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# On Chains and Sperner $k$-Families in Ranked Posets

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A ranked poset $P$ has the Sperner property if the sizes of the largest rank and of the largest antichain in $P$ are equal. A natural strengthening of the Sperner property is condition $S$: For all $k$, the set of elements of the $k$ largest ranks in $P$ is a Sperner $k$-family. $P$ satisfies condition $T$ if for all $k$ there exist disjoint chains in $P$ each of which meets the $k$ largest ranks and which covers the $k$th largest rank. It is proven here that if $P$ satisfies $S$, it also satisfies $T$, and that the converse, although in general false, is true for posets with unimodal Whitney numbers. Conditions $S$ and $T$ and the Sperner property are compared here with two other conditions on posets concerning the existence of certain partitions of $P$ into chains.

It is a famous theorem of Dilworth [1] that every finite poset (partially ordered set) $P$ can be partitioned into $d_1(P)$ chains, where $d_1(P)$ is the size of the largest antichain (set of incomparable elements) in $P$. This was generalized by Greene and Kleitman [2], who showed that for all $k$ there is a natural relationship between the size of the largest $k$-family in a finite poset $P$ and the existence of certain partitions of $P$ into chains (Theorem 1).

For a given ranked poset $P$, a question usually asked is whether $P$ has the Sperner property (condition $S_1$ below): The set of elements of the largest rank in $P$ is a maximum-sized antichain. This has a natural generalization to $k$-families (condition $S$ below). On the other hand, many important ranked posets satisfy nice conditions on the existence of sets of disjoint chains with certain properties (conditions $A$, $R$, $T$ below). This is particularly true if the ranks are of a special form. For example, a ranked poset with symmetric unimodal ranks may be a symmetric chain order.

In this paper we find all implications among these conditions $A$, $R$, $S$, $S_1$, and $T$, relating conditions of Sperner-type to conditions on the existence of certain chains. The most interesting results are that if $P$ satisfies $S$, it satisfies $T$, and that for posets with unimodal ranks, $S$ and $T$ are equivalent.

Following the theorems and their proofs, examples are given to show that there are no further implications among the conditions. The paper concludes



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with a summary of what these results mean for ranked posets in general and for those of some special forms.

We begin by reviewing the basic definitions, concepts, and notation. A finite poset $P$ is *ranked* (or graded) if for every $x \in P$ every maximal chain with $x$ as top element has the same length, denoted $r(x)$. Here the length of a chain with $k$ elements is $k - 1$. If $P$ is ranked the function $r$, called the *rank function*, is zero for minimal elements of $P$ and $r(y) = r(x) + 1$ if $x, y \in P$ and $y$ covers $x$. The *rank of* $P$, denoted $r(P)$, is the maximum value of $r(x)$ for $x \in P$.

A *k-family* in a finite poset is a set of elements such that no $k + 1$ lie on any single chain. Thus a 1-family is just an antichain. The union of $k$ antichains is a $k$-family, and, conversely, $k$-families can be expressed as the union of at most $k$ antichains. A *Sperner k-family* is a $k$-family of maximum size. The size of Sperner $k$-families in a poset $P$ is denoted by $d_k(P)$ or just by $d_k$ if it is understood what $P$ is.

In a ranked poset $P$, $N_i(P)$ or just $N_i$ denotes the number of elements of rank $i$, $0 \leqslant i \leqslant r(P)$. $(N_0, N_1, ..., N_{r(P)})$ is called the *sequence of Whitney numbers* of $P$. $M_i(P)$ or just $M_i$ denotes the $(i + 1)$st largest Whitney number in $P$, so that $(M_0, M_1, ..., M_{r(P)})$ is the sequence of Whitney numbers arranged in nonincreasing order.

The elements of a given rank in $P$ form an antichain. So the set of elements of any $k$ ranks is a $k$-family. By taking the $k$ largest ranks we obtain a lower bound of $\sum_{i=0}^{k-1} M_i$ on $d_k$. $P$ has the *Sperner property* if this bound is correct for $k = 1$, that is, $d_1 = M_0$. This idea can be extended to $k$-families as follows:

DEFINITION. A ranked poset $P$ satisfies *condition* $S_k$ if $d_k = \sum_{i=0}^{k-1} M_i$. $P$ satisfies *condition* $S$ if $P$ satisfies $S_k$ for all $k$.

