in \cap_{k \in J} A_k} (x, y)$ , for  $J \subset I$ . Since we are assuming  $\pm \sum_{x \in \cap_{k \in J} A_k} (x, y)$ , lie in the span of  $v_{(a,b)}$  it follows that  $l_I = \sum_{x \in \cap_{k=1}^s A_k} (x, y)$  does as well.

□

**Lemma 12.16.** *For all  $J \subset [n]$ , with  $|J| \leq i$ , we have  $l_J$  lies in the span of  $v_{(a,b)}$ .*

*Proof.* We shall prove this by induction on the size of  $I$ . First, we know  $h_{[n]}$  lies in the span of  $v_{(a,b)}$  and so have completed the base case  $|I| = 0$  since  $h_{[n]} = l_\emptyset$ . So, to complete the proof, it suffices to show that if we know this Lemma holds for all  $|s| > j$  then it holds for  $|s| = j$ . This is exactly what 12.15 proves.

□

**Lemma 12.17.** *For  $|x| = i$ , we have  $l_x = \sum_{y \supset x} (y, x)$ .*

*Proof.* By definition,  $l_{\bar{x}} = \sum_{\bar{x} \subset x, \bar{x} \cap x = \bar{x}, x \subset y} (x, y) = \sum_{\bar{x} \subset y} (\bar{x}, y)$ . Replacing  $\bar{x}$  by  $x$  gives the result.

□

**Notation 12.18.** Define  $m_a = \sum_{x \subset a} l_x$ .

**Lemma 12.19.** *With  $m_a$  as defined above, we have  $m_a = \sum_{x \subset a} (x, y)$ .*

*Proof.* By 12.17, we obtain

$$m_a = \sum_{x \subset a} l_x = \sum_{x \subset a} \sum_{y \supset x} (y, x) = \sum_{x \subset a} (x, y)$$

□

**Notation 12.20.** Denote  $r_a = \sum_{b, b \subset a} v_{(a,b)}$ .

**Lemma 12.21.** *With  $r_a$  as defined above, we have*

$$r_a = ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y).$$

*Proof.*

$$\begin{aligned}
r_a &= \sum_{b \supset a} v_{(a,b)} \\
&= \sum_{b \supset a} \left( \beta \sum_{y \subset a} (x, y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \right) \\
&= \sum_{b \supset a} \beta \sum_{y \subset a} (x, y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \\
&= (i+1)\beta \sum_{y \subset a} (x, y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) \\
&= (i+1)\beta \sum_{y \subset a} (x, y) + \sum_{x \subset a, y \not\subset a} (x, y) \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \left( \sum_{x \subset a, y \not\subset a} (x, y) + \sum_{y \subset a} (x, y) \right) \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y).
\end{aligned}$$

□

**Notation 12.22.** Assuming we do not have  $(i+1)\beta = 1$ , define  $t_a = \frac{r_a - m_a}{(i+1)\beta - 1}$

**Lemma 12.23.** With  $t_a$  as defined above,  $t_a = \sum_{y \subset a} (x, y)$

*Proof.* By 12.21 and 12.19, we have

$$\begin{aligned}
t_a &= \frac{r_a - m_a}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x, y) + \sum_{x \subset a} (x, y) - \sum_{x \subset a} (x, y)}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x, y)}{(i+1)\beta - 1} \\
&= \sum_{y \subset a} (x, y)
\end{aligned}$$

□

**Notation 12.24.** Assuming  $\beta(i+1) \neq 1$ , let  $q_{a,b} = v_{(a,b)} - \beta t_a$ .

**Lemma 12.25.** We have  $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)$ .

*Proof.* Using 12.23

$$\begin{aligned}
q_{a,b} &= v_{(a,b)} - \beta t_a \\
&= \beta \sum_{y \subset a} (x, y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y) - \beta \sum_{y \subset a} (x, y) \\
&= \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)
\end{aligned}$$

