PECKNESS OF EDGE POSETS

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Abstract. For any graded poset P, we define a new graded poset, $\mathcal{E}(P)$, whose elements are the edges in the Hasse diagram of P. For any group, G, acting on the boolean algebra, B_n, we conjecture that $\mathcal{E}(B_n/G)$ is Peck. We prove that the conjecture holds for "common cover transitive" actions. We give some infinite families of common cover transitive actions and show that the common cover transitive actions are closed under direct and semidirect products.

1. Introduction

Let P be a finite graded poset of rank n. In this paper we study the structure of the edges in the Hasse diagram of P. To this end, we define an endofunctor $\mathcal{E}$ on the category of finite graded posets with rank-preserving morphisms as follows

Definition 1.1. For $\mathcal{P}$ the category of graded posets, define the *functor of edges* $\mathcal{E}\colon \mathcal{P} \to \mathcal{P}$ as follows. Given $P \in \mathcal{P}$, the elements of the graded poset $\mathcal{E}(P)$ are pairs (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. Define the covering relation $\lessdot_{\mathcal{E}}$ on $\mathcal{E}(P)$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$. Then define the relation $\leq_{\mathcal{E}}$ on $\mathcal{E}(P)$ to be the transitive closure of $\lessdot_{\mathcal{E}}$.

Let Q be a finite graded poset of rank n. Given a morphism $f\colon P \to Q$, define $\mathcal{E}(f)\colon \mathcal{E}(P) \to \mathcal{E}(Q)$ by $\mathcal{E}(f)(x, y) = (f(x), f(y))$.

We will show that $\mathcal{E}(P)$ is a well-defined graded poset in Section 3. Note that an edge in the Hasse diagram of P can be identified with a pair $(x, y) \in P \times P$ such that $x \lessdot y$, and the edges in the Hasse diagram are in bijection with elements $(x, y) \in \mathcal{E}(P)$ via this identification. With this in mind, we will frequently refer to $\mathcal{E}(P)$ as the *edge poset* of P.

Example 1.2. We give an example of an edge poset in Figure 1. Note that it is important we declare the relation $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$. If instead we defined a relation $\leq_{\mathcal{E}'}$ on $\mathcal{E}(P)$ by $(x, y) \leq (a, b)$ if $x \leq a, y \leq b$, then $\mathcal{E}(P)$ would not necessarily be a graded poset. In Figure 1 we give an example of a poset P for which $\mathcal{E}(P)$ is not graded with the relation $\leq_{\mathcal{E}'}$. In Figure 1 it is clear that $\mathcal{E}(P)$ is a graded poset, with $\mathrm{rk}(x, y) = \mathrm{rk}(x)$, but the Hasse diagram on the right represents a poset which does not have a grading.

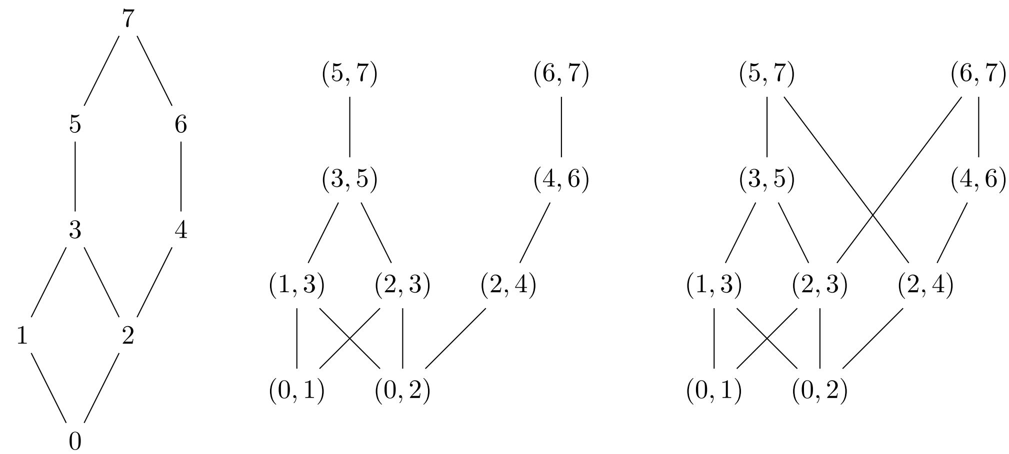
We observe that when P has a nice structure, $\mathcal{E}(P)$ commonly has a nice structure as well. In particular we examine the *boolean algebra of rank* n, denoted B_n, which is defined to be the poset whose elements are subsets of $\{1, \ldots, n\}$ with the relation given by containment, namely, for all $x, y \in B_n$, $x \leq y$ if x is a subset of y.

Throughout the paper we say a group G acts on P if it acts on the elements of P and that action is order-preserving and rank-preserving, that is, for all $g \in G$ we have $x \leq y \Leftrightarrow gx \leq gy$ and $\mathrm{rk}(gx) = \mathrm{rk}(x)$. By Theorem 2.6 and the fact that B_n is unitary peck, if G is any action on B_n, then B_n/G is Peck. We conjecture the following.

Conjecture 1.3. *If $G \subseteq \mathrm{Aut}(B_n)$, then $\mathcal{E}(B_n/G)$ is Peck.*

We prove this conjecture holds whenever the group action of G on B_n has the following property.

Definition 1.4. A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$ there exists some $g \in \mathrm{Stab}(z)$ such that $g \cdot x = y$.



P $\mathcal{E}(P)$ $\mathcal{E}(P)$ with relation $\leq_{\mathcal{E}'}$ (not graded)

FIGURE 1. Examples of $\mathcal{E}$

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

A large number of group actions on B_n have the CCT property. We first prove that some basic group actions on B_n are CCT. Throughout the paper we let a subgroup $G \subseteq S_n$ act on B_n by letting it act on the elements within subsets of $[n] := \{1, \ldots, n\}$, i.e. $g \cdot x = \{g \cdot i : i \in x\}$ for all $g \in G$, $x \in B_n$. We also embed the dihedral group D_{2n} into S_n by letting it act as rotations and reflections on the vertices of an n-gon.

Proposition 1.6. *The following actions are CCT.*

(1) *The action of S_n on B_n,*
(2) *The action of D_{2p} on B_p,*
(3) *The action of D_{4p} on B_{2p},*

where p is prime.

We further show that cover transitivity is preserved under semidirect products, allowing us to describe several large families of CCT actions in Subsection 4.2.

Proposition 1.7. *Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

The paper is organized as follows. In Section 2 we cover the necessary background for posets and Peck posets. In Section 3 we show that $\mathcal{E}$ is well-defined and prove Theorem 1.5 regarding CCT actions along with various other nice properties of $\mathcal{E}$. Section 4 contains the proofs of Propositions 1.6 and 1.7 as well as some examples of families of group actions shown to be CCT by these propositions. In Section 5, we obtain a very different proof of [4, Theorem 1.1], in the case that $r = 1$.

2. BACKGROUND

In this section we review necessary background for this paper.

A *graded poset* P is a poset with a rank function $\mathrm{rk}\colon P \to \mathbb{Z}_{\geq 0}$ satisfying the following conditions.

(1) If $x \in P$ and $x \lessdot y$, then $rk(x) + 1 = \mathrm{rk}(y)$,
(2) If $x < y$ then $\mathrm{rk}(x) < \mathrm{rk}(y)$

We denote the ith rank of P by $P_i = \{x \in P\colon \operatorname{rk}(x) = i\}$. Additionally, if for all $x \in P$, we have $0 \le \operatorname{rk}(x) \le n$, and there exists x, y with $\operatorname{rk}(x) = 0, \operatorname{rk}(y) = n$, we say that P is a graded poset of *rank* n.

Throughout the paper we write $x \le_P y$ to denote that x is less than or equal to y under the relation defined on the poset P. When the poset is clear we omit the P and simply write $x \le y$.

Let $P, Q \in \mathcal{P}$ be two finite graded posets. A map $f\colon P \to Q$ is a *morphism* from P to Q if it is rank-preserving and order preserving, in other words, for all $x, y \in P$, $x \le_P y$ implies $f(x) \le_Q f(y)$ and $\operatorname{rk}(x) = \operatorname{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Remark 2.1. Note that we do not require the implication that $f(x) \le_Q f(y)$ implies $x \le_P y$ in order for f to be a morphism. In particular this means that a bijective morphism f need not be an isomorphism, since it will not necessarily have a two-sided inverse.

In what follows, let P be a poset of rank n, and write $p_i = |P_i|$. If we have

$$p_0 \le p_1 \le \ldots \le p_k \ge p_{k+1} \ge \ldots \ge p_n$$

for some $0 \le k \le n$, then P is *rank-unimodal.* If $p_i = p_{n-i}$ for all $1 \le i \le n$, then P is *rank-symmetric.* An *antichain* in P is a set of elements in P that are pairwise incomparable. If no antichain in P is larger than the largest rank of P, then P is *Sperner.* More generally, P is *k-Sperner* if no union of k antichains in P is larger than the union of the largest k ranks of P. We say that P is *strongly Sperner* if it is k-Sperner for all $1 \le k \le n$.

Definition 2.2. P is *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x : x \in P\}$ and $\{x : x \in P_i\}$ respectively. Note that we will frequently abuse notation and write P and P_i for $V(P)$ and $V(P_i)$ when what we mean is clear from context. In determining whether P is Peck, it is often useful to consider certain linear transformations on $V(P)$.

