List of Todos

ARTIN GROUP PRESENTATIONS ARISING FROM CLUSTER ALGEBRAS

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Abstract. In 2003, Fomin and Zelevinsky proved that finite-type cluster algebras can be classified by Dynkin diagrams. Then in 2013, Barot and Marsh defined the presentation of a reflection group associated to a Dynkin diagram in terms of an edge-weighted, oriented graph, and proved that this group is invariant (up to isomorphism) under diagram mutations. In this paper, we extend Barot and Marsh's results to Artin group presentations, defining new generator relations and showing mutation-invariance for these presentations.

1. Introduction & Motivation

In [FZ02], Fomin and Zelevinsky first introduced the concept of cluster algebras in order to make further strides in the areas of representation theory, Lie theory, and total positivity. Since then, the study of cluster algebras has provided a motivation for applications in various other areas of mathematics, including quiver representations. Of particular interest were *finite-type* cluster algebras, that is, cluster algebras whose variables are generated through mutation on a finite number of *seeds*. In the sequel to their introductory paper ([FZ03]), Fomin and Zelevinsky introduce the concept of *mutation equivalence* between diagrams, proving that a connected graph is mutation equivalent to an oriented Dynkin diagram if and only if all mutation equivalent graphs have edge weights not exceeding 3. In particular, this proves that finite-type cluster algebras can be classified by Dynkin diagrams.

Barot and Marsh extended Fomin and Zelevinsky's results in [BM13], providing a presentation of the reflection group associated to a Dynkin diagram

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with generators that correspond to elements of a companion basis associated to a seed of a finite-type cluster algebra. They also proved that this group presentation is invariant up to isomorphism under the mutation equivalence relation.That is, given a diagram Γ and a diagram mutation equivalent to Γ, denoted $\Gamma' = \mu_k(\Gamma)$, they proved that $W_\Gamma \cong W_{\Gamma'}$, where W_Γ and $W_{\Gamma'}$ are the group representations corresponding to Γ and Γ', respectively.

In our paper, we define A_Γ to be the Artin group presentation arising from a cluster algebra, where Γ is the diagram associated to the cluster algebra. We provide the necessary relations for the generators of the group, and show that these relations hold under mutations of chordless cycles in a diagram. Our main result is to show that this Artin group presentation is invariant up to isomorphism under the mutation equivalence relation. We state the result here, but present the detailed proof in Section 6.

Theorem 1.1 (Theorem 6.1)**.** *Let Γ be a diagram of finite type, and let $\Gamma' = \mu_k(\Gamma)$ be the mutation of Γ at vertex k. Then $A_\Gamma \cong A_{\Gamma'}$.*

Section 2 provides the necessary definitions and fundamental results from [BM13] to motivate our own results. For further definitions and references on the topic, we refer the reader to [FZ02]. Section 3 will review theory from [FZ02], [FZ03] as well as review the classifications (from [BM13]) of mutations of diagrams and their oriented chordless cycles. In Section 4, we define the appropriate relations for our Artin group presentations. Section 5 specifies how certain relations in chordless cycles imply other relations in those chordless cycles. Finally, Section 6 will provide the proof that the Artin group defined for a diagram Γ is invariant up to isomorphism under mutations of Γ.

2. Background

We begin by introducing some preliminary notations and definitions which will aid the reader in understanding the results in the following sections. For further references on cluster algebras, we refer the reader to [FZ02] and [FZ03] and for a more detailed description of Artin group presentations, we direct attention to [FN61]. We also provide references to several lemmas and propositions from [BM13] which were helpful in formulating our own results.

A *cluster algebra* is an integral domain which can be generated by a set of elements called *cluster variables* that satisfy certain exchange relations. Following the style of [FZ02] and [BM13], we will define cluster algebras in terms of *skew-symmetrisable* matrices (that is, a matrix B such that there exists a diagonal matrix D of the same size with $D_{ii} > 0$ such that DB is skew-symmetric). Let $\mathbb{F} = \mathbb{Q}(u_1, u_2, \dots, u_n)$ be the field of rational functions in n indeterminates over $\mathbb{Q}$. We will define an *intial seed* for the cluster algebra to be a fixed pair $(\mathbf{x}, B)$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ is a free generating set of $\mathbb{F}$ and B is an $n \times n$ skew-symmetric matrix. Define $x'_k \in \mathbb{F}$ by the *exchange relation*

$$x'_k x_k = \prod_{B_{ik}>0} x_i^{B_{ik}} + \prod_{B_{ik}<0} x_i^{-B_{ik}}$$

Then, given an initial seed $(\mathbf{x}, B)$ and $k \in 1, 2, \ldots, n$, we can define a *mutation* of the seed at k, denoted $\mu_k(\boldsymbol{x}, B) = (\mathbf{x}', B')$ where:

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\ B_{ij} + \frac{|B_{ik}|B_{kj} + B_{ik}|B_{kj}|}{2} & \text{otherwise.} \end{cases}$$

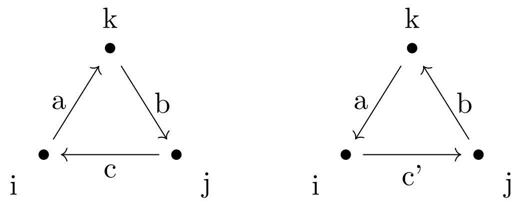
and $\mathbf{x}' = x_1, x_2, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n$. Such a mutation or a sequence of such mutations generate *seeds* which in turn generate all cluster variables in that, for each $\mathbf{x} = x_1, \ldots, x_n$ corresponding to a seed of the cluster algebra, the entries x_i are the cluster variables.

A cluster algebra is said to be of *finite type* if the number of cluster variables that generate it is finite (if it has finitely many seeds). For each finite type cluster algebra, we can associate to its corresponding skew-symmetrisable matrix an edge-weighted, oriented graph, called a *diagram*. We will often denote this diagram by Γ, and the vertex set of Γ by $V(\Gamma)$. We will denote two connected vertices by $i \to j$, or by $i -\ j$ if the orientation is not specified. The diagram is determined by, for $i, j \in V(\Gamma)$, $i \xrightarrow{w} j$ if and only if $B_{ij} > 0$ and $w = |B_{ij}B_{ji}|$ is the weight of the edge. A skew-symmetrisable matrix B is *2-finite* if $|B_{ij}B_{ji}| \leq 3$ for $i, j \in 1, \ldots, n$. By [FZ02, 7.5], we have that if B is 2-finite, all 3-cycles in the unoriented graph underlying our diagram must be oriented cyclically.

Just as we can define mutations of the seeds of a cluster variable, we can also define mutations of a diagram associated to a cluster algebra of finite type by the following set of rules:

Proposition 2.1. *[BM13, Proposition 1.4] Let B be a 2-finite skew-symmetrisable matrix. Then $\Gamma(\mu_k(B))$ is uniquely determined by $\Gamma(B)$ as follows:*

- *Reverse the orientations of all edges in $\Gamma(B)$ incident with k (leaving the weights unchanged)*
- *For any path in $\Gamma(B)$ of form $i \xrightarrow{a} k \xrightarrow{b} j$ (i.e. with a, b positive), let c be the weight on the edge $j \to i$, taken to be zero if there is no such arrow. Let c' be determined by $c' \geq 0$ and $c + c' = max(a, b)$. Then $\Gamma(B)$ changes as in Figure 2, taking the case $c' = 0$ to mean no arrow between i and j.*



Notation 2.2. We notate this mutation of $\Gamma(B)$ at vertex k by $\mu_k(\Gamma)$.

