Peckness of Edge Posets

David Hemminger[1], Aaron Landesman[2], Zijian Yao[3]

[1]Duke University , [2]Harvard University , [3]Brown University

August 7, 2014

Outline of Talk

Basic Definitions

Definition

Let P be a finite graded poset of rank n. That is:

- Elements of P are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks*
- If $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$
- Define $\mathrm{rk}(x) = k$, where $x \in P_k$.

Definition

A map $f\colon P \to Q$ is a *morphism* from P to Q if $x \leq_P y \implies f(x) \leq_Q f(y)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Peck Posets

Definition

Write $p_i = |P_i|$. P is

- *Rank-symmetric* if $p_i = p_{n-i}$ for all $1 \leq i \leq n$
- *Rank-unimodal* if for some $0 \leq k \leq n$ we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

- *k-Sperner* if no disjoint union of k antichains (sets of pairwise incomparable elements) in P is larger than the disjoint union of the largest k ranks of P
- *Strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$.
- *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Definition

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x|x \in P\}$ and $\{x|x \in P_i\}$

Lemma (Stanley, 1982)

P is Peck if and only if there exists an linear transformation $U\colon V(P) \to V(P)$ such that

- *For every basis element $x \in P$,*

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

- *For all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition

If the Lefschetz map defined by

$$L(x) = \sum_{y \gtrdot x} y$$

satisfies the second condition in the previous lemma, then P is *unitary Peck*.

Outline of Talk

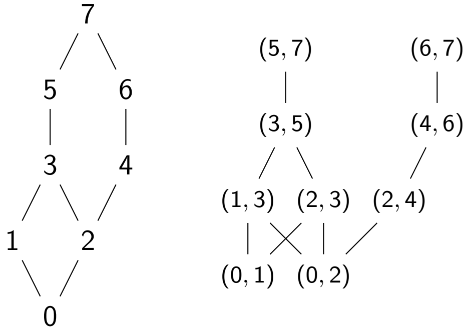
Definition of the Edge Poset

Definition

For P a finite graded poset, its *edge poset* $\mathcal{E}(P)$ is the finite graded poset defined as follows.

- Elements of $\mathcal{E}(P)$ are ordered pairs $(x, y) \in P \times P$ where $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$
- Define $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$
- Define $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$.

Basic Example



P $\mathcal{E}(P)$

Conjecture on the Peckness of Edge Posets

Definition

The *boolean algebra of rank n* is the poset whose elements are subsets of $[n]$ with order given by containment, i.e. for $x, y \in B_n$, $x \leq y$ if $x \subseteq y$.

Conjecture (Hemminger, Landesman, and Yao 2014)

Let $G \subseteq \mathrm{Aut}(B_n)$. Then $\mathcal{E}(B_n/G)$ is Peck.

Outline of Talk

Main Result

Definition

A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in \mathrm{Stab}_G(z)$ such that $g \cdot x = y$.

Theorem (Hemminger, Landesman, and Yao 2014)

If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.

Definition

Given a group action of G on P, define a group action of G on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

Definition

Given a group action of G on P, define a group action of G on $\mathcal{E}(P)$ by letting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$.

Proposition

The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism. Furthermore, q is also injective if and only if the action of G on P is CCT.

Lemma

If $f : P \to Q$ is a bijective morphism and P is Peck then Q is Peck.

Theorem (Stanley, 1984; Harper, 1984; Pouzet and Rosenberg, 1986)

If P is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then P/G is Peck.

It suffices to show that $\mathcal{E}(B_n)$ is unitary Peck. Our proof of this is complicated. Instead, we construct a unitary Peck poset $\mathcal{H}(B_n)$ such that there is a bijective morphism $\mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$.

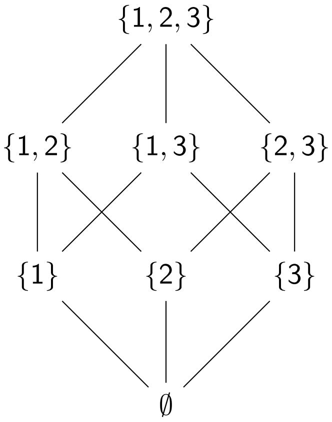
Definition of $\mathcal{H}(P)$

Definition

For P a finite graded poset, define the graded poset $\mathcal{H}(P)$ as follows.

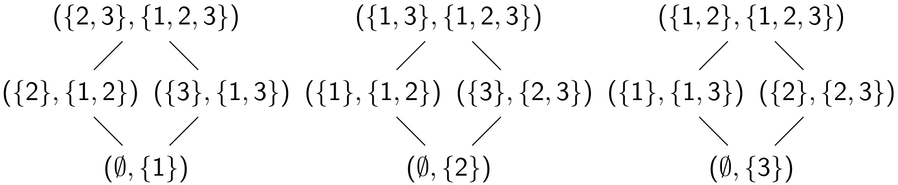
- Elements are pairs $(x, y) \in P \times P$ such that $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{H}} (x', y')$ if $x \lessdot_P x', y \lessdot_P y'$ **and** $\mathbf{y} \neq \mathbf{x}'$
- Define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$
- Define $rk_{\mathcal{H}}(x, y) = rk_P(x)$.

