List of Todos

(1) Verify 2.5
(2) Verify that the relations (a),(b) are equivalent in 2.2 and 2.4

It's not immediately obvious which relations analogous to the original R3 relations we want to choose. Here we try out several possible relations, in each case trying to prove an analogous lemma to Lemma 4.1 in Barot and Marsh's paper.

1. Attempts

1.1. Here we test out the set of relations R3' where if $i_{a_0}, i_{a_1}, \ldots, i_{a_{d-1}}, i_{a_0}$ is a chordless cycle with all weights 1, then we add in the relation

$$s_0 s_1^{-1} s_2 s_3^{-1} \cdots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \cdots s_4^{-1} s_3 s_2^{-1} s_1 = s_1^{-1} s_2 s_3^{-1} \cdots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \cdots s_4^{-1} s_3 s_2^{-1} s_1 s_0$$

We then have

$$\begin{aligned}
&s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_0 \\
&= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0
\end{aligned}$$

...so I don't think this direction is going to work from here, simply because I don't see a way to flip all of the generators in the middle to their inverses.

Lemma 1.1. *(Analogue of 4.6) Let W_Γ be generated by $s_1, \ldots, s_n$. Then $s_1^{-1}, \ldots, s_n^{-1}$ satisfy the relations (R2') and (R3') in $W_{\Gamma^{op}}$.*

Proof. One can see that the elements satisfy (R2') in $W_{\Gamma^{op}}$ by taking the inverse of both sides of the relation in W_Γ. To see that the elements satisfy (R3') in $W_{\Gamma^{op}}$, note that for a chordless cycle in Γ with all weights equal to one, we have

$$s_0^{-1} \ldots s_{d-2}^{-1} s_{d-1} s_{d-2} \ldots s_0 = s_1^{-1} \ldots s_{d-2}^{-1} s_{d-1} s_{d-2} \ldots s_1$$

by the relation r(0,1) in (R3') in W_Γ. But then applying relations from (R2'), we have that

$$s_0^{-1} \ldots s_{d-1} s_{d-2} s_{d-1}^{-1} \ldots s_0 = s_1^{-1} \ldots s_{d-1} s_{d-2} s_{d-1}^{-1} \ldots s_1,$$

and since the cycle in chordless, we then have

$$s_0^{-1} s_{d-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_{d-1}^{-1} s_0 = s_{d-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_{d-1}^{-1}.$$

Repeating this process, we find that

$$s_0^{-1} s_{d-1} s_{d-2} \ldots s_2 s_1 s_2^{-1} \ldots s_{d-2}^{-1} s_{d-1}^{-1} s_0 = s_{d-1} s_{d-2} \ldots s_2 s_1 s_2^{-1} \ldots s_{d-2}^{-1} s_{d-1}^{-1}.$$

But this can only occur if $s_1^{-1}, \ldots, s_n^{-1}$ satisfies the relation r(0, d-1) in $W_{\Gamma^{Op}}$. Therefore, we are done. $\square$

An R3 that seems promising:

Definition 1.2. Using the notation as in Barot - Marsh, given a chordless cycle C in Γ so that all edges in C have weight 1, we define $r(a, a+1) = s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a$.
An equivlent relation is as follows
$s_a^{-1} s_{a+1}^{-1} \cdots s_{a-1}^{-1} s_{a-2} \cdots s_a s_{a+1}^{-1} \cdots s_{a-2}^{-1} s_{a-1} \cdots s_{a+1}$

Lemma 1.3. *The relation r(a,a+1) for some vertex $a \in C$ implies the relation r(b, b+1) for all $b \in C$.*

Proof. As in Barot-Marsh, it suffices to prove that the relation r(0, 1) implies the relation r(d-1, 0). So suppose W_γ satisfies the relation r(0, 1). Then we have

$$\begin{aligned}
& s_{d-1} s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 \\
&= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_1^{-1} \dots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \dots s_1 s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_0 s_1^{-1} \dots s_{d-3}^{-1} s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \dots s_1) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \dots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \dots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \dots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1} s_0 s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 s_{d-1} s_0^{-1} s_0 \\
&= s_0^{-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_0 s_{d-1}
\end{aligned}$$

as required. Note that line 3 is equal to 4 and line 7 is equal to line 8 since the cycle is chordless, meaning that s_{d-1} commutes with every element except s_0 and s_{d-2}. □