□

**Theorem 12.26.** *For  $n > 2$ , the matrix  $M^{n-2i-1}$  is invertible.*

*Proof.* We saw that if  $\beta(i+1) \neq 1$ , we have that  $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x, y)$  lie in the span of  $v_{(a,b)}$ . It is always the case that  $\beta \geq 1, i \geq 0$ . The only time in which  $\beta = 1, i = 0$  is when  $n = 2$ . Therefore,  $q_{a,b}$  are always defined for  $n > 2$ . However,  $q_{a,b}$  are exactly the rows of  $L^{n-2i-1}$  as defined in Theorem 12.3. Using Theorem ??, we know  $\mathcal{H}^1(B_n)$  is unitary Peck, and therefore the rows of  $L^{n-2i-1}$  are independent. Therefore, the rows of  $M^{n-2i-1}$  span an independent set in a vector space of the same dimension. So, the rows of  $M^{n-2i-1}$  are independent. Hence,  $M^{n-2i-1}$  is indeed an isomorphism, when  $n > 2$ . □

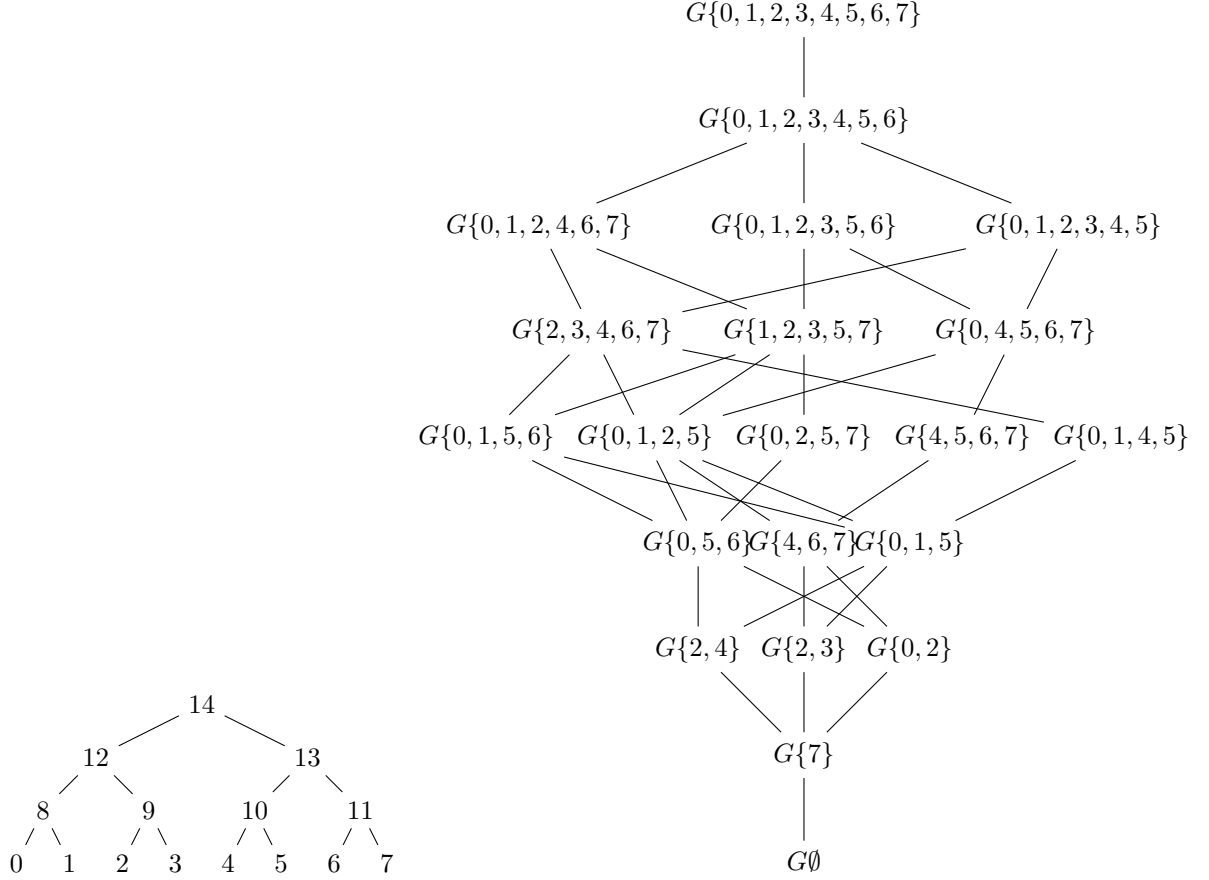
**Corollary 12.27.** *The poset  $\mathcal{E}(B_n)$  is unitary Peck.*

