UNIMODALITY IDEAS

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1. Directions to move

(1) Look at generalising p_i^r for general r.
(2) Generalizing to q analog of cyclic group.
(3) Try relating p_i, q_i.
(4) Coding which groups G we have $p_i = q_i$.
(5) When are $p_i = q_i$.
(6) Try to compute q_i.
(7) Look at simple groups, and maybe solvable groups, try quotienting by normal subgroups?
(8) Are there any ways to combine G_1, G_2 where G_i are groups with $p_i = q_i$.
(9) Are there some characterisations of groups with q_i, p_i.
(10) How to use sage, what can we do with groups?
(11) Which edge poset definition do we want? Do we include edges containing y or exclude them?
(12) Look at $B_n(q)$.
(13) Look at generalizing $F_r(B_n)$ to arbitrary posets
(14) Try relating Wilson's Normal Form to our posets?

2. The Functor of Faces

Remark 2.0.1. *We assume all posets are ranked posets, and G actions are rank preserving, order preserving actions.*

Definition 2.0.2. *Define the poset category $\mathcal{P}_r$, where the objects $P \in \mathcal{P}_r$ are ranked poset, and the morphisms $Mor(P, Q)$ are rank preserving, order preserving maps, which send all B_{r+1} to other B_{r+1}.*

Definition 2.0.3. *Define the Functor of Faces $\mathcal{F} : \mathcal{P} \to \mathcal{P}$, that associates to any poset $P \in \mathcal{P}$ a poset $\mathcal{F}(P)$, where the vertices of $\mathcal{F}(P)$ are pairs $(x, y) \in P \times P, x \lessdot y$, and the ordering in $\mathcal{F}(P)$ is given by $(x, y) \leq (a, b)$ if $x \leq a, y \leq b$. For $f \in Mor(P, Q)$, the corresponding morphism $\mathcal{F}(f) : \mathcal{F}(P) \to \mathcal{F}(Q)$, given by $\mathcal{F}(f)(x, y) = (f(x), f(y))$.*

Conjecture 1. *The Functor of faces $\mathcal{F}$ is fully faithful.*

3. Quotient Edges

This section will build up to the theorem that for a symmetric poset P, with $\mathcal{F}(P)$ having injective order raising maps for rank $i < \frac{n}{2}$, we will also have $\mathcal{F}(P/G)$ has injective order raising maps for $i < \frac{n-1}{2}$.

Notation 3.0.4. *Let $U_i : P_i \to P_{i+1}$ be the raising operator for the poset P. Then, we obtain an induced map*

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \to P_{i+1} \otimes P_{i+1}, x \otimes y \mapsto U(x) \otimes U(y).$$

Notation 3.0.5. *We also have the natural inclusions*

$$\begin{aligned} k_i : \mathcal{F}(P)_i &\to P_i \otimes P_{i+1}, \\ x \otimes y &\mapsto x \otimes y \\ k_i^{G \times G} : \mathcal{F}(P/G)_i &\to (P/G)_i \otimes (P/G)_{i+1}, \\ Gx \otimes Gy &\mapsto Gx \otimes Gy, \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Notation 3.0.6. *Next, we define the map*

$$\begin{aligned} j_i : (P/G)_i \otimes (P/G)_{i+1} &\to P_i \otimes P_{i+1}, \\ Gx \otimes Gy &\mapsto \frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy. \end{aligned}$$

where x is an arbitrary representative of Gx and y is an arbitrary representative of Gy

Lemma 3.0.7. *The map j_i is well defined.*

Proof. It suffices to check that if x, z are two representatives of Gx, then $\sum_{g \in G} gx = \sum_{h \in G} gz$. This is clear because by definition $Gx = Gz$ means $z = g_1 x$ for some $g_1 \in G$, and so we can reorder the sum. □

Notation 3.0.8. *Define the map*

$$\begin{aligned} p_i : P_i \otimes P_{i+1} &\to (P/G)_i \otimes (P/G)_{i+1}, \\ x \otimes y &\mapsto Gx \otimes Gy. \end{aligned}$$

Notation 3.0.9. *Define the map*

$$\begin{aligned} (U_i \otimes U_{i+1})^{G \times G} : P_i \otimes P_{i+1} &\to P_{i+1} \otimes P_{i+1}, \\ Gx \otimes Gy &\mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx \otimes Gy). \end{aligned}$$