Given a diagram Γ, Barot and Marsh define for $i, j \in V(\Gamma)$,

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}$$

Then, they define $W(\Gamma)$ to be the group generated by s_i, for $i \in V(\Gamma)$, under the following relations. Note that e will denote the identity element of $W(\Gamma)$.

(1) $s_i^2 = e$ for all i;
(2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$;
(3) For any chordless cycle C in Γ, where

$$C = i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$$

and all of the weights w_k are 1 or $w_0 = 2$, we have

$$(s_{i_0} s_{i_1} \cdots s_{i_{d-2}} s_{i_{d-1}} s_{i_{d-2}} \cdots s_{i_1})^2 = e.$$

Maybe it's worth presenting definition 2.4 before this, and then saying it is equivalent to removing certain the (R3)(b) relations. Don't they actually present the definition you refer to in Definition 2.4 as the presentation? You say it is an alteration there, but I think it's the original definition

Using this group presentation, Barot and Marsh state the following result:

Theorem 2.3. *[BM13, Theorem A] Let Γ be the diagram associated to a seed in a cluster algebra of finite type. Then $W(\Gamma)$ is isomorphic to the corresponding reflection group.*

In Section 3 of [BM13], Barot and Marsh provide an alteration of the group $W(\Gamma)$ in order to extend the group definition to any diagram of finite type. The group they define is as follows:

Definition 2.4. Let W_Γ be the group with generators $s_i, i = 1, 2, \ldots, n$, subject to the following relations:

- (R1) $s_i^2 = e$ for all i
- (R2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$

Furthermore, for a chordless cycle $C : i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$ and for $a = 0, 1, 2, \ldots, d-1$, define $r(i_a, i_{a+1}) = s_{i_a} s_{i_{a+1}} \cdots s_{i_{a+d-1}} s_{i_{a+d-2}} \cdots s_{i_{a+1}}$.

Then we have the following relations:

- (R3)(a) If all the weights in the edges of C are 1, then $r(i_a, i_{a+1})^2 = e$
- (R3)(b) If C has some edges of weight 2, then $r(i_a, i_{a+1})^k = e$ where $k = 4 - w_a$ and w_a is the weight of the edge $i_a - \ i_{a-1}$

Defining the group W_Γ with relations as shown above allows them to prove certain characteristics of the interaction between the relations in this group for the chordless cycles underlying the diagrams in question. In particular, they prove the following result.

Theorem 2.5. *[BM13, Theorem 5.4a] Let* Γ *be a diagram of finite type and* $\Gamma' = \mu_k(\Gamma)$ *the mutation of* Γ *at vertex* k*. Then* $W_\Gamma \cong W_{\Gamma'}$*.*

The rest of the paper will be devoted to building up analogous relations, defined in 4.4 to prove a similar result in the case of Artin groups. For Γ a diagram of finite type, we define the associated Artin Group as follows.

Definition 2.6. The associated Artin group A_Γ is generated by s_i, where there is one s_i for each vertex i in Γ. These generators are subject to the following relations

(R2) With m_{ij} as defined in 2, for all $i \neq j$, we add the relations $\langle s_i, s_j\rangle^{m_{ij}} = \langle s_j, s_i\rangle^{m_{ij}}$.

(R3) For every ordered chordless cycle of the form

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1} \longrightarrow i_0$$

such that either

(1) All edges in the chordless cycle are of weight 1 or 2 and the edge $i_{d-1} \to i_0$ has weight 2.

(2) All edges in the chordless cycle have weight 1.

we include the relation

$$s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a,$$

where subscripts are taken $\pmod d$. We will denote this relation by r(a, a+1).

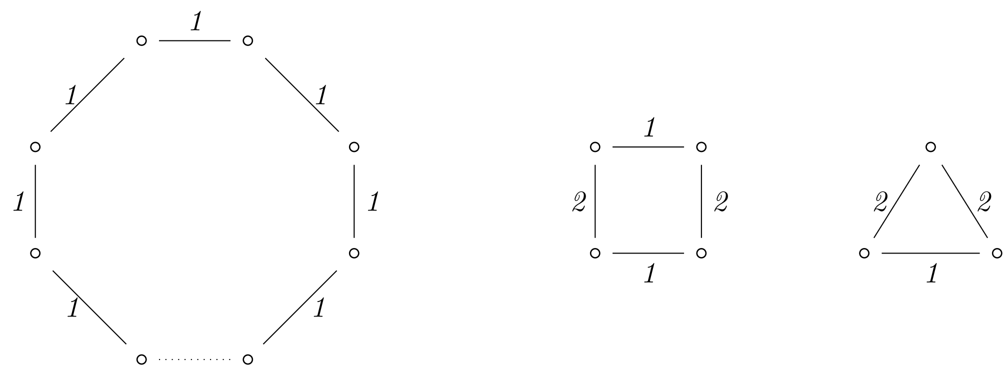
We will use these group relations to prove the mutation invariance of A_Γ in Section 6

3. Diagrams of Finite Type

In this section, we shall review the structure of diagrams of finite type, and how their cycles are affected by mutation. This section is simply a recap of [BM13, Section 2]. First, in Proposition 3.2, all types of chordless cycles in diagrams of finite type are classified. Second, in Corollary 3.3 all possible local pictures between a mutated vertex and two adjacent vertices are drawn. Finally, in 3.4, all chordless cycles introduced from a mutation are drawn. These three lemmas will be crucial in proving the main result Theorem 6.1, as they will allow us to inspect precisely which relations are added and removed after mutating at a prescribed vertex.

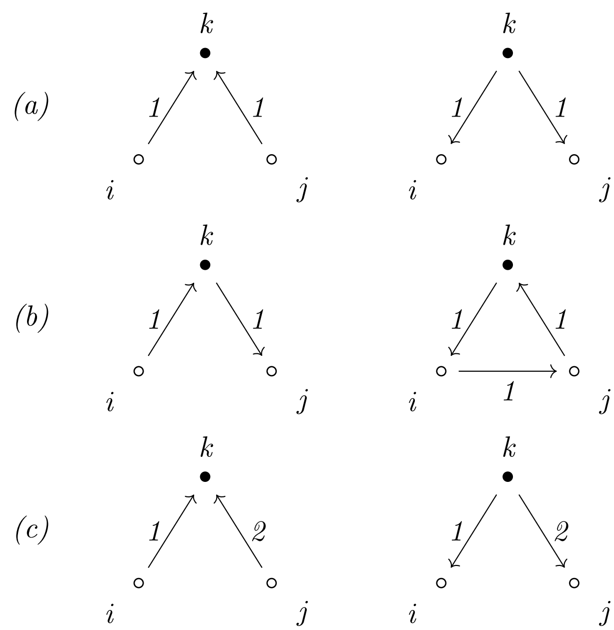
Definition 3.1. A *chordless cycle* of an unoriented graph G is a connected subgraph $H \subset G$ such that the number of vertices in H is equal to the number of edges in H, and the edges in H form a single cycle.