"$S_k$ holds in $P$" means that the $k$ largest ranks in $P$ form a Sperner $k$-family. $S$ is a strong condition which requires that $P$ have the Sperner property $(S_1)$ and its analogs $S_k$ for $k$-families for all $k > 1$. $S$ is known to be satisfied by many important posets, including LYM posets and symmetric chain orders. Verifying whether $P$ satisfies $S$ is simplified by the observation that $P$ itself is a $k$-family for $k \geqslant r(P) + 1$, so that $P$ trivally satisfies $S_k$ for $k \geqslant r(P) + 1$.

We aim to relate condition $S$, which concerns the $k$-families of $P$, to conditions on chain decompositions of $P$. Of results of this type, the fundamental one, known as Dilworth's theorem [1], says that for any finite poset $P$, ranked or not, there exists a partition of $P$ into just $d_1$ chains. That is, there exists a collection $\{C_1, ..., C_{d_1}\}$ of disjoint chains in $P$ which contains every element of $P$. At least $d_1$ chains are required in any such decomposition because a chain can intersect an antichain at most once and $P$ contains an antichain of size $d_1$.

More generally, if $C$ is any chain in $P$ and $F$ is a $k$-family, then $|C \cap F| \leqslant \mathrm{Min}(|C|, k)$. So if $\mathscr{C} = \{C_1, ..., C_n\}$ is a partition of $P$ into chains and $F$ is a $k$-family, then

$$|F| = \sum_i |C_i \cap F| \leqslant \sum_i \mathrm{Min}(|C_i|, k).$$

In particular, if $F$ is a Sperner $k$-family, then $|F| = d_k$ and

$$d_k \leqslant \sum_i \mathrm{Min}(|C_i|, k),$$

which gives a bound on $d_k$ purely in terms of the chain sizes in $\mathscr{C}$.

DEFINITION. A partition $\mathscr{C} = \{C_1, ..., C_n\}$ of a finite poset $P$ into chains $C_i$ is *k-saturated* if $\sum_i \mathrm{Min}(|C_i|, k) = d_k$.

In these terms Dilworth's theorem says that every finite poset has a 1-saturated partition. Greene and Kleitman extended this by showing that for all $k$ every finite poset has a $k$-saturated partition. They actually obtained an even stronger result which we shall require to prove Theorem 3:

THEOREM 1. (Greene and Kleitman [2]). *For any finite poset P and any integer k, there exists a partition of P into chains which is both k-saturated and* $(k+1)$*-saturated.*

Note that this applies to all finite posets, ranked or not. Dilworth's theorem can be proven readily by induction on $|P|$. But Theorem 1 is a much deeper result. The proof requires a careful study of the structure of Sperner $k$-families and depends on showing that the Sperner $k$-families form a lattice with suitably defined join and meet operations. Merely showing that $P$ has a $k$-saturated partition is difficult.

DEFINITION. A partition $\mathscr{C}$ of a poset $P$ into chains is *completely saturated* if it is $k$-saturated for all $k$. A ranked poset $P$ satisfies *condition A* if it has a completely saturated partition.

Example 1 below shows that a ranked poset $P$ need not have a completely saturated partition, even though Theorem 1 implies that, for all $k$, $P$ has partitions that are simultaneously $k$-saturated and $(k+1)$-saturated. First we discuss the problem of determining the values of the $d_k$ and whether $P$ has a completely saturated partition.

To prove that $d_k$ takes on a certain value, call it $d$, it is not necessary to examine all $k$-families: It suffices to exhibit any $k$-family $F$ of size $d$ and any partition $\mathscr{C}$ into chains such that $\sum_i \min(|C_i|, k) = d$. $|F| = d$ shows that $d_k \geqslant d$ and $\mathscr{C}$ shows that $d_k \leqslant d$. It follows that $d_k = d$, $F$ is a Sperner $k$-family, and $\mathscr{C}$ is $k$-saturated. This method will always work, even if $P$ is not ranked, since such $F$ and $\mathscr{C}$ always exist, although they may be difficult to find.

To determine whether $P$ has a completely saturated partition $\mathscr{C}$, it is often useful to determine the distribution of chain lengths such a partition would be required to have. These can be computed from the $d_k$.

DEFINITION. For a finite poets $P$, let $\Delta_k = d_k - d_{k-1}$, where by convention $d_0 = 0$.

For a completely saturated partition $\mathscr{C}$ the following must hold for all $k$:

$$\Delta_k = \sum_i [\min(| C_i |, k) - \min(| C_i |, k - 1)].$$

The right side merely counts the number of chains in $\mathscr{C}$ containing at least $k$ elements, i.e., with length $\geqslant k - 1$. Hence $\Delta_{k+1} - \Delta_{k+2}$ counts the number of chains in $\mathscr{C}$ with length precisely $k$. So our problem is reduced to determining whether $P$ has a partition with these chain lengths.

