

*Artin Group Presentations Arising from  
Cluster Algebras*

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# Outline

## REU Problem 6

### Question

Develop variations of relations for the Coxeter group associated to a diagram of finite type provided in Barot-Marsh to define the Artin group  $A_\Gamma$  corresponding to a diagram  $\Gamma$ . Prove that for  $\Gamma' = \mu_k(\Gamma)$ , we get  $A_\Gamma \cong A_{\Gamma'}$ .

### *Definition*

We say a matrix  $B$  is *skew-symmetrisable* if there exists a diagonal matrix  $D$  of the same size such that  $|D_{ii}| > 0$  and  $DB$  is skew-symmetric.

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### *Definition*

A skew-symmetrisable matrix  $B$  is *2-finite* if  $|B_{ij}B_{ji}| \leq 3$  for all  $i, j \in \{1, \dots, n\}$ .

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From the skew-symmetrisable matrix associated to a cluster algebra of finite type, we can associate an diagram  $\Gamma$  as follows: For  $i, j \in V(\Gamma)$ ,  $i \xrightarrow{w} j$  if and only if  $B_{ij} > 0$  and  $w = |B_{ij}B_{ji}|$  is the weight of the edge.

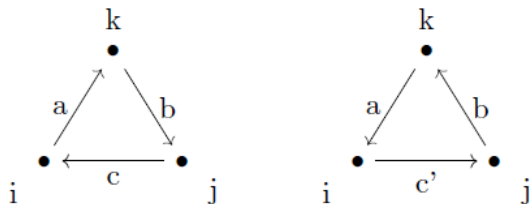
# Mutation Rules

## Proposition

Let  $B$  be a 2-finite skew-symmetrisable matrix. Then  $\Gamma(\mu_k(B))$  is uniquely determined by  $\Gamma(B)$  as follows:

- ▶ Reverse the orientations of all edges in  $\Gamma(B)$  incident with  $k$  (leaving the weights unchanged)
- ▶ For any path in  $\Gamma(B)$  of form  $i \xrightarrow{a} k \xrightarrow{b} j$  (i.e. with  $a, b$  positive), let  $c$  be the weight on the edge  $j \rightarrow i$ , taken to be zero if there is no such arrow. Let  $c'$  be determined by  $c' \geq 0$  and  $c + c' = \max(a, b)$ . Then  $\Gamma(B)$  changes in a predetermined way, taking the case  $c' = 0$  to mean no arrow between  $i$  and  $j$ .

# Mutation Rules



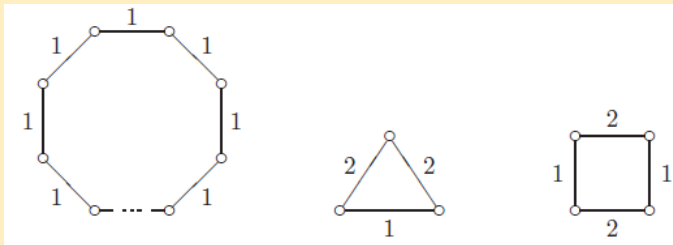
*Figure 1:* Predetermined method for mutation at  $k$



# Unoriented structures underlying $\Gamma$

## Proposition

*Any chordless cycle in  $\Gamma$  must have an unoriented structure that is one of the following. Furthermore, it must be cyclically oriented.*



*Figure 2:* Possible chordless cycles in a diagram

# Unoriented structures underlying $\Gamma$

## Lemma

For  $\Gamma$ , if we have a subdiagram of  $\Gamma$  with three connected vertices, then the unoriented graph underlying the subdiagram must be one of the following:

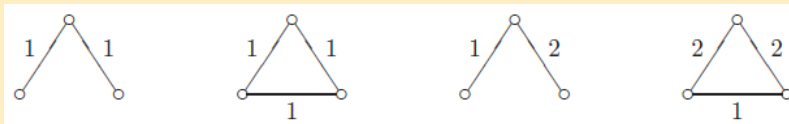


Figure 3: Unoriented 3-vertex connected subdiagrams

# Possible mutations for $i - k - j$

## Corollary

If in  $\Gamma$  we have  $i, j, k \in V(\Gamma)$ ,  $i \neq k \neq j$ , and we have  $i - k - j$ , then the only possible mutations for this connected path between the three vertices are the following:

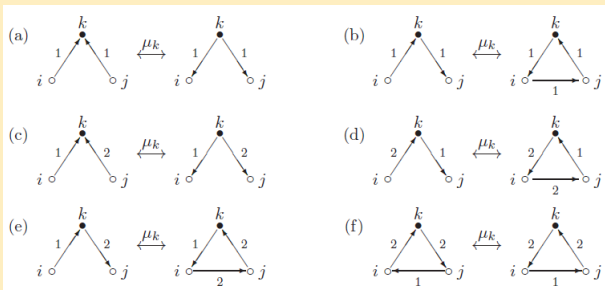


Figure 4: Mutation of three connected vertices

### *Definition*

For a diagram  $\Gamma$  and  $i, j \in V(\Gamma)$ , define

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}$$

This definition will allow us to present generator relations in the group  $W_\Gamma$ .