The Boolean Algebra B_3



B_3

$\mathcal{H}(B_3)$ is unitary Peck



$\mathcal{H}(B_3)$

Definition

As before, for G acting on $\mathcal{H}(P)$, define $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Remark

Since $\mathcal{E}(P)$ and $\mathcal{H}(P)$ have the same elements and $(x, y) \leq_{\mathcal{H}} (x', y') \implies (x, y) \leq_{\mathcal{E}} (x', y')$, there is a natural bijective morphism $\mathcal{H}(P)/G \to \mathcal{E}(P)/G$.

Proof of Main Result.

$\mathcal{H}(B_n)$ unitary Peck $\implies$ $\mathcal{H}(B_n)/G$ Peck $\implies$ $\mathcal{E}(B_n)/G$ Peck $\implies$ $\mathcal{E}(B_n/G)$ Peck. $\square$

Outline of Talk

CCT actions

Lemma

Let G be a group acting on a graded poset P. The following are equivalent:

1. *The action of G on P is CCT.*
2. *Whenever $w \lessdot x, w \lessdot y$, and $x \in Gy$, there exists some $g \in Stab(w)$ with $gx = y$.*
3. *The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x,z)) = (Gx, Gz)$ is a bijective morphism (but not necessarily an isomorphism).*
4. *For all i there is an equality $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$*

Some examples of CCT actions

The building blocks

1. The trivial group;

Some examples of CCT actions

The building blocks

1. The trivial group;
2. The group S_n acting on B_n;

Some examples of CCT actions

The building blocks

1. The trivial group;
2. The group S_n acting on B_n;
3. The group D_{2n} acting on B_n when $n = p$ or $n = 2p$, and p is a prime;

Some examples of CCT actions

The building blocks

1. The trivial group;
2. The group S_n acting on B_n;
3. The group D_{2n} acting on B_n when $n = p$ or $n = 2p$, and p is a prime;
4. The elementary 2-group $(\mathbb{Z}/2\mathbb{Z})^k$ with any action on B_n induced by an action on $[n]$.

The direct product

Lemma

For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.

The semi-direct product

Proposition

Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.

The wreath product

Corollary

If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action is also CCT.

The wreath product

Corollary

If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action is also CCT.

Corollary

The action $S_m \wr S_l \times B_n \to B_n$ is CCT, where $n = ml$.

The wreath product

Corollary

If $\psi : G \times P \to P$ *is CCT, then* $\phi : G \wr S_l \times P^l \to P^l$ *where* ϕ *is the induced action is also CCT.*

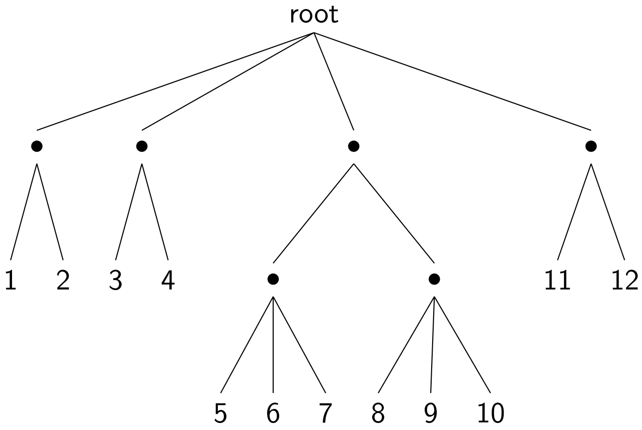
Corollary

The action $S_m \wr S_l \times B_n \to B_n$ *is CCT, where* $n = ml$*.*

Remark

This recovers a special case of a theorem obtained by Pak & Panova:
The poset $\mathcal{E}(B_n/S_m \wr S_l)$ is rank symmetric and rank unimodal.
(Furthermore, it is Peck!)

The automorphism of rooted trees



Automorphism of rooted trees

Proposition

Let P be a rooted tree. Then,

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

Automorphism of rooted trees

Proposition

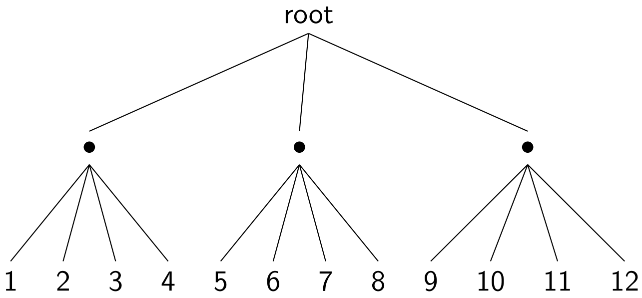
Let P be a rooted tree. Then,

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}),$$

Corollary

Let P be a rooted tree with leaves $L(P)$, and let $n = |L(P)|$, then the action of $Aut(P)$ on B_n induced from the action of $Aut(P)$ on $L(P)$ is CCT.

