

LIST OF TODOS

■ verify this	11
-------------------------	----

(1) Verify 2.5

(2) Verify that the relations (a),(b) are equivalent in 2.2 and 2.4

It's not immediately obvious which relations analogous to the original R3 relations we want to choose. Here we try out several possible relations, in each case trying to prove an analogous lemma to Lemma 4.1 in Barot and Marsh's paper.

1. ATTEMPTS

1.1. Here we test out the set of relations R3' where if $i_{a_0}, i_{a_1}, \dots, i_{a_{d-1}}, i_{a_0}$ is a chordless cycle with all weights 1, then we add in the relation

$$s_0 s_1^{-1} s_2 s_3^{-1} \dots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \dots s_4^{-1} s_3 s_2^{-1} s_1 = s_1^{-1} s_2 s_3^{-1} \dots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \dots s_4^{-1} s_3 s_2^{-1} s_1 s_0$$

We then have

$$\begin{aligned} & s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \dots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \dots s_4 s_3^{-1} s_2 s_1^{-1} s_0 \\ &= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \dots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \dots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0 \\ &= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1 s_2^{-1} s_3 \dots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \dots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0 \end{aligned}$$

...so I don't think this direction is going to work from here, simply because I don't see a way to flip all of the generators in the middle to their inverses.

An R3 that seems promising:

Definition 1.1. Using the notation as in Barot - Marsh, given a chordless cycle C in Γ so that all edges in C have weight 1, we define $r(a, a+1) = s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a$.

An equivalent relation is as follows

$$s_a^{-1} s_{a+1}^{-1} \dots s_{a-1}^{-1} s_{a-2} \dots s_a s_{a+1}^{-1} \dots s_{a-2}^{-1} s_{a-1} \dots s_{a+1}$$

Lemma 1.2. (Analogue of 4.1) The relation $r(a, a+1)$ for some vertex $a \in C$ implies the relation $r(b, b+1)$ for all $b \in C$.

Proof. As in Barot-Marsh, it suffices to prove that the relation $r(0, 1)$ implies the relation $r(d-1, 0)$. So suppose W_γ satisfies the relation $r(0, 1)$. Then we

have

$$\begin{aligned}
& s_{d-1}s_0^{-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_0 \\
&= s_0^{-1}s_0s_{d-1}s_0^{-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_{d-1}^{-1}s_{d-1}s_0 \\
&= s_0^{-1}s_{d-1}^{-1}s_0s_{d-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_{d-1}^{-1}s_{d-1}s_0 \\
&= s_0^{-1}s_{d-1}^{-1}s_0s_1^{-1}\dots s_{d-3}^{-1}s_{d-1}s_{d-2}s_{d-1}^{-1}s_{d-3}\dots s_1s_{d-1}s_0 \\
&= s_0^{-1}s_{d-1}^{-1}(s_0s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-1}s_{d-2}s_{d-3}\dots s_1)s_{d-1}s_0 \\
&= s_0^{-1}s_{d-1}^{-1}(s_1^{-1}\dots s_{d-2}^{-1}s_{d-1}s_{d-2}s_{d-3}\dots s_0)s_{d-1}s_0 \\
&= s_0^{-1}s_{d-1}^{-1}(s_1^{-1}\dots s_{d-3}^{-1}s_{d-1}s_{d-2}s_{d-1}^{-1}s_{d-3}\dots s_0)s_{d-1}s_0 \\
&= s_0^{-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_{d-1}^{-1}s_0s_{d-1}s_0 \\
&= s_0^{-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_0s_{d-1}s_0^{-1}s_0 \\
&= s_0^{-1}s_1^{-1}\dots s_{d-3}^{-1}s_{d-2}s_{d-3}\dots s_1s_0s_{d-1}
\end{aligned}$$

as required. Note that line 3 is equal to 4 and line 7 is equal to line 8 since the cycle is chordless, meaning that s_{d-1} commutes with every element except s_0 and s_{d-2} . \square

Definition 1.3. (Possible relations for the weight 2 triangle) Given the cycle in Figure 8 of Barot-Marsh, we define the following relations

- (a) $s_1s_2^{-1}s_3s_2 = s_2^{-1}s_3s_2s_1$
- (b) $s_2s_3^{-1}s_1s_3 = s_3^{-1}s_1s_3s_2$
- (c) $s_3^{-1}s_1^{-1}s_2s_1s_3s_1^{-1}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_1 = e$

Lemma 1.4. (Analogue of 4.2) In the above definition, if the generators satisfy the relations (R2'), then the relations (a) and (b) are equivalent, and the relations (a) and (b) imply the relation (c).