Definition 2.3. A linear map $U\colon V(P) \to V(P)$ is an *order-raising operator* if $U(V(P_n)) = 0$ and for all $0 \le i \le n-1$, $x \in P_i$ we have

$$U(x) = \sum_{y \gtrdot x} c_{x,y}\, y$$

for some constants $c_{x,y} \in \mathbb{C}$. We say that U is the *Lefschetz map* if all $c_{x,y}$ on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

Lemma 2.4 ([5], Lemma 1.1)**.** *P is Peck if and only if there exists an order-raising operator U such that for all $0 \le i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition 2.5. If the Lefschetz map satisfies the condition for U in Lemma 2.4, then P is *unitary Peck.*

Note that a group G acts on P if the action defines an embedding $G \hookrightarrow \operatorname{Aut}(P)$. We define the *quotient poset* P/G to be the poset whose elements are the orbits of G, with the relation $\mathcal{O} \le \mathcal{O}'$ if there exist $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ such that $x \le_P x'$. We will use the following result in the paper.

Theorem 2.6 ([6], Theorem 1)**.** *If P is unitary Peck and $G \subseteq \operatorname{Aut}(P)$, then P/G is Peck.*

3. The Edge Poset Construction

In Subsection 3.1 we show that $\mathcal{E}$ as described in Definition 1.1 is well-defined and prove some useful properties of $\mathcal{E}$. In Subsection 3.2 we prove that $\mathcal{E}$ sends self-dual posets to self-dual posets. In Subsection 3.3 we give several equivalent definitions for CCT actions, and Subsection 3.4 is devoted to the proof of Theorem 1.5.

3.1. **Functoriality of $\mathcal{E}$ and Group Actions.** First we show that $\mathcal{E}$ is well-defined in Lemmas 3.1, 3.2, and 3.3. After showing that $\mathcal{E}$ is well-defined we then define a natural G action on $\mathcal{E}(P)$ and define a surjection $\mathcal{E}(P)/G \to \mathcal{E}(P/G)$ that will be important for the proof of Theorem 1.5.

When the poset P is clear, we will use $\leq_{\mathcal{E}}$ and $\lessdot_{\mathcal{E}}$ to refer to $\leq_{\mathcal{E}(P)}$ and $\lessdot_{\mathcal{E}(P)}$. Similarly, in Subsection 3.4, we define posets $\mathcal{H}(B_n)$, and will use $\leq_{\mathcal{H}}$ and $\lessdot_{\mathcal{H}}$ in place of $\leq_{\mathcal{H}(B_n)}$ and $\lessdot_{\mathcal{H}(B_n)}$.

Lemma 3.1. *The relation $\leq_{\mathcal{E}}$ defines a partial order on $\mathcal{E}(P)$.*

Proof. We have that $(x, y) \leq_{\mathcal{E}} (x, y)$ and that $\leq_{\mathcal{E}}$ is transitive by definition. It remains to be shown that $\leq_{\mathcal{E}}$ is antisymmetric. Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$ and $(x', y') \leq_{\mathcal{E}} (x, y)$. Then $x \leq_P x' \leq_P x$ and $y \leq_P y' \leq_P y$, so $x = x'$ and $y = y'$ by antisymmetry of $\leq_P$, hence $(x, y) = (x', y')$. □

Lemma 3.2. *For P a graded poset, the object $\mathcal{E}(P)$ is a graded poset.*

Proof. To show $\mathcal{E}(P)$ is graded, we must show that $(x, y) \lessdot_{\mathcal{E}} (x', y') \implies \mathrm{rk}(x, y) + 1 = \mathrm{rk}(x', y')$. This fact follows immediately from the definition of $\lessdot_{\mathcal{E}}$ and the definition $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$. □

Lemma 3.3. *Let $f : P \to Q$ be a morphism of finite graded posets, and define a map $\mathcal{E}(f) : \mathcal{E}(P) \to \mathcal{E}(Q)$ by $\mathcal{E}(f)(x, y) = (f(x), f(y))$ for all $(x, y) \in \mathcal{E}(P)$. Then*

1. *$\mathcal{E}(f)$ is a morphism of finite graded posets*
2. *$\mathcal{E}(\mathrm{id}_P) = \mathrm{id}_{\mathcal{E}(P)}$*
3. *If $g : Q \to R$ is a morphism of finite graded posets, then $\mathcal{E}(g \circ f) = \mathcal{E}(g) \circ \mathcal{E}(f)$.*

Proof.
Part (1) First, $\mathcal{E}(f)$ is rank-preserving, since for all $(x, y) \in \mathcal{E}(P)$ we have

$$\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x) = \mathrm{rk}_P(f(x)) = \mathrm{rk}_{\mathcal{E}}(\mathcal{E}(f)(x, y)).$$

Suppose $(x, y) \lessdot_{\mathcal{E}} (x', y')$. Then $x \lessdot_P x'$, $y \lessdot_P y'$, and since f is order-preserving, it follows that $f(x) \lessdot_P f(x')$, $f(y) \lessdot_P f(y')$. Hence $\mathcal{E}(f)(x, y) \lessdot_{\mathcal{E}} \mathcal{E}(f)(x', y')$. Since $\leq_{\mathcal{E}}$ is the transitive closure of $\lessdot_{\mathcal{E}}$, we similarly obtain $\mathcal{E}(f)$ is order-preserving and hence a morphism of finite graded posets.

Part (2) This is trivial.

Part (3) For all $(x, y) \in \mathcal{E}(P)$ we have

$$\mathcal{E}(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}(g) \circ \mathcal{E}(f))(x, y).$$

□

Remark 3.4. By Lemmas 3.1, 3.2, and 3.3, the edge poset construction $\mathcal{E}$ defines an endofunctor on the category of finite graded posets with rank-preserving morphisms.

Given a group action of G on P, we can now easily define a natural group action of G on $\mathcal{E}(P)$ using Lemma 3.3. For all $g \in G$ we have that multiplication by g is an automorphism of P, so it follows that $\mathcal{E}(g)$ is an automorphism of $\mathcal{E}(P)$ and furthermore that this gives a well-defined group action by Lemma 3.3.

Definition 3.5. Given a G-action on P, define a G-action on $\mathcal{E}(P)$ by $g \cdot (x, y) = \mathcal{E}(g)(x, y) = (gx, gy)$.

We then have a well-defined quotient poset $\mathcal{E}(P)/G$. It is natural to ask whether the operation of quotienting out by G commutes with $\mathcal{E}$, that is, whether $\mathcal{E}(P/G) \cong \mathcal{E}(P)/G$. Unfortunately the two posets are rarely isomorphic, but there is always a surjection $\mathcal{E}(P)/G \to \mathcal{E}(P/G)$, and this surjection is also an injection precisely when the G-action on P is CCT, as will be shown in Lemma 3.14.

Proposition 3.6. *The map $q : \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism.*

Proof. Note that q is well defined because if $(x', y') = g(x, y) = (g \cdot x, g \cdot y)$ for some $g \in G$, then $x' \in Gx$ and $y' \in Gy$. Clearly q is rank-preserving and surjective, so it suffices to show that q is order-preserving. Suppose that $G(x, y) \lessdot_{\mathcal{E}(P)/G} G(w, z)$. Then there exist some $(x_0, y_0) \in G(x, y)$, $(w_0, z_0) \in G(w, z)$ such that $x_0 \lessdot_P w_0$ and $y_0 \lessdot_P z_0$. We then have that $(Gx, Gy) \lessdot_{\mathcal{E}(P/G)} (Gw, Gz)$ by definition. Since $\leq_{\mathcal{E}(P/G)}$ is the transitive closure of $\lessdot_{\mathcal{E}(P/G)}$, q is order-preserving. □

3.2. **The Opposite Functor and Self Dual Posets.** Next, we introduce the notion of a dual poset, given by applying the opposite functor, op, to a graded poset. We will show that op commutes with $\mathcal{E}$. This will imply that $\mathcal{E}(P)$ is self-dual if P is, which, in turn, will imply that $\mathcal{E}(B_n/G)$ is self-dual for any group action of G on B_n.

Definition 3.7. Let $\mathcal{P}$ be the category of graded posets and let $\mathrm{op}\colon \mathcal{P} \to \mathcal{P}$ be the opposite functor, defined on posets as follows. For P a poset, the elements of P^{op} are the same as those of P with order relation $\leq_{P^{\mathrm{op}}}$ defined by $x \leq_{P^{\mathrm{op}}} y \Leftrightarrow x \geq_P y$. Induced maps on morphisms are given as follows: for P, Q graded posets with $f\colon P \to Q$, then $f^{\mathrm{op}}\colon P^{\mathrm{op}} \to Q^{\mathrm{op}}$ is defined by $f^{\mathrm{op}}(x) = f(x)$. The poset P^{op} is called the *dual* poset of P. A poset P is *self-dual* if there is an isomorphism of posets $P \cong P^{\mathrm{op}}$.

Remark 3.8. Note that it is easy to check $\mathrm{op}\colon \mathcal{P} \to \mathcal{P}$ is indeed a covariant functor. In more abstract terms, if we view P as a category, P^{op} is the opposite category. Additionally, op as defined in this way is actually a endofunctor on the category of all finite posets, which restricts to a functor on the subcategory of graded posets.