Given a diagram  $\Gamma$  of finite type, Barot and Marsh define the Coxeter group  $W_\Gamma$  with generators  $s_i, i = 1, 2, \dots, n$ , subject to the following relations:

- ▶ (R1)  $s_i^2 = e$  for all  $i$
- ▶ (R2)  $(s_i s_j)^{m_{ij}} = e$  for all  $i \neq j$

Furthermore, for a chordless cycle  $C : i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{d-1} \rightarrow i_0$  and for  $a = 0, 1, 2, \dots, d-1$ , define

$$r(i_a, i_{a+1}) = s_{i_a} s_{i_{a+1}} \cdots s_{i_{a+d-1}} s_{i_{a+d-2}} \cdots s_{i_{a+1}}.$$

Then we have the following relations:

- ▶ (R3)(a) If all the weights in the edges of  $C$  are 1, then  $r(i_a, i_{a+1})^2 = e$
- ▶ (R3)(b) If  $C$  has some edges of weight 2, then  $r(i_a, i_{a+1})^k = e$  where  $k = 4 - w_a$  and  $w_a$  is the weight of the edge  $i_a - i_{a-1}$

Given a diagram  $\Gamma$  and the corresponding Coxeter group  $W_\Gamma$ , Barot and Marsh prove that this group is invariant (up to isomorphism) under mutation of  $\Gamma$ .

### *Theorem*

1. Let  $\Gamma$  be a diagram of finite type and  $\Gamma' = \mu_k(\Gamma)$  the mutation of  $\Gamma$  at vertex  $k$ . Then  $W_\Gamma \cong W_{\Gamma'}$ .
2. Let  $\mathcal{A}$  be a cluster algebra of finite type. Then the groups  $W_\Gamma$  associated to the diagrams  $\Gamma$  arising from the seeds of  $\mathcal{A}$  are all isomorphic (to the reflection group associated to the Dynkin diagram associated to  $\mathcal{A}$ ).

Let

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod{2} \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod{2} \end{cases}$$

That is,  $\langle x_i, x_j \rangle$  is just an alternating sequence of  $x_i$  and  $x_j$  of length  $k$ . We also write  $\langle x_i, x_j \rangle^{-k}$  to denote  $(\langle x_i, x_j \rangle^k)^{-1}$ .

Let  $(i_0, \dots, i_{d-1})$  be an ordered tuple such that the subgraph of  $\Gamma$  on the vertices  $i_0, \dots, i_{d-1}$  is a chordless cycle, with edges of nonzero weight from  $i_k$  to  $i_{k+1}$ , where subscripts are taken (mod  $d$ ). Then, denote

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \dots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \dots s_{i_{a+1}} s_{i_a}^{-1}.$$

Now we have

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod{2} \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod{2} \end{cases}$$

And

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \dots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \dots s_{i_{a+1}} s_{i_a}^{-1}.$$

Additionally, let

$$t(i_a, i_{a+1}) = s_{i_a} p(i_a, i_{a+1}) p(i_a, i_{a+1})^{-1} s_{i_a}^{-1}.$$

These definitions will allow us to present generator relations in the group  $A_\Gamma$ .



Given a diagram  $\Gamma$  of finite type, we define the Artin group  $A_\Gamma$  with generators  $s_i, i = 1, 2, \dots, n$ , subject to the following relations:

- ▶ (T2) With  $m_{ij}$  as defined previously, for all  $i \neq j$ , we add the relations  $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$ .
- ▶ (T3) Let  $(i_0, i_1, \dots, i_{d-1})$  be an ordered tuple as before. If additionally one of the following two conditions hold:
  1. All edges in the chordless cycle are of weight 1 or 2 and the edge  $i_{d-1} \rightarrow i_0$  has weight 2.
  2. All edges in the chordless cycle have weight 1.then we include the relation  $t(i_0, i_1) = e$ .

### *Remark*

*Each Artin group has an associated Coxeter group defined by adding in the additional relations  $s_i^2 = 1$  for all  $i$ . An Artin group is said to be of finite type if its associated Coxeter group is of finite type. To each Artin group of finite type we can assign it the same Dynkin diagram which is assigned to the Coxeter group associated to the Artin group.*

Given a diagram  $\Gamma$  and the corresponding Artin group  $A_\Gamma$ , we prove this group is invariant (up to isomorphism) under mutation of  $\Gamma$ .

*Theorem*

*Let  $\Gamma$  be a diagram of finite type, and let  $\Gamma' = \mu_k(\Gamma)$  be the mutation of  $\Gamma$  at vertex  $k$ . Then  $A_\Gamma \cong A_{\Gamma'}$ .*