Proof. I haven't worked out the equivalence of (a) and (b) yet, so I'll leave it for someone else.

In showing that (a) and (b) imply (c), I will underline the terms being manipulated in each line for emphasis.

$$\begin{aligned}
& s_1 s_3 s_1 (\underline{s_3^{-1} s_1^{-1} s_2 s_1 s_3 s_1^{-1} s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_1}) \underline{s_1^{-1} s_3^{-1} s_1^{-1}} \\
&= s_1 \underline{s_3 s_1 s_3^{-1} s_1^{-1} s_2 s_1 s_3 s_1^{-1} s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1^{-1}} \\
&= \underline{s_1 s_1^{-1} s_3^{-1} s_1 s_3 s_2 s_1 s_3 s_1^{-1} s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_2 \underline{s_3^{-1} s_1 s_3 s_1 s_3 s_1^{-1} s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_2 s_1 s_3 \underline{s_1 s_3^{-1} s_3 s_1^{-1} s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_2 \underline{s_1 s_3 s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_2 s_3 \underline{s_3^{-1} s_1 s_3 s_2 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= \underline{s_2 s_3 s_2 s_3^{-1} s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_3 s_2 s_3 \underline{s_3^{-1} s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_3 s_2 s_1 \underline{s_1^{-1} s_3^{-1} s_1 s_3 s_2^{-1} s_3^{-1} s_1^{-1}} \\
&= s_3 \underline{s_2 s_1^{-1} s_3^{-1} s_1 s_3 s_3^{-1} s_1^{-1}} \\
&= \underline{s_3 s_3^{-1} s_1 s_1^{-1}} \\
&= e
\end{aligned}$$

□

Lemma 1.5. (*Analogue of 4.6*) Let W_Γ be generated by s_1, \dots, s_n . Then $s_1^{-1}, \dots, s_n^{-1}$ satisfy the relations (R2') and (R3') in $W_{\Gamma^{op}}$.

Proof. One can see that the elements satisfy (R2') in $W_{\Gamma^{op}}$ by taking the inverse of both sides of the relation in W_Γ . To see that the elements satisfy (R3') in $W_{\Gamma^{op}}$, note that for a chordless cycle in Γ with all weights equal to one, we have

$$s_0^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} \dots s_0 = s_1^{-1} \dots s_{d-2}^{-1} s_{d-1} s_{d-2} \dots s_1$$

by the relation $r(0,1)$ in (R3') in W_Γ . But then applying relations from (R2'), we have that

$$s_0^{-1} \dots s_{d-1} s_{d-2} s_{d-1}^{-1} \dots s_0 = s_1^{-1} \dots s_{d-1} s_{d-2} s_{d-1}^{-1} \dots s_1,$$

and since the cycle is chordless, we then have

$$s_0^{-1} s_{d-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_{d-1}^{-1} s_0 = s_{d-1} s_1^{-1} \dots s_{d-3}^{-1} s_{d-2} s_{d-3} \dots s_1 s_{d-1}^{-1}.$$

Repeating this process, we find that

$$s_0^{-1} s_{d-1} s_{d-2} \dots s_2 s_1 s_2^{-1} \dots s_{d-2}^{-1} s_{d-1}^{-1} s_0 = s_{d-1} s_{d-2} \dots s_2 s_1 s_2^{-1} \dots s_{d-2}^{-1} s_{d-1}^{-1}.$$

But this can only occur if $s_1^{-1}, \dots, s_n^{-1}$ satisfies the relation $r(0, d-1)$ in $W_{\Gamma^{op}}$. Therefore, we are done. □

Definition 1.6. Let Γ be a weighted diagram of finite type, and suppose $\Gamma' = \mu_k(\Gamma)$. If s_1, \dots, s_n are the generators in W_Γ , define:

$$t_i = \begin{cases} s_k s_i s_k^{-1} & \text{if there is arrow from } i \text{ to } k \\ s_i & \text{otherwise} \end{cases}$$

Lemma 1.7 (Generalized Lemma 5.1). *Let i and j be vertices of Γ .*

(a) *If $i = k$ or $j = k$, then $(t_i t_j)^{m'_{ij}/2} = (t_j t_i)^{m'_{ij}/2}$.*