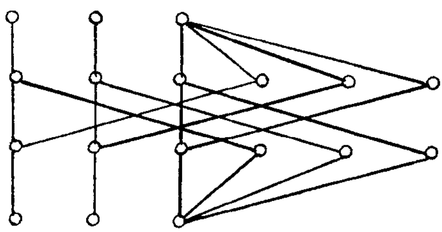

FIGURE 1

EXAMPLE 1. Let $P_1$ be the poset shown in Fig. 1. For $P_1$, $M_0 = M_1 = 6$ and $M_2 = M_3 = 3$. $P_1$ can be partitioned into six chains of length 2, from which it follows that $d_1 = 6$, $d_2 = 12$ and $P_1$ satisfies $S_1$ and $S_2$. There is another partition of $P_1$ into three chains of length 3 and six chains of length 0. This shows that $d_3 = 15$ and $P_1$ satisfies $S_3$. Hence $P_1$ satisfies $S$. Trivially, $d_4 = d_5 = 18$. Thus $\Delta_1 = \Delta_2 = 6$, $\Delta_3 = \Delta_4 = 3$, and $\Delta_5 = 0$. A completely saturated partition would require three chains of length 3 and three chains of length 1. It is easy to verify that this is impossible, so that $P_1$ does not satisfy $A$.

All examples in this paper, such as the one above, have been constructed so that not only $P$ but also its dual (obtained by reversing the ordering) are ranked and so that the rank of an element $x$ in the dual is just $r(P) - r(x)$. The duals of the examples satisfy the same conditions as the examples themselves.

Now we introduce a condition concerning the existence of certain chains in ranked posets.

DEFINITION. A ranked poset $P$ satisfies *condition* $T_k$, $1 \leqslant k \leqslant r(P)$, if there exist $M_k$ disjoint chains in $P$ which each intersect each of the $k+1$ largest ranks. $P$ satisfies *condition* $T$ if it satisfies *condition* $T_k$ for all $k$.

Thus condition $T$ means that for all $k$ there are disjoint chains which cover the $(k+1)$st largest rank and intersect every larger rank. To be precise in defining $T_k$ it is necessary to order ranks of equal size to decide which is "larger." But here we will be interested only in $T$, so that all $T_k$ must hold, and equal-sized ranks can be ordered arbitrarily. For example, $P_1$ in Example 1 satisfies $T$: The six diagonal lines determine chains covering the two middle ranks, so $P_1$ satisfies $T_1$, while the three vertical chains of length 3 show that $P_1$ satisfies $T_2$ and $T_3$.

Symmetric chain orders and skew chain orders immediately satisfy $T$, and we shall discuss them in more detail later on. LYM orders can be shown to satisfy it. The definition of $T$ was motivated by the recent result of Stanley [5], which says, in these terms, that the poset of order ideals of the direct product of two chains satisfies $T$.

We now show how $S$ and $T$ are related.

THEOREM 2. *If $S$ holds in $P$, then $T$ holds in $P$.*

*Proof.* The theorem follows immediately from this stronger result that we now prove: For any $k$, $1 \leqslant k \leqslant r(P)$, if $S_k$ holds in $P$, then $T_k$ holds in $P$. To prove this, let $P$ be a ranked poset satisfying $S_k$. Let $P'$ be the poset consisting of the $k+1$ largest ranks in $P$ ordered as in $P$, so that for $x, y \in P'$, $x \leqslant y$ in $P' \Leftrightarrow x \leqslant y$ in $P$. (Arbitrarily choose from among the ranks of size $M_k$ to obtain the $k+1$ largest ranks if $M_k = M_{k+1}$.) Then $P'$ is also ranked, $r(P') = k$, and $P'$ satisfies $S_k$. Let $\mathscr{C} = \{C_1, C_2, \ldots\}$ be a $k$-saturated partition of $P'$, which exists by Theorem 1. Then,

$$\begin{aligned} \sum_i \min(|C_i|, k) &= d_k(P') \\ &= M_0 + M_1 + \cdots + M_{k-1} \\ &= |P'| - M_k, \end{aligned}$$

so $\mathscr{C}$ must contain $M_k$ chains of length $k$. These chains then each intersect each of the $k+1$ largest ranks in $P$. It follows that $P$ satisfies $T_k$. ∎

The converse of Theorem 2 is not true in general. $P$ may satisfy T without satisfying $S$.