Lemma 3.9. *The functor* $\mathrm{op}\colon \mathcal{P} \to \mathcal{P}$ *commutes with the functor* $\mathcal{E}\colon \mathcal{P} \to \mathcal{P}$. *That is,* $\mathcal{E}(P^{\mathrm{op}}) \cong \mathcal{E}(\mathcal{P})^{\mathrm{op}}$.

Proof. Observe that $\mathcal{E}(P^{\mathrm{op}})$ is canonically isomorphic to $\mathcal{E}(P)^{\mathrm{op}}$, as given by the morphism $F\colon \mathcal{E}(P^{\mathrm{op}}) \to \mathcal{E}(P)^{\mathrm{op}}, (x, y) \mapsto (x, y)$. The inverse to F is given by $G\colon \mathcal{E}(P)^{\mathrm{op}} \to \mathcal{E}(P^{\mathrm{op}}), (x, y) \mapsto (x, y)$. These maps are well defined because $\mathcal{E}(P)^{\mathrm{op}}$ and $\mathcal{E}(P^{\mathrm{op}})$ are the same as sets, and it follows from the definitions that these are both morphisms of graded posets, and so F defines an isomorphism. $\square$

Proposition 3.10. *If P is a self-dual poset, then $\mathcal{E}(P)$ is also self-dual.*

Proof. Since P is self-dual, there is an isomorphism $f\colon P \to P^{\mathrm{op}}$. By functoriality of $\mathcal{E}$, as shown in Lemma 3.3, we obtain that $\mathcal{E}(f)\colon \mathcal{E}(P) \to \mathcal{E}(P^{\mathrm{op}})$ is an isomorphism. By Lemma 3.9, there is an isomorphism $\mathcal{E}(P^{\mathrm{op}}) \cong \mathcal{E}(P)^{\mathrm{op}}$. Then, letting $F\colon \mathcal{E}(P^{\mathrm{op}}) \to \mathcal{E}(P)^{\mathrm{op}}, (x, y) \mapsto (x, y)$ be the same isomorphism defined in the proof of Lemma 3.9, the composition $F \circ \mathcal{E}(f)\colon \mathcal{E}(P) \to \mathcal{E}(P)^{\mathrm{op}}$ defines an isomorphism, so $\mathcal{E}(P)$ is self-dual. $\square$

Example 3.11. While $\mathcal{E}(P)$ is commonly Peck when P is Peck, $\mathcal{E}(P)$ need not be Peck in general. Furthermore, adding the condition that P be self-dual does not change this fact. In Figure 3.2 we give an example of a poset P such that P is unitary Peck and self-dual, but $\mathcal{E}(P)$ is not rank-unimodal, hence not Peck.

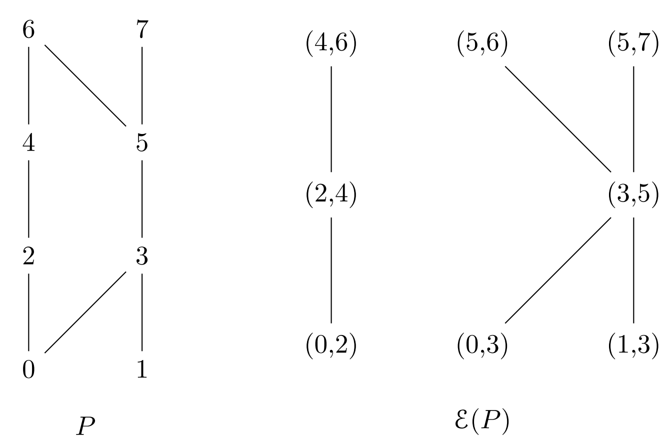


FIGURE 2. P is self-dual and unitary Peck, but $\mathcal{E}(P)$ is not Peck

Remark 3.12. Whenever there is an action $\psi\colon G\times[n]\to[n]$, we obtain an induced action $\phi : G\times B_n \to B_n$ defined by

$$\phi(g,\{x_1,\dots,x_k\}) = \{\psi(g,x_1),\dots,\psi(g,x_k)\}.$$

It is easy to see that any action $\phi\colon G\times B_n\to B_n$ arises in this way. That is, for any action ϕ of G on B_n there exists an action ψ of G on $[n]$ such that $\phi(g,\{x_1,\dots,x_k\}) = \{\psi(g,x_1),\dots,\psi(g,x_k)\}$. This fact, that all actions on B_n are induced by actions on $[n]$ follows from the more general fact about atomic lattices. Recall, L is an *atomic lattice* if there exists a minimum element of L and a subset of L called "atoms" such that any other element of L can be expressed as a join of atoms. Then, for L an atomic lattice, any poset automorphism $f\colon L\to L$ restricts to an automorphism of the atoms of L, and f is uniquely determined by this restriction to atoms. In the case of B_n, the atoms are precisely the singletons, and hence can be identified with $[n]$. Whenever an action ψ of G on $[n]$ is given, we refer to the action ϕ defined above as the *induced action* on B_n.

Corollary 3.13. *For any action $\phi\colon G\times B_n\to B_n$, both $\mathcal{E}(B_n/G)$ and $\mathcal{E}(B_n)/G$ are self-dual. In particular, they are both rank-symmetric.*

Proof. By Remark 3.12, any action $\phi\colon G\times B_n\to B_n$ is induced by an action $\psi\colon G\times[n]\to[n]$. Using this, observe that for any ϕ, the poset B_n/G is self-dual, as there is an isomorphism $f\colon B_n/G\to(B_n/G)^{\mathrm{op}}, G\cdot x\mapsto G\cdot([n]\setminus x)$. This map is well defined on G orbits because every action on B_n is induced by an action on $[n]$. Then, by Proposition 3.10, it follows that $\mathcal{E}(B_n/G)$ is self-dual.

It only remains to prove that $\mathcal{E}(B_n)/G$ is self-dual. However, from Proposition 3.10, $\mathcal{E}(B_n)$ is self-dual, with the isomorphism given by $\mathcal{E}(f)\colon \mathcal{E}(B_n)\to\mathcal{E}(B_n^{\mathrm{op}})\cong\mathcal{E}(B_n)^{\mathrm{op}}$, that is, the map sending a $(x,y)\mapsto([n]\setminus y,[n]\setminus x)$. Once again, since the action on B_n is induced by an action on $[n]$, this isomorphism descends to an isomorphism $\mathcal{E}(f)^G\colon\mathcal{E}(B_n)/G\to(\mathcal{E}(B_n)/G)^{\mathrm{op}}$, and so $\mathcal{E}(B_n)/G$ is self-dual. $\square$

3.3. **Equivalent Definitions of CCT Actions.** We next give four equivalent definitions of CCT actions. The equivalence of (1) and (2) in the following Lemma 3.14 uses the notion of dual posets, while the equivalence of (1), (3), and (4) use the fact that $q\colon\mathcal{E}(P)/G\to\mathcal{E}(P/G), G(x,y)\mapsto(Gx,Gy)$ is always a surjection.

Lemma 3.14. *Let G be a group acting on a graded poset P. The following are equivalent:*

1. *The action of G on P is CCT.*
2. *Whenever $x\lessdot y, x\lessdot z$, and $y\in Gz$, there exists some $g\in\mathrm{Stab}(x)$ with $gx=z$.*
3. *The map $q\colon\mathcal{E}(P)/G\to\mathcal{E}(P/G)$ defined by $q(G(x,y))=(Gx,Gy)$ is a bijective morphism (but not necessarily an isomorphism).*
4. *For all i there is an equality $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$.*

Proof. First, we show (1) $\Leftrightarrow$ (2). There is an isomorphism $f\colon B_n\to B_n^{\mathrm{op}}, x\mapsto[n]\setminus x$. This defines a bijection between triples (x,y,z) with $x\lessdot z, y\lessdot z$ with $x\in Gy$, and triples (a,b,c) with $c\lessdot a, c\lessdot b$ with $a\in Gb$, given by $(x,y,z)\mapsto(f(x),f(y),f(z))$. Furthermore, some $g\in G$ satisfies $g\in\mathrm{Stab}(z)$ and $gx=y$, if and only if it also satisfies $g\in\mathrm{Stab}(f(z))$ and $g\cdot f(x)=f(y)$. If an action is CCT, all triples (x,y,z) satisfy these properties, and condition (2) says all triples (a,b,c) satisfy these properties. So, the above shows that (1) is equivalent to (2).

Second, we show (1) $\Leftrightarrow$ (3). Observe that q is a bijection exactly when there do not exist distinct orbits $G(x,y)\neq G(x',y')$ with $x'\in Gx$, $y'\in Gy$. Fix $(x,y),(x',y')\in\mathcal{E}(P)$ such that $x'\in Gx$ and $y'\in Gy$. Pick a $g\in G$ such that $g\cdot y'=y$. Then $(g\cdot x',y)\in G(x',y')$, so $G(x,y)=G(x',y')$ if and only if there exists some $g'\in G$ such that $g'\cdot x=g\cdot x'$ and $g'\cdot y=y$. Hence q is a bijection if and only if the G action is CCT.

