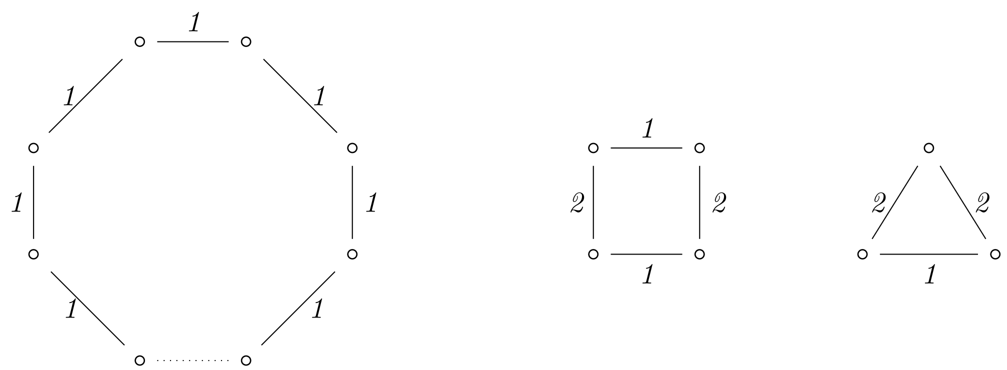
List of Todos

1. Diagrams of Finite Type

In this section, we shall review the structure of diagrams of finite type, and how their cycles are effected by mutation. This section is simply a recap of [BM13, Section 2].

Definition 1.1. A *chordless cycle* of an unoriented graph G is a connected subgraph $H \subset G$ such that the number of vertices in H is equal to the number of edges in H, and the edges in H form a single cycle.

Proposition 1.2. *Let Γ be a diagram of finite type. Then, a chordless cycle in the unoriented graph of Γ is cyclically oriented in Γ. Furthermore, the unoriented graph underlying the cycle must either be a cycle such that all edges have weight 1, a triangle with two edges of weight 2 and one of weight 1, or a square with two opposite edges of weight 2 and two opposite edges of weight 1, as pictured below.*



Proof. See [BM13, Proposition 2.1]. ☐

Lemma 1.3. *Let Γ be a diagram of finite type with $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. Below, we list induced subdiagrams in Γ on the left and the resulting induced subdiagrams in Γ' with chordless cycles C' on the right, after mutation at k. We draw the diagrams so that C' always has a clockwise cycle. Furthermore, in case (i), we assume C' has at least three vertices, while in case (j), we assume C' has at least four vertices. Every chordless cycle in Γ' is of one of these types.*

(a)

(b)

(c)

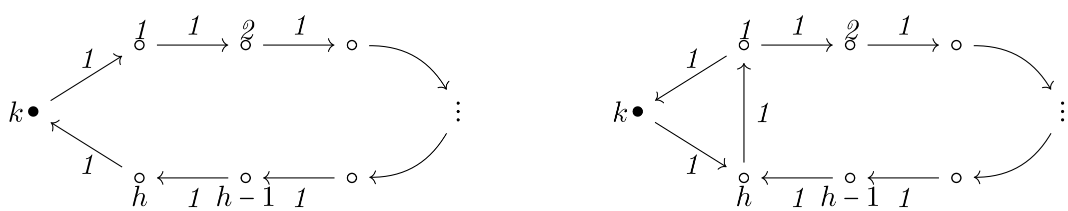

(d)

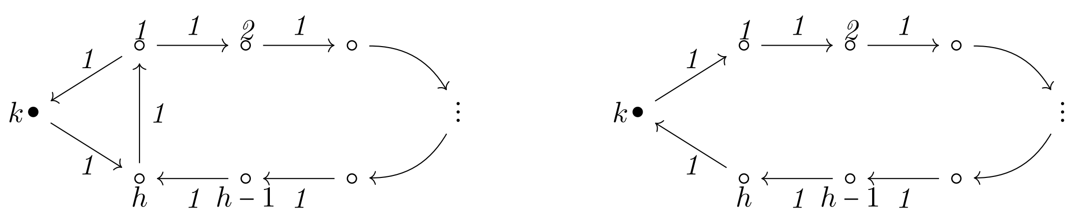

(e)


(f)


(g)


(h)


(i)


(j)


(k) C is an oriented cycle in Γ not connected to k and C' is the corresponding cycle in Γ'.

(l) *C is an oriented cycle in Γ with exactly one vertex in C connected to k by an edge of either weight 1 or 2. Then, C' is the corresponding cycle in Γ'.*

Proof. See [BM13, Lemma 2.5]. ☐

2. The Group of a Diagram in an Artin Group

Definition 2.1. For Γ a diagram of finite type, we define the associated Artin Group as follows. The associated artin group W_Γ is generated by s_i, where there is one s_i for each vertex i in Γ. These generators are subject to the relations

(R2') For all $i \neq j$, we add the relations

$$\begin{cases} s_i s_j = s_j s_i, & \text{if there is no edge between } i \text{ and } j \\ s_i s_j s_i = s_j s_i s_j & \text{if there is an edge of weight 1 between } i \text{ and } j. \\ s_i s_j s_i s_j = s_j s_i s_j s_i & \text{if there is an edge of weight 2 between } i \text{ and } j. \\ s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i & \text{if there is an edge of weight 3 between } i \text{ and } j. \end{cases}$$

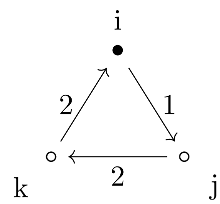
(R3')(a) For every chordless cycle of the form

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1} \longrightarrow i_0$$

such that all edges have weight 1, for all i, with $0 \leq a \leq d-1$, we include the relation

$$s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a.$$

Where subscripts are taken $\pmod{d}$.

