Peckness of Edge Posets

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Basic Definitions

Definition

Let P be a finite graded poset of rank n. That is:

- Elements of P are a disjoint union of $P_0, P_1, \ldots, P_n$, called the *ranks*
- If $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$
- Define $\mathrm{rk}(x) = k$, where $x \in P_k$.

Peck Posets

Definition

Write $p_i = |P_i|$. P is

- *Rank-symmetric* if $p_i = p_{n-i}$ for all $1 \leq i \leq n$
- *Rank-unimodal* if for some $0 \leq k \leq n$ we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

- *k-Sperner* if no union of k antichains (sets of pairwise incomparable elements) in P is larger than the union of the largest k ranks of P
- *Strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$.
- *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

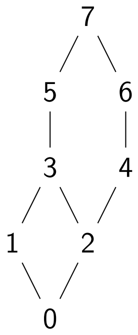
Definition of the Edge Poset

Definition

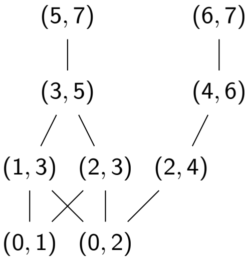
For P a finite graded poset, its *edge poset* $\mathcal{E}(P)$ is the finite graded poset defined as follows.

- Elements of $\mathcal{E}(P)$ are ordered pairs $(x, y) \in P \times P$ where $x \lessdot y$
- Define $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$
- Define $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$
- Define $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$.

Basic Example



P



$\mathcal{E}(P)$

Conjecture on the Peckness of Edge Posets

Definition

The *boolean algebra of rank n*, denoted B_n, is the poset whose elements are subsets of $[n]$ with order given by containment, i.e. for $x, y \in B_n$, $x \leq y$ if $x \subseteq y$.

Conjecture (Hemminger, Landesman, and Yao 2014)

Let $G \subseteq \mathrm{Aut}(B_n)$*. Then* $\mathcal{E}(B_n/G)$ *is Peck.*

Main Result

Definition

A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$, there exists some $g \in \mathrm{Stab}_G(z)$ such that $g \cdot x = y$.

Theorem (Hemminger, Landesman, and Yao 2014)

If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.

Some examples of CCT actions

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CCT Actions

1. The trivial group;
2. The group S_n acting on B_n;
3. The group D_{2n} acting on B_n when $n = p$ or $n = 2p$, and p is a prime;
4. The elementary 2-group $(\mathbb{Z}/2\mathbb{Z})^k$ with any action on B_n.

The direct product and semi-direct product

Lemma

For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product $\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$ is also CCT.

Proposition

Let $G \subseteq \mathrm{Aut}(P)$, $H \lhd G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.

The wreath product

Corollary

If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action is also CCT.

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The action $S_m \wr S_l \times B_n \to B_n$ is CCT, where $n = ml$.

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Remark

This recovers a special case of [IP13, Theorem 1.1]:
The poset $\mathcal{E}(B_n/S_m \wr S_l)$ is rank symmetric and rank unimodal.
(Furthermore, it is Peck!)

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