Finally, we check (3) $\Leftrightarrow$ (4). Again using Proposition 3.6, the morphism q is always surjective. Since a morphism is always rank preserving, it must map $(\mathcal{E}(P)/G)_i$ surjectively onto $(\mathcal{E}(P/G))_i$. However, since the posets are finite, this surjection is a bijection if and only if the sets have the same cardinality. $\square$

Remark 3.15. While q is a bijection if and only if the action of G on P is CCT, it is *not* true that if the action of G on P is CCT, then q is an isomorphism. For example, take $G=D_{20}\subset S_{10}$ acting by reflections and rotations on $\{1,2,\dots,10\}$ and consider the induced action on B_{10}. From Proposition 1.6, this action is CCT. However, consider $x = \{2,4\}, y = \{1,2,4\}, a = \{2,4,7\}, b = \{2,4,6,7\}$. Then we may observe $(x,y),(a,b) \in \mathcal{E}(B_{10})$ and $Gx < Ga, Gy < Gb$, so $(Gx,Gy) <_{\mathcal{E}} (Ga,Gb)$. However, it is not true that $G(x,y) <_{\mathcal{E}} G(a,b)$.

3.4. **Proof of Theorem 1.5.** In this section we prove Theorem 1.5, which we recall here:

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

The proof is largely based on the following Lemma.

Lemma 3.16. *Let P, Q two graded posets with a morphism $f\colon P \to Q$ that is a bijection (but not necessarily an isomorphism). If P is Peck, then Q is Peck.*

Proof. Let $\mathrm{rk}(P) = \mathrm{rk}(Q) = n$. Since P is Peck there exists an order-raising operator U such that $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism. Since f is a poset morphism, it follows that the map $f \circ U \circ f^{-1}$ is an order-raising operator on Q. We then have that $f \circ U^{n-2i} \circ f^{-1} = \left(f \circ U \circ f^{-1}\right)^{n-2i} : V(Q_i) \to V(Q_{n-i})$ is an isomorphism since $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism and f is a bijection. □

By Lemma 3.16 and Proposition 3.6, in order to prove Theorem 1.5 it suffices to prove that $\mathcal{E}(B_n)/G$ is Peck. One way to do this is to prove that $\mathcal{E}(B_n)$ is unitary Peck and then apply Theorem 2.6. In fact, this approach generalizes to an arbitrary poset P.

Theorem 3.17. *If the action of G on P is CCT and $\mathcal{E}(P)$ is unitary Peck, then $\mathcal{E}(P/G)$ is Peck.*

Proof. Since the G-action is CCT, there is a bijection $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ by Lemma 3.14. Since $\mathcal{E}(P)$ is unitary Peck we have that $\mathcal{E}(P)/G$ is Peck by Theorem 2.6, hence $\mathcal{E}(P/G)$ is Peck by Lemma 3.16. □

We prove that $\mathcal{E}(B_n)$ is unitary Peck in [3, Section 8], but unfortunately the proof is technical and computational. Note that by Theorem 3.17, this immediately implies Theorem 1.5. Fortunately there is a cleaner – albeit less direct – route to proving Theorem 1.5. In order to avoid showing that $\mathcal{E}(B_n)$ is unitary Peck, we define a graded poset $\mathcal{H}(B_n)$ for all n such that $\mathcal{H}(B_n)$ is easily seen to be unitary Peck in Corollary 3.26. Furthermore, we define $\mathcal{H}(B_n)$ such that a group action of G on B_n induces a group action on $\mathcal{H}(B_n)$ (Lemma 3.23) and there is always a bijective morphism $f\colon \mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$ (Lemma 3.24). By the above discussion, Theorem 1.5 readily follows.

Definition 3.18. For P a graded poset, define the graded poset $\mathcal{H}(P)$ as follows. Let the elements $(x,y) \in \mathcal{H}(P)$ be pairs $(x,y) \in P \times P$ such that $x \lessdot y$. Define $(x,y) \lessdot_{\mathcal{H}} (x',y')$ if $x \lessdot x', y \lessdot y'$ and $x' \neq y$. Then, define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$, and define $\mathrm{rk}_{\mathcal{H}}(x,y) = \mathrm{rk}_P(x)$.

Example 3.19. We give an example of the poset $\mathcal{H}(B_3)$ in Figure 3. Observe that $\mathcal{H}(B_3)$ can be written as a disjoint union of three copies of B_2. This is a single case of the more general phenomenon proven in Proposition 3.25.

Remark 3.20. Note that by definition $(x,y) \lessdot_{\mathcal{H}} (x',y')$ precisely when $(x,y) \lessdot_{\mathcal{E}} (x',y')$ and $x' \neq y$, hence $(x,y) \lessdot_{\mathcal{H}} (x',y') \Rightarrow (x,y) \lessdot_{\mathcal{E}} (x',y')$. In other words, $\mathcal{H}(P)$ has the same elements as $\mathcal{E}(P)$ but with a weaker partial order.

Lemma 3.21. *For P a graded poset, the object $\mathcal{H}(P)$, as defined in Definition 3.18, is a graded poset.*

Proof. This follows immediately from Remark 3.20 and the fact that $\mathcal{E}(P)$ is graded. □

Remark 3.22. While $\mathcal{E}\colon \mathcal{P} \to \mathcal{P}$ is a functor, $\mathcal{H}$ is not a functor. In particular, it is not possible to define $\mathcal{H}(f)$ for f a morphism. This is illustrated in Figure 4. For example, suppose we took $f\colon P \to Q$ defined by $f(1) = 1, f(2) = f(3) = 2, f(4) = 3$. It is easy to see that there is no possible morphism $\mathcal{H}(f)\colon \mathcal{H}(P) \to \mathcal{H}(Q)$ because there are no morphisms $\mathcal{H}(P) \to \mathcal{H}(Q)$.

Given an action of a group G on P, we define an action of G on $\mathcal{H}(P)$ as we did for $\mathcal{E}(P)$ by again defining $g \cdot (x,y) = (gx, gy)$ for all $(x,y) \in P$. We will then have a well-defined quotient poset $\mathcal{H}(P)/G$ with the same elements as $\mathcal{E}(P)/G$.

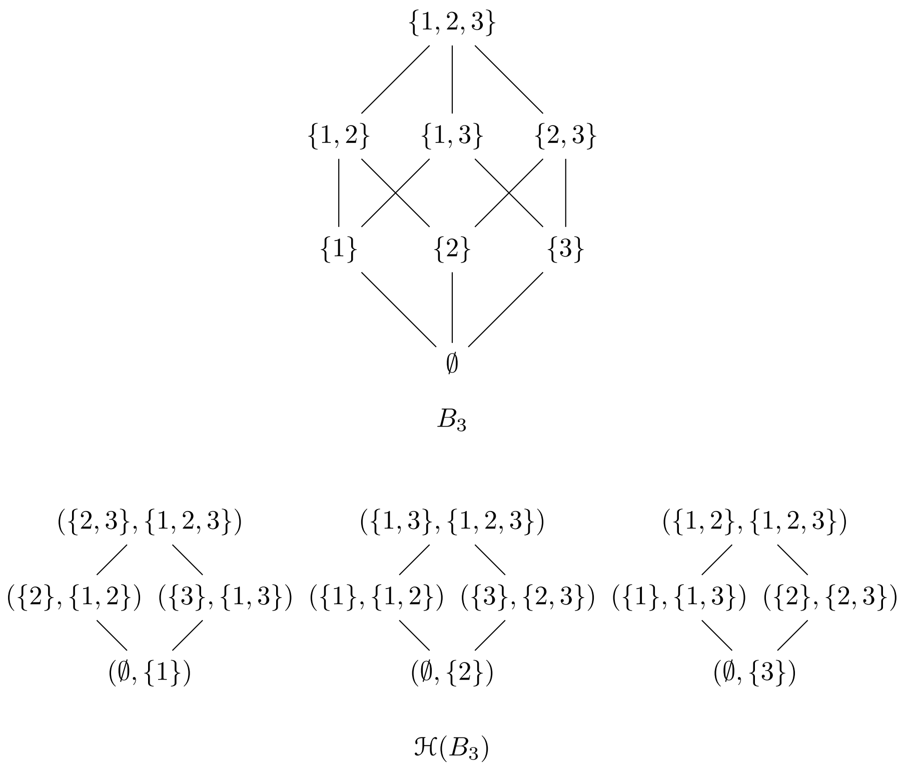


FIGURE 3. B_3 and $\mathcal{H}(B_3)$

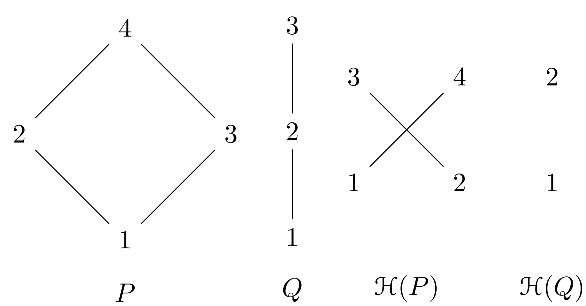


FIGURE 4.