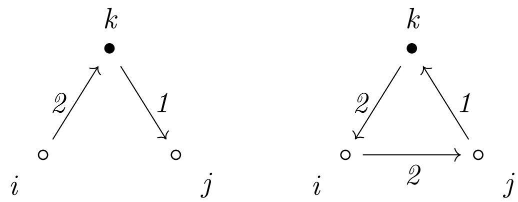
Proposition 3.2. *Let Γ be a diagram of finite type. Then, a chordless cycle in the unoriented graph of Γ is cyclically oriented in Γ. Furthermore, the unoriented graph underlying the cycle must either be a cycle such that all edges have weight 1, a triangle with two edges of weight 2 and one of weight 1, or a square with two opposite edges of weight 2 and two opposite edges of weight 1, as pictured below.*

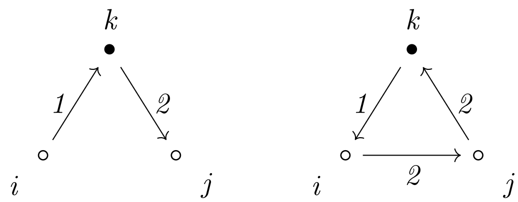


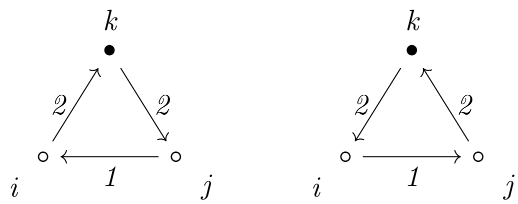
Proof. See [BM13, Proposition 2.1]. □

Corollary 3.3. *Let Γ be a graph of finite type and suppose there are three vertices, labeled i, j, k with both i, j connected to k. Then mutation at k on the induced subdiagram appear as in one of the following figures, either from left to right or right to left, up to switching i and j,*



(d) 

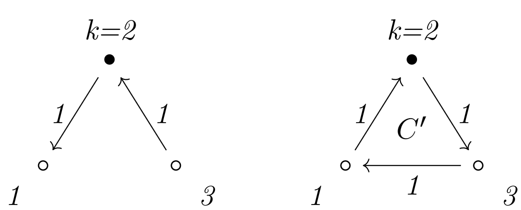
(e) 

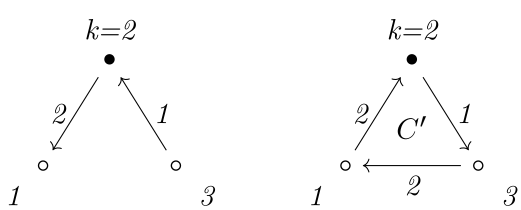
(f) 

Proof. See [BM13, Corollary 2.3]. □

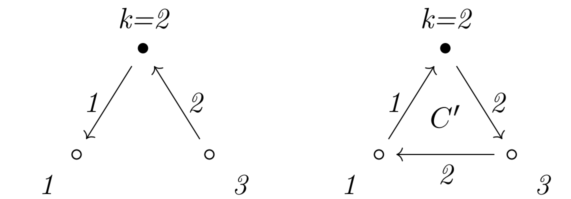
directions of arrows within squares?

Lemma 3.4. *Let* Γ *be a diagram of finite type with* $\Gamma' = \mu_k(\Gamma)$ *the mutation of* Γ *at vertex* k*. Below, we list induced subdiagrams in* Γ *on the left and the resulting induced subdiagrams in* Γ' *with chordless cycles* C' *on the right, after mutation at* k*. We draw the diagrams so that* C' *always has a clockwise cycle. Furthermore, in case* (i)*, we assume* C' *has at least three vertices, while in case* (j)*, we assume* C' *has at least four vertices. Every chordless cycle in* Γ' *is of one of these types.*

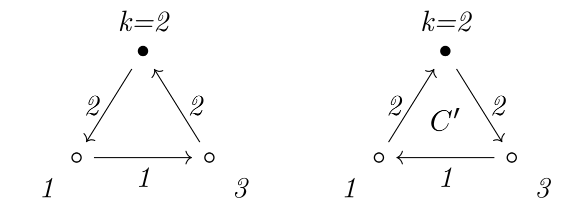
(a) 

(b) 

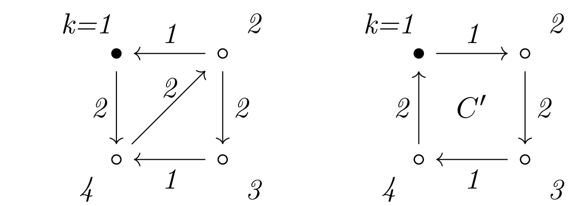
(c)


k=2
1
2
1
3
k=2
1
C′
2
2
1
3

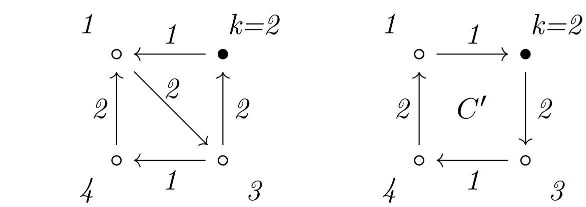
(d)


k=2
2
2
1
1
3
k=2
2
C′
2
1
1
3

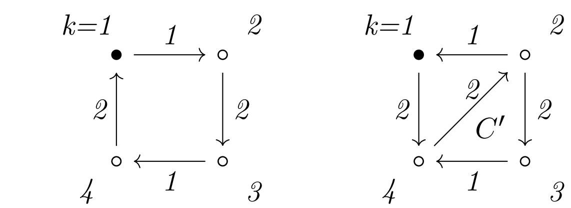
(e)


k=1
1
2
2
2
2
1
4
3
k=1
1
2
2
C′
2
1
4
3

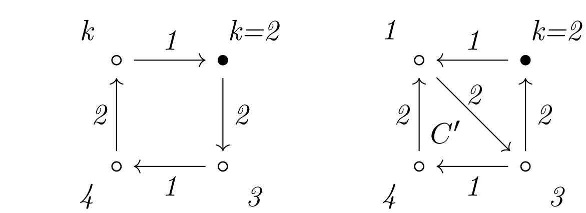
(f)


1
1
k=2
2
2
2
1
4
3
1
1
k=2
2
C′
2
1
4
3

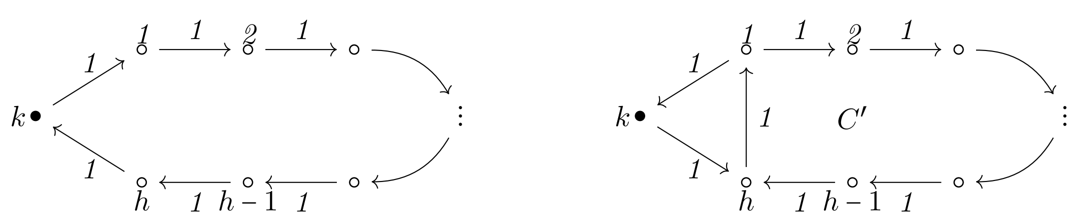
(g)

