

UNIMODALITY IDEAS

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1. DIRECTIONS TO MOVE

- (1) Look at generalising p_i^r for general r .
- (2) Generalizing to q analog of cyclic group.
- (3) Try relating p_i, q_i .
- (4) Coding which groups G we have $p_i = q_i$.
- (5) When are $p_i = q_i$.
- (6) Try to compute q_i .
- (7) Look at simple groups, and maybe solvable groups, try quotienting by normal subgroups?
- (8) Are there any ways to combine G_1, G_2 where G_i are groups with $p_i = q_i$.
- (9) Are there some characterisations of groups with q_i, p_i .
- (10) How to use sage, what can we do with groups?
- (11) Which edge poset definition do we want? Do we include edges containing y or exclude them?
- (12) Look at $B_n(q)$.
- (13) Look at generalizing $F_r(B_n)$ to arbitrary posets
- (14) Try relating Wilson's Normal Form to our posets?

2. EDGE FUNCTOR

Remark 2.0.1. *We assume all posets are ranked posets, and G actions are rank preserving, order preserving actions.*

Definition 2.0.2. *Define the poset category \mathcal{P}_r , where $P \in \mathcal{P}_r$ is a ranked poset, and the morphisms $\text{Mor}(P, Q)$ are rank preserving, order preserving maps, which send all B_{r+1} to other B_{r+1} .*

Definition 2.0.3. *Define the Faces functors (there is one for each r) $\mathcal{F}_r : \mathcal{P}_r \rightarrow \mathcal{P}_r$, which takes a poset to the poset of its i faces. That is, for each B_k subalgebra of P_r , we associate a point. We say a point $p \leq q$ if p and q are nonintersecting boolean subalgebras, and the bottommost point of the cube representing p is right below the bottommost point of the cube representing q . It takes a map of posets to the induced map on cubes, by definition of the morphisms in \mathcal{P}_r . For ease of notation, we shall use \mathcal{F} for \mathcal{F}_1 .*

3. THE PICTURE FOR B_n

Theorem 3.0.4. *For B_n the boolean algebra, $\mathcal{F}(B_n)$ is unitary peck.*

Proof. $\mathcal{F}(B_n)$ is actually just a disjoint union of n copies of B_{n-1} , where each copy is indexed as corresponding to the set of pairs $B_{n-1} \cong (\mathcal{F}(B_n))_{(i)} = \{(y, x) | y \geq x, y/x = i\}$, where $i \in [n]$. \square

Notation 3.0.5. Let $\Delta(G) \subset G \times G$ denote the diagonal subgroup. Define $X_G(P) = \text{Ind}_{\Delta(G)}^{G \times G}(\mathcal{F}(V(P)))/(G \times G)$

Notation 3.0.6. Let $V(P)$ denote the graded vector space with basis $p \in P$. The grading is given by the rank of p .

Theorem 3.0.7. We then have an isomorphism of graded vector spaces $\mathcal{F}(V(P)/G) \cong X_G(P)$.

Proof. The basis for $\mathcal{F}(V(P)/G)$ is exactly given by the edges of $V(P)/G$. By definition, we have an edge (Gx, Gy) in $V(P)/G$ if and only if $\exists g, h \in G$ with $gx < hy$. Next, $\text{Ind}_{\Delta(G)}^{G \times G}(\mathcal{F}(V(P)))$ is precisely the set of all edges of the form gx, hy for $g, h \in G$. And hence, we have a natural $G \times G$ action on it. Then, by definition, if we quotient by the $G \times G$ action, we obtain the exact same set of edges as in $\mathcal{F}(V(P)/G)$. Since ranks are always preserved under these maps, we obtain the claimed isomorphism of graded vector spaces. \square

Lemma 3.0.8. The poset corresponding to the graded vector space $\text{Ind}_{\Delta(G \times G)}^{G \times G}(\mathcal{F}(V(P)))$ is unitary peck.

Proof. First, by 3.0.4, we know $\mathcal{F}(V(P))$ is unitary peck. Then, induction simply makes $|G|$ disjoint copies of $\mathcal{F}(V(P))$. Therefore, we can take the corresponding block diagonal raising operators for each disjoint copy, and they obviously provide isomorphisms from level i to $n - i$. \square

Theorem 3.0.9. (Stanley) The quotient of a unitary peck poset by an order preserving, rank preserving group action G , is peck.

Corollary 3.0.10. The $X_G(P)$ are vector spaces with an underlying peck poset structure.

Proof. By 3.0.8, we have $\text{Ind}_{\Delta(G)}^{G \times G}(\mathcal{F}(V(P)))$ is unitary peck. But then, since $X_G(P) = \text{Ind}_{\Delta(G)}^{G \times G}(\mathcal{F}(V(P)))/(G \times G)$, by 3.0.9 we obtain the poset corresponding to $X_G(P)$ is peck. \square

Corollary 3.0.11. The poset of edges $\mathcal{F}(V(P)/G)$ is unitary peck.

Proof. By 3.0.10 we know $X_G(P)$ is peck, but by 3.0.7 we have $\mathcal{F}(V(P)/G) \cong X_G(P)$, and so $\mathcal{F}(V(P)/G)$ is peck as well. \square

Notation 3.0.12. For any poset P , define p_i to be the number of edges from the i th level set P_i to the $i + 1$ th level set P_{i+1} ,

Corollary 3.0.13. The sequence p_i is unimodal and symmetric.

Proof. By definition, the rank of the i th level set of $\mathcal{F}(V(P)/G)$ is exactly the number of edges from levels i to $i+1$ of P . That is $\dim(\mathcal{F}(V(P)/G)_i) = p_i$. Since by 3.0.11, $\mathcal{F}(V(P)/G)$ is peck, it is in particular symmetric and unimodal, and so the p_i are symmetric and unimodal. \square