*Proof.* By definition, a poset  $P, \text{rk}(P) = t$  is unitary peck if and only if the Lefschetz map  $L : P \rightarrow P, x \mapsto \sum_{y \succ x} y$ , satisfies  $L^{t-2i} : P_i \rightarrow P_{t-i}$  is an isomorphism. In the case  $P = \mathcal{E}(B_n), \text{rk}(P) = n - 1$ , we have that  $M$  is the Lefschetz map and by Theorem 12.26,  $M^{n-1-2i} : \mathcal{E}(B_n)_i \rightarrow \mathcal{E}(B_n)_{n-1-i}$  is indeed an isomorphism. Hence,  $\mathcal{E}(B_n)$  is unitary Peck. □

### 13. ODDS, ENDS, AND FAILED ATTEMPTS

*In this section, we include various remarks, as well as failed attempts, which don't particularly belong in other sections.*

**13.1. Quotients by Automorphism Groups of Rooted Trees.** *In Subsection 5.2.1, we saw that the full automorphism groups of rooted trees are CCT. It is well known that the quotient  $B_{ml}/(S_m \wr S_l)$  is a distributive lattice. Hence, we were curious whether  $B_{ml}/G$  is a distributive lattice as well, where  $G$  is the automorphism group of a rooted tree. The answer is a definitive no, as can be seen in the case  $G = (S_2 \wr S_2) \wr S_2$ , where  $G$  acts on a the binary rooted tree  $P$  with  $|L(P)| = 8$ , whose vertices are labeled  $0, \dots, 7$  from left to right. In this case, it is easy to see that  $G\{0, 5, 6\} \leq G\{0, 1, 5, 6\}, G\{0, 5, 6\} \leq G\{0, 1, 2, 5\}, G\{0, 1, 5\} \leq G\{0, 1, 5, 6\}, G\{0, 1, 5\} \leq G\{0, 1, 2, 5\}$ . Hence,  $G\{0, 5, 6\}, G\{0, 1, 5\}$  do not have a well defined join, and so  $B_{ml}/G$  is not a lattice.*



**13.2. A Failed Attempt for Showing  $\mathcal{E}^1(B_n/G)$  is unimodal.** In this section, we describe one path we were pursuing in order to show  $\mathcal{E}^1(B_n/G)$  is unimodal. We were attempting to do this by trying to show there were injective order raising maps  $U_i : \mathcal{E}^1(B_n/G)_i \rightarrow \mathcal{E}^1(B_n/G)_{i+1}$ .

We shall now define several maps, so that we can draw a certain commuting diagram in Remark 13.2

**Notation 13.1.** Let  $U_i : P_i \rightarrow P_{i+1}$  be the raising operator for the poset  $P$ . Then, we obtain an induced map

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \rightarrow P_{i+1} \otimes P_{i+1}, (x, y) \mapsto U(x) \otimes U(y).$$

We also have the natural inclusions

$$\begin{aligned} k_i : \mathcal{E}^1(P)_i &\rightarrow P_i \otimes P_{i+1}, \\ (x, y) &\mapsto (x, y) \\ k_i^{G \times G} : \mathcal{E}^1(P/G)_i &\rightarrow (P/G)_i \otimes (P/G)_{i+1}, \\ (Gx, Gy) &\mapsto (Gx, Gy), \end{aligned}$$

where we have  $x \leq y$  and  $Gx \leq Gy$ . The maps above are defined on a basis, and are extended by linearity.

Next, we define the map

$$\begin{aligned} j_i : (P/G)_i \otimes (P/G)_{i+1} &\rightarrow P_i \otimes P_{i+1}, \\ (Gx, Gy) &\mapsto \left( \frac{1}{|G|} \sum_{g \in G} gx, \frac{1}{|G|} \sum_{h \in G} hy \right) \end{aligned}$$

where  $x$  is an arbitrary representative of  $Gx$  and  $y$  is an arbitrary representative of  $Gy$ .  
Then, define the map

$$\begin{aligned} p_i : P_i \otimes P_{i+1} &\rightarrow (P/G)_i \otimes (P/G)_{i+1}, \\ (x, y) &\mapsto (Gx, Gy). \end{aligned}$$

Further, define the map

$$\begin{aligned} (U_i \otimes U_{i+1})^{G \times G} : P_i \otimes P_{i+1} &\rightarrow P_{i+1} \otimes P_{i+1}, \\ (Gx, Gy) &\mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx, Gy). \end{aligned}$$

We also have the projections inclusions

$$\begin{aligned} \pi_i : P_i \otimes P_{i+1} &\rightarrow \mathcal{E}^1(P)_i, \\ (x, y) &\mapsto \begin{cases} (x, y), & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases} \\ \pi_i^{G \times G} : (P/G)_i \otimes (P/G)_{i+1} &\rightarrow \mathcal{E}^1(P/G)_i, \\ (Gx, Gy) &\mapsto \begin{cases} (Gx, Gy), & \text{if } Gx \leq Gy \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where we have  $x \leq y$  and  $Gx \leq Gy$ . The maps above are defined on a basis, and are extended by linearity.