Notation 3.0.10. *We also have the projections inclusions*

$$\begin{aligned} \pi_i : P_i \otimes P_{i+1} &\to \mathcal{F}(P)_i, \\ x \otimes y &\mapsto \begin{cases} x \otimes y, & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases} \\ \pi_i^{G \times G} : (P/G)_i \otimes (P/G)_{i+1} &\to \mathcal{F}(P/G)_i, \\ Gx \otimes Gy &\mapsto \begin{cases} Gx \otimes Gy, & \text{if } Gx \lessdot Gy \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Notation 3.0.11. *We denote*

$$\begin{aligned}\mathcal{F}(U)_i : \mathcal{F}(P)_i &\to \mathcal{F}(P)_{i+1}\\ x \otimes y &\mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x \otimes y)\\ \mathcal{F}(U)_i^{G\times G} : \mathcal{F}(P/G)_i &\to \mathcal{F}(P/G)_{i+1}\\ Gx \otimes Gy &\mapsto k_i^{G\times G} \circ (U \otimes U)^{G\times G} \circ \pi_{i+1}^{G\times G}(Gx \otimes Gy)\end{aligned}$$

where it is defined above on a basis and we extend to the whole space by linearity.

Remark 3.0.12. *For $i < \frac{n}{2}$ we obtain the following (almost commuting, but $j_{i+1} \circ p_{i+1} \neq \mathrm{id}$.) diagram*

(1)

$$\begin{tikzcd}
\mathcal{F}(P)_i \arrow[r, hook, "k_i"] \arrow[rrr, hook, bend left, "\mathcal{F}(U)_i"] & P_i \otimes P_{i+1} \arrow[r, hook, "U_i \otimes U_{i+1}"] & P_{i+1} \otimes P_{i+2} \arrow[r, two heads, "\pi_{i+1}"] \arrow[d, two heads, shift right, "p_{i+1}"'] & \mathcal{F}(P)_{i+1} \\
\mathcal{F}(P/G)_i \arrow[r, hook, "k_i^{G\times G}"] \arrow[rrr, hook, bend right, "\mathcal{F}(U)_i^{G\times G}"'] & (P/G)_i \otimes (P/G)_{i+1} \arrow[r, hook, "(U_i \otimes U_{i+1})^{G\times G}"] \arrow[u, hook, "j_i"'] & (P/G)_{i+1} \otimes (P/G)_{i+2} \arrow[r, two heads, "\pi_{i+1}^{G\times G}"] \arrow[u, hook, shift right, "j_{i+1}"'] & \mathcal{F}(P/G)_{i+1}
\end{tikzcd}$$

Lemma 3.0.13. *The map p_i is a left inverse for j_i. That is, $p_i \circ j_i = \mathrm{id}$.*

Proof. For $Gx \otimes Gy \in (P/G)_i \otimes (P/G)_{i+1}$, we have

$$\begin{aligned}p_i \circ j_i(Gx \otimes Gy) &= p_i(\frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h\in G} hy)\\ &= \frac{1}{|G|}\sum_{g\in G} Gx \otimes \frac{1}{|G|}\sum_{h\in G} Gy\\ &= Gx \otimes Gy\end{aligned}$$

□

Lemma 3.0.14. *The central square commutes. That is, $j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} = (U_i \otimes U_{i+1}) \circ j_i$.*

Proof. By definition, $(U_i \otimes U_{i+1})^{G\times G} = p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i$. Therefore, we have $j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} = j_{i+1} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i$.

So, in order to complete the lemma, it suffices to show that $j_{i+1} \circ p_{i+1}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)} = \mathrm{id}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)}$. Indeed, any element $v \in \mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)$ must be of the form

$$v = \sum_{x \otimes y} c_{x \otimes y} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right).$$

In order to show

$$j_{i+1} \circ p_{i+1}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)} = \mathrm{id}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)},$$

it suffices to check it on a basis. That is, we only have to show

$$j_{i+1} \circ p_{i+1} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right) = \frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy,$$

or equivalently

$$j_{i+1} \circ p_{i+1} \left(\sum_{g \in G} gx \otimes \sum_{h \in G} hy \right) = \sum_{g \in G} gx \otimes \sum_{h \in G} hy.$$

However,

$$\begin{aligned} j_{i+1} \circ p_{i+1} \left(\sum_{g \in G} gx \otimes \sum_{h \in G} hy \right) &= j_{i+1} \left(\sum_{g \in G} Gx \otimes \sum_{h \in G} Gy \right) \\ &= j_{i+1} \left(|G| \cdot Gx \otimes |G| \cdot Gy \right) \\ &= \frac{1}{|G|} \sum_{g \in G} |G| gx \otimes \frac{1}{|G|} \sum_{h \in G} |G| hy \\ &= \sum_{g \in G} gx \otimes \sum_{h \in G} hy \end{aligned}$$

□

Lemma 3.0.15. *If $\mathcal{F}(U)_i$ is injective, then* $\ker(\pi_i)|_{Im((U_i \otimes U_{i+1}) \circ k_i)} = 0$.

