UNIMODALITY IDEAS

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1. Directions to move

(1) Look at generalising p_i^r for general r.
(2) Generalizing to q analog of cyclic group.
(3) Try relating p_i, q_i.
(4) Coding which groups G we have $p_i = q_i$.
(5) When are $p_i = q_i$.
(6) Try to compute q_i.
(7) Look at simple groups, and maybe solvable groups, try quotienting by normal subgroups?
(8) Are there any ways to combine G_1, G_2 where G_i are groups with $p_i = q_i$.
(9) Are there some characterisations of groups with q_i, p_i.
(10) How to use sage, what can we do with groups?
(11) Which edge poset definition do we want? Do we include edges containing y or exclude them?
(12) Look at $B_n(q)$.
(13) Look at generalizing $F_r(B_n)$ to arbitrary posets
(14) Try relating Wilson's Normal Form to our posets?

2. The Functor of Faces

Remark 2.0.1. *We assume all posets are ranked posets, and G actions are rank preserving, order preserving actions.*

Definition 2.0.2. *Define the poset category $\mathcal{P}$, where the objects $P \in \mathcal{P}$ are ranked poset, and the morphisms $Mor(P, Q)$ are rank preserving, order preserving maps.*

Definition 2.0.3. *Define the Functor of Faces $\mathcal{F} : \mathcal{P} \to \mathcal{P}$, that associates to any poset $P \in \mathcal{P}$ a poset $\mathcal{F}(P)$, where the vertices of $\mathcal{F}(P)$ are pairs $(x, y) \in P \times P, x \lessdot y$, and the ordering in $\mathcal{F}(P)$ is given by $(x, y) \lessdot (a, b)$ if $x \lessdot a, y \lessdot b$. For $f \in Mor(P, Q)$, the corresponding morphism $\mathcal{F}(f) : \mathcal{F}(P) \to \mathcal{F}(Q)$, given by $\mathcal{F}(f)(x, y) = (f(x), f(y))$.*

Conjecture 1. *The Functor of faces $\mathcal{F}$ is fully faithful.*

Definition 2.0.4. *Define the weak poset category $\mathcal{W}$, where the objects $P \in \mathcal{W}$ are ranked poset, and the morphisms $f \in Mor(P, Q)$ are rank preserving, order preserving maps, and if $a \lessdot b, a \lessdot c$, then $f(b) \neq f(c)$.*

Definition 2.0.5. *Define the Weak Functor of Faces $\} : \mathcal{W} \to \mathcal{W}$, that associates to any poset $P \in \mathcal{W}$ a poset $\mathcal{G}(P)$, where the vertices of $\mathcal{G}(P)$ are pairs $(x, y) \in P \times P, x \lessdot y$, and the ordering in $\mathcal{F}(P)$ is given by $(x, y) \lessdot (a, b)$ if $x \lessdot a, y \lessdot b, a \neq y$. For $f \in Mor(P, Q)$, the corresponding morphism $\mathcal{G}(f) : \mathcal{G}(P) \to \mathcal{G}(Q)$, given by $\mathcal{G}(f)(x, y) = (f(x), f(y))$.*

3. Quotient Edges

This section will build up to the theorem that for a symmetric poset P, with $\mathcal{F}(P)$ having injective order raising maps for rank $i < \frac{n}{2}$, we will also have $\mathcal{F}(P/G)$ has injective order raising maps for $i < \frac{n-1}{2}$.

Notation 3.0.6. *Let $U_i : P_i \to P_{i+1}$ be the raising operator for the poset P. Then, we obtain an induced map*

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \to P_{i+1} \otimes P_{i+1}, x \otimes y \mapsto U(x) \otimes U(y).$$