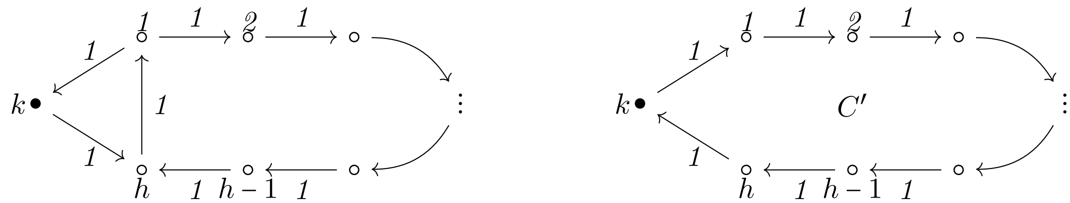

k=1
1
2
2
2
1
4
3
k=1
1
2
2
2
C′
2
1
4
3

(h)


k
1
k=2
2
2
1
4
3
1
1
k=2
2
2
C′
2
1
4
3

(i)


k
1
1
1
1
2
1
1
h
1
h − 1
1
k
1
1
1
1
2
1
1
C′
h
1
h − 1
1

(j) 

(k) C is an oriented cycle in Γ not connected to k and C' is the corresponding cycle in Γ'.

(l) C is an oriented cycle in Γ with exactly one vertex in C connected to k by an edge of either weight 1 or 2. Then, C' is the corresponding cycle in Γ'.

Proof. See [BM13, Lemma 2.5]. □

4. The Artin Group of a Diagram

In order to prove our main result, Theorem 6.1, we must first define the Artin Group associated to a finite type diagram. This definition will be similar to that made in [BM13] at the beginning of Section 3, except that we shall not require $s_i^2 = e$. Since Artin Groups are very similar to coxeter groups, with the caveat that the generators are not involutions, we will be able to use these modified relations to great effect.

4.1. **Artin Groups.**

Notation 4.1. Let

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod 2 \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

That is, $\langle x_i, x_j \rangle$ is just an alternating sequence of x_i and x_j of length k.

Definition 4.2. For $M \in M_{n\times n}(\mathbb{Z} \cup \infty)$ a matrix which whose entries can take values in the real numbers or infinity, satisfying $m_{i,j} = m_{j,i}$ we define the associated *Artin Group* in terms of generators and relations by

$$A = \langle x_1, \dots, x_n | \langle x_i, x_j \rangle^{M_{i,j}} \forall i, j \text{ with } m_{i,j} < \infty \rangle,$$

Remark 4.3. Each Artin group has an associated Coxeter Group defined by adding in the additional relations $s_i^2 = 1$ for all i. An Artin group is said to be of *finite type* if its associated Coxeter group is of finite type. To each Artin group of finite type we can assign it the same Dynkin diagram which is assigned to the Coxeter group associated to the Artin group.

4.2. **The Group associated to Diagram.**

Definition 4.4. Recall from Section 2 that for Γ a diagram of finite type, we define the associated Artin Group as follows. The associated Artin group A_Γ is generated by s_i, where there is one s_i for each vertex i in Γ. These generators are subject to the following relations

(R2) With m_{ij} as defined in 2, for all $i \neq j$, we add the relations $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$.

(R3) For every ordered chordless cycle of the form

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1} \longrightarrow i_0$$

such that either

(1) All edges in the chordless cycle are of weight 1 or 2 and the edge $i_{d-1} \to i_0$ has weight 2.

(2) All edges in the chordless cycle have weight 1.

we include the relation

$$s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a,$$

where subscripts are taken $\pmod d$. We will denote this relation by r(a, a+1).

Remark 4.5. In the above definition, the chordless cycle is ordered, which means that two cycles are considered different if they are cyclic rotations of each other.

Remark 4.6. We purposely include relations $(R2), (R3)$ but not $(R1)$ in order to make our relation labeling analogous to that of [BM13] at the beginning of Section 3. Note that if we add the additional relation $(R1)$ as defined at the beginning of Section 3 of [BM13] (namely, if we add $s_i^2 = e$ for all vertices i in Γ), then we will precisely obtain the group W_Γ as defined at the beginning of section 3 in [BM13].

Remark 4.7. Throughout the remainder of the paper, we shall frequently discuss relations on one diagram of finite type, Γ, and another diagram of finite type Γ'. In order to distinguish the relations in these two groups, we shall refer to the relations on Γ as $(R2), (R3)$ and the relations on Γ' as $(R2'), (R3')$.

Example 4.8. The relations $(R2), (R3)$ in that Γ is a square with all edges of weight 1 are as follows:

$$\Gamma = \begin{tikzcd} 1 \arrow[r, dash, "1"] \arrow[d, dash, "1"'] & 2 \arrow[d, dash, "1"] \\ 4 \arrow[r, dash, "1"'] & 3 \end{tikzcd}$$

(R2)
- $\langle s_1, s_2 \rangle^3 = \langle s_2, s_21 \rangle^3$
- $\langle s_2, s_3 \rangle^3 = \langle s_3, s_2 \rangle^3$
- $\langle s_3, s_4 \rangle^3 = \langle s_4, s_3 \rangle^3$
- $\langle s_4, s_1 \rangle^3 = \langle s_1, s_4 \rangle^3$

(R3)
- $s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 = s_2^{-1} s_3^{-1} s_4 s_3 s_2 s_1$
- $s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3 = s_3^{-1} s_4^{-1} s_1 s_4 s_3 s_2$
- $s_3 s_4^{-1} s_1^{-1} s_2 s_1 s_4 = s_4^{-1} s_1^{-1} s_2 s_1 s_4 s_3$

- $s_4 s_1^{-1} s_2^{-1} s_3 s_2 s_1 = s_1^{-1} s_2^{-1} s_3 s_2 s_1 s_4$

Remark 4.9. Note that if Γ is the graph associated to a Dynkin diagram, then W_Γ as we have defined it is precisely the corresponding Artin group corresponding to that Dynkin diagram. This occurs because, in this case, we have no cycles in Γ, and so we only have relation of the form $(R2')$, which define the Artin Group.

5. Symmetry among (R3) Relations

Given the relations (R2), many of the relations in (R3) become redundant. For example,

Lemma 5.1. *Let Γ be a diagram of finite type which contains a chordless cycle C:*

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1} \longrightarrow i_0$$

so that all edges have weight 1. Then if W is a group generated by $s_1, \dots, s_n$ satifying the relations (R2) and $r(i_a, i_{a+1})$ for some $a \in \{1, \dots, d\}$, all of the relations in (R3) hold for C.