Lemma 3.23. *The automorphism defined by $g \cdot (x, y) = (gx, gy)$ for all $g \in G$, $(x, y) \in \mathcal{H}(P)$ yields a well-defined group action of G on $\mathcal{H}(P)$.*

Proof. Let $g \in G$. Since $\leq_{\mathcal{H}}$ is the transitive closure of $\lessdot_{\mathcal{H}}$ it suffices to show that for all $(x, y), (x', y') \in \mathcal{H}(P)$ we have $(x, y) \lessdot_{\mathcal{H}} (x', y') \Leftrightarrow g(x, y) \lessdot_{\mathcal{H}} g(x', y')$. Since g is an automorphism of P, we have $x \leq_P x' \Leftrightarrow gx \leq_P gx'$, $y \leq_P y' \Leftrightarrow gy \leq_P gy'$, and $y \neq x' \Leftrightarrow gy \neq gx'$, so the result follows from the definition of $\leq_{\mathcal{H}}$. $\square$

Lemma 3.24. *The map $f : \mathcal{H}(P)/G \to \mathcal{E}(P)/G$ defined by $G(x, y) \mapsto G(x, y)$ is a bijective morphism for any group action of G on P.*

Proof. The elements of $\mathcal{H}(P)/G, \mathcal{E}(P)/G$ are the same by definition, so it suffices to show that f is a morphism. Since f is clearly rank-preserving, it suffices to show f is order-preserving. This is immediate from Remark 3.20. $\square$

The remaining step in the proof of Theorem 1.5 is to show that $\mathcal{H}(B_n)$ is unitary Peck, which we do by generalizing Example 3.19 and showing that $\mathcal{H}(B_n)$ is isomorphic to the disjoint union of boolean algebras.

Proposition 3.25. *$\mathcal{H}(B_n)$ is isomorphic to n disjoint copies of B_{n-1}.*

Proof. Let the n disjoint copies of B_{n-1} be labeled $B_{n-1}^{(i)}$, $1 \le i \le n$, with the elements of $B_{n-1}^{(i)}$ labeled $x^{(i)}$, $x \subseteq \{1, \dots, n-1\}$. We will show that the map

$$\begin{aligned} f \colon \mathcal{H}(B_n) &\longrightarrow \bigcup_{i=1}^{n} B_{n-1}^{(i)} \\ (x, x \cup i) &\longmapsto x^{(i)} \end{aligned}$$

is an isomorphism. Suppose we have $(x, y), (x', y') \in \mathcal{H}(B_n)$ with $(x, y) \lessdot_{\mathcal{H}} (x', y')$. Let $j \in [n]$ such that $y' = y \cup \{j\}$, and let $i \in [n]$ such that $x' = x \cup \{i\}$. If $i \neq j$, then $x' = y$, contradicting the assumption that $(x, y) \lessdot_{\mathcal{H}} (x', y')$. Thus $x' = x \cup \{i\}$ and $y' = y \cup \{i\}$ for some $i \in [n]$.

Conversely we can easily check that if $i \notin y$, then $(x, y) \lessdot_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$. It follows that for all subsets $w \subset [n]$ such that $|w| = 1$, there is an isomorphism from the subposet of elements $\{(x, y) \colon y \setminus x = w\}$ to B_{n-1} defined by $(x, y) \mapsto (x \setminus w, y \setminus w)$. Furthermore if $y \setminus x \neq y' \setminus x'$, then (x, y) and (x', y') are incomparable, so these subposets indexed by w are pairwise disjoint, and $\mathcal{H}(B_n)$ is isomorphic to n copies of B_n. □

Corollary 3.26. *$\mathcal{H}(B_n)$ is unitary Peck for all $n \ge 0$.*

Proof. This follows immediately from Proposition 3.25 and the fact that B_{n-1} is unitary Peck. Indeed, B_n is shown to be unitary Peck, in [6, Theorem 2a] by noting that $B_k = (B_1)^k$ and that B_1 is clearly unitary Peck. □

Corollary 3.27. *$\mathcal{H}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. This follows from Corollary 3.26 and Theorem 2.6. □

The next corollary will not be particularly relevant in proving Theorem 1.5, but let us note it as an aside.

Corollary 3.28. *Both $\mathcal{E}(B_n)$ and $\mathcal{H}(B_n)$ have symmetric chain decompositions (SCD).*

Proof. $\mathcal{H}(B_n)$ has an SCD by Proposition 3.25 and the fact that B_{n-1} has an SCD, as shown in [2]. By Lemma 3.24 there is a bijective morphism $f \colon \mathcal{H}(B_n) \to \mathcal{E}(B_n)$, and since a bijective morphism takes an SCD to an SCD it follows that $\mathcal{E}(B_n)$ has an SCD. □

Corollary 3.29. *$\mathcal{E}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. By Corollary 3.27, $\mathcal{H}(B_n)/G$ is Peck. By Lemma 3.24, the map $f \colon \mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G, G(x, y) \mapsto G(x, y)$ is a bijective morphism. Then, by Lemma 3.16, it follows that $\mathcal{E}(B_n)/G$ is Peck. □

We now deduce Theorem 1.5.

Proof of Theorem 1.5. By Corollary 3.29, $\mathcal{E}(B_n)/G$ is Peck for any group action of G on B_n. Since the G-action is CCT, there is a bijective morphism from $\mathcal{E}(B_n)/G$ to $\mathcal{E}(B_n/G)$ by Lemma 3.14. Hence $\mathcal{E}(B_n/G)$ is Peck by Lemma 3.16. □

Note that we have also developed several generalizations of $\mathcal{E}$, for which many similar results hold. For more information, see [[3], Subsection 3.3]

4. Common Cover Transitive Actions

In this section, we develop the theory of CCT actions ϕ where G is a group, P is a poset, and $\phi : G \times P \to P$ is an action. Recall Definition 1.4, that ϕ is CCT if whenever $x, y, z \in P, x \lessdot z, y \lessdot z, x \in Gy$ then there exists $g \in \operatorname{Stab}(z)$ with $gx = y$. We show that the CCT property is closed under semidirect products, in the appropriate sense. From Proposition 1.6, which will be proven in Subsubsection 4.2.4, the action of S_n on B_n and the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

Example 4.1. Two rather trivial examples of CCT actions are $\phi \colon S_n \times B_n \to B_n$ and $\psi \colon G \times B_n \to B_n$ where G is arbitrary, ϕ is the action induced by S_n permuting the elements of $[n]$, and ψ is the trivial action. In the former case, $\mathcal{E}(B_n/S_n)$ is simply a chain with n points, and so is $\mathcal{E}(B_n)/S_n$, since all (x, y) are identified under the S_n action. In the latter case, since G acts trivially by ϕ, both $\mathcal{E}(B_n/G) \cong \mathcal{E}(B_n)$ and $\mathcal{E}(B_n)/G \cong \mathcal{E}(B_n)$. So again, ψ is CCT.

4.1. Preservation Under Semidirect Products.

Lemma 4.2. *Let $G \subseteq \operatorname{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. We then have a well defined group action $K \times P/H \to P/H, (k, Hx) \mapsto H(k \cdot x)$.*

Proof. Note that if $x, x' \in Hx$, we have $x' = h \cdot x$ for some $h \in H$. Since H is normal in G, we have that for all $k \in G$ there exists $h' \in H$ so that $khk^{-1} = h'$. So

$$k \cdot x' = kh \cdot x = k(k^{-1}h'k) \cdot x = h' \cdot (k \cdot x)$$

Hence $k \cdot x$ and $k \cdot x'$ are in the same H-orbit, so we have a well-defined group action of K on P/H defined by $k \cdot Hx = H(k \cdot x)$. □

Recall Proposition 1.7, as stated in the introduction, which says that the CCT property is preserved under semidirect products. We will use Proposition 1.7 to construct more examples of CCT group actions, in particular using it to give a simple proofs that CCT actions are preserved under direct products and wreath products.

Proposition 1.7. *Let $G \subseteq \operatorname{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

Proof. Since $G = H \rtimes K$, every element $g \in G$ can be written uniquely as a product hk for some $h \in H$, $k \in K$. Let $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$ and such that there exists some $h_0k_0 \in G$ such that $h_0k_0 \cdot x = y$. It suffices to show that there exists some $g \in \operatorname{Stab}(z)$ such that $g \cdot x = y$.

The orbits $Hx, Hy, Hz \in P/H$ satisfy $Hx \lessdot Hz$, $Hy \lessdot Hz$ such that $k_0 \cdot Hx = Hy$, so since the action of K on P/H is CCT there exists some $k_1 \in K$ such that $k_1 \in \operatorname{Stab}(Hz)$ and $k_1 \cdot Hx = Hy$. It follows that there exists some $h_1 \in H$ such that $h_1k_1h_0 \in \operatorname{Stab}(z)$ and $h_1k_1h_0 \cdot x \in Hy$.

Write $x' = h_1k_1h_0 \cdot x$. Since the group action of G must be order-preserving by definition, we have that $x' \lessdot z$. We already had that $y \lessdot z$ and $x' \in Hy$, hence there exists some $h_2 \in \operatorname{Stab}(z)$ such that $h_2 \cdot x' = y$ by the fact that the action of H on P is CCT. Then we have that $h_2h_1k_1h_0 \cdot x = h_2 \cdot x' = y$ and $h_2h_1k_1h_0 \cdot z = h_2 \cdot z = z$, as desired. □

Proposition 4.3. *For $\phi \colon G \times P \to P, \psi \colon H \times Q \to Q$ two CCT actions, then the direct product*

$$\phi \times \psi \colon (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$$

is also CCT.