(b) *If at most one of i, j is connected to k , then $(t_i t_j)^{m'_{ij}/2} = (t_j t_i)^{m'_{ij}/2}$*

Proof. We begin with case (a). Suppose that $i=k$. As in Barot-Marsh, we have that $m_{ij} = m'_{ij}$, and the only nontrivial case is when there is an arrow from j to k . Note that m_{ij} cannot be 2 in this case since there is an arrow from j to k , so suppose $m_{ij} = 3$. Then we have

$$s_j s_k s_j = s_k s_j s_k,$$

and so

$$s_k s_k s_j s_k = s_j s_k s_j s_k.$$

Multiplying both sides on the right by s_k^{-1} , we then obtain that

$$s_k s_k s_j = s_k s_j s_k s_k^{-1},$$

and so we find that

$$t_i t_j t_i = s_k s_k s_j s_k^{-1} s_k = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = t_j t_i t_j.$$

Now suppose $m_{ij} = 4$. Then

$$s_k s_j s_k s_j = s_j s_k s_j s_k$$

which means that

$$s_k s_k s_j s_k s_j = s_k s_j s_k s_j s_k.$$

This then gives us that

$$s_k s_k s_j s_k s_j s_k^{-1} = s_k s_j s_k s_j,$$

and so

$$t_i t_j t_i t_j = s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k = t_j t_i t_j t_i.$$

Finally, suppose $m_{ij} = 6$. Then we know

$$s_k s_j s_k s_j s_k s_j = s_j s_k s_j s_k s_j s_k,$$

and so

$$s_k s_k s_j s_k s_j s_k s_j = s_k s_j s_k s_j s_k s_j s_k.$$

Then

$$s_k s_k s_j s_k s_j s_k s_j m_k^{-1} = s_k s_j s_k s_j s_k s_j,$$

and so

$$t_i t_j t_i t_j t_i t_j = s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} = s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k s_k s_j s_k^{-1} s_k = t_j t_i t_j t_i t_j t_i.$$

A similar argument holds when $j=k$. This concludes (a).

Now considering case (b), suppose i is connected to k . We once again have $m_{ij} = m'_{ij}$, and the only nontrivial case is when the arrow points from i to k . Suppose $m_{ij} = 2$. Then we have

$$s_i s_j = s_j s_i,$$

and so we have

$$s_k s_i s_j = s_k s_j s_i = s_j s_k s_i$$

since j is not connected to k by assumption. But then

$$t_i t_j = s_k s_i s_k^{-1} s_j = s_k s_i s_j s_k^{-1} = s_j s_k s_i s_k^{-1} = t_j t_i.$$

If $m_{ij} = 3$, then we have $s_j s_k = s_k s_j$, and so we have $s_k^{-1} s_j s_k = s_j$. This means that

$$s_i s_k^{-1} s_j s_k s_i = s_i s_j s_i = s_j s_i s_j,$$

and we then have that

$$t_i t_j t_i = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = s_k s_j s_i s_j s_k^{-1} = s_j s_k s_i s_k^{-1} s_j = t_j t_i t_j.$$

If $m_{ij} = 4$, we have $s_i s_j s_i s_j = s_j s_i s_j s_i$. Then

$$s_i s_k^{-1} s_j s_k s_i s_j = s_i s_k^{-1} s_k s_j s_i s_j = s_j s_i s_k^{-1} s_k s_j s_i = s_j s_i s_k^{-1} s_j s_k s_i.$$

But then conjugating the right and left sides by s_k , we find that

$$s_k s_i s_k^{-1} s_j s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} s_j s_k s_i s_k^{-1},$$

and thus by the commutativity of s_j and s_k , we have

$$t_i t_j t_i t_j = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j = s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = t_j t_i t_j t_i.$$

Finally, if $m_{ij} = 6$, we have $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$. Thus conjugating by s_k and using the commutativity of s_k and s_j , we find that

$$s_k s_i s_j s_i s_j s_i s_k^{-1} s_j = s_j s_k s_i s_j s_i s_j s_i s_k^{-1}.$$

But this occurs if and only if

$$t_i t_j t_i t_j t_i t_j = s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j = s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} = t_j t_i t_j t_i t_j t_i.$$

A similar argument holds when j is connected to k , so we are done. \square

Lemma 1.8. *[Proposition 5.2 Analog] The elements t_i , for i a vertex of Γ , satisfy the relations (R2) and (R3).*

After Lemma 1.7 we have left to check the relations (R2) when both i and j are connected to k and the relations (R3).