Notation 3.0.7. *We also have the natural inclusions*

$$\begin{aligned} k_i : \mathcal{F}(P)_i &\to P_i \otimes P_{i+1}, \\ x \otimes y &\mapsto x \otimes y \\ k_i^{G\times G} : \mathcal{F}(P/G)_i &\to (P/G)_i \otimes (P/G)_{i+1}, \\ Gx \otimes Gy &\mapsto Gx \otimes Gy, \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Notation 3.0.8. *Next, we define the map*

$$\begin{aligned} j_i : (P/G)_i \otimes (P/G)_{i+1} &\to P_i \otimes P_{i+1}, \\ Gx \otimes Gy &\mapsto \frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy. \end{aligned}$$

where x is an arbitrary representative of Gx and y is an arbitrary representative of Gy

Lemma 3.0.9. *The map j_i is well defined.*

Proof. It suffices to check that if x, z are two representatives of Gx, then $\sum_{g\in G} gx = \sum_{h \in G} gz$. This is clear because by definition $Gx = Gz$ means $z = g_1 x$ for some $g_1 \in G$, and so we can reorder the sum. □

Notation 3.0.10. *Define the map*

$$\begin{aligned} p_i : P_i \otimes P_{i+1} &\to (P/G)_i \otimes (P/G)_{i+1}, \\ x \otimes y &\mapsto Gx \otimes Gy. \end{aligned}$$

Notation 3.0.11. *Define the map*

$$\begin{aligned} (U_i \otimes U_{i+1})^{G\times G} : P_i \otimes P_{i+1} &\to P_{i+1} \otimes P_{i+1}, \\ Gx \otimes Gy &\mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx \otimes Gy). \end{aligned}$$

Notation 3.0.12. *We also have the projections inclusions*

$$\begin{aligned} \pi_i : P_i \otimes P_{i+1} &\to \mathcal{F}(P)_i, \\ x \otimes y &\mapsto \begin{cases} x \otimes y, & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases} \\ \pi_i^{G\times G} : (P/G)_i \otimes (P/G)_{i+1} &\to \mathcal{F}(P/G)_i, \\ Gx \otimes Gy &\mapsto \begin{cases} Gx \otimes Gy, & \text{if } Gx \lessdot Gy \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Notation 3.0.13. *We denote*

$$\begin{aligned}\mathcal{F}(U)_i : \mathcal{F}(P)_i &\to \mathcal{F}(P)_{i+1}\\ x \otimes y &\mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x \otimes y)\\ \mathcal{F}(U)_i^{G\times G} : \mathcal{F}(P/G)_i &\to \mathcal{F}(P/G)_{i+1}\\ Gx \otimes Gy &\mapsto k_i^{G\times G} \circ (U \otimes U)^{G\times G} \circ \pi_{i+1}^{G\times G}(Gx \otimes Gy)\end{aligned}$$

where it is defined above on a basis and we extend to the whole space by linearity.

Remark 3.0.14. *For $i < \frac{n}{2}$ we obtain the following (almost commuting, but $j_{i+1} \circ p_{i+1} \neq \mathrm{id}$.) diagram*

(1)

$$\begin{tikzcd}
\mathcal{F}(P)_i \arrow[r, hook, "k_i"] \arrow[rrr, bend left, hook, "\mathcal{F}(U)_i"] & P_i \otimes P_{i+1} \arrow[r, hook, "U_i \otimes U_{i+1}"] & P_{i+1} \otimes P_{i+2} \arrow[r, two heads, "\pi_{i+1}"] \arrow[d, shift right, two heads, "p_{i+1}"'] & \mathcal{F}(P)_{i+1} \\
\mathcal{F}(P/G)_i \arrow[r, hook, "k_i^{G\times G}"] \arrow[rrr, bend right, hook, "\mathcal{F}(U)_i^{G\times G}"'] & (P/G)_i \otimes (P/G)_{i+1} \arrow[r, hook, "(U_i \otimes U_{i+1})^{G\times G}"] \arrow[u, hook, "j_i"'] & (P/G)_{i+1} \otimes (P/G)_{i+2} \arrow[r, two heads, "\pi_{i+1}^{G\times G}"] \arrow[u, shift right, hook, "j_{i+1}"'] & \mathcal{F}(P/G)_{i+1}
\end{tikzcd}$$

Lemma 3.0.15. *The map p_i is a left inverse for j_i. That is, $p_i \circ j_i = \mathrm{id}$.*