Proof. Since $\mathcal{F}(U)_i$ is injective, $\ker \mathcal{F}(U)_i = 0$. Therefore, since $\mathcal{F}(U)_i = \pi_i \circ (U_i \otimes U_{i+1}) k_i$, we have $\ker(\pi_i)|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ k_i)} = 0$. □

Lemma 3.0.16. *If $\mathcal{F}(U)_i$, are injective, then* $\ker \pi_{i+1} \cap Im((U \otimes U) \circ j_i \circ k_i^{G \times G}) = 0$.

Proof. Any element $v \in \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G})$ can be written in the form

$$v = \sum_{x \otimes y} c_{x \otimes y} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right).$$

Therefore, if $v \in \ker \pi_{i+1}$, we must have $\pi_{i+1} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right) = 0$ for all pairs (x, y), such that $x \in Gx, y \in Gy$, because distinct orbits are disjoint. Hence, it suffices to show that we cannot have $\pi_{i+1} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right) = 0$.

We know that if $\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \in \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G \times G})$, then there must exist some $x \in Gx, y \in Gy$ for which $x \lessdot y$. However, this implies that $x \otimes y \in Supp\left(\pi_{i+1}\left(\frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h\in G} hy\right)\right)$, and so in particular $\pi_{i+1}\left(\frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h\in G} hy\right) \neq 0$. $\square$

Lemma 3.0.17. *We have* $\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G})$

Proof. Once, again, starting with an arbitrary $v \in \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$, we can write

$$v = \sum_{x\otimes y} c_{x\otimes y}\left(\frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h\in G} hy\right).$$

where $x \lessdot y$, for all $x \otimes y$ in the support of the above sum. The aim is to show that $v \in \ker \pi \implies v \in \ker(\pi_i^{G\times G} \circ p_i)$ Since distinct G orbits are disjoint, we can assume $v = \frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h\in G} hy$. In this case, if $v \in \ker \pi$, this means $x \not\lessdot y$ for any $x \in Gx, y \in Gy$. But this means $Gx \not\lessdot Gy$, and so $v \in \ker(\pi_i^{G\times G} \circ p_i)$. $\square$

Corollary 3.0.18. *If $\mathcal{F}(U)_i$, are injective, then* $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$.

Proof. By 3.0.17, we know $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$. But by 3.0.16 we know $\ker \pi_{i+1} \cap \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$. Therefore, $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$. $\square$

Lemma 3.0.19. *If $\mathcal{F}(U)_i, U_i, U_{i+1}$ are injective, then so is $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$.*

Proof. Since U_i, U_{i+1} are both injective, $U_i \otimes U_{i+1}$ is as well. It is always the case that $j_i, k_i^{G\times G}$ are injective. Therefore, $(U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$ is injective. In order to show $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$ is injective, it suffices to show that $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \mathrm{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$, which is precisely true by 3.0.18 $\square$

Lemma 3.0.20. *We have an equality of linear transformations $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \mathcal{F}(U)_i$.*

Proof. By 3.0.14, we have
$\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \pi_{i+1}^{G\times G} \circ p_{i+1} \circ j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} \circ k_i^{G\times G}$.

Then, by **??**, we obtain

$$\begin{aligned}\pi_{i+1}^{G\times G} \circ p_{i+1} \circ j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} \circ k_i^{G\times G} &= \pi_{i+1}^{G\times G} \circ (U_i \otimes U_{i+1})^{G\times G} \circ k_i^{G\times G} \\ &= \mathcal{F}(U)_i\end{aligned}$$

Therefore,

$$\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \mathcal{F}(U)_i.$$

$\square$

Theorem 3.0.21. *If $\mathcal{F}(U)_i, U_i, U_{i+1}$ are injective, then $\mathcal{F}(U)_i$ is injective.*

Proof. By 3.0.19, we know $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$ is injective. But by 3.0.20, $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \mathcal{F}(U)_i$. Therefore, $\mathcal{F}(U)_i$ is injective. $\square$

4. The object $\mathcal{F}(B_n)$.