Finally, denote

$$\begin{aligned} \mathcal{E}^1(U)_i : \mathcal{E}^1(P)_i &\rightarrow \mathcal{E}^1(P)_{i+1} \\ (x, y) &\mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x, y) \\ \mathcal{E}^1(U)_i^{G \times G} : \mathcal{E}^1(P/G)_i &\rightarrow \mathcal{E}^1(P/G)_{i+1} \\ (Gx, Gy) &\mapsto k_i^{G \times G} \circ (U \otimes U)^{G \times G} \circ \pi_{i+1}^{G \times G}(Gx, Gy) \end{aligned}$$

where it is defined above on a basis and we extend to the whole space by linearity.

*Remark 13.2.* For  $i < \frac{n}{2}$  we obtain the following (almost commuting, but  $j_{i+1} \circ p_{i+1} \neq \text{id.}$ ) diagram

$$\begin{array}{ccccccc} & & & \mathcal{E}^1(U)_i & & & \\ & \swarrow & & \searrow & & & \\ \mathcal{E}^1(P)_i & \xrightarrow{k_i} & P_i \otimes P_{i+1} & \xrightarrow{U_i \otimes U_{i+1}} & P_{i+1} \otimes P_{i+2} & \xrightarrow{\pi_{i+1}} & \mathcal{E}^1(P)_{i+1} \\ & & \uparrow j_i & & \downarrow p_{i+1} & \uparrow j_{i+1} & \\ \mathcal{E}^1(P/G)_i & \xrightarrow{k_i^{G \times G}} & (P/G)_i \otimes (P/G)_{i+1} & \xrightarrow{(U_i \otimes U_{i+1})^{G \times G}} & (P/G)_{i+1} \otimes (P/G)_{i+2} & \xrightarrow{\pi_{i+1}^{G \times G}} & \mathcal{E}^1(P/G)_{i+1} \\ & \searrow & & \swarrow & & & \\ & & & \mathcal{E}^1(U)_i^{G \times G} & & & \end{array}$$

Unfortunately, in general, with the above definitions of the maps,

$$\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \not\subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}).$$

However, if we did have

$$\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}),$$



using the fact that  $\mathcal{E}^1(U)_i, U_i, U_{i+1}$  are all injective, a fairly simple diagram chase would reveal  $\mathcal{E}^1(U)_i^{G \times G}$  is injective. This, in turn, would imply  $\mathcal{E}^1(B_n/G)_i$  is symmetric, unimodal, and sperner. We have tried several variations on these exact maps, but were never quite able to obtain the desired  $\ker(\pi_{i+1}^{G \times G} \circ p_{i+1}) \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G}) \subset \ker \pi_{i+1} \cap \text{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G})$ .

### 13.3. Further Questions.

**13.3.1. Category Theoretic Properties.** First, it is worth noting that the functor  $\mathcal{E} : \mathcal{P} \rightarrow \mathcal{P}$  is neither essentially surjective nor fully faithful. Let  $P$  be the poset with points  $a, b, c, d$  and relations  $a < c, b < c, a < d$ . It is easy to see there is no poset  $A$  with  $\mathcal{E}(A) = P$ . This shows  $\mathcal{E}$  is not essentially surjective. Next, let  $Q$  be the poset whose elements are  $e, f, g$  with  $e < f, e < g$ , and let  $R$  be the poset whose elements are  $h, i, j, k$  with  $h < i, j < k$ . It is clear that  $\mathcal{E}(Q) \cong \mathcal{E}(R)$  are both a disjoint union of two points, but  $Q \not\cong R$ . So,  $\mathcal{E}$  is also not fully faithful.