Proof. For $Gx \otimes Gy \in (P/G)_i \otimes (P/G)_{i+1}$, we have

$$\begin{aligned}p_i \circ j_i(Gx \otimes Gy) &= p_i(\frac{1}{|G|}\sum_{g\in G} gx \otimes \frac{1}{|G|}\sum_{h \in G} hy)\\ &= \frac{1}{|G|}\sum_{g\in G} Gx \otimes \frac{1}{|G|}\sum_{h\in G} Gy\\ &= Gx \otimes Gy\end{aligned}$$

□

Lemma 3.0.16. *The central square commutes. That is, $j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} = (U_i \otimes U_{i+1}) \circ j_i$.*

Proof. By definition, $(U_i \otimes U_{i+1})^{G\times G} = p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i$. Therefore, we have $j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} = j_{i+1} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i$.

So, in order to complete the lemma, it suffices to show that $j_{i+1} \circ p_{i+1}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)} = \mathrm{id}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)}$. Indeed, any element $v \in \mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)$ must be of the form

$$v = \sum_{x \otimes y} c_{x \otimes y} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right).$$

In order to show

$$j_{i+1} \circ p_{i+1}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)} = \mathrm{id}|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ j_i)},$$

it suffices to check it on a basis. That is, we only have to show

$$j_{i+1} \circ p_{i+1} \left(\frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy \right) = \frac{1}{|G|} \sum_{g \in G} gx \otimes \frac{1}{|G|} \sum_{h \in G} hy,$$

or equivalently

$$j_{i+1} \circ p_{i+1} \left(\sum_{g \in G} gx \otimes \sum_{h \in G} hy \right) = \sum_{g \in G} gx \otimes \sum_{h \in G} hy.$$

However,

$$\begin{aligned} j_{i+1} \circ p_{i+1} \left(\sum_{g \in G} gx \otimes \sum_{h \in G} hy \right) &= j_{i+1} \left(\sum_{g \in G} Gx \otimes \sum_{h \in G} Gy \right) \\ &= j_{i+1} \left(|G| \cdot Gx \otimes |G| \cdot Gy \right) \\ &= \frac{1}{|G|} \sum_{g \in G} |G| gx \otimes \frac{1}{|G|} \sum_{h \in G} |G| hy \\ &= \sum_{g \in G} gx \otimes \sum_{h \in G} hy \end{aligned}$$

$\square$

Lemma 3.0.17. *If $\mathcal{F}(U)_i$ is injective, then* $\ker(\pi_i)|_{Im((U_i \otimes U_{i+1}) \circ k_i)} = 0$.

Proof. Since $\mathcal{F}(U)_i$ is injective, $\ker \mathcal{F}(U)_i = 0$. Therefore, since $\mathcal{F}(U)_i = \pi_i \circ (U_i \otimes U_{i+1}) k_i$, we have $\ker(\pi_i)|_{\mathrm{Im}((U_i \otimes U_{i+1}) \circ k_i)} = 0$. $\square$

THE FOLLOWING PROOF IS WRONG, ITS NOT EXACTLY THE G invariant bases, there are really sums of the G invariant basis. TO FIX THIS, WE REALLY HAVE TO GO UP FIRST, then go right. We can take an averaging when we go up, or we can make an arbitrary choice, it won't matter since U commutes with G. What bothers me is that we only seem to have used that $\mathcal{F}(U)_i$ is injective, even though it seems that sometimes the quotients aren't injective at the middle.

ALSO, WE MIGHT BE ABLE TO GET PECK POSETS IN GENERAL. I think it has to do with $U_i \otimes U_{i+1}$ being injective from the appropriate levels.

Lemma 3.0.18. *If $\mathcal{F}(U)_i$, are injective, then* $\ker \pi_{i+1} \cap Im((U \otimes U) \circ j_i \circ k_i^{G \times G}) = 0$.