Definition 1.4. Let Γ be a weighted diagram of finite type, and suppose Γ' = $\mu_k(\Gamma)$. If $s_1, \dots, s_n$ are the generators in W_Γ, define:

$$t_i = \begin{cases} s_k s_i s_k^{-1} & \text{if there is arrow from i to k} \\ s_i & \text{otherwise} \end{cases}$$

Lemma 1.5 (Generalized Lemma 5.1)**.** *Let i and j be vertices of* Γ*.*
(a) If i = k or j = k, then $(t_i t_j)^{m'_{ij}/2} = (t_j t_i)^{m'_{ij}/2}$.
(b) If at most one of i,j is connected to k, then $(t_i t_j)^{m'_{ij}/2} = (t_j t_i)^{m'_{ij}/2}$

Proof. We begin with case (a). Suppose that i=k. As in Barot-Marsh, we have that $m_{ij} = m'_{ij}$, and the only nontrivial case is when there is an arrow from j to k. Note that m_{ij} cannot be 2 in this case since there is an arrow from j to k, so suppose $m_{ij} = 3$. Then we have

$$s_j s_k s_j = s_k s_j s_k,$$

and so

$$s_k s_k s_j s_k = s_j s_k s_j s_k.$$

Multiplying both sides on the right by s_k^{-1}, we then obtain that

$$s_k s_k s_j = s_k s_j s_k s_j s_k^{-1},$$

and so we find that

$$t_i t_j t_i = s_k s_k s_j s_k^{-1} s_k = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = t_j t_i t_j.$$

Now suppose $m_{ij} = 4$. Then

$$s_k s_j s_k s_j = s_j s_k s_j s_k$$

which means that

$$s_k s_k s_j s_k s_j = s_k s_j s_k s_j s_k.$$

This then gives us that

$$s_k s_k s_j s_k s_j s_k^{-1} = s_k s_j s_k s_j,$$

and so

$$t_i t_j t_i t_j = s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k = t_j t_i t_j t_i.$$

Finally, suppose $m_{ij} = 6$. Then we know

$$s_k s_j s_k s_j s_k s_j = s_j s_k s_j s_k s_j s_k,$$

and so

$$s_k s_k s_j s_k s_j s_k s_j = s_k s_j s_k s_j s_k s_j s_k.$$

Then

$$s_k s_k s_j s_k s_j s_k s_j m_k^{-1} = s_k s_j s_k s_j s_k s_j,$$

and so

$$t_i t_j t_i t_j t_i t_j = s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k = t_j t_i t_j t_i t_j t_i.$$

A similar argument holds when j=k. This concludes (a).

Now considering case (b), suppose i is connected to k. We once again have $m_{ij} = m'_{ij}$, and the only nontrivial case is when the arrow points from i to k. Suppose $m_{ij} = 2$. Then we have

$$s_i s_j = s_j s_i,$$

and so we have

$$s_k s_i s_j = s_k s_j s_i = s_j s_k s_i$$

since j is not connected to k by assumption. But then

$$t_i t_j = s_k s_i s_k^{-1} s_j = s_k s_i s_j s_k^{-1} = s_j s_k s_i s_k^{-1} = t_j t_i.$$

If $m_{ij} = 3$, then we have $s_j s_k = s_k s_j$, and so we have $s_k^{-1} s_j s_k = s_j$. This means that

$$s_i s_k^{-1} s_j s_k s_i = s_i s_j s_i = s_j s_i s_j,$$

and we then have that

$$t_i t_j t_i = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = s_k s_j s_i s_j s_k^{-1} = s_j s_k s_i s_k^{-1} s_j = t_j t_i t_j.$$

If $m_{ij} = 4$, we have $s_is_js_is_j = s_js_is_js_i$. Then

$$s_is_k^{-1}s_js_ks_is_j = s_is_k^{-1}s_ks_js_is_j = s_js_is_k^{-1}s_ks_js_i = s_js_is_k^{-1}s_js_ks_i.$$

But then conjugating the right and left sides by s_k, we find that

$$s_ks_is_k^{-1}s_js_ks_is_js_k^{-1} = s_ks_js_is_k^{-1}s_js_ks_is_k^{-1},$$

and thus by the commutativity of s_j and s_k, we have

$$t_it_jt_it_j = s_ks_is_k^{-1}s_js_ks_is_k^{-1}s_j = s_js_ks_is_k^{-1}s_js_ks_is_k^{-1} = t_jt_it_jt_i.$$