Proof. As in Barot-Marsh, it suffices to prove that the relation r(0, 1) implies the relation r(d-1, 0). So suppose A_Γ satisfies the relation r(0, 1). Then we have

$$\begin{aligned}
&s_{d-1} s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 \\
&= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_1^{-1} \dots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \dots s_1 s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_0 s_1^{-1} \dots s_{d-3}^{-1} s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \dots s_1) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \dots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \dots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \dots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_0 s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 s_{d-1} s_0^{-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 s_{d-1}
\end{aligned}$$

as required. Note that line 3 is equal to 4 and line 7 is equal to line 8 since the cycle is chordless, meaning that s_{d-1} commutes with every element except s_0 and s_{d-2}. □

Furthermore, we obtain similar results for cycles containing edges of weight 2.

Lemma 5.2. *Let Γ be a diagram of finite type containing the following 3-cycle:*

$$\begin{tikzcd} & k\circ \arrow[dr, "2"] & \\ j\circ \arrow[ur, "2"] & & \circ i \arrow[ll, "1"] \end{tikzcd}$$

and let A be the group with generators $s_1, \ldots, s_n$ defined by Γ. Then the relations r(i, j) and r(k, i) are equivalent.

Proof. The lemma follows from the fact that

$$\begin{aligned}
& s_k^{-1} s_j (s_i s_j^{-1} s_k s_j s_i^{-1} s_j^{-1} s_k^{-1} s_j) s_j^{-1} s_k \\
&= s_k^{-1} s_j s_i s_j^{-1} s_k s_j s_i^{-1} s_j^{-1} \\
&= s_k^{-1} s_i^{-1} s_j s_i s_k s_i^{-1} s_j^{-1} s_i
\end{aligned}$$

□

In the setting of the previous lemma, we also obtain the following relation, which will play an important role in later proofs.

Lemma 5.3. *Suppose Γ contains a 3-cycle with edges of weight 2, labeled as in 5.2, and suppose that A_Γ is generated by $s_1, \ldots s_n$. Then we have that*

$$s_j s_k^{-1} s_i s_k s_j s_k^{-1} s_i^{-1} s_k s_j^{-1} s_k^{-1} s_i^{-1} s_k = e.$$

Proof. We first show that

$$s_j^{-1} s_k^{-1} s_i s_k s_j s_k^{-1} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_k = e.$$

The result then follows by inverting the relation and conjugating by s_j. In the following computation, we will underline the terms being manipulated in each line for emphasis.

$$
\begin{aligned}
&s_k s_j s_k (s_j^{-1} s_k^{-1} s_i s_k s_j s_k^{-1} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_k) s_k^{-1} s_j^{-1} s_k^{-1} \\
&= s_k \underline{s_j s_k s_j^{-1} s_k^{-1}} s_i s_k s_j s_k^{-1} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} \underline{s_k s_k^{-1}} s_j^{-1} s_k^{-1} \\
&= \underline{s_k s_k^{-1}} \underline{s_j^{-1} s_k s_j s_i} s_k s_j s_k^{-1} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_i \underline{s_j^{-1} s_k s_j s_k} s_j s_k^{-1} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_i s_k s_j \underline{s_k s_j^{-1} s_j s_k^{-1}} s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_i \underline{e} s_k s_j s_i s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_i s_j \underline{s_j^{-1} s_k s_j s_i} s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= \underline{s_i s_j s_i} s_j^{-1} s_k s_j s_k s_j^{-1} s_k^{-1} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_j s_i \underline{s_j s_j^{-1}} s_k \underline{s_j s_k s_j^{-1} s_k^{-1}} s_i^{-1} s_j^{-1} s_k^{-1} \\
&= s_j s_i \underline{s_k s_k^{-1}} \underline{s_j^{-1} s_k s_j s_i^{-1}} s_j^{-1} s_k^{-1} \\
&= s_j \underline{s_i s_i^{-1}} s_j^{-1} s_k \underline{s_j s_j^{-1}} s_k^{-1} \\
&= \underline{s_j s_j^{-1}} \underline{s_k s_k^{-1}} \\
&= e
\end{aligned}
$$

□

Lemma 5.4. *Let Γ be a diagram of finite type containing the following 4-cycle:*

$$
\begin{tikzcd}
i\circ \arrow[r, "1"] & \circ j \arrow[d, "2"] \\
l\circ \arrow[u, "2"] & \circ k \arrow[l, "1"]
\end{tikzcd}
$$

and let A be the group with generators $s_1, \dots, s_n$ defined by Γ. Then the relations r(i, j) and r(k, l) are equivalent.

Proof. We have that

$$
\begin{aligned}
&s_k^{-1} s_l^{-1} s_j (s_i s_j^{-1} s_k^{-1} s_l s_k s_j s_i^{-1} s_j^{-1} s_k^{-1} s_l^{-1} s_k s_j) s_j^{-1} s_l s_k \\
&= s_k^{-1} s_l^{-1} (s_j s_i s_j^{-1})(s_k^{-1} s_l s_k)(s_j s_i^{-1} s_j^{-1}) s_k^{-1} s_l^{-1} s_k s_l s_k \\
&= s_k^{-1} s_l^{-1} s_i^{-1} s_j s_i s_l s_k s_l^{-1} s_i^{-1} s_j^{-1} s_i s_k^{-1} (s_l^{-1} s_k s_l) s_k \\
&= s_k^{-1} s_l^{-1} s_i^{-1} s_j s_i s_l s_k s_l^{-1} s_i^{-1} s_j^{-1} s_i s_l
\end{aligned}
$$

□

Finally, we conclude the section by establishing a relationship between the groups defined by Γ and Γ^{op}, the diagram obtained by reversing all arrows in Γ.

Lemma 5.5. *Let A_Γ be generated by $s_1, \dots, s_n$. Then $s_1^{-1}, \dots, s_n^{-1}$ satisfy the relations (R2) and (R3) in $A_{\Gamma^{op}}$. In particular, $A_\Gamma \cong A_{\Gamma^{op}}$.*

Proof. One can see that the inverse elements satisfy (R2) in $A_{\Gamma^{op}}$ by taking the inverse of both sides of the corresponding relation in A_Γ. To see that the elements satisfy (R3) in $A_{\Gamma^{op}}$, note that for a chordless cycle in Γ with all weights equal to one, we have

$$s_0^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} \dots s_0 = s_1^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} \dots s_1$$

by the relation r(0,1) in (R3) in A_Γ. But then applying relations from (R2), we have that

$$s_0^{-1} \dots s_{d-1} s_{d-2} s_{d-1}^{-1} \dots s_0 = s_1^{-1} \dots s_{d-1} s_{d-2} s_{d-1}^{-1} \dots s_1,$$

and since the cycle is chordless, we then have

$$s_0^{-1} s_{d-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_{d-1}^{-1} s_0 = s_{d-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1}.$$

Repeating this process, we find that

$$s_0^{-1} s_{d-1} s_{d-2} \dots s_2 s_1 s_2^{-1} \dots s_{d-2}^{-1} s_{d-1}^{-1} s_0 = s_{d-1} s_{d-2} \dots s_2 s_1 s_2^{-1} \dots s_{d-2}^{-1} s_{d-1}^{-1}.$$

But this occurs if and only if $s_1^{-1}, \dots, s_n^{-1}$ satisfies the relation r(0, d-1) in $A_{\Gamma^{op}}$.