Proof. Any element $v \in \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$ can be written in the form

$$v = \sum_{x\otimes y} c_{x\otimes y} \left(\frac{1}{|G|} \sum_{g\in G} gU(x) \otimes \frac{1}{|G|} \sum_{h\in G} hU(y) \right).$$

Therefore, if $v \in \ker \pi_{i+1}$, we must have $\pi_{i+1} \left(\frac{1}{|G|} \sum_{g\in G} gU(x) \otimes \frac{1}{|G|} \sum_{h\in G} hU(y) \right) = 0$ for all pairs (x, y), such that $x \in Gx, y \in Gy$, because distinct orbits are disjoint. BUT WHEN YOU APPLY U THEY ARE NOT DISJOINT! Hence, it suffices to show that we cannot have $\pi_{i+1} \left(\frac{1}{|G|} \sum_{g\in G} gU(x) \otimes \frac{1}{|G|} \sum_{h\in G} hU(y) \right) = 0$.

We know that if $\frac{1}{|G|} \sum_{g\in G} gx \otimes \frac{1}{|G|} \sum_{h\in G} hy \in \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$, then there must exist some $x \in Gx, y \in Gy$ for which $x \lessdot y$. However, this implies that $x \otimes y \in Supp \left(\pi_{i+1} \left(\frac{1}{|G|} \sum_{g\in G} gx \otimes \frac{1}{|G|} \sum_{h\in G} hy \right) \right)$, and so in particular $\pi_{i+1} \left(\frac{1}{|G|} \sum_{g\in G} gx \otimes \frac{1}{|G|} \sum_{h\in G} hy \right) \neq 0$. □

Lemma 3.0.19. *We have* $\ker \left(\pi_{i+1}^{G\times G} \circ p_{i+1} \right) \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G})$

Proof. Once, again, starting with an arbitrary $v \in \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$, we can write

$$v = \sum_{x\otimes y} c_{x\otimes y} \left(\frac{1}{|G|} \sum_{g\in G} gx \otimes \frac{1}{|G|} \sum_{h\in G} hy \right).$$

where $x \lessdot y$, for all $x \otimes y$ in the support of the above sum. The aim is to show that $v \in \ker \pi \implies v \in \ker(\pi_i^{G\times G} \circ p_i)$ Since distinct G orbits are disjoint, we can assume $v = \frac{1}{|G|} \sum_{g\in G} gx \otimes \frac{1}{|G|} \sum_{h\in G} hy$. In this case, if $v \in \ker \pi$, this means $x \not\lessdot y$ for any $x \in Gx, y \in Gy$. But this means $Gx \not\lessdot Gy$, and so $v \in \ker(\pi_i^{G\times G} \circ p_i)$. □

Corollary 3.0.20. *If $\mathcal{F}(U)_i$, are injective, then* $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap Im((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$.

Proof. By 3.0.19, we know $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$. But by 3.0.18 we know $\ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$. Therefore, $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$. □

Lemma 3.0.21. *If $\mathcal{F}(U)_i, U_i, U_{i+1}$ are injective, then so is* $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$.

Proof. Since U_i, U_{i+1} are both injective, $U_i \otimes U_{i+1}$ is as well. It is always the case that $j_i, k_i^{G\times G}$ are injective. Therefore, $(U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$ is injective. In order to show $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G}$ is injective, it suffices to show that $\ker(\pi_{i+1}^{G\times G} \circ p_{i+1}) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) = 0$, which is precisely true by 3.0.20 □

Lemma 3.0.22. *We have an equality of linear transformations* $\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \mathcal{F}(U)_i^{G\times G}$.

Proof. By 3.0.16, we have

$$\pi_{i+1}^{G\times G} \circ p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i \circ k_i^{G\times G} = \pi_{i+1}^{G\times G} \circ p_{i+1} \circ j_{i+1} \circ (U_i \otimes U_{i+1})^{G\times G} \circ k_i^{G\times G}.$$

Then, by 3.0.15, we obtain

$$\begin{aligned}\pi_{i+1}^{G\times G}\circ p_{i+1}\circ j_{i+1}\circ (U_i\otimes U_{i+1})^{G\times G}\circ k_i^{G\times G} &= \pi_{i+1}^{G\times G}\circ (U_i\otimes U_{i+1})^{G\times G}\circ k_i^{G\times G}\\ &= \mathcal{F}(U)_i^{G\times G}\end{aligned}$$