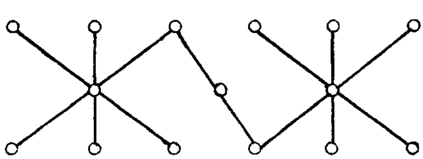

FIGURE 2

EXAMPLE 2. For $P_2$ as shown in Fig. 2, $d_1 = 7 > M_0$, so $P_2$ satisfies neither $S_1$ nor $S$. But it can be shown to satisfy $T$ and $A$.

However, in the case that the sequence of Whitney numbers of $P$ is unimodal, so that, for some I,

$$N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_I \geqslant N_{I+1} \geqslant \cdots \geqslant N_{r(P)},$$

it is true that $P$ satisfies $S \Leftrightarrow P$ satisfies $T$.

THEOREM 3. *Let $P$ be a poset with a unimodal sequence of Whitney numbers. If $P$ satisfies $T$ then $P$ satisfies $S$.*

*Proof.* Suppose $P$ has unimodal Whitney numbers and satisfies $T$. For $0 \leqslant i \leqslant r(P)$ let $a(i)$ denote the $(i + 1)$st largest rank, so that $N_{a(i)} = M_i$. When two or more ranks have the same size, define $a(i)$ in such a way that, for all $k$, the $k$ largest ranks form a connected set of ranks. Thus for each $k$ there is an $i$ such that $\{a(0), a(1),..., a(k)\} = \{i, i + i,..., i + k\}$.

We prove that $P$ satisfies $S_k$ by induction on $k$. First we establish that $P$ satisfies $S_1$. It suffices to show that $P$ can be partitioned into just $M_0$ chains. This can be done as follows: For $j > a(0)$ match rank $j$ into rank $j - 1$ and for $j < a(0)$ match rank $j$ into rank $j + 1$. These matchings exist for two reasons. First, by unimodality, rank $j$ is being matched into a rank at least as large. Second, since $P$ satisfies $T$—Specifically, $T_{a^{-1}(j)}$—there are $N_j$ disjoint chains intersecting both ranks. These matchings are now linked for all $j$ to form the desired chains.

Now suppose $P$ satisfies $S_k$. We must show that $P$ also satisfies $S_{k+1}$ to complete the induction. $P$ trivially satisfies $S_i$ for $i > r(P)$, so we can assume that $1 \leqslant k < r(P)$. It suffices to find a parition $\mathscr{C}$ such that

$$\sum_i \min(|C_i|, k + 1) = M_0 + \cdots + M_k. \tag{1}$$

We assume that $a(k) > a(0)$. (That $P$ satisfies $S_{k+1}$ if $a(k) < a(0)$ can be shown by essentially the dual of the argument presented here for $a(k) > a(0)$. We omit the details.)

There are two cases depending on whether $a(k) = k$ or $a(k) > k$. The second case is more difficult because it turns out to be tricky to extend chains with the desired properties below the $k$ largest ranks. The full power of Theorem 1 is required: The existence of partitions that are not merely $k$-saturated, but also $(k + 1)$-saturated is used.

*Case* 1. $a(k) = k$. Thus, $\{a(0), a(1),..., a(k)\} = \{0, 1,..., k\}$. Let $P'$ consist of the elements of ranks $0, 1,..., k$ in $P$ with the same ordering. $P'$ has rank $k$

and satisfies $S_k$ and $T$ since $P$ does. By Theorem 1, $P'$ has a $k$-saturated partition, $\mathscr{C}' = \{C_1', C_2', ...\}$. So

$$\sum_i \min(|C_i'|, k) = d_k(P')$$
$$= M_0 + \cdots + M_{k-1}$$
$$= |P'| - M_k .$$

Hence $\mathscr{C}'$ has $M_k$ chains of length $k$, and every element of rank $k$ lies on a chain of length $k$.

For $j > k$, rank $j$ in $P$ can be matched into rank $j - 1$, as in proving $P$ satisfies $S_1$. Link these matchings in $P$ for all $j > k$ with the chains from $\mathscr{C}'$ in $P'$ to form a partition $\mathscr{C} = \{C_1, C_2, ...\}$. Now every element of rank $\geqslant k + 1$ lies on a chain of length $\geqslant k + 1$ in $\mathscr{C}$. Hence, (1) holds for $\mathscr{C}$, which shows that $P$ satisfies $S_{k+1}$. Indeed, this partition shows that $P$ satisfies $S_{k+2}, S_{k+3}, ...$.