Rooted trees and the wreath product



Outline of Talk

Unimodality of ranks of certain edge posets

Lemma

Let C_n be the cyclic group which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|\mathcal{E}(B_n)/C_n|_i = \binom{n-1}{i}.$$

Unimodality of ranks of certain edge posets

Lemma

Let C_n be the cyclic group which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|\mathcal{E}(B_n)/C_n|_i = \binom{n-1}{i}.$$

Lemma

Let C_p be the cyclic group with prime order p, then

$$|(\mathcal{E}(B_p)/C_p)_i| - |\mathcal{E}(B_p/C_p)_i| = \frac{p-1}{2}.$$

Unimodality of ranks of certain edge posets

Lemma

Let C_n be the cyclic group which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|\mathcal{E}(B_n)/C_n|_i = \binom{n-1}{i}.$$

Lemma

Let C_p be the cyclic group with prime order p, then

$$|(\mathcal{E}(B_p)/C_p)_i| - |\mathcal{E}(B_p/C_p)_i| = \frac{p-1}{2}.$$

Proposition

The poset $\mathcal{E}(B_n/C_n)$ is rank symmetric and rank unimodal.

Unimodality of ranks of certain edge posets

Lemma

Let D_{2n} be the dihedral group of size $2n$ which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/D_{2n}$ is

$$|\mathcal{E}(B_n)/D_{2n}|_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+1)}+1]\cdot\binom{\lceil n/2\rceil - 1}{\lceil (i+1)/2\rceil - 1}\right)$$

Unimodality of ranks of certain edge posets

Lemma

Let D_{2n} be the dihedral group of size $2n$ which acts naturally on B_n, then size of the i^{th} rank of the poset $\mathcal{E}(B_n)/D_{2n}$ is

$$|\mathcal{E}(B_n)/D_{2n}|_i = \frac{1}{2}\left(\binom{n-1}{i}+\frac{1}{2}[(-1)^{n(i+1)}+1]\cdot\binom{\lceil n/2\rceil-1}{\lceil (i+1)/2\rceil-1}\right)$$

Proposition

The poset $\mathcal{E}(B_n/D_{2n})$ is rank symmetric and rank unimodal.

Outline of Talk

The q analog of the problem

The q-Boolean algebra

Let $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, and $G < Gln(\mathbb{F}_q)$. We consider $\mathcal{E}(B_n(q))/G$ and $\mathcal{E}(B_n(q)/G)$.

The q analog of the problem

The q-Boolean algebra

Let $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, and $G < Gln(\mathbb{F}_q)$. We consider $\mathcal{E}(B_n(q))/G$ and $\mathcal{E}(B_n(q)/G)$.

Lemma

Let q be a prime, then

$$|\mathcal{F}^1(B_n(q))/C_n(q)|_i = \binom{n-1}{i}_q.$$

The q analog of the problem

The q-Boolean algebra

Let $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, and $G < Gln(\mathbb{F}_q)$. We consider $\mathcal{E}(B_n(q))/G$ and $\mathcal{E}(B_n(q)/G)$.

Lemma

Let q be a prime, then

$$|\mathcal{F}^1(B_n(q))/C_n(q)|_i = \binom{n-1}{i}_q.$$

Questions

Is $\mathcal{E}\left(B_n(q)/G\right)$ Peck? or more weakly, is it rank unimodal?

Some Generalizations of $\mathcal{E}$ and Remarks

$\mathcal{E}^r(P)$

Similarly we can define the $\mathcal{E}^r(P)$ on a graded poset P. The elements of $\mathcal{E}^r(P)$ are (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + r$. Define the covering relation $\lessdot_{\mathcal{E}}$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$.

Some Generalizations of $\mathcal{E}$ and Remarks

$\mathcal{E}^r(P)$

Similarly we can define the $\mathcal{E}^r(P)$ on a graded poset P. The elements of $\mathcal{E}^r(P)$ are (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + r$. Define the covering relation $\lessdot_{\mathcal{E}}$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$.

Other generalizations

$\mathcal{H}^r(P)$; $\mathcal{E}^{\vec{r}}(P)$; $\mathcal{H}^{\vec{r}}(P)$.

References

Acknowledgements

1. Thanks to Dr. Vic Reiner and Elise DelMas for mentoring and TAing this project. We also want to thank Ka Yu Tam for helpful comments.
2. The work for this project took place at the Minnesota at Twin Cities REU. Thanks to Dr. Gregg Musiker and the University of Minnesota School of Mathematics for coordinating and hosting the REU.
3. This research was supported by the RTG grant NSF/DMS-1148634.