For a triangle labeled as in 5.2, by the relation r(k, i) we have

$$s_k s_i^{-1} s_j s_i s_k^{-1} = s_i^{-1} s_j s_i.$$

Hence

$$s_k s_j s_i s_j^{-1} s_k^{-1} = s_j s_i s_j^{-1}.$$

But as before, this can occur if and only if $s_i^{-1}, s_j^{-1}, s_k^{-1}$ satisfy the relation r(k, j) in $A_{\Gamma^{op}}$.

Finally, given a square labeled as in 5.4 and the relations r(1, 2) and r(3, 4), we have

$$\begin{aligned}
&s_j s_i s_l s_k s_l^{-1} s_i^{-1} \\
&= s_i s_i^{-1} s_j s_i s_l s_k s_l^{-1} s_i^{-1} \\
&= s_i s_j s_i s_j^{-1} s_k^{-1} s_l s_k s_i^{-1} \\
&= s_i s_j (s_i s_j^{-1} s_k^{-1} s_l s_k s_j) s_j^{-1} s_i^{-1} \\
&= s_i s_j s_j^{-1} (s_k^{-1} s_l s_k) s_j (s_i s_j^{-1} s_i^{-1}) \\
&= s_i s_l s_k s_l^{-1} s_j s_j^{-1} s_i^{-1} s_j \\
&= s_i s_l s_k s_l^{-1} s_i^{-1} s_j
\end{aligned}$$

But this relation holds if and only if $s_i^{-1}, \dots, s_l^{-1}$ satisfy r(j, i) in $A_{\Gamma^{op}}$. Therefore, we are done. $\square$

6. Proof of Main Result

In this section we prove our main result:

Theorem 6.1. *Let Γ be a diagram of finite type, and let $\Gamma' = \mu_k(\Gamma)$ be the mutation of Γ at vertex k. Then $A_\Gamma \cong A_{\Gamma'}$*

Throughout this section we will fix a diagram of finite type Γ, a vertex k of Γ, and write $\Gamma' = \mu_k(\Gamma)$. We will write s_i, r_i, q_i, and u_i for the generators corresponding to vertex i of A_Γ, $A_{\Gamma'}$, $A_{\Gamma^{op}}$, and $A_{(\Gamma')^{op}}$, respectively. We will also abuse notation and write u_i for the generators of $A_{(\Gamma^{op})'}$ as well, making use of the fact that $(\Gamma')^{op} = (\Gamma^{op})'$.

In the proof of Theorem 6.1 we will use Lemma 5.5 along with the following proposition, whose proof we will defer until after the proof of Theorem 6.1 and then split into Lemmas 6.3, 6.4, and 6.5.

Proposition 6.2. *The map $\varphi\colon A_{\Gamma'} \to A_\Gamma$ defined by*

$$\varphi(r_i) = \begin{cases} s_k s_i s_k^{-1} & \textit{if there is a (possibly weighted) arrow } i \to k \textit{ in } \Gamma \\ s_i & \textit{otherwise} \end{cases}$$

is a group homomorphism.

do we define m'_{ij} somewhere?

Proof of Theorem 6.1. By Proposition 6.2 $\varphi\colon A_{\Gamma'} \to A_\Gamma$ is a group homomorphism. We also abuse notation and write $\varphi\colon A_{\Gamma^{op}} \to A_{(\Gamma^{op})'}$ by letting $\varphi(q_i)$ equal $u_k u_i u_k^{-1}$ if there is an arrow $i \to k$ in $(\Gamma^{op})'$, and u_i otherwise, and this is a homomorphism by Proposition 6.2 as well. By Lemma 5.5, the homormorphism Δ that sends $s_i \in A(\Gamma)$ to $q_i^{-1} \in A(\Gamma^{op})$ and $u_i \in A_{(\Gamma')^{op}}$ to $r_i \in A_{\Gamma'}$ is well-defined. We then have a homomorphism

$$\psi = \Delta \circ \varphi \circ \Delta\colon A(\Gamma) \to A(\Gamma^{op}) \to A((\Gamma^{op})') \to A(\Gamma')$$

Suppose that there is an arrow $i \to k$ in Γ. Then there will be an arrow $k \to i$ in Γ^{op} and hence an arrow $i \to k$ in $(\Gamma^{op})'$, so we have that

$$\psi \circ \varphi(r_i) = \Delta(\varphi(\Delta(\varphi(r_i)))) = \Delta(\varphi(\Delta(s_k s_i s_k^{-1}))) = \Delta(\varphi(q_k^{-1} q_i^{-1} q_k)) = \Delta(u_i^{-1}) = r_i$$

Similarly if there is an arrow $k \to i$ or no arrow between i and k in Γ then there will be an arrow $k \to i$ or no arrow between i and k in $(\Gamma^{op})'$, respectively. In each of these cases we have that

$$\psi \circ \varphi(r_i) = \Delta(\varphi(\Delta(\varphi(r_i)))) = \Delta(\varphi(\Delta(s_i))) = \Delta(\varphi(q_i^{-1})) = \Delta(u_i^{-1}) = r_i$$

Thus $\psi \circ \varphi$ is the identity map on $A(\Gamma')$. By a similar argument $\varphi \circ \psi$ is the identity map on $A(\Gamma)$, and hence $A(\Gamma) \cong A(\Gamma'))$. $\square$

We prove Proposition 6.2 by showing that that the elements $\varphi(r_i) \in A_\Gamma$ satisfy the $(R2')$ and $(R3')$ relations in $A_{\Gamma'}$. Proofs that the t_i satisfy these relations are divided among Lemmas 6.3, 6.4, and 6.5. Throughout the proofs we write $t_i = \varphi(r_i)$ for convenience.

Lemma 6.3. *Let i, j be distinct vertices of Γ.*
(a) If $i = k$ or $j = k$, then $\langle t_i t_j \rangle^k = e$.
(b) If at most of i, j is connected to k in Γ, then $\langle t_i t_j \rangle^k = e$.

Proof. For case (a), suppose without loss of generality that $i = k$. Note that $m'_{ij} = m_{ij}$. The only nontrivial case is when there is an arrow $j \to k = i$. Since i and j are connected in this case, m_{ij} is one of 3,4, or 6.

Case $m_{ij} = 3$. Here we have $s_j s_i s_j = s_i s_j s_i$, so $s_i s_j = s_j s_i s_j s_i^{-1}$, so

$$t_i t_j t_i = s_i s_i s_j s_i^{-1} s_i = s_i s_i s_j = s_i s_j s_i s_j s_i^{-1} = t_j t_i t_j$$

Case $m_{ij} = 4$. Here we have $s_i s_j s_i s_j = s_j s_i s_j s_i$, so $s_i s_i s_j s_i s_j s_i^{-1} = s_i s_j s_i s_j$ and therefore

$$t_i t_j t_i t_j = s_i s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} = s_i s_i s_j s_i s_j s_i^{-1} = s_i s_j s_i s_j = s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} s_i = t_j t_i t_j t_i$$

Case $m_{ij} = 6$. Here we have $s_i s_i s_j s_i s_j s_i s_j s_i^{-1} = s_i s_j s_i s_j s_i s_j$. As in the previous case, we add an remove pairs $s_i s_i^{-1}$ as necessary, giving

$$t_i t_j t_i t_j t_i t_j = s_i s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} = s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} s_i s_i s_j s_i^{-1} s_i = t_j t_i t_j t_i t_j t_i$$

For case (b), the only nontrivial case is when there is an arrow $i \to k$ or $j \to k$. Without loss of generality, suppose there is an arrow $i \to k$. Since j is not connected to k, we know that $s_j s_k = s_k s_j$.