Therefore,

$$\pi_{i+1}^{G\times G}\circ p_{i+1}\circ (U_i\otimes U_{i+1})\circ j_i\circ k_i^{G\times G} = \mathcal{F}(U)_i^{G\times G}.$$

□

Theorem 3.0.23. *If $\mathcal{F}(U)_i, U_i, U_{i+1}$ are injective, then $\mathcal{F}(U)_i^{G\times G}$ is injective.*

Proof. By 3.0.21, we know $\pi_{i+1}^{G\times G}\circ p_{i+1}\circ (U_i\otimes U_{i+1})\circ j_i\circ k_i^{G\times G}$ is injective. But by 3.0.22, $\pi_{i+1}^{G\times G}\circ p_{i+1}\circ (U_i\otimes U_{i+1})\circ j_i\circ k_i^{G\times G} = \mathcal{F}(U)_i$. Therefore, $\mathcal{F}(U)_i$ is injective. □

4. The object $\mathcal{F}(B_n)$.

Notation 4.0.24. *For this section, we will use $M^{n-2i-1} = \mathcal{F}(U)^{n-2i-1}$, where $\mathcal{F}(U)^{n-2i-1} : \mathcal{F}(B_n)_i \to \mathcal{F}(B_n)_{n-i-1}$ is the lefchetz map, as defined in 3.0.13.*

Notation 4.0.25. *For the remainder of this section, we shall take $|a| = n-i-1, |b| = n-i, |x| = i, |y| = i+1$. Additionally, whenever we write an expression of the form $x\otimes y$ or $a\otimes b$, it is assumed that $x \lessdot y, a \lessdot b$.*

Theorem 4.0.26. *The map M satisfies*

$$M^{n-2i-1}(x\otimes y) = (2^k-1)(k-1)!\sum_{y\not\subset a, x\subset a, y\subset b} a\otimes b + k!\sum_{|b|=n-i,|a|=n-i-1,a\lessdot b,y\subset a} a\otimes b$$

Proof. COME BACK TO THIS THEOREM (MAYBE SOMEONE ELSE CAN WRITE IT UP.) □

Theorem 4.0.27. *The poset $\mathcal{G}(B_n)$ is isomorphic to a disjoint union of n copies of B_{n-1}. In particular, it is unitary peck.*

Proof. COME BACK TO THIS THEOREM (MAYBE SOMEONE ELSE CAN WRITE IT UP.) □

Theorem 4.0.28. *Defining L to be the lefchetz map $\mathcal{G}(B_n) \to \mathcal{G}(B_n)$, we have $L^{n-2i-1} : \mathcal{G}(B_n)_i \to \mathcal{G}(B_n)_{n-i-1}$, and explicitly*

$$L^{n-2i-1}(x\otimes y) = k!\sum_{|b|=n-i,|a|=n-i-1,a\lessdot b,y\subset a} a\otimes b.$$

Proof. COME BACK TO THIS THEOREM (MAYBE SOMEONE ELSE CAN WRITE IT UP.) □

4.1. **Proof that B_n is peck.**

The aim of this proof will be to show that the matrix produced above is invertible.

5. The proof from the row point of view

In this section, we will show the rows of M form a basis by showing we can make a change of basis to a map which is a Lefchetz map on a disjoint union of n copies of B_{n-1}. Since B_{n-1} is unitary peck, it will follow that M is an isomorphism.

Notation 5.0.1. *Let $\beta = \frac{2^k-1}{k}$. Denote*

$$v_{a\otimes b} = \beta \sum_{y\subset a} x\otimes y + \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y.$$

Note that $v_{a\otimes b}$ are simply the rows of M^{n-2i-1}, each divided by the constant $k!$.