Beginning with the relations (R2), and following cases a-f from Corollary 2.3 in Barot and Marsh:

a) i)

$$t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

b) i)

$$\begin{aligned}
t_i t_j t_i &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} \\
&= s_k s_i s_j s_k s_j^{-1} s_i s_k^{-1} \\
&= s_k s_j s_i s_k s_i s_j^{-1} s_k^{-1} \\
&= s_k s_j s_k s_i s_k s_j^{-1} s_k^{-1} \\
&= s_j s_k s_j s_i s_j^{-1} s_k^{-1} s_j \\
&= s_j s_k s_i s_k^{-1} s_j \\
&= t_i t_j t_i
\end{aligned}$$

$$\begin{aligned}
t_j t_k^{-1} t_i t_k &= s_j s_k^{-1} s_k s_i s_k^{-1} s_k \\
&= s_j s_i \\
&= s_i s_j \\
&= t_k^{-1} t_i t_k t_j
\end{aligned}$$

ii)

$$t_i t_j = s_i s_k s_j s_k^{-1} = s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i = s_k s_j s_k^{-1} s_i = t_j t_i$$

c) i)

$$t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

d) i)

$$\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_j s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_k s_j s_k s_i s_k s_i^{-1} s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_j s_i s_k s_i s_k s_i^{-1} s_k^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} \\
&= e
\end{aligned}$$

We also have

$$t_j t_k^{-1} t_i t_k = s_j s_k^{-1} s_k s_i s_k^{-1} s_k = s_i s_j = t_k^{-1} t_i t_k t_j$$

ii)

$$\begin{aligned}
t_i t_j &= s_i s_k s_j s_k^{-1} \\
&= s_i s_j^{-1} s_k s_j \\
&= s_j^{-1} s_k s_j s_i \\
&= s_k s_j s_k^{-1} s_i \\
&= t_j t_i
\end{aligned}$$

e)

f) i)

$$\begin{aligned}
s_k^{-1} t_i t_j t_i t_j^{-1} t_i^{-1} t_j^{-1} &= s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k \\
&= e
\end{aligned}$$

$$\begin{aligned}
t_i t_j^{-1} t_k t_j t_i^{-1} t_j^{-1} s_k^{-1} t_j &= s_k s_i s_k^{-1} s_j^{-1} s_k s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_k^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_k^{-1} \\
&= s_k s_j s_k s_j^{-1} s_i s_i^{-1} s_j s_k^{-1} s_j^{-1} s_k \\
&= e
\end{aligned}$$

ii) This follows from part (i) by symmetry

Next we check the (R3) relations, following the cases from Figure 6.

2. RELATIONS FOR 2,2,1 TRIANGLE, TALK TO AARON IF YOU THINK THERE'S AN ERROR

Remark 2.1. I am proposing to add the following relations whenever we have the situation

$$a \xrightarrow{2} b \xrightarrow{1} c, \text{ or } a \xrightarrow{1} b \xrightarrow{2} c :$$

Include $s_i s_{i+1}^2 = s_{i+1}^2 s_i$, and $s_i s_{i-1}^2 = s_{i-1}^2 s_i$, for all $i = 1, 2, 3$. I believe these mutate correctly, as in cases b,c,d in Barot-Marsh. Note that in the above diagrams, it is fine if we have a 2 edges from a to c or c to a.

Lemma 2.2. (4.2 Analog) Assuming the additional relation $s_2 s_3 s_3 = s_3 s_3 s_2$ from 2.1, the two relations $s_1^{-1} s_2^{-1} s_3^{-1} s_2 s_1 s_2^{-1} s_3 s_2 = e$ and $s_2^{-1} s_3^{-1} s_1^{-1} s_3 s_2 s_3^{-1} s_1 s_3 = e$ are equivalent and they imply $s_3 s_1 s_2 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 = e$.