Proof. First note that if either G or H acts trivially, then it can be easily checked that the action of $G \times H$ is CCT. Next, observe that $G \times H$ can be viewed as the semidirect product of $(G \times \{e\}) \rtimes (\{e\} \times H)$. Since the action of G on P is CCT, the action of $G \times \{e\}$ on $P \times Q$ is CCT. Also, since the action of H on Q is CCT, it follows that the action of $\{e\} \times H$ on $(P/G) \times Q$ is CCT. Therefore, the action of $(G \times \{e\}) \rtimes (\{e\} \times H)$ satisfies the conditions of Proposition 1.7 and so the action of $G \times H$ is CCT. □

Next, we use Proposition 1.7 to prove in Proposition 4.8 that the CCT property is preserved under wreath products with the symmetric group. First, we need some definitions of wreath product.

Definition 4.4. For G, H groups, with $H \subset S_l$, the *wreath product*, denoted by $G \wr H$, is the group whose elements are pairs $(g, h) \in G^l \times H$ with multiplication defined by

$$((g'_1, \ldots, g'_l), h') \cdot ((g_1, \ldots, g_l), h) = ((g'_{h'(1)} g_1, \ldots, g'_{h'(l)} g_l), hh')$$

where H acts on $[l]$ via the embedding of H into S_l.

In other words, $G \wr H$ can be viewed as a certain semidirect product of $G^l \rtimes H$.

Definition 4.5. For any group G with a given action $\psi\colon G \times P \to P$, we obtain an induced action of $G \wr H$, $\phi\colon G \wr H \times P^l \to P^l$ defined by

$$((g_1, \ldots, g_l), h)(a_1, \ldots, a_l) = (g_{h^{-1}(1)} \cdot a_{h^{-1}(1)}, \ldots, g_{h^{-1}(l)} \cdot a_{h^{-1}(l)}).$$

Remark 4.6. Heuristically, one may think of the above action as obtained by first having G act separately on the l distinct copies of P, and then letting H act by permuting the copies.

Lemma 4.7. *For P a graded poset, the action*

$$\phi\colon S_l \times P^l \to P^l, \sigma \cdot (x_1, \ldots, x_l) = (x_{\sigma(1)}, \ldots, x_{\sigma(l)})$$

is CCT.

Proof. For $a \in P^l$ notate $a = (a_1, \ldots, a_l)$. Suppose $x, y, z \in P^l$ with $x \lessdot z, y \lessdot z$, and $x \in S_l y$. This means there is a unique i such that $x_i \lessdot z_i, x_k = z_k$ for $k \neq i$. Additionally, there is a unique j for which $y_j \lessdot z_j, y_k = z_k$ for $k \neq j$. Since $x \in S_l y$, we obtain the equality of multisets $\{x_1, \ldots, x_l\} = \{y_1, \ldots, y_l\}$. But for $k \neq i, j$ we have $x_k = z_k = y_k$, so we also obtain equality of sets $\{x_i, x_j\} = \{y_i, y_j\}$. Since $\mathrm{rk}(y_j) \lessdot \mathrm{rk}(x_j)$, we obtain $y_j = x_i, y_i = x_j$. Then, taking the transposition $\sigma = (ij) \in S_l$, it follows that $\sigma \in \mathrm{Stab}(z)$ and $\sigma \cdot x = y$. □

Proposition 4.8. *If $\psi\colon G \times P \to P$ is CCT, let $\phi\colon G \wr S_l \times P^l \to P^l$ where ϕ is the induced action defined in Definition 4.5. Then ϕ is also CCT.*

Proof. Note that the wreath product $G \wr S_l$ can be viewed as a semidirect product $G^l \rtimes S_l$. Since the action of G on P is CCT we obtain that the action of G^l on P^l is CCT by Proposition 4.3. Furthermore the action $S_l \times (P/G)^l \to (P/G)^l$ defined by $(\sigma, (x_1, \ldots, x_l)) \mapsto (x_{\sigma(1)} \ldots, x_{\sigma(l)})$ for $\sigma \in S_l$ and $x_i \in P/G$ is CCT by Lemma 4.7. Since $P^l/G^l \cong (P/G)^l$, it follows that the action ϕ satisfies the conditions of Proposition 1.7, so ϕ is CCT. □

4.2. **Examples of CCT Actions.** In this subsection, we describe several classes of CCT actions. First, we show that the automorphism group of any rooted tree is CCT. Second, we show linear automorphisms of simplices and octahedra are CCT. Third, we show that the left multiplication action is CCT if and only if the group is $\mathbb{Z}_2^k$, and that any action of $\mathbb{Z}_2^k$ on $[n]$ induces a CCT action on B_n. In the end of this subsection, we prove Proposition 1.6, which shows certain symmetric group and dihdral group actions are CCT.

4.2.1. *An application to rooted trees.* In this subsubsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Proposition 4.8 and Proposition 4.3, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT. Our examination of of rooted trees automorphisms here, together with the examination of polytopes in Subsubsection 4.2.2 was motivated by [1, Section 5].

Definition 4.9. A graded poset P is a *rooted tree* if there is a unique element $x \in P$ of maximal rank, called the *root*, and for all $x \in P$, other than the root, there exists a unique $y \in P$ with $y \gtrdot x$.

Example 4.10. In Figure 5 and Figure 6, we give two examples of rooted trees.

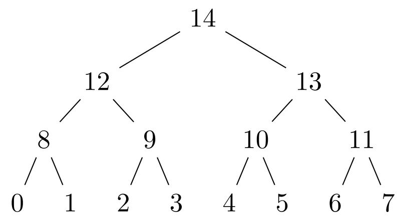


FIGURE 5. An example of a rooted tree with 8 leaves.

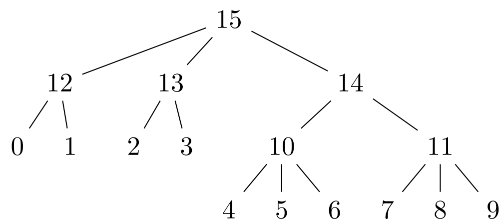


FIGURE 6. An example of a rooted tree with 10 leaves.

Definition 4.11. For P a rooted tree, an element $x \in P$ is a *leaf* if there is no $z \in P$ with $x > z$. Denote the set of all leaves of P by $L(P)$.

Lemma 4.12. *Let P be a rooted tree. Then, the action of $\mathrm{Aut}(P)$ on P induces an action of $\mathrm{Aut}(P)$ on $L(P)$. Furthermore, there is also an induced action of $\mathrm{Aut}(P)$ on B_n where $n = |L(P)|$.*

Proof. First, we must show that $\mathrm{Aut}(P)$ induces an action on $L(P)$. That is, we must show that for any $g \in \mathrm{Aut}(P), x \in L(P)$, then $gx \in L(P)$. If $x \in L(P)$ but $gx \notin L(P)$, then there exists $y < gx$. However, then $g^{-1}y < x$, contradicting the assumption that x is a leaf. We then obtain the induced action $\mathrm{Aut}(P) \times L(P) \to L(P), (g, x) \mapsto gx$.

Finally, to obtain the induced action of $\mathrm{Aut}(P)$ on B_n identify $L(P) \cong [n]$ as sets, where $|L(P)| = n$. From this we get of action of $\mathrm{Aut}(P)$ on B_n. □

Notation 4.13. For the rest of this section only, fix a rooted tree P and denote by G the group of automorphisms $\mathrm{Aut}(P)$. Let G act on B_n, where $n = |L(P)|$, by the induced action $\phi\colon G \times L(P) \to L(P)$ described in the proof of Lemma 4.12.

Notation 4.14. For $x \in P$, denote $D(x) = \{y \in P : y \leq x\}$, so $D(x)$ is the maximal subposet of P with maximum element x.

Proposition 4.15. *Let P be a rooted tree with root vertex labeled 0. Let $\{A_1, \ldots, A_m\}$ denote the set of isomorphism classes of $\{D(x) : x \lessdot 0\}$, and let i_k denote the number of subtrees in $\{D(x) : x \lessdot 0\}$ in the isomorphism class A_k. For $A_k \in \{A_1, \ldots, A_m\}$, denote $G_k = \mathrm{Aut}(A_k)$. Then,*

$$\mathrm{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}) \tag{4.1}$$

In particular, $\mathrm{Aut}(P)$ can be expressed as a sequence of direct products and wreath products of symmetric groups.

Proof. We proceed by induction on the rank of P. It is clear that if P is rank 0, then $\mathrm{Aut}(P)$ is trivial. If the rank of P is greater than 0, label the vertices of P by $\{0, 1, \ldots, s\}$ such that the root is labeled 0 and the vertices just below the root are labeled $1, \ldots, k$. Let $A_1, \ldots, A_m$ denote the distinct isomorphism classes of trees in the set $\{D(1), \ldots, D(k)\}$. For $A_k \in \{A_1, \ldots, A_m\}$ denote $G_k = \mathrm{Aut}(A_k)$. Let $T_j = \{t : t \lessdot 0, D(t) \cong A_j\}$. Then, letting Q_j be the subtree of P whose elements lie in the set $\{0\} \cup (\cup_{t \in T_j} D(t))$, we have that $\mathrm{Aut}(Q_j) \cong G_j \wr S_{i_j}$, because after choosing a permutation of the elements of T_j, we are free to choose any element of G_j to permute each $D(t), t \in T_j$. If $t_1 \lessdot 0, t_2 \lessdot 0, g \cdot t_1 = t_2$, then it must be that $g \cdot D(t_1) = D(t_2)$. Therefore, $\mathrm{Aut}(P)$ must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$\mathrm{Aut}(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \tag{4.2}$$

Since each G_j is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (4.2) that so is $\mathrm{Aut}(P)$. □

Example 4.16. Let P_1 be the rooted tree in Figure 5 and P_2 be the rooted in Figure 6, the proposition says that $\mathrm{Aut}(P_1) = (S_2 \wr S_2) \wr S_2$; and $\mathrm{Aut}(P_2) = (S_2 \wr S_2) \times (S_3 \wr S_2)$.

