List of Todos

It's not immediately obvious which relations analogous to the original R3 relations we want to choose. Here we try out several possible relations, in each case trying to prove an analogous lemma to Lemma 4.1 in Barot and Marsh's paper.

1. Attempts

1.1. Here we test out the set of relations R3' where if $i_{a_0}, i_{a_1}, \ldots, i_{a_{d-1}}, i_{a_0}$ is a chordless cycle with all weights 1, then we add in the relation

$$s_0 s_1^{-1} s_2 s_3^{-1} \cdots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \cdots s_4^{-1} s_3 s_2^{-1} s_1 = s_1^{-1} s_2 s_3^{-1} \cdots s_{d-2}^{(-1)^{d-2}} s_{d-1} s_{d-2}^{(-1)^{d-1}} \cdots s_4^{-1} s_3 s_2^{-1} s_1 s_0$$

We then have

$$\begin{aligned}
&s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_0 \\
&= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1 s_2^{-1} s_3 \cdots s_{d-3}^{(-1)^{d-2}} s_{d-2} s_{d-3}^{(-1)^{d-3}} \cdots s_4 s_3^{-1} s_2 s_1^{-1} s_{d-1} s_{d-1}^{-1} s_0
\end{aligned}$$

...so I don't think this direction is going to work from here, simply because I don't see a way to flip all of the generators in the middle to their inverses.

An R3 that seems promising:

Definition 1.1. Using the notation as in Barot - Marsh, given a chordless cycle C in γ so that all edges in C have weight 1, we define $r(a, a+1) = s_a s_{a+1}^{-1} s_{a+2}^{-1} \ldots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \ldots s_{a+1} = s_{a+1}^{-1} \ldots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \ldots s_{a+1} s_a$.

Lemma 1.2. *The relation r(a,a+1) for some vertex $a \in C$ implies the relation r(b, b+1) for all $b \in C$.*

Proof. As in Barot-Marsh, it suffices to prove that the relation r(0, 1) implies the relation r(d-1, 0). So suppose W_γ satisfies the relation r(0, 1). Then we have

$$\begin{aligned}
&s_{d-1} s_0^{-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_0 \\
&= s_0^{-1} s_0 s_{d-1} s_0^{-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_{d-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_{d-1}^{-1} s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} s_0 s_1^{-1} \ldots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \ldots s_1 s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_0 s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \ldots s_1) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \ldots s_{d-2}^{-1} s_{d-1} s_{d-2} s_{d-3} \ldots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_{d-1}^{-1} (s_1^{-1} \ldots s_{d-3}^{-1} s_{d-1} s_{d-2} s_{d-1}^{-1} s_{d-3} \ldots s_0) s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_{d-1}^{-1} s_0 s_{d-1} s_0 \\
&= s_0^{-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_0 s_{d-1} s_0^{-1} s_0 \\
&= s_0^{-1} s_1^{-1} \ldots s_{d-3}^{-1} s_{d-2} s_{d-3} \ldots s_1 s_0 s_{d-1}
\end{aligned}$$

as required. Note that line 3 is equal to 4 and line 7 is equal to line 8 since the cycle is chordless, meaning that s_{d-1} commutes with every element except s_0 and s_{d-2}. $\square$