Proof. We only have to show the first two imply the third. We can use the first two identities to deduce $s_2 s_3^{-1} = s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1}$, $s_2^{-1} s_3^{-1} = s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_3^{-1} s_1$. Indeed, we can see

$$\begin{aligned}
e &= s_2 s_3 s_2 s_3^{-1} s_2^{-1} s_3^{-1} \\
&= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1}) (s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_3^{-1} s_1) \\
&= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1}) (s_3^{-1} s_1^{-1} s_3 s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= s_2 s_3 (s_3^{-1} s_1 s_3 s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= s_2 s_3 (s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1)
\end{aligned}$$

Next, I claim $s_2 s_3 = s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_3^{-1} s_1^{-1}$. This should be easy to verify by manipulating the equation $s_2^{-1} s_3^{-1} s_1^{-1} s_3 s_2 s_3^{-1} s_1 s_3 = e$, if we can use the relation $s_2 s_3 s_3 = s_3 s_3 s_2$. To verify this, note:

$$\begin{aligned}
e &= s_2^{-1} s_3^{-1} s_1^{-1} s_3 s_2 s_3^{-1} s_1 s_3 \iff \\
e &= s_2^{-1} s_3^{-1} s_1^{-1} s_3^{-1} s_2 s_3 s_1 s_3 \iff \\
e &= s_3^{-1} s_1^{-1} s_3^{-1} s_2^{-1} s_3 s_1 s_3 s_2 \iff \\
e &= s_3^{-1} s_2^{-1} s_3 s_1 s_3 s_2 s_3^{-1} s_1^{-1} \iff \\
s_2 s_3 &= s_3 s_1 s_3 s_1 s_2 s_3^{-1} s_1^{-1} \iff \\
s_2 s_3 &= s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_3^{-1} s_1^{-1} \iff
\end{aligned}$$

Where an equivalent form of $s_2 s_3 s_3 = s_3 s_3 s_2$. is used in going from the first line to the second in the above computation.

Plugging this in yields

$$\begin{aligned}
e &= (s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= (s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_1 s_1^{-1} s_3^{-1} s_1^{-1}) (s_1 s_3 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= (s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_1) (s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_1 s_3^{-1} s_1^{-1}) (s_1 s_3 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= (s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_1) (s_3^{-1} s_1^{-1} s_2 s_1^{-1}) (s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1) \\
&= (s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_1) (s_3^{-1} s_1^{-1} s_2 s_1^{-1}) (s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1)
\end{aligned}$$

So, since $e = s_1^{-1} s_3 s_1 s_3 s_1 s_2 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1 s_1^{-1} s_3^{-1} s_1$, cancelling three terms from both sides, we get $e = s_3 s_1 s_2 s_1 s_3^{-1} s_1^{-1} s_2 s_1^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1$, as claimed. \square

Proposition 2.3. *For the case of the $(2,1,2,1)$ square, we have $s_2^{-1} s_3^{-1} s_2 s_4^{-1} s_1^{-1} s_4 = s_4^{-1} s_1^{-1} s_4 s_2^{-1} s_3^{-1} s_2$. By inverting both sides of this relation, we also obtain $s_4^{-1} s_1 s_4 s_2^{-1} s_3 s_2 = s_3^{-1} s_3 s_2 s_4^{-1} s_1 s_4$.*

Proof. Using the cyclic $r(1, 2)$ relation we have

$$\begin{aligned}
e &= s_1^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 \\
&= s_1^{-1} s_2^{-1} s_4 s_3^{-1} s_4^{-1} s_2 s_1 s_2^{-1} s_4 s_3 s_4^{-1} s_2 \\
&= s_1^{-1} s_4 s_2^{-1} s_3^{-1} s_2 s_4^{-1} s_1 s_4 s_2^{-1} s_3 s_2 s_4^{-1} \iff \\
e &= s_4^{-1} s_1^{-1} s_4 s_2^{-1} s_3^{-1} s_2 s_4^{-1} s_1 s_4 s_2^{-1} s_3 s_2 \iff \\
s_2^{-1} s_3^{-1} s_2 s_4^{-1} s_1^{-1} s_4 &= s_4^{-1} s_1^{-1} s_4 s_2^{-1} s_3^{-1} s_2
\end{aligned}$$

□

Lemma 2.4. (Analog of 4.4) For the case of the $(2, 1, 2, 1)$ square, assuming the relations from 2.1 we get that the analogs of $r(1, 2)^2 = e, r(3, 4)^2 = e$ are equivalent, and they imply the following relation, which is an analog of R3b:

$$s_2^{-1} s_3 s_4^{-1} s_1 s_4 s_3 s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4 s_3 = e.$$

Proof. Hailee said she showed that the analogs of $r(1, 2), r(3, 4)$ are equivalent.