(R3')(a) For every chordless cycle of the form  we include the three relations

(1) $s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i$
(2) $s_j s_k^{-1} s_i s_k = s_k^{-1} s_i s_k s_j$
(3) $s_k^{-1} s_i^{-1} s_j s_i s_k s_i^{-1} s_j s_i s_k^{-1} s_i^{-1} s_j^{-1} s_i = e$.

Remark 2.2. Note that if Γ is the graph associated to a dynikin diagram, then W_Γ as we have defined it is precisely the corresponding Artin group corresponding to that dynkin diagram. This is the case since in this case we have no cycles in Γ, and so we only have relation of the form $(R2')$, which define the Artin Group.

Lemma 2.3. *[Proposition 5.2 Analog] The elements t_i, for i a vertex of Γ, satisfy the relations $(R2)$ and $(R3)$.*

After Lemma **??** we have left to check the relations $(R2)$ when both i and j are connected to k and the relations $(R3)$. Beginning with the relations $(R2)$, and following cases a-f from Corollary 2.3 in Barot and Marsh:

a) i)

$$t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

b) i)

$$\begin{aligned}
t_i t_j t_i &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} \\
&= s_k s_i s_j s_k s_j^{-1} s_i s_k^{-1} \\
&= s_k s_j s_i s_k s_i s_j^{-1} s_k^{-1} \\
&= s_k s_j s_k s_i s_k s_j^{-1} s_k^{-1} \\
&= s_j s_k s_j s_i s_j^{-1} s_k^{-1} s_j \\
&= s_j s_k s_i s_k^{-1} s_j \\
&= t_i t_j t_i
\end{aligned}$$

$$\begin{aligned}
t_j t_k^{-1} t_i t_k &= s_j s_k^{-1} s_k s_i s_k^{-1} s_k \\
&= s_j s_i \\
&= s_i s_j \\
&= t_k^{-1} t_i t_k t_j
\end{aligned}$$

ii)

$$t_i t_j = s_i s_k s_j s_k^{-1} = s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i = s_k s_j s_k^{-1} s_i = t_j t_i$$

c) i)

$$t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

d) i)

$$\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_j s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_k s_j s_k s_i s_k s_i^{-1} s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_j s_i s_k s_i s_k s_i^{-1} s_k^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} \\
&= e
\end{aligned}$$

We also have

$$t_j t_k^{-1} t_i t_k = s_j s_k^{-1} s_k s_i s_k^{-1} s_k = s_i s_j = t_k^{-1} t_i t_k t_j$$

ii)

$$\begin{aligned}
t_i t_j &= s_i s_k s_j s_k^{-1} \\
&= s_i s_j^{-1} s_k s_j \\
&= s_j^{-1} s_k s_j s_i \\
&= s_k s_j s_k^{-1} s_i \\
&= t_j t_i
\end{aligned}$$

e) i)

$$
\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_i^{-1} s_k s_i s_j s_i^{-1} s_k s_i s_j s_i^{-1} s_k^{-1} s_i s_j^{-1} s_i^{-1} s_k^{-1} s_i s_j^{-1} \\
&= s_i^{-1} s_k s_j s_k s_j s_k^{-1} s_j^{-1} s_k^{-1} s_j^{-1} s_i \\
&= e
\end{aligned}
$$

We also have

$$
t_j t_k^{-1} t_i t_k = s_j s_k^{-1} s_k s_i s_k^{-1} s_k = s_j s_i = s_i s_j = t_i t_j
$$

ii)

$$
\begin{aligned}
s_k^{-1} t_i t_j t_i^{-1} t_j^{-1} s_k &= s_k^{-1} s_i s_k s_j s_k^{-1} s_i^{-1} s_k s_j^{-1} \\
&= s_i s_k s_i^{-1} s_j s_i s_k^{-1} s_i^{-1} s_j^{-1} \\
&= e
\end{aligned}
$$

f) i)

$$
\begin{aligned}
s_k^{-1} t_i t_j t_i t_j^{-1} t_i^{-1} t_j^{-1} &= s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k \\
&= e
\end{aligned}
$$

$$
\begin{aligned}
t_i t_j^{-1} t_k t_j t_i^{-1} t_j^{-1} s_k^{-1} t_j &= s_k s_i s_k^{-1} s_j^{-1} s_k s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_k^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} s_j \\
&= s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_k^{-1} \\
&= s_k s_j s_k s_j^{-1} s_i s_i^{-1} s_j s_k^{-1} s_j^{-1} s_k \\
&= e
\end{aligned}
$$

ii) This follows from part (i) by symmetry

References

[BM13] Michael Barot and Robert J. Marsh. Reflection group presentations arising from cluster algebras. arXiv: 1112.2300v2, 2013.