Case $m_{ij} = 2$. Here $s_i s_j = s_j s_i$, so we have that

$$t_i t_j = s_k s_i s_k^{-1} s_j = s_j s_k s_i s_k^{-1} = t_j t_i$$

Case $m_{ij} = 3$. Here $s_i s_j s_i = s_j s_i s_j$, so we have that

$$t_i t_j t_i = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = s_k s_i s_j s_i s_k^{-1} = s_k s_j s_i s_j s_k^{-1} = s_j s_k s_i s_k^{-1} s_j = t_j t_i t_j$$

Case $m_{ij} = 4$. Here $s_i s_j s_i s_j = s_j s_i s_j s_i$, so we have that

$$t_i t_j t_i t_j = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j = s_k s_i s_j s_i s_j s_k^{-1} = s_k s_j s_i s_j s_i s_k^{-1} = s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = t_j t_i t_j t_i$$

Case $m_{ij} = 6$. Here $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$, so we have that

$$t_i t_j t_i t_j t_i t_j = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j = s_k s_i s_j s_i s_j s_i s_j s_k^{-1} = s_k s_j s_i s_j s_i s_j s_i s_k^{-1} = t_j t_i t_j t_i t_j t_i$$

$\square$

Lemma 6.4. *Let i, j be distinct vertices of Γ such that i and j are connected. Then $\langle t_i t_j \rangle^k = e$.*

Proof. The possibilities for the subdiagram induced by i, j, and k are enumerated in Corollary 3.3. We show that t_i and t_j satisfy the $(R2')$ relations by checking each case. We also check that the corresponding $(R3')$ relations hold in cases (b), (d), (e), and (f). Within each case we view the subdiagram on the left as a subdiagram of Γ in the subcase (i) and as a subdiagram of Γ' in the subcase (ii).

Disaster. Open to suggestions for how to make this more clear

Throughout the proof we will make frequent use of the fact that if m and n are vertices of Γ, then

$$s_m s_n s_m = s_n s_m s_n \Leftrightarrow s_m s_n s_m^{-1} = s_n^{-1} s_m s_n \Leftrightarrow s_m s_n^{-1} s_m^{-1} = s_n^{-1} s_m^{-1} s_n \Leftrightarrow s_m^{-1} s_n^{-1} s_m^{-1} = s_n^{-1} s_m^{-1} s_n^{-1}$$

When helpful, we underline the sections of an expression that are about to be manipulated. We also frequently combine two applications of A_Γ relations when one manipulation is simply commuting pairs of variables.

a) i) We have $t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$.
   ii) We have $t_i t_j = s_i s_j = s_j s_i = t_j t_i$.

b) i) We have

$$\begin{aligned}
t_i t_j t_i &= s_k s_i \underline{s_k^{-1} s_j s_k} s_i s_k^{-1} \\
&= s_k \underline{s_i s_j} s_k \underline{s_j^{-1} s_i} s_k^{-1} \\
&= s_k s_j \underline{s_i s_k s_i} s_j^{-1} s_k^{-1} \\
&= \underline{s_k s_j s_k} s_i \underline{s_k s_j^{-1} s_k^{-1}} \\
&= s_j s_k \underline{s_j s_i s_j^{-1}} s_k^{-1} s_j \\
&= s_j s_k s_i s_k^{-1} s_j \\
&= t_i t_j t_i
\end{aligned}$$

   ii) We have $t_i t_j = s_i \underline{s_k s_j s_k^{-1}} = \underline{s_i s_j^{-1} s_k s_j} = \underline{s_j^{-1} s_k s_j} s_i = s_k s_j s_k^{-1} s_i = t_j t_i$

c) i) We have $t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$
   ii) We have $t_i t_j = s_i s_j = s_j s_i = t_j t_i$

d) i) We have

$$\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i \underline{s_k^{-1} s_j s_k} s_i^{-1} \underline{s_k^{-1} s_j^{-1} s_k} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} \underline{s_j s_k s_i s_j} s_k \underline{s_j^{-1} s_i^{-1} s_j} s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k \underline{s_i s_k^{-1} s_k s_j} s_k s_i s_k s_i^{-1} s_k^{-1} \underline{s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1}} \\
&= s_k s_j \underline{s_i s_k s_i s_k s_i^{-1} s_k^{-1} s_i^{-1} s_k^{-1}} s_j^{-1} s_k^{-1} \\
&= e
\end{aligned}$$

   ii) We have $t_i t_j = s_i s_k s_j s_k^{-1} = s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i = s_k s_j s_k^{-1} s_i = t_j t_i$

e) i) We have

$$\begin{aligned} t_it_jt_it_jt_i^{-1}t_j^{-1}t_i^{-1}t_j^{-1} &= \underline{s_ks_is_k^{-1}}s_j\underline{s_ks_is_k^{-1}}s_j\underline{s_ks_i^{-1}s_k^{-1}}s_j^{-1}\underline{s_ks_i^{-1}s_k^{-1}}s_j^{-1} \\ &= s_i^{-1}s_k\underline{s_is_js_i^{-1}}s_k\underline{s_is_js_i^{-1}}s_k^{-1}\underline{s_is_j^{-1}s_i^{-1}}s_k^{-1}\underline{s_is_j^{-1}} \\ &= s_i^{-1}\underline{s_ks_js_ks_js_k^{-1}s_j^{-1}s_k^{-1}s_j^{-1}}s_i \\ &= e \end{aligned}$$

ii) Since $s_js_k^{-1}s_is_k = s_k^{-1}s_is_ks_j$, we have that $s_js_k^{-1}s_i^{-1}s_k = s_k^{-1}s_i^{-1}s_ks_j$, hence

$$t_it_jt_i^{-1}t_j^{-1} = s_is_k\underline{s_js_k^{-1}s_i^{-1}s_k}s_j^{-1}s_k^{-1} = s_is_ks_k^{-1}s_i^{-1}s_ks_js_j^{-1}s_k^{-1} = e$$

f) i) We have that

$$\begin{aligned} s_k^{-1}t_it_jt_it_j^{-1}t_i^{-1}t_j^{-1}s_k &= s_is_k^{-1}s_js_ks_is_k^{-1}s_j^{-1}s_ks_i^{-1}s_k^{-1}s_j^{-1}s_k \\ &= e \end{aligned}$$

Where the second equality follows from Lemma 5.3.

ii) This follows from part (i) by symmetry

□

Lemma 6.5. *The elements t_i satisfy the $(R3')$ relations in Γ'.*

Proof. We know that every chordless cycle in Γ' arises from a subdiagram of Γ in the form of one of the cases of Lemma 3.4, so we simply need to check that a cycle relation holds in each case. We follow the labeling of the vertices used in Lemma 3.4.