*Case* 2. $a(k) > k$. Let $j = a(k) - k - 1$ and let $q = a^{-1}(j)$. So $j \geqslant 0$, $j$ is the $(q + 1)$st largest rank, $q > k$, and $\{a(0), a(1), ..., a(k)\} = \{j + 1, j + 2, ..., j + k + 1\}$. Let $P'$ consist of the elements of ranks $j, j + 1, ..., j + k + 1$ in $P$ with the same ordering. $P'$ has rank $k + 1$ and satisfies $S_k$. For $i \leqslant k$, $P$ satisfies $T_i \Rightarrow P'$ satisfies $T_i$, since $P$ and $P'$ have the same $i + 1$ largest ranks. Further, since $P$ satisfies $T_q$, there exist disjoint chains in $P'$ covering rank 0 and intersecting every other rank, so that $P'$ satisfies $T_{k+1}$. Hence $P'$ satisfies $T$.

We next show that $P'$ satisfies $S_{k+1}$. As noted above there exist disjoint chains in $P'$ covering rank 0 and intersecting each rank. These chains have length $k + 1$, and there are $N_0(P') = N_j(P) = M_q(P)$ of them. These chains together with the remaining elements of $P'$, as length 0 chains, determine a partition $\mathscr{C}' = \{C_1', C_2', ...\}$ of $P'$ which satisfies

$$\sum_i \min(|C_i'|, k + 1) = M_0 + \cdots + M_k .$$

Hence $d_{k+1}(P') = M_0 + \cdots + M_k$, and $P'$ satisfies $S_{k+1}$.

We will use $P'$ to construct a partition of $P$ showing that $P$, like $P'$, satisfies $S_{k+1}$. By Theorem 1, $P'$ has a partition $\mathscr{C}'' = \{C_1'', C_2'', ...\}$ which is both $k$-saturated and $(k + 1)$-saturated. Thus,

$$\sum_i \min(|C_i''|, k) = M_0 + \cdots + M_{k-1}, \tag{2}$$

$$\sum_i \min(|C_i''|, k + 1) = M_0 + \cdots + M_k, \tag{3}$$

$$|P'| = M_0 + \cdots + M_k + N_0(P'). \tag{4}$$

By (3) and (4), $\mathscr{C}''$ must contain $N_0(P')$ disjoint chains of length $k+1$. These chains cover rank 0. Then, by (2) and (3), $\mathscr{C}''$ must contain $M_k - N_0(P')$ more disjoint chains of length $k$. These must each intersect rank $1, 2, \ldots, k+1$ since rank 0 is already covered. Together these $M_k$ chains of length $\geqslant k$ cover ranks 0 and $k+1$ in $P'$. Now apply the chains in $\mathscr{C}''$ to partition ranks $j, j+1, \ldots, j+k+1$ in $P$. For $i > a(k) = j+k+1$, rank $i$ in $P$ can be matched into rank $i-1$, as in the proof that $P$ satisfies $S_1$. Similarly, for $i < a(q) = j$, rank $i$ can be matched into rank $i+1$. Link together these matchings for all $i > j+k+1$ and for all $i < j$ with the chains from $\mathscr{C}''$. This forms a partition $\mathscr{C}$ of $P$ in which every element of ranks $a(k+1)$, $a(k+2), \ldots$ lies on some chain of length $\geqslant k+1$. Hence $\mathscr{C}$ satisfies (1), and $P$ satisfies $S_{k+1}$. ∎

There is one more nice condition we introduce here which relates to the existence of disjoint chains in ranked posets. Condition $T$ requires that for all $k$ there exist disjoint chains which cover the $(k+1)$st largest rank and intersect all larger ranks. A natural strengthening of this condition is to require that there exist some partition which does this for all $k$ simultaneously.

DEFINITION. A ranked poset $P$ satisfies *condition $R$* if it has a partition $\mathscr{C}$ into chains such that for all $k$, $1 \leqslant k \leqslant r(P)$, $M_k$ chains in $\mathscr{C}$ each intersect each rank of size $\geqslant M_k$.

Posets with symmetric unimodal Whitney numbers satisfying R are just symmetric chain orders. These are well known to have the Sperner property and its analogs for $k$-families and to have completely saturated partitions. We now show that these conditions characterize the posets satisfying $R$ regardless of their Whitney numbers.

THEOREM 4. *$R$ holds in $P$ if and only if $S$ and $A$ both hold in $P$.*

*Proof.* Order the ranks of $P$ by size so that rank $a(i)$ is the $(i+1)$st largest rank, $0 \leqslant i \leqslant r(P)$. Then $N_{a(i)} = M_i$ for all $i$.