Finally, if $m_{ij} = 6$, we have $s_is_js_is_js_is_j = s_js_is_js_is_js_i$. Thus conjugating by s_k and using the commutativity of s_k and s_j, we find that

$$s_ks_is_js_is_js_is_k^{-1}s_j = s_js_ks_is_js_is_js_is_k^{-1}.$$

But this occurs if and only if

$$t_it_jt_it_jt_it_j = s_ks_is_k^{-1}s_js_ks_is_k^{-1}s_js_ks_is_k^{-1}s_j = s_js_ks_is_k^{-1}s_js_ks_is_k^{-1}s_js_ks_is_k^{-1} = t_jt_it_jt_it_jt_i.$$

A similar argument holds when j is connected to k, so we are done. $\square$

Lemma 1.6. *[Proposition 5.2 Analog] The elements t_i, for i a vertex of Γ, satisfy the relations $(R2)$ and $(R3)$.*

After Lemma 1.5 we have left to check the relations $(R2)$ when both i and j are connected to k and the relations $(R3)$.

Beginning with the relations $(R2)$, and following cases a-f from Corollary 2.3 in Barot and Marsh:

a) i)

$$t_it_j = s_ks_is_k^{-1}s_ks_js_k^{-1} = s_ks_is_js_k^{-1} = s_ks_js_is_k^{-1} = t_jt_i$$

ii)

$$t_it_j = s_is_j = s_js_i = t_jt_i$$

b) i)

$$\begin{aligned} t_it_jt_i &= s_ks_is_k^{-1}s_js_ks_is_k^{-1} \\ &= s_ks_is_js_ks_j^{-1}s_is_k^{-1} \\ &= s_ks_js_is_ks_is_j^{-1}s_k^{-1} \\ &= s_ks_js_ks_is_ks_j^{-1}s_k^{-1} \\ &= s_js_ks_js_is_j^{-1}s_k^{-1}s_j \\ &= s_js_ks_is_k^{-1}s_j \\ &= t_it_jt_i \end{aligned}$$

ii)

$$t_it_j = s_is_ks_js_k^{-1} = s_is_j^{-1}s_ks_j = s_j^{-1}s_ks_js_i = s_ks_js_k^{-1}s_i = t_jt_i$$

c) i)

$$t_it_j = s_ks_is_k^{-1}s_ks_js_k^{-1} = s_ks_is_js_k^{-1} = s_ks_js_is_k^{-1} = t_jt_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

d) i)

$$\begin{aligned} t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\ &= s_k s_i s_k^{-1} s_j s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\ &= s_k s_i s_k^{-1} s_k s_j s_k s_i s_k s_i^{-1} s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\ &= s_k s_j s_i s_k s_i s_k s_i^{-1} s_k^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} \\ &= e \end{aligned}$$

2. Relations for 2,2,1 triangle, talk to Aaron if you think there's an error

Remark 2.1. I am proposing to add the following relations whever we have the situation

$a \xrightarrow{2} b \xrightarrow{1} c$, or $a \xrightarrow{1} b \xrightarrow{2} c$:

Include $s_i s_{i+1}^2 = s_{i+1}^2 s_i$, and $s_i s_{i-1}^2 = s_{i-1}^2 s_i$, for all $i = 1, 2, 3$. I believe these mutate correctly, as in cases b,c,d in Barot-Marsh. Note that in the above diagrams, it is fine if we have a 2 edges from a to c or c to a.

Lemma 2.2. *(4.2 Analog) Assuming the additional relation $s_2 s_3 s_3 = s_3 s_3 s_2$ from 2.1, the two relations $s_1^{-1} s_2^{-1} s_3^{-1} s_2 s_1 s_2^{-1} s_3 s_2 = e$ and $s_2^{-1} s_3^{-1} s_1^{-1} s_3 s_2 s_3^{-1} s_1 s_3 = e$ are equivalent and they imply $s_3 s_1 s_2 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 = e$.*

Proof. We only have to show the first two imply the third. We can use the first two identities to deduce $s_2 s_3^{-1} = s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1}, s_2^{-1} s_3^{-1} = s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_3^{-1} s_1$. Indeed, we can see

$$\begin{aligned} e &= s_2 s_3 s_2 s_3^{-1} s_2^{-1} s_3^{-1} \\ &= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1})(s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_3^{-1} s_1) \\ &= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1})(s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\ &= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1})(s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\ &= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1})(s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\ &= s_2 s_3 (s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1})(s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \end{aligned}$$