Notation 5.0.2. *For any set s of size at least $n-i$,*

$$z_s = \frac{1}{\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}} \sum_{b\subset s, a\subset b} v_{a\otimes b}$$

Lemma 5.0.3. *For any set s of size at least $n-i$, we have $z_s = \sum_{y\subset s} x\otimes y$. In particular, $\sum_{y\subset s} x\otimes y$ lies in the span of $v_{a\otimes b}$*

Proof. We have

$$\begin{aligned}
&\left(\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}\right) z_s \\
&= \sum_{b\subset s, a\subset b} v_{a\otimes b} \\
&= \sum_{b\subset s, a\subset b}\left(\beta\sum_{y\subset a} x\otimes y + \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y.\right) \\
&= \sum_{b\subset s, a\subset b}\beta\sum_{y\subset a} x\otimes y + \sum_{b\subset s, a\subset b}\ \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y. \\
&= \beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)\sum_{y\subset s} x\otimes y + \sum_{b\subset s, a\subset b}\ \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y. \\
&= \beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)\sum_{y\subset s} x\otimes y + \binom{|s|-i-1}{n-2i-1}\sum_{y\subset s} x\otimes y \\
&= \left(\frac{1}{\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}}\right)\sum_{y\subset s} x\otimes y.
\end{aligned}$$

In going between the fourth line and the fifth line, I counted the number of y satisfying $y\subset a\subset b\subset s$. If we fix s, a there are $(|s|-n+i+1)$ choices for the element b, since it can add any element in s but not in a. Then, we need to count the number of a with $y\subset a\subset s$. This is exactly $\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)$.

In going from the fifth line to the sixth line, for y, s fixed, we count the number of b with $y\subset b\subset s$. This is exactly $\binom{|s|-i-1}{n-2i-1}$. □

Notation 5.0.4. *For any set s of size at least $n-i$, let $w_s = \sum_{a\subset s, t\notin s} v_{a\otimes a\cup\{t\}}$.*

Lemma 5.0.5. *We have*

$$w_s = \sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} x \otimes x \cup \{t\} + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} x \otimes x \cup \{t\} \right).$$

Proof.

$$\begin{aligned} w_s &= \sum_{a \subset s, t \notin s} v_{a \otimes a \cup \{t\}} \\ &= \sum_{a \subset s, t \notin s} \left(\beta \sum_{y \subset a} x \otimes y + \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} x \otimes y \right) \\ &= \sum_{t \notin s} \left(\sum_{a \subset s} \beta \sum_{y \subset a} x \otimes y + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} x \otimes y \right) \\ &= \sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} x \otimes y + \sum_{a \subset s} \sum_{y \subset b, x \subset a, y \not\subset a \cup \{t\}} x \otimes y \right) \\ &= \sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} x \otimes y + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} x \otimes x \cup \{t\} \right) \end{aligned}$$

The equalities between lines three and four, and four and five hold for similar reasons as the equalities between lines four and five, and five and six in 5.0.3 $\square$

Notation 5.0.6. *Let* $u_s = \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2} z_s}{\binom{|s|-i}{n-2i-1}} + z_s$.

Lemma 5.0.7. *We have* $u_s = \sum_{x \subset s, y \supset x} x \otimes y$. *In particular,* $\sum_{x \subset s, y \supset x} x \otimes y$ *lies in the span of* $v_{a \otimes b}$.

Proof. By 5.0.5 and 5.0.3 we have

$$\begin{aligned} u_s &= \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2} z_s}{\binom{|s|-i}{n-2i-1}} + z_s \\ &= \frac{\sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} x \otimes y + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} x \otimes x \cup \{t\} \right) - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} x \otimes y}{\binom{|s|-i}{n-2i-1}} + \sum_{y \subset s} x \otimes y \\ &= \frac{\sum_{t \notin s} \left(\sum_{x \subset s} x \otimes x \cup \{t\} \right)}{\binom{|s|-i}{n-2i-1}} + \sum_{y \subset s} x \otimes y \\ &= \sum_{t \notin s} \left(\binom{|s|-i}{n-2i-1} \sum_{x \subset s} x \otimes x \cup \{t\} \right) + \sum_{y \subset s} x \otimes y \\ &= \sum_{x \subset s, y \supset x} x \otimes y \end{aligned}$$

The penultimate line is equal to the ultimate line because any element $x \otimes y$ with $x \subset s$ must either have $y \subset s$ or else $y \subset s \cup \{t\}$ for $t \notin s$. The two terms on the penultimate line cover precisely these two cases. $\square$

Notation 5.0.8. *For any* $s \subset [n]$, *with* $|s| \leq n - i$, *define* $h_s = z_{[n]} - u_s$.