However, there are many aspects of the functor  $\mathcal{E}$ , and indeed  $\mathcal{E}^{\vec{\tau}}$  which are still unexplored. First, note that it is worth noting that  $\mathcal{E}$  also determines a endofunctor on the category of (ungraded) posets, which we shall now notate  $\mathcal{Q}$ , as well as an endofunctor on the category of graded posets  $\mathcal{P}$ . In general, the points of a poset  $P$  can be identified with the set  $\text{Hom}_{\mathcal{Q}}(B_0, P)$ , and the edges of  $P$  can be identified with the set  $\text{Hom}_{\mathcal{Q}}(B_1, P)$ . It is simple to see additionally that the points of  $\mathcal{E}(P)$  are  $\text{Hom}_{\mathcal{Q}}(B_0, \mathcal{E}(P)) \cong \text{Hom}_{\mathcal{Q}}(B_1, P)$ . Additionally, the edges of  $\mathcal{E}(P)$  are  $\text{Hom}_{\mathcal{Q}}(B_1, \mathcal{E}(P)) \cong \text{Hom}_{\mathcal{Q}}(B_2, P)$ . These relations may be a reasonable alternate way to think of the functor of points, and are further suggestive of the existence of an adjoint functor.

**Question 13.3.** Does the functor  $\mathcal{E}$  have a left or right adjoint?

**13.3.2. Unitary Peck Generalizations.** Additionally, there are several posets which we would like to be unitary Peck, as this would allow us to apply much of the theory developed in this paper. They would also make several of the generalizations of  $\mathcal{E}$  potentially more interesting. Two posets we are particularly interested in are  $\mathcal{E}^{\vec{\tau}}(B_n)$  and its  $q$ -analog,  $\mathcal{E}^{\vec{\tau}}(B_n(q))$ .

**Question 13.4.** Is  $\mathcal{E}^{\vec{\tau}}(B_n)$  unitary Peck?

As mentioned in the introduction, a natural extension of Conjecture 1.3 would be an analogous result for  $q$ -analogs. We suspect that a proof similar to that of Corollary 12.27 may also prove Question 13.5.

**Question 13.5.** Is  $\mathcal{E}(B_n(q))$  unitary peck? What about  $\mathcal{E}^{\vec{\tau}}(B_n(q))$ ?

**13.3.3. A  $q$ -analog of Conjecture 1.3.** In particular, if the answer to the previous Question 13.5 is affirmative, it would immediately hold that  $\mathcal{E}(B_n(q))/G$  is Peck, and if  $G$  is CCT then  $\mathcal{E}(B_n(q))/G$  is Peck. Hence, we pose the following question:

**Question 13.6.** For  $G$  a group with a CCT action on  $B_n(q)$  is  $\mathcal{E}(B_n(q))/G$  Peck?

Even more generally, we wonder if the  $q$ -analog of Conjecture 1.3 holds.

**Question 13.7.** For  $G$  a group acting on  $B_n(q)$ , is  $\mathcal{E}(B_n(q))/G$  Peck? If not, is  $\mathcal{E}(B_n(q))/G$  rank unimodal?

**13.3.4. Additional CCT Actions.** We also found several additional interesting examples of CCT actions, and we are curious whether they generalize. Once such action is the linear automorphism of the  $n$ -cube. Using computers, we found that for  $n \leq 3$ , the linear automorphisms of the  $n$ -cube is CCT. We wonder if this generalizes.

**Question 13.8.** Does the group of linear automorphism of an  $n$ -cube in  $\mathbb{R}^n$ , whose vertices lie at  $(\pm 1, \dots, \pm 1)$  induce a common cover transitive action on  $\mathbb{R}^n$ ?

Also, we found using computers that the group of invertible linear maps on  $\mathbb{F}_2^3$  acting on the the seven nonzero points of  $\mathbb{F}_2^3$  induce an action on  $B_7$  which is CCT. We wonder if this generalizes to other groups of invertible linear maps on finite fields.

**Question 13.9.** Is the action of  $GL_n(\mathbb{F}_q)$  on  $B_{q^n-1}$  (induced by the action of  $GL_n(\mathbb{F}_q)$  on  $(\mathbb{F}_q^n)_*$ ) CCT? What about the action of  $PGL_n(\mathbb{F}_q)$  on  $B_{q^n-1}$ ? If not, what about the action of  $PGL_n(\mathbb{F}_2)$  on  $B_{2^n-1}$ ?

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