Next,I claim $s_2 s_3 = s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_3^{-1} s_1^{-1}$. This should be easy to verify by manipulating the equation $s_2^{-1} s_3^{-1} s_1^{-1} s_3 s_2 s_3^{-1} s_1 s_3 = e$, if we can use the relation $s_2s_3s_3 = s_3s_3s_2$. To verify this, note:

$$
\begin{aligned}
e &= s_2^{-1}s_3^{-1}s_1^{-1}s_3s_2s_3^{-1}s_1s_3 \iff\\
e &= s_2^{-1}s_3^{-1}s_1^{-1}s_3^{-1}s_2s_3s_1s_3 \iff\\
e &= s_3^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_3s_1s_3s_2 \iff\\
e &= s_3^{-1}s_2^{-1}s_3s_1s_3s_2s_3^{-1}s_1^{-1} \iff\\
s_2s_3 &= s_3s_1s_3s_1s_2s_3^{-1}s_1^{-1} \iff\\
s_2s_3 &= s_1^{-1}s_3s_1s_3s_1s_1s_2s_3^{-1}s_1^{-1} \iff
\end{aligned}
$$

Where an equivalent form of $s_2s_3s_3 = s_3s_3s_2$. is used in going from the first line to the second in the above computation.

Plugging this in yields

$$
\begin{aligned}
e &= (s_1^{-1}s_3s_1s_3s_1s_2s_3^{-1}s_1^{-1})(s_1s_3s_1s_3^{-1}s_1^{-1}s_2s_1^{-1}s_1s_3^{-1}s_1^{-1})(s_1s_3s_1^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1)\\
&= (s_1^{-1}s_3s_1s_3s_1s_2s_1s_1^{-1}s_3^{-1}s_1^{-1})(s_1s_3s_1s_3^{-1}s_1^{-1}s_2s_1^{-1}s_1s_3^{-1}s_1^{-1})(s_1s_3s_1^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1)\\
&= (s_1^{-1}s_3s_1s_3s_1s_2s_1)(s_3^{-1}s_1^{-1}s_2s_1^{-1}s_1s_3^{-1}s_1^{-1})(s_1s_3s_1^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1)\\
&= (s_1^{-1}s_3s_1s_3s_1s_2s_1)(s_3^{-1}s_1^{-1}s_2s_1^{-1})(s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1)\\
&= (s_1^{-1}s_3s_1s_3s_1s_2s_1)(s_3^{-1}s_1^{-1}s_2s_1^{-1})(s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1)
\end{aligned}
$$

So, since $e = s_1^{-1}s_3s_1s_3s_1s_2s_1s_3^{-1}s_1^{-1}s_2s_1^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1s_1^{-1}s_3^{-1}s_1$, cancelling three terms from both sides, we get $e = s_3s_1s_2s_1s_3^{-1}s_1^{-1}s_2s_1^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1$, as claimed. □

Proposition 2.3. *For the case of the (2,1,2,1) square, we have* $s_2^{-1}s_3^{-1}s_2s_4^{-1}s_1^{-1}s_4 = s_4^{-1}s_1^{-1}s_4s_2^{-1}s_3^{-1}s_2$. *By inverting both sides of this relation, we also obtain* $s_4^{-1}s_1s_4s_2^{-1}s_3s_2 = s_3^{-1}s_3s_2s_4^{-1}s_1s_4$.

Proof. Using the cyclic $r(1,2)$ relation we have

$$
\begin{aligned}
e &= s_1^{-1}s_2^{-1}s_3^{-1}s_4^{-1}s_3s_2s_1s_2^{-1}s_3^{-1}s_4s_3s_2\\
&= s_1^{-1}s_2^{-1}s_4s_3^{-1}s_4^{-1}s_2s_1s_2^{-1}s_4s_3s_4^{-1}s_2\\
&= s_1^{-1}s_4s_2^{-1}s_3^{-1}s_2s_4^{-1}s_1s_4s_2^{-1}s_3s_2s_4^{-1} \iff\\
e &= s_4^{-1}s_1^{-1}s_4s_2^{-1}s_3^{-1}s_2s_4^{-1}s_1s_4s_2^{-1}s_3s_2 \iff\\
s_2^{-1}s_3^{-1}s_2s_4^{-1}s_1^{-1}s_4 &= s_4^{-1}s_1^{-1}s_4s_2^{-1}s_3^{-1}s_2
\end{aligned}
$$

□

Lemma 2.4. *(Analog of 4.4) For the case of the (2,1,2,1) square, assuming the relations from 2.1 we get that the analogs of* $r(1,2)^2 = e, r(3,4)^2 = e$ *are equivalent, and they imply the following relation, which is an analog of R3b:*

$$
s_2^{-1}s_3s_4^{-1}s_1s_4s_3s_2s_3^{-1}s_4^{-1}s_1s_4s_3^{-1}s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4s_3 = e.
$$

Proof. Hailee said she showed that the analogs of $r(1,2), r(3,4)$ are equivalent.