Corollary 4.17. *For P the automorphism group of a rooted tree, $\mathrm{Aut}(P)$ is CCT.*

Proof. By Proposition 4.8, wreath products with symmetric groups preserve the CCT property, and by Proposition 4.3 the direct product of two CCT groups is again CCT. Therefore, by Proposition 4.15, all groups of the form $\mathrm{Aut}(P)$ are built up from these operations, and so $\mathrm{Aut}(P)$ is also CCT. □

4.2.2. *Automorphisms of Polytopes.* As another class of CCT actions, we describe several linear automorphism groups of polytopes whose actions are CCT. In particular, we prove that the linear automorphism groups of simplices and octahedra are CCT. Once we prove Proposition 1.6 in Subsubsection 4.2.4 we will also see that the action of the dihedral group on a regular n-gon is CCT for $n = p, 2p$. Since the dihedral group is the group of all linear automorphisms of the regular n-gon, this action gives another example of the linear automorphism group of a polytope being CCT.

Definition 4.18. Let M be a polytope, with a particular embedding in $\mathbb{R}^n$. The *group of linear automorphisms of M* is the subgroup of GL_n whose elements are $\{g \in \mathrm{GL}_n : g \cdot M = M\}$.

First, we look at linear automorphisms of simplices. Let G be the group linear automorphisms of the $(n-1)$-simplex, whose vertices lie at the standard basis vectors in $\mathbb{R}^n$. The action of G on the $(n-1)$-simplex induces an action on $[n]$, given by identifying $[n]$ with the n vertices of the $(n-1)$-simplex. Hence, it induces an action on B_n.

Example 4.19. The action of the group of linear automorphisms of the $(n-1)$-simplex, acting on B_n, as defined above, is CCT. To see this, observe that the group of linear automorphisms in this case induces the usual action of S_n on B_n, because any permutation matrix defines a linear map on $\mathbb{R}^n$. However, we know the action of S_n on B_n is CCT from Example 4.1.

Next, we look at linear automorphisms of octahedrons. Let G be the group of linear automorphisms of the n-octahedron, embedded inside $\mathbb{R}^n$, whose vertices are located at $\pm e_i$, where $e_1, \ldots e_n$ are the standard basis vectors of $\mathbb{R}^n$. Then, the action of G on the octahedron induces an action of G on the $2n$ vertices of the octahedron, and hence on B_{2n}.

Proposition 4.20. *The induced action of the group of linear automorphisms of the n-octahedron on B_{2n} is CCT.*

Proof. It is simple to see that the group of linear automorphisms of the n-octahedron is the hyperoctahedral group, since it is generated by the permutation matrices, together with the matrix A with $A_{1,1} = -1, A_{i,i} = 1, A_{j,k} = 0$ where $i \neq 1, j \neq k$.[1] It is well known that they hyperoctahedral group can be written as $S_2 \wr S_n$. Then, by Proposition 4.8, it follows that $S_2 \wr S_n$ is CCT. □

Remark 4.21. Let us give a brief recap of which linear automorphisms of polytopes are known to induce actions on B_n which are CCT. First, by the above lemmas, for octahedrons and simplices, the induced action is CCT. By Proposition 1.6 and Proposition 4.24 the linear automorphism group of an n-gon induces a CCT action on B_n if and only if one of $n \in \{1, p, 2p\}$ for p a prime. Additionally, computers have verified that automorphisms of the 3-cube, with vertices at $(\pm 1, \pm 1, \pm 1)$ induces a CCT action. It is still unknown whether the linear automorphisms of n-cubes is CCT for $n > 3$, and also whether the remaining five exceptional regular polytopes (namely the dodecahedron and icosahedron in $\mathbb{R}^3$ as well as the 24-cell, 120-cell, and 600-cell polytopes in $\mathbb{R}^4$) induce CCT actions. These questions are repeated in Question 6.6 and Question 6.7.

[1]The hyperoctahedral group is commonly denoted by B_n, since it is the type B Coexeter group. We do not use this notation here to avoid confusing it with the boolean algebra.

4.2.3. *CCT Actions of* $\mathbb{Z}_2^k$. In this subsubsection, we show that any embedding of $\mathbb{Z}_2^k$ into S_n defines an action on B_n which is CCT. In particular, this implies that the left multiplication action of $\mathbb{Z}_2^k$ is CCT. However, it turns out that this is the only class of groups for which the left multiplication action is CCT.

Proposition 4.22. *Recall that G is an elementary abelian 2-group if $G \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$.*

(1) *For any $n \in \mathbb{N}$, and G an elementary abelian 2-group, every G-action $\phi : G \times B_n \to B_n$ is CCT.*
(2) *For every finite group G which is not an elementary abelian 2-group, there exists at least one G action which is not CCT, namely the action of G on B_n induced by the left-regular action of G on itself, where $n = |G|$.*

Proof. Part (1) Let $x, y, z \in B_n$ such that $x \lessdot z, y \lessdot z$, and $x = gy$ for some $g \in G$. Since $x \neq y$ we have $z = x \cup y$. Furthermore, since every element in $\mathbb{Z}_2^k$ has order 2 we have that $gy = g^2x = x$ and thus $gz = gx \cup gy = y \cup x = z$. Hence $g \in \mathrm{Stab}(z)$ and thus ϕ is CCT.

Part (2) First, let us show $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$. The forward implication is obvious. To see the converse, first note that if $\forall g \in G, g^2 = e$, then G is abelian because $aba^{-1}b^{-1} = abba = a^2 = e$. Then, G is an abelian group, all of whose elements have order two. The structure theorem of finite abelian groups tells us $G \cong \mathbb{Z}_2^k$.

So, Suppose $G \ncong \mathbb{Z}_2^k$. Then, there exists $g \in G, g^2 \neq e$. Clearly $\{e\} \lessdot \{e \cup g\}, \{g\} \lessdot \{e \cup g\}, \{g\} \in G\{e\}$. So, to show the induced action $\phi \colon G \times B_n \to B_n$ is not CCT, it suffices to show there is no $h \in G, h \in \mathrm{Stab}(\{e \cup g\}), h \cdot \{e\} = \{g\}$. If $h \in \mathrm{Stab}(\{e \cup g\})$ then $h \cdot e = e$ or $h \cdot e = g$. In both cases, it is simple to see that $g^2 = e$. Therefore, there does not exist such an h and left multiplication is not CCT. □

4.2.4. *The proof of 1.6.*

Notation 4.23. For the remainder of this section, we abuse notation by writing (Gx, Gy) in place of $(Gx, Gy) \cap \mathcal{E}(B_n)$ as sets, and hence, viewing $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}(B_n) \subset \mathcal{E}(B_n)$. Namely,

$$(Gx, Gy) = \{(\sigma x, \tau y) \in \mathcal{E}(B_n) : \sigma, \tau \in G;\ \sigma x \lessdot \tau y\}.$$

Similarly, we notate

$$G(x, y) = \{(gx, gy) \in \mathcal{E}(B_n) \colon g \in G\}.$$

Proposition 1.6. *The following actions are CCT.*

(1) *The action of S_n on B_n,*
(2) *The action of D_{2p} on B_p,*
(3) *The action of D_{4p} on B_{2p},*

where p is prime.

Proof. We have already seen in 4.1 that (1) holds trivially. We prove part (2). The proof of part (3) is similar.

We wish to prove that given $G(x, y) \in \mathcal{E}(B_p)$ such that $\sigma x \lessdot y$ where $\sigma \in D_{2p}$, there exists some $\tau \in D_{2p}$ such that $\tau x = \sigma x$ and $\tau y = y$.