First suppose $P$ satisfies $R$. Let $\mathscr{C}$ be a partition satisfying the conditions in the definition of $R$. Fix $k$, $1 \leqslant k \leqslant r(P)$. For every element $x$ of $P$ not in ranks $a(0), a(1), \ldots, a(k-1)$, the chain in $\mathscr{C}$ which passes through $x$ also intersects ranks $a(0), \ldots, a(k-1)$. So this chain has length $\geqslant k$. It follows that

$$\sum_i \min(|C_i|, k) = M_0 + \cdots + M_{k-1} = d_k \tag{5}$$

so that $P$ satisfies $S_k$ and $\mathscr{C}$ is $k$-saturated. As $k$ was arbitrary, $P$ satisfies both $S$ and $A$.

Conversely, suppose $S$ and $A$ both hold in $P$. Let $\mathscr{C}$ be a completely saturated partition of $P$. Then for all $k$, $1 \leqslant k \leqslant r(P)+1$, (5) holds. So

$\Delta_k = M_{k-1}$ and the number of chains of length $k$ in $\mathscr{C}$ is $M_k - M_{k+1}$. (See the discussion following the definition of $\Delta_k$.) The $M_{r(P)}$ chains of length $r(P)$ cover the smallest rank, $a(r(P))$, and meet every other rank. If $M_0 = M_{r(P)}$, then we already have that $P$ satisfies $R$. Otherwise let rank $a(j)$ be the smallest rank strictly larger than rank $a(r(P))$, i.e., $M_j > M_{j+1} = M_{r(P)}$. There are $M_j - M_{r(P)}$ chains of length $j$. They meet each rank $a(0)$, $a(1)$,..., $a(i)$ since the other ranks are already covered by the longer chains in $\mathscr{C}$. Hence the $M_j$ chains of length $\geqslant j$ meet each rank of size $\geqslant M_j$. Continuing this argument for successively larger ranks makes it evident that for all $k$, the $M_k$ chains of length $\geqslant k$ meet each rank of size $\geqslant M_k$. Therefore, $P$ satisfies $R$. ∎

Example 1 shows that a poset satisfying $S$ need not satisfy $R$, even if its Whitney numbers are symmetric and unimodal. However, if $P$ has monotonic Whitney numbers, this cannot happen.

THEOREM 5. *Let $P$ be a poset with a monotonic sequence of Whitney numbers. If $P$ satisfies $T$, then it also satisfies $R$.*

*Proof.* Suppose $P$ has monotonic Whitney numbers. Indeed, suppose that $N_0 \geqslant N_1 \geqslant \cdots \geqslant N_{r(P)}$. The argument is similar if instead the Whitney numbers are increasing upwards. Suppose $P$ satisfies T. For every $k$ there are $N_k$ disjoint chains which each meet each rank 0, 1,..., $k$. In particular, there is a matching from rank $k$ into rank $k - 1$. The chain partition of $P$ obtained by combining these matchings for all $k$ shows that $P$ satisfies $R$. ∎

COROLLARY 6. *Conditions $R$, $S$, and $T$ are equivalent for posets with monotonic Whitney numbers.*

COROLLARY 7. *Conditions $R$, $S$, $T$, and $S_1$ are equivalent for posets with equal Whitney numbers.*

*Proof.* Suppose $P$ has all Whitney numbers equal to $M_0$. It suffices to show that if $S_1$ holds in $P$, then $R$ also holds in $P$. If $P$ satisfies $S_1$, then $d_1 = M_0$. By Dilworth's theorem, there is a partition of $P$ into $M_0$ chains Each chain must intersect each rank since all ranks contain $M_0$ elements. It follows that $P$ satisfies $R$. ∎

Before discussing in more detail what these theorems say for posets with Whitney numbers that are arbitrary, unimodal, symmetric and unimodal, monotonic, or equal, we give some more examples which show that no further implications are possible among the conditions $S$, $S_1$, $A$, $T$, and $R$. It is convenient for these examples to define the following two simple operations on posets.

DEFINITION. If $P$ and $Q$ are disjoint posets, let $P \cup Q$ and $P/Q$ denote posets with elements in the union of $P$ and $Q$ and with orderings given by:

$$x \leqslant y \text{ in } P \cup Q \Leftrightarrow (x, y \in P \text{ and } x \leqslant y \text{ in } P) \text{ or } (x, y \in Q \text{ and } x \leqslant y \text{ in } Q);$$

$$x \leqslant y \text{ in } P/Q \quad \Leftrightarrow (x, y \in P \text{ and } x \leqslant y \text{ in } P) \text{ or } (x, y \in Q \text{ and } x \leqslant y \text{ in } Q) \text{ or } (x \in Q \text{ and } y \in P).$$

So in the Hasse diagram for $P \cup Q$, the diagrams for $P$ and $Q$ are side by side, whereas for $P/Q$ the diagram for $P$ is over the diagram for $Q$.