So, I will only show that these relations imply the analog of R3b.

$$\begin{aligned}
e &= s_2^{-1} s_3 s_4^{-1} s_1 s_4 s_3 s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4 s_3 \iff \\
e &= (s_2^{-1} s_3 s_4^{-1} s_1 s_4 s_3)(s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3^{-1})(s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4 s_3) \iff \\
e &= s_3(s_2^{-1} s_3 s_4^{-1} s_1 s_4 s_3)s_3^{-1} s_3(s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3^{-1})s_3^{-1} s_3(s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4 s_3)s_3^{-1} \\
&= (s_3 s_2^{-1} s_3 s_4^{-1} s_1 s_4)(s_3 s_2 s_3^{-1} s_4^{-1} s_1 s_4)(s_3 s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4)
\end{aligned}$$

Now observe that

$$s_2^{-1} s_3 s_2 s_3^{-1} s_2^{-1} = s_3 s_2 s_2^{-1} s_2^{-1} = s_3 s_2^{-1} s_3,$$

by 2.1 in the second equality. Also,

$$s_2 s_3^{-1} s_2^{-1} s_3 s_2 = s_2^{-1} s_3^{-1} s_2 s_3 s_2 = s_3 s_2 s_3^{-1}$$

again using 2.1 in the first equality. And finally,

$$s_2^{-1} s_3^{-1} s_2^{-1} s_3^{-1} s_2 = s_3^{-1} s_2^{-1} s_3^{-1}.$$

So, substituting these three equations in the above, we have

$$\begin{aligned}
e &= s_3(s_2^{-1} s_3 s_4^{-1} s_1 s_4 s_3)s_3^{-1} s_3(s_2 s_3^{-1} s_4^{-1} s_1 s_4 s_3^{-1})s_3^{-1} s_3(s_2^{-1} s_3^{-1} s_4^{-1} s_1^{-1} s_4 s_3)s_3^{-1} \\
&= ((s_3 s_2^{-1} s_3)s_4^{-1} s_1 s_4)((s_3 s_2 s_3^{-1})s_4^{-1} s_1 s_4)((s_3 s_2^{-1} s_3^{-1})s_4^{-1} s_1^{-1} s_4) \\
&= ((s_2^{-1} s_3 s_2 s_3 s_2^{-1})s_4^{-1} s_1 s_4)((s_2 s_3^{-1} s_2^{-1} s_3 s_2)s_4^{-1} s_1 s_4)((s_2^{-1} s_3^{-1} s_2^{-1} s_3 s_2)s_4^{-1} s_1^{-1} s_4)
\end{aligned}$$

Now, using 2.3 three times, we have

$$\begin{aligned}
&((s_2^{-1} s_3 s_2 s_3 s_2^{-1})s_4^{-1} s_1 s_4)((s_2 s_3^{-1} s_2^{-1} s_3 s_2)s_4^{-1} s_1 s_4)((s_2^{-1} s_3^{-1} s_2^{-1} s_3 s_2)s_4^{-1} s_1^{-1} s_4) \\
&= (s_2^{-1} s_3(s_4^{-1} s_1 s_4 s_2 s_3 s_2^{-1}))(s_2^{-1} s_3(s_4^{-1} s_1 s_4 s_2^{-1} s_3 s_2))(s_2^{-1} s_3^{-1}(s_4^{-1} s_1^{-1} s_4 s_2^{-1} s_3^{-1} s_2)) = (s_2^{-1} s_3 s_4^{-1} s_1 s_2 s_4 s_3 s_2^{-1}) \\
&= s_2^{-1} s_3 s_4^{-1} s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} s_4 s_3^{-1} s_2
\end{aligned}$$

Therefore,

$$\begin{aligned} e &= s_2^{-1} s_3 s_4^{-1} s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} s_4 s_3^{-1} s_2 \iff \\ e &= s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} \end{aligned}$$

Which is of course one of our basic R2 relations. \square

Theorem 2.5. The relations described in 2.1 are permuted (and hence preserved) after mutation at any vertex. verify this

Proof. This is what additionally needs to be checked in 1.8 by adding 2.1 in the cases corresponding to cycles with edges of weight 2 in the paper. I.e. it has to be checked in cases (b,c,d,e,f,g,h). It seems like a lot of casework, but there's a fair amount of symmetry to make it a bit easier, and I checked this in a few cases, and it seemed to work out. \square