a) We have $t_1t_2^{-1}t_3t_2 = s_1s_2^{-1}s_2s_3s_2^{-1}s_2 = s_1s_3 = s_3s_1 = t_2^{-1}t_3t_2t_1$
b) We have $t_1t_2^{-1}t_3t_2 = s_1s_2^{-1}s_2s_3s_2^{-1}s_2 = s_1s_3 = s_3s_1 = t_3t_1$
c) We have $t_1t_2^{-1}t_3t_2 = s_1s_2^{-1}s_2s_3s_2^{-1}s_2 = s_3s_1 = t_2^{-1}t_3t_2t_1$
d) We have

$$\begin{aligned} t_3t_1^{-1}t_2t_1t_3^{-1}t_1^{-1}s_2^{-1}t_1 &= s_2s_3s_2^{-1}s_1^{-1}s_2s_1s_2s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1}s_1 \\ &= s_2s_3s_1s_2s_1^{-1}s_2^{-1}s_2s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1}s_1 \\ &= s_2s_3s_1s_2s_1^{-1}s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1}s_1 \\ &= s_2s_3s_1s_2s_1^{-1}s_3^{-1}s_1s_2^{-1}s_1^{-1}s_2^{-1} \\ &= s_2s_1s_2s_1^{-1}s_3s_3^{-1}s_1s_2^{-1}s_1^{-1}s_2 \\ &= e \end{aligned}$$

e) We have

$$\begin{aligned}
t_1t_2^{-1}t_3^{-1}t_4t_3t_2 &= (s_1s_2^{-1}s_1^{-1}s_1s_2s_1^{-1})s_1s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4s_3s_1s_2s_1^{-1}\\
&= s_1s_2^{-1}s_1^{-1}\underline{s_1s_2s_1s_2^{-1}s_1^{-1}}s_3^{-1}s_4s_3s_1s_2s_1^{-1}\\
&= s_1s_2^{-1}s_1^{-1}\underline{s_2s_3^{-1}s_4s_3}s_1s_2s_1^{-1}\\
&= s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4s_3\underline{s_2s_1s_2s_1^{-1}}\\
&= s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4s_3s_1s_2s_1^{-1}s_1\\
&= t_2^{-1}t_3^{-1}t_4t_3t_2t_1
\end{aligned}$$

f) We have

$$\begin{aligned}
t_1t_2^{-1}t_3^{-1}t_4t_3t_2 &= s_1\underline{s_2^{-1}s_2}s_3^{-1}\underline{s_2^{-1}s_4s_2}s_3\underline{s_2^{-1}s_2}\\
&= s_1s_3^{-1}s_4s_3\\
&= s_3^{-1}s_4s_3s_1\\
&= (s_2^{-1}s_2)s_3^{-1}(s_2^{-1}\underline{s_2)s_4(s_2^{-1}}s_2)s_3(s_2^{-1}s_2s_1\\
&= s_2^{-1}2s_3^{-1}s_2^{-1}s_4s_2s_3s_2^{-1}s_2s_1\\
&= t_2^{-1}t_3^{-1}t_4t_3t_2t_1
\end{aligned}$$

g) We have

$$\begin{aligned}
t_3t_4^{-1}t_2t_4t_3^{-1}t_4^{-1}t_2^{-1}t_4 &= \underline{s_3s_1}s_4^{-1}s_1^{-1}s_2s_1s_4\underline{s_1^{-1}s_3^{-1}s_1}s_4^{-1}s_1^{-1}s_2^{-1}s_1s_4s_1^{-1}\\
&= s_1\underline{s_3s_4^{-1}s_1^{-1}s_2s_1s_4s_3^{-1}s_4^{-1}s_1^{-1}s_2^{-1}s_1s_4}s_1^{-1}\\
&= s_1s_1^{-1}\\
&= e
\end{aligned}$$

h) We have

$$\begin{aligned}
t_3t_4^{-1}t_1t_4 &= s_3s_4^{-1}\underline{s_2s_1s_2^{-1}}s_4\\
&= s_3s_4^{-1}s_1^{-1}s_2s_1s_4\\
&= s_4^{-1}\underline{s_1^{-1}s_2s_1}s_4s_3\\
&= s_4^{-1}s_2s_1s_2^{-1}s_4s_3\\
&= t_4^{-1}t_1t_4t_3
\end{aligned}$$

i) We have

$$\begin{aligned}
&t_1t_2^{-1}t_3^{-1}\cdots t_{h-1}^{-1}t_ht_{h-1}\cdots t_2t_1^{-1}t_2^{-1}\cdots t_{h-1}^{-1}t_h^{-1}t_{h-1}\cdots t_2\\
&= s_1s_2^{-1}s_3^{-1}\cdots s_{h-1}^{-1}\underline{s_ks_hs_k^{-1}}s_{h-1}\cdots s_2s_1^{-1}s_2^{-1}\cdots s_{h-1}^{-1}\underline{s_ks_h^{-1}s_k^{-1}}s_{h-1}\cdots s_2\\
&= s_1s_2^{-1}s_3^{-1}\cdots s_{h-1}^{-1}s_h^{-1}s_ks_hs_{h-1}\cdots s_2s_1^{-1}s_2^{-1}\cdots s_{h-1}^{-1}s_h^{-1}s_k^{-1}s_hs_{h-1}\cdots s_2\\
&= e
\end{aligned}$$

j) We have

$$\begin{aligned}
&t_ht_k^{-1}t_1^{-1}t_2^{-1}\cdots t_{h-2}^{-1}t_{h-1}t_{h-2}\cdots t_1t_kt_h^{-1}t_k^{-1}t_1^{-1}t_2^{-1}\cdots t_{h-2}^{-1}t_{h-1}^{-1}t_{h-2}\cdots t_2t_1t_k\\
&= s_h\underline{s_k^{-1}s_k}s_1^{-1}\underline{s_k^{-1}}s_2^{-1}\cdots s_{h-2}^{-1}s_{h-1}s_{h-2}\cdots \underline{s_k}s_1\underline{s_k^{-1}s_k}s_h^{-1}\underline{s_k^{-1}s_k}s_1^{-1}\underline{s_k^{-1}}s_2^{-1}\cdots s_{h-2}^{-1}s_{h-1}^{-1}s_{h-2}\cdots s_2\underline{s_k}s_1\underline{s_k^{-1}s_k}\\
&= s_hs_1^{-1}s_2^{-1}\cdots s_{h-2}^{-1}s_{h-1}s_{h-2}\cdots s_2s_1s_h^{-1}s_1^{-1}s_2^{-1}\cdots s_{h-2}^{-1}s_{h-1}^{-1}s_{h-2}\cdots s_2s_1\\
&= e
\end{aligned}$$

□

Proof of Proposition 6.2. Lemma 6.3 and Lemma 6.4 show that the elements t_i satisfy the $(R2')$ relations for $A_{\Gamma'}$. Lemma 6.5 shows that they satisfy the $(R3')$ relations. Since these are all of the relations defining $A_{\Gamma'}$, it follows that φ defines a group homomorphism $A_{\Gamma'} \to A_{\Gamma}$. □

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