So, I will only show that these relations imply the analog of $R3b$.

$$
\begin{aligned}
&e = s_2^{-1}s_3s_4^{-1}s_1s_4s_3s_2s_3^{-1}s_4^{-1}s_1s_4s_3^{-1}s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4s_3 \iff \\
&e = (s_2^{-1}s_3s_4^{-1}s_1s_4s_3)(s_2s_3^{-1}s_4^{-1}s_1s_4s_3^{-1})(s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4s_3) \iff \\
&e = s_3(s_2^{-1}s_3s_4^{-1}s_1s_4s_3)s_3^{-1}s_3(s_2s_3^{-1}s_4^{-1}s_1s_4s_3^{-1})s_3^{-1}s_3(s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4s_3)s_3^{-1} \\
&\quad = (s_3s_2^{-1}s_3s_4^{-1}s_1s_4)(s_3s_2s_3^{-1}s_4^{-1}s_1s_4)(s_3s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4)
\end{aligned}
$$

Now observe that

$$s_2^{-1}s_3s_2s_3^{-1}s_2^{-1} = s_3s_2s_2^{-1}s_2^{-1} = s_3s_2^{-1}s_3,$$

by 2.1 in the second equality. Also,

$$s_2s_3^{-1}s_2^{-1}s_3s_2 = s_2^{-1}s_3^{-1}s_2s_3s_2 = s_3s_2s_3^{-1}$$

again using 2.1 in the first equality. And finally,

$$s_2^{-1}s_3^{-1}s_2^{-1}s_3^{-1}s_2 = s_3^{-1}s_2^{-1}s_3^{-1}.$$

So, substituting these three equations in the above, we have

$$
\begin{aligned}
e &= s_3(s_2^{-1}s_3s_4^{-1}s_1s_4s_3)s_3^{-1}s_3(s_2s_3^{-1}s_4^{-1}s_1s_4s_3^{-1})s_3^{-1}s_3(s_2^{-1}s_3^{-1}s_4^{-1}s_1^{-1}s_4s_3)s_3^{-1} \\
&= ((s_3s_2^{-1}s_3)s_4^{-1}s_1s_4)((s_3s_2s_3^{-1})s_4^{-1}s_1s_4)((s_3s_2^{-1}s_3^{-1})s_4^{-1}s_1^{-1}s_4) \\
&= ((s_2^{-1}s_3s_2s_3s_2^{-1})s_4^{-1}s_1s_4)((s_2s_3^{-1}s_2^{-1}s_3s_2)s_4^{-1}s_1s_4)((s_2^{-1}s_3^{-1}s_2^{-1}s_3s_2)s_4^{-1}s_1^{-1}s_4)
\end{aligned}
$$

Now, using 2.3 three times, we have

$$
\begin{aligned}
&((s_2^{-1}s_3s_2s_3s_2^{-1})s_4^{-1}s_1s_4)((s_2s_3^{-1}s_2^{-1}s_3s_2)s_4^{-1}s_1s_4)((s_2^{-1}s_3^{-1}s_2^{-1}s_3s_2)s_4^{-1}s_1^{-1}s_4) \\
&= (s_2^{-1}s_3(s_4^{-1}s_1s_4s_2s_3s_2^{-1}))(s_2^{-1}s_3(s_4^{-1}s_1s_4s_2^{-1}s_3s_2))(s_2^{-1}s_3^{-1}(s_4^{-1}s_1^{-1}s_4s_2^{-1}s_3^{-1}s_2)) \quad = (s_2^{-1}s_3s_4^{-1}s_1s_2s_4s_3s_2^{-1} \\
&= s_2^{-1}s_3s_4^{-1}s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1}s_4s_3^{-1}s_2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&e = s_2^{-1}s_3s_4^{-1}s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1}s_4s_3^{-1}s_2 \iff \\
&e = s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1}
\end{aligned}
$$

Which if of course one of our basic R2 relations. ☐

verify this

Theorem 2.5. *The relations described in 2.1 are permuted (and hence preserved) after mutation at any vertex.*

Proof. This is what additionally needs to be checked in 1.6 by adding 2.1 in the cases corresponding to cycles with edges of weight 2 in the paper. I.e. it has to be checked in cases (b,c,d,e,f,g,h). It seems like a lot of casework, but there's a fair amount of symmetry to make it a bit easier, and I checked this in a few cases, and it seemed to work out. ☐