EXAMPLE 3. For a trivial example of a poset which satisfies all of the conditions in this paper ($R$, $S$, $S_1$, $T$, and $A$), let $P$ be any nonempty chain. All of the Whitney numbers of such a poset $P$ equal one.

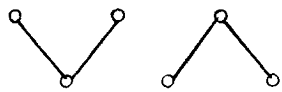

FIGURE 3

EXAMPLE 4. Let $P_4$ be as shown in Fig. 3. $P_4$ has equal Whitney numbers and satisfies $A$. As $d_1 = 4 > M_0$, it does not satisfy $S_1$, and hence it does not satisfy $R$, $S$, or $T$.

EXAMPLE 5. $P_1$ as shown in Fig. 4 has monotonic Whitney numbers (4, 3, 3) and satisfies $A$ and $S_1$, but not $R$, $S$ or $T$.

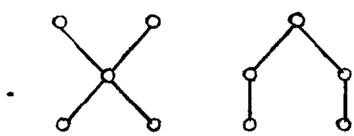

FIGURE 4

EXAMPLE 6. $P_6$ as shown in Fig. 5 has symmetric and unimodal Whitney numbers: (3. 4, 4, 3) and satisfies $A$ and $S_1$, but not $R$, $S$, or $T$.

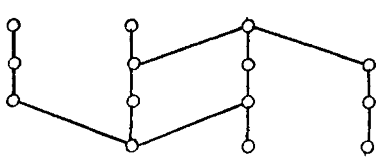

FIGURE 5

EXAMPLE 7. Let $P_7 = P_1 \cup P_6$, where $P_1$ and $P_6$ are the posets in Examples 1 and 6. $P_7$ has symmetric and unimodal Whitney numbers (6, 10, 10, 6) and satisfies only $S_1$.

EXAMPLE 8. Let $P_8 = P_1 \cup Q$, where $P_1$ is Example 1 and $Q$ is Fig. 6. $P_8$ has equal Whitney numbers (7, 7, 7, 7) but fails to satisfy any of the conditions $R$, $S$, $S_1$, $T$, or $A$.

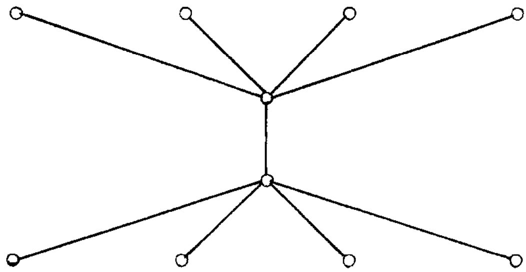

FIGURE 1

EXAMPLE 9. Let $P_9 = P_8/Q$, where $P_8$ is Example 8 and $Q$ consists of 10 totally unordered points. Then $P$ has monotonic Whitney numbers (10, 7, 7, 7, 7) and satisfies only $S_1$.

EXAMPLE 10. Let $P_{10} = P_1/P_2$, where $P_1$ and $P_2$ are Examples 1 and 2. $P_{10}$ has Whitney numbers (6, 3, 6, 3, 6, 6, 3) and satisfies only $T$.

EXAMPLE 11. Let $P_{11} = P_2/Q$, where $P_2$ is Example 2 and $Q$ consists of eight totally unordered points. The Whitney numbers for $P_{11}$ are (8, 6, 3, 6). $P_{11}$ satisfies $S_1$, $T$, and $A$, but not $R$ or $S$.

EXAMPLE 12. Let $P_{12} = P_{10}/Q$, where $Q$ is eight unordered points. $P_{12}$ is also just $P_1/P_{11}$. Its sequence of Whitney numbers is (8, 6, 3, 6, 3, 6, 6, 3) and it satisfies only $T$ and $S_1$.

The results of this paper are summarized below. Among the five conditions, $R$, $S$, $S_1$, $T$, and $A$, only certain collections can be satisfied by ranked posets, with even fewer possibilities if restrictions are placed on the Whitney numbers. In each case, the theorems, corollaries, and remarks in the paper reduce the possible sets of satisfied conditions to those listed here, while for each possible set some example above actually satisfies precisely these conditions.

EQUAL WHITNEY NUMBERS $N_0 = N_1 = \cdots = N_{r(P)}$. There are only three possibilities in this case: $P$ satisfies all five conditions, $R$, $S$, $S_1$, $T$, and $A$; $P$ satisfies only $A$; or $P$ satisfies none of the conditions at all.