Lemma 5.0.9. *With h_s as defined above $h_s = \sum_{x \not\subset s} x \otimes y$*

Proof. Using 5.0.3 and 5.0.7

$$\begin{aligned} h_s &= z_{[n]} - u_s \\ &= \sum_{x \subset y} x \otimes y - \sum_{x \subset s, x \subset y} x \otimes y \\ &= \sum_{x \not\subset s} x \otimes y \end{aligned}$$

□

Notation 5.0.10. *For $I \subset [n]$, let $I^c = [n] \setminus I$, the complement of I in $[n]$.*

Notation 5.0.11. *For I, J with $I \subset J \subset [n]$, and $I \neq \emptyset$. such that $|J| \leq i$, define $l_{J,I} = \sum_{x \supset J, J \cap x = I} x \otimes y$.*

Lemma 5.0.12. *Assuming $i \neq 0$, for $|s| = n-1$, it follows that $h_s = l_{s^c,s^c}$.*

Proof. By 5.0.9,

$$\begin{aligned} h_s &= \sum_{x \not\subset s} x \otimes y \\ &= \sum_{s^c \subset x} x \otimes y \\ &= l_{s^c,s^c} \end{aligned}$$

In equating the first line with the second line, we are crucially using that $|s^c| = 1$, because then $x \not\subset s \Leftrightarrow s^c \subset x$. □

Lemma 5.0.13. $l_{s^c,s^c} = h_s - \sum_{I \subset s^c, I \neq s^c, I \neq \emptyset} l_{s^c,I}$.

Proof. We have

$$\begin{aligned} h_s &= \sum_{x \not\subset s} x \otimes y \\ &= \sum_{x \cap s^c \neq \emptyset} x \otimes y \\ &= \sum_{I \subset s^c, I \neq \emptyset, x \cap s^c = I} x \otimes y \\ &= \sum_{x \cap s^c = s^c} x \otimes y + \sum_{I \subset s^c, I \neq s^c, I \neq \emptyset, x \cap s^c = I} x \otimes y \\ &= l_{s^c,s^c} + \sum_{I \subset s^c, I \neq s^c, I \neq \emptyset} l_{s^c,I} \end{aligned}$$

□

Lemma 5.0.14. *For all $J \subset [n]$, with $|J| \leq i$, we have $l_{J,J}$ lies in the span of $v_{a \otimes b}$.*

Proof. By 5.0.12 we have that for any $s, |s| = n-1$, we have l_{s^c,s^c} lies in the span of $v_{a\otimes b}$. Inductively assume that we have shown l_{s^c,s^c} lies in the span of $v_{a\otimes b}$ for $|s| > j$. Then, for $|s| = j$ we can use 5.0.13 to write l_{s^c,s^c} in terms of h_s and l_{r^c,r^c} where $|r| > j$. Since we know both h_s and l_{r^c,r^c} lie in the span of $v_{a\otimes b}$, so does l_{s^c,s^c} for all $s, |s| > n-i$. □

Lemma 5.0.15. *For $|x| = i$, we have $l_{x,x} = \sum_{y\supset x} y \otimes x$.*

Proof. By definition, $l_{\bar{x},\bar{x}} = \sum_{\bar{x}\subset x, \bar{x}\cap x = \bar{x}, x\subset y} x\otimes y = \sum_{\bar{x}\subset y} \bar{x}\otimes y$. Replacing $\bar{x}$ by x gives the result. □

Notation 5.0.16. *Define $m_a = \sum_{x\subset a} l_{x,x}$.*

Lemma 5.0.17. *With m_a as defined above, we have $m_a = \sum_{x\subset a} x\otimes y$.*