The action of D_{2p} on B_p is induced by the action of D_{2p} on $[p]$ where $[p]$ is identified with vertices of the regular p-gon. Note that any element in D_{2p} is either some reflection r by one of the lines of symmetry of the polygon, or some rotation σ_0^d where σ_0 is the generator $\sigma_0 = (12 \cdots p)$ and d is some integer. Hence we only need to show the claim when $\sigma = r$ or $\sigma = \sigma_0^d$. It is clear that the claim holds for $\sigma = r$: if $x \lessdot y$ and $r \cdot x \lessdot y$, then $r \cdot y = y$, since r is of order 2. Now suppose $\sigma_0^d \cdot x \lessdot y$ for some $G(x, y) \in \mathcal{E}(B_p)_i$. It is fairly straightforward to see that (x, y) is of form $x = \{s, s+d, ..., s+(i-1) \cdot d\}$ for some starting point $s \in [n]$ and $y = \{s, s+d, ..., s+(i-1)d, s+i \cdot d\}$. Now let r_0 be the reflection that sends $x \mapsto (2s + i \cdot d) - x$ for all $x \in [n]$, reducing mod n whenever necessary. Then $r_0 x = \sigma_0^d x$ and r_0 fixes y by construction.Therefore, the action D_{2p} on B_p is CCT. □

Remark 4.24. It is easy to see that if $n \neq p, n \neq 2p, n > 8$ for any prime p, then the action of D_{2n} on B_n is not CCT. To do this, we give an example of a non-CCT pair. Assume $n \neq p, 2p$, then $n = mk$ for some $m \geq k \geq 3$. Let us consider elements x, y, z where $z = \{1, m+1, 2m+1, ..., (k-1)m+1, 2, m+2\}$. Let $x = z \backslash \{m+2\}$ and $y = z \backslash \{2\}$. We immediately have that $x, y \lessdot z$, and $x \in D_{2n}y$ since x is sent to

y by the permutation $(12\cdots n)^m \in D_{2n}$. It is also clear that there is no $g \in D_{2n}$ translating x to y while fixing z, from the asymmetry of the element z. Therefore, the action of D_{2n} on B_n as described is CCT if and only if $n = p$ or $n = 2p$ for some prime p.

Hence, a complete list of n for which D_{2n} is CCT is given by $n = p, n = 2p, n = 1, n = 8$ where p varies over all primes.

Remark 4.25. There are several other results related to $\mathcal{E}(C_n), \mathcal{E}(D_{2n})$, with $C_n = \mathbb{Z}/n\mathbb{Z}$, which are proven in [3, Section 7]. Notably,

(1) For G any group of order n, acting transitively on $[n]$, the induced action of G on B_n defines a quotient poset with $|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}$.
(2) For all n, $\mathcal{E}(B_n/C_n)$ is symmetric and unimodal.
(3) For all n, $\mathcal{E}(B_n/D_{2n})$ is symmetric and unimodal.

5. A Unimodality Result

In this section, we prove a result similar to that of [4, Theorem 1.1]. We construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. This construction gives an alternate proof of [4, Theorem 1.1] in the case that $r = 1$.

Notation 5.1. For this section, fix l, m with $n = l \cdot m$ and fix $G = S_m \wr S_l$. Let S_m act on B_m by the permutation representation, and then let G act on $B_m^l \cong B_{m\cdot l}$ by the action defined in Definition 4.5.

5.1. **Restatement of the Unimodality Result.** We first review the necessary definitions and then state [4, Theorem 1.1]:

A *partition* λ of n is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and that $\sum_{i=1}^k \lambda_i = n$, denoted by $\lambda \vdash n$. Let $P_n(l, m)$ denote the set of partitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, such that $\lambda_1 \leq m, k \leq l$. That is, $P_n(l, m)$ is the partitions which fit inside an $l \times m$ rectangle.

Notation 5.2. [4, Section 1] For λ a partition, let $\nu(\lambda)$ be the number of distinct nonzero part sizes of λ. Let $p_k(l, m, r) = \sum_{\lambda \in P_k(l,m)} \binom{\nu(\lambda)}{r}$.

Theorem 5.3. [4, Theorem 1.1] *The sequence $p_r(l, m, r), p_{r+1}(l, m, r), \dots, p_{l\cdot m}(l, m, r)$ is unimodal and symmetric.*

5.2. **A Proof of Theorem 5.3 for $r = 1$.** Now that we have stated Pak and Panova's Theorem, we give an alternative proof of Theorem 5.3 in the case of $r = 1$. In fact, we do better, by realizing $p_i(l, m, 1)$ as ranks of a Peck poset.

Proposition 5.4. *There is an equality $|(\mathcal{E}(B_n)/G)_i| = |\mathcal{E}(B_n/G)_i| = p_{1+i}(l, m, 1)$. In particular, Theorem 5.3 holds in the case $r = 1$.*

Proof. First, observe that $S_m \wr S_l$ can be described as the automorphism group of a rooted tree of rank 2 with l elements at rank 1 and $m \cdot l$ elements at rank 2, such that each element at rank 1 is above m elements at rank 2. Then, by Corollary 4.17, it follows that the action of G on $B_{m\cdot l}$ is CCT and so $\mathcal{E}(B_n/G)$ is Peck.

Next, note that each equivalence class in B_n/G has a unique representative which is a Young Tableau. The correspondence is given by sending an equivalence class, to the representative which is left and bottom justified, when elements of $B_n = B_{l\cdot m}$ are thought of as subsets of the $l \cdot m$ boxes in an $l \times m$ rectangle. For a complete proof, see, for example, [[3], Lemma 5.11].

Now, let Gx, Gy be two G orbits with $\bar{x}$ the Young Tableau corresponding to x, and $\bar{y}$ the young tableau corresponding to y. Suppose $Gx \lessdot Gy$. Then, $\bar{x}$ must be contained in $\bar{y}$, but with a single box removed. Since $\bar{x}, \bar{y}$ are both Young Tableau, the removed box must be one of the corners of $\bar{y}$. Observe that the number of corners of a partition is precisely the number of distinct part sizes, and so

$|\{Gx|Gx \lessdot Gy\}| = \nu(\bar{y})$. Therefore,

$$\begin{aligned}
|\mathcal{E}(B_n/G)_i| &= \sum_{(Gx,Gy)\in\mathcal{E}(B_n/G)_i} 1 \\
&= \sum_{Gy\in(B_n/G)_{i+1}} \left(\sum_{Gx\lessdot Gy} 1\right) \\
&= \sum_{Gy\in(B_n/G)_{i+1}} \nu(\bar{y}) \\
&= \sum_{\lambda\in P_{i+1}(l,m)} \nu(\lambda).
\end{aligned}$$

Therefore, $|(\mathcal{E}(B_n)/G)_i| = |\mathcal{E}(B_n/G)_i| = p_{1+i}(l,m,1)$. Since $\mathcal{E}(B_n)/G$ is Peck, $p_1(l,m,1), p_2(l,m,1), \ldots, p_{l\cdot m}(l,m,1)$ is unimodal and symmetric, and hence Theorem 5.3 holds in the case $r = 1$. □

6. Final Remarks

In this section, we discuss several related results and list further questions.

Remark 6.1. Observe that Theorem 1.5 trivially implies that $\mathcal{E}(B_n)$ is Peck. In fact, it is true that $\mathcal{E}(B_n)$ is unitary Peck. For a proof, see [3, Section 8].

Definition 6.2. Let $B_n(q)$, the *q-analog of the boolean algebra*, be the graded poset whose elements are linear subspaces $V \subset \mathbb{F}_q^n$ with $V \leq W$ if $V \subset W$.

As mentioned in the introduction, a natural extension of Conjecture 1.3 would be an analogous result for q-analogs. We suspect the method used in [3, Section 8] to prove $\mathcal{E}(B_n)$ is unitary peck may solve Question 6.3.

Question 6.3. Is $\mathcal{E}(B_n(q))$ unitary peck?

In particular, if the answer to the previous Question 6.3 is affirmative, it would immediately hold that $\mathcal{E}(B_n(q))/G$ is Peck, and if G is CCT then $\mathcal{E}(B_n(q)/G)$ is Peck. Hence, we pose the following question:

Question 6.4. For G a group with a CCT action on $B_n(q)$ is $\mathcal{E}(B_n(q)/G)$ Peck?

Even more generally, we wonder if the q-analog of Conjecture 1.3 holds.

Question 6.5. For G a group acting on $B_n(q)$, is $\mathcal{E}(B_n(q)/G)$ Peck? If not, is $\mathcal{E}(B_n(q)/G)$ rank-unimodal?

We found several additional interesting examples of CCT actions. Once such action is the linear automorphism of the n-cube. Using computers, we found that for $n \leq 3$, the linear automorphisms of the n-cube induces a CCT action on B_{2^n}. We wonder if this generalizes.

Question 6.6. Does the group of linear automorphism of an n-cube in $\mathbb{R}^n$, whose vertices lie at $(\pm 1, \ldots, \pm 1)$ induce a CCT action on B^{2^n}?

There is also the question of which exceptional regular polytopes induce CCT actions. We have shown it holds for the octahedron, the simplex (tetrahedron) in Subsubsection 4.2.2. We also checked using the computer that it holds for the cube. We wonder whether it holds for all exceptional regular polytopes.

Question 6.7. Do the remaining five exceptional regular polytopes (namely the dodecahedron and icosahedron in $\mathbb{R}^3$ as well as the 24-cell, 120-cell, and 600-cell polytopes in $\mathbb{R}^4$) induce CCT actions?

We found using computers that the group of invertible linear maps on $\mathbb{F}_2^3$ acting on the the seven nonzero points of $\mathbb{F}_2^3$ induces an action on B_7 which is CCT. We wonder if this generalizes to other groups of invertible linear maps on finite fields.

Question 6.8. Is the action of $\mathrm{GL}_n(\mathbb{F}_q)$ on B_{q^n-1} (induced by the action of $\mathrm{GL}_n(\mathbb{F}_q)$ on $(\mathbb{F}_q^n)^\times$) CCT? What about the action of $\mathrm{PGL}_n(\mathbb{F}_q)$ on $B_n(q)$? If not, what about the action of $\mathrm{PGL}_n(\mathbb{F}_2)$ on $B_n(2)$?

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