SYMMETRIC UNIMODAL WHITNEY NUMBERS $N_0 = N_{r(P)} \leqslant N_1 = N_{r(P)-1} \leqslant \cdots \leqslant N_{\lfloor r(P)/2 \rfloor} = N_{\lceil r(P)/2 \rceil}$. With this condition on the rank sizes $P$ may satisfy all five conditions; $S$, $S_1$, and $T$; $S_1$ and $A$; $S_1$ alone; $A$ alone; or no conditions at all.

$P$ with symmetric unidmodal Whitney numbers satisfies $R$ if and only if it is a *symmetric chain order*: This means $P$ can be partitioned into chains $C_i$ such that each chain is consecutive and symmetric about middle rank, i.e., consists of elements of each rank $k$ through $r(P) - k$, for some $k$. Such orders include Boolean algebras, the lattice of subspaces of a finite vector space, and the lattice of divisors of an integer. Symmetric chain orders are useful in extensions of Sperner's theorem [4].

An *LYM poset* is a ranked poset $P$ which has the property that for all antichains $A \subseteq P$,

$$\sum_{x \in A} \frac{1}{N_{r(x)}} \leqslant 1.$$

It follows easily from this definition that LYM posets satisfy $S$. LYM posets include *uniform posets*: These have the property that for all $i$, every element of rank $i$ covers the same number $a_i$ of elements of rank $(i - 1)$ and is covered by the same number $b_i$ of elements of rank $(i + 1)$. It can be shown that if an LYM poset has symmetric unimodal Whitney numbers, then it is a symmetric chain order [3]. It would be nice to show that in general LYM posets (or at least uniform posets) satisfy $R$.

A poset studied by Stanley [5], called $L(m, n)$, is the set of order ideals of the direct product of two chains of lengths $m - 1$ and $n - 1$. Equivalently, $L(m, n)$ is the set of integer sequences $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m \leqslant n$, ordered by $(a_1 ,..., a_m) \leqslant (b_1 ,..., b_m) \Leftrightarrow a_i \leqslant b_i$ for all $i$. $L(m, n)$ has rank $mn$. It can be shown that $L(m, n)$ has symmetric unimodal Whitney numbers and satisfies T, but no structly combinatorial proof is known. It is apparently still an open problem to determine whether $L(m, n)$ is always a symmetric chain order.

MONOTONIC WHITNEY NUMBERS $N_0 \leqslant N_1 \leqslant \cdots \leqslant N_{r(P)}$ or $N_0 \geqslant N_1 \geqslant \cdots \geqslant N_{r(P)}$. In this case $P$ may satisfy all five conditions; $S_1$ and $A$; $S_1$ alone; $A$ alone; or no conditions at all.

If $N_0 \geqslant N_1 \geqslant \cdots \geqslant N_{r(P)}$ (respectively, $N_0 \leqslant \cdots \leqslant N_{r(P)}$) and $P$ can be partitioned into consecutive chains which each intersect rank 0 (respectively, rank $r(P)$), then $P$ is called a skew chain order (antiskew chain order). Skew chain orders and antiskew chain orders are precisely the posets with monotonic Whitney numbers which satisfy $R$. Like symmetric chain orders, they are useful in extending Sperner's theorem [6]. LYM posets with monotonic Whitney numbers satisfy $R$. A nice example of a skew chain order (which is also LYM) is the poset $I_n$ of all intervals of a chain of length $n$, ordered by inclusion. Equivalently, $I_n$ has elements $\{(i, j) : 0 \leqslant i \leqslant j \leqslant n\}$ ordered by $(a_1, a_2) \leqslant (b_1, b_2) \Leftrightarrow b_1 \leqslant a_1 \leqslant a_2 \leqslant b_2$.

UNIMODAL WHITNEY NUMBERS. For some $I$, $N_0 \leqslant N_1 \leqslant \cdots \leqslant N_I \geqslant N_{I+1} \geqslant \cdots \geqslant N_{r(P)}$. The conditions $P$ may satisfy are the same as for the more restrictive case above with symmetric unimodal Whitney numbers. The key result here is clearly Theorem 3 that $P$ satisfies $T \Rightarrow P$ satisfies $S$. This is quite easy to show if $P$ has unimodal Whitney numbers which are also symmetric. It is nice that the result still holds without the symmetry.

ARBITRARY WHITNEY NUMBERS. For the general case, there are 10 possibilities. $P$ may satisfy all five conditions $R$, $S$, $S_1$, $T$, and $A$; $S$, $S_1$, and $T$; or any subset of the three conditions $\{S_1, T, A\}$.

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