Proof. By 5.0.15, we obtain

$$m_a = \sum_{x\subset a} l_{x,x} = \sum_{x\subset a}\sum_{y\supset x} y\otimes x = \sum_{x\subset a} x\otimes y$$

□

Notation 5.0.18. *Denote $r_a = \sum_{b, b\subset a} v_{a\otimes b}$.*

Lemma 5.0.19. *With r_a as defined above, we have $r_a = ((i+1)\beta - 1)\sum_{y\subset a} x\otimes y + \sum_{x\subset a} x\otimes y$.*

Proof.

$$\begin{aligned}
r_a &= \sum_{b\supset a} v_{a\otimes b} \\
&= \sum_{b\supset a}\left(\beta\sum_{y\subset a} x\otimes y + \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y.\right) \\
&= \sum_{b\supset a}\beta\sum_{y\subset a} x\otimes y + \sum_{b\supset a}\sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y. \\
&= (i+1)\beta\sum_{y\subset a} x\otimes y + \sum_{b\supset a}\sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y. \\
&= (i+1)\beta\sum_{y\subset a} x\otimes y + \sum_{x\subset a, y\not\subset a} x\otimes y. \\
&= ((i+1)\beta - 1)\sum_{y\subset a} x\otimes y + \left(\sum_{x\subset a, y\not\subset a} x\otimes y + \sum_{y\subset a} x\otimes y\right). \\
&= ((i+1)\beta - 1)\sum_{y\subset a} x\otimes y + \sum_{x\subset a} x\otimes y.
\end{aligned}$$

□

Notation 5.0.20. *Assuming we do not have $(i+1)\beta = 1$, define $t_a = \frac{r_a - m_a}{(i+1)\beta - 1}$*

Lemma 5.0.21. *With t_a as defined above, $t_a = \sum_{y\subset a} x\otimes y$*

Proof. By 5.0.19 and 5.0.17, we have

$$\begin{aligned} t_a &= \frac{r_a - m_a}{(i+1)\beta - 1} \\ &= \frac{((i+1)\beta - 1)\sum_{y\subset a} x\otimes y + \sum_{x\subset a} x\otimes y - \sum_{x\subset a} x\otimes y}{(i+1)\beta - 1} \\ &= \frac{((i+1)\beta - 1)\sum_{y\subset a} x\otimes y}{(i+1)\beta - 1} \\ &= \sum_{y\subset a} x\otimes y \end{aligned}$$

□

Notation 5.0.22. *Assuming $\beta(i+1) \neq 1$, let $q_{a,b} = v_{a\otimes b} - \beta t_a$.*

Lemma 5.0.23. *We have $q_{a,b} = \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y$.*

Proof. Using 5.0.21

$$\begin{aligned} q_{a,b} &= v_{a\otimes b} - \beta t_a \\ &= \beta \sum_{y\subset a} x\otimes y + \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y - \beta \sum_{y\subset a} x\otimes y \\ &= \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y \end{aligned}$$

□

Theorem 5.0.24. *For $n > 2$, the matrix M^{n-2i-1} is invertible.*

Proof. We saw that if $\beta(i+1) \neq 1$, we have that $q_{a,b} = \sum_{y\subset b, x\subset a, y\not\subset a} x\otimes y$ lie in the span of $v_{a\otimes b}$. It is the case that $\beta \geq 1, i \geq 0$. The only time in which $\beta = 1, i = 0$ is when $n = 2$. Therefore, $q_{a,b}$ are always defined for $n > 2$. However, $q_{a,b}$ are exactly the rows of L^{n-2i-1} as defined in Theorem 4.0.28. Using Theorem 4.0.27, we know $\mathcal{G}(B_n)$ is unitary peck, and therefore the rows of L^{n-2i-1} are independent. Therefore, the rows of M^{n-2i-1} span an independent set in a vector space of the same dimension. So, the rows of M^{n-2i-1} are indepenent. Hence, M^{n-2i-1} is indeed an isomorphism, when $n > 2$. □