Artin Group Presentations Arising from Cluster Algebras

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Outline

Brief History

Fomin and Zelevinsky introduced *cluster algebras*, generated from seed variables. These algebras are of finite type if they are generated from a finite number of cluster variables. Fomin and Zelevinsky also showed that cluster algebras of finite type can be classified by Dynkin diagrams.

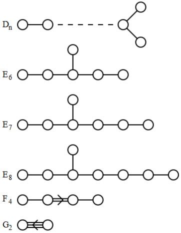

Figure 1: Dynkin diagrams

Example

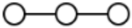

Figure 2: Dynkin diagram of type A_3

Coxeter group relations:

- $s_i^2 = e$: $s_1^2 = s_2^2 = s_3^2 = e$
- $(s_i s_j)^{m_{ij}} = e$: $(s_1 s_2)^3 = (s_2 s_3)^3 = e$
- $(s_i s_j)^{m_{ij}} = e$: $(s_1 s_3)^2 = e$

Artin group relations:

- $s_1 s_2 s_1 = s_2 s_1 s_2$
- $s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$

Barot and Marsh extended the Coxeter group presentations to diagrams of finite type (making an allowance for chordless cycles).

Theorem (Theorem 5.4, Barot-Marsh)

1. *Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. Then $W_\Gamma \cong W_{\Gamma'}$.*
2. *Let $\mathscr{A}$ be a cluster algebra of finite type. Then the groups W_Γ associated to the diagrams Γ arising from the seeds of $\mathscr{A}$ are all isomorphic (to the reflection group associated to the Dynkin diagram associated to $\mathscr{A}$).*

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Theorem

(HHLP, 14) Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. Then $A_\Gamma \cong A_{\Gamma'}$.

Definition

We say a matrix B is *skew-symmetrisable* if there exists a diagonal matrix D of the same size such that $D_{ii} > 0$ and DB is skew-symmetric.

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A skew-symmetrisable matrix B is *2-finite* if $|B_{ij}B_{ji}| \leq 3$ for all $i, j \in \{1, \ldots, n\}$.

From the skew-symmetrisable matrix associated to a cluster algebra of finite type, we can associate an diagram Γ as follows: For $i, j \in V(\Gamma)$, $i \xrightarrow{w} j$ if and only if $B_{ij} > 0$ and $w = |B_{ij}B_{ji}|$ is the weight of the edge.

Mutation Rules

Proposition (Proposition 1.4, Barot-Marsh)

Let B be a 2-finite skew-symmetrisable matrix. Then $\Gamma(\mu_k(B))$ is uniquely determined by $\Gamma(B)$ as follows:

- *Reverse the orientations of all edges in $\Gamma(B)$ incident with k (leaving the weights unchanged)*
- *For any path in $\Gamma(B)$ of form $i \xrightarrow{a} k \xrightarrow{b} j$ (i.e. with a, b positive), let c be the weight on the edge $j \rightarrow i$, taken to be zero if there is no such arrow. Let c' be determined by $c' \geq 0$ and $c + c' = max(a, b)$. Then $\Gamma(B)$ changes in a predetermined way, taking the case $c' = 0$ to mean no arrow between i and j.*

Mutation Rules

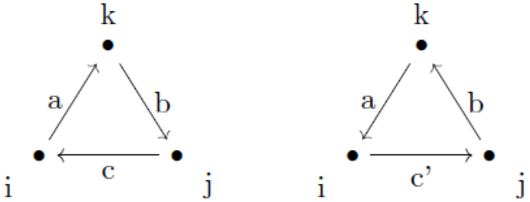


Figure 3: Predetermined method for mutation at k

Unoriented structures underlying Γ

Proposition (Proposition 9.7, Fomin-Zelevinsky II)

Any chordless cycle in Γ *must have an unoriented structure that is one of the following. Furthermore, it must be cyclically oriented.*

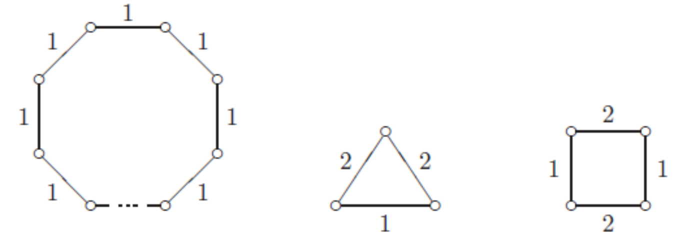


Figure 4: Possible chordless cycles in a diagram

Coxeter group presentation

Definition

For a diagram Γ and $i, j \in V(\Gamma)$, define

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}$$

This definition will allow us to present generator relations in the group W_Γ.

Coxeter group presentation

Given a diagram Γ of finite type, Barot and Marsh define the Coxeter group W_Γ with generators $s_i, i = 1, 2, \ldots, n$, subject to the following relations:

- (R1) $s_i^2 = e$ for all i
- (R2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$

Furthermore, for a chordless cycle $C : i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$ and for $a = 0, 1, 2, \ldots, d-1$, define
$r\left(i_a, i_{a+1}\right) = s_{i_a} s_{i_{a+1}} \cdots s_{i_{a+d-1}} s_{i_{a+d-2}} \cdots s_{i_{a+1}}$.
Then we have the following relations:

- (R3)(a) If all the weights in the edges of C are 1, then $r(i_a, i_{a+1})^2 = e$
- (R3)(b) If C has some edges of weight 2, then $r(i_a, i_{a+1})^k = e$ where $k = 4 - w_a$ and w_a is the weight of the edge $i_a - \ i_{a-1}$

Coxeter group invariance

Given a diagram Γ and the corresponding Coxeter group W_Γ, Barot and Marsh prove that this group is invariant (up to isomorphism) under mutation of Γ.

Theorem (Theorem 5.4, Barot-Marsh)

1. *Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. Then $W_\Gamma \cong W_{\Gamma'}$.*
2. *Let $\mathscr{A}$ be a cluster algebra of finite type. Then the groups W_Γ associated to the diagrams Γ arising from the seeds of $\mathscr{A}$ are all isomorphic (to the reflection group associated to the Dynkin diagram associated to $\mathscr{A}$).*

Artin group presentation

Let

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod 2 \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

Let $(i_0, \ldots, i_{d-1})$ be an ordered tuple such that the subgraph of Γ on the vertices $i_0, \ldots, i_{d-1}$ is a chordless cycle, with edges of nonzero weight from i_k to i_{k+1}, where subscripts are taken $\pmod d$. Then, denote

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \ldots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \ldots s_{i_{a+1}}.$$

Artin group presentation

Now we have

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod 2 \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

And

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \dots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \dots s_{i_{a+1}}.$$

Additionally, let

$$t(i_a, i_{a+1}) = s_{i_a} p(i_a, i_{a+1}) s_{i_a}^{-1} p(i_a, i_{a+1})^{-1}.$$

These definitions will allow us to present generator relations in the group A_Γ.

Artin group presentation

Given a diagram Γ of finite type, we define the Artin group A_Γ with generators $s_i, i = 1, 2, \ldots, n$, subject to the following relations (noting that $t(i_a, i_{a+1}) = s_{i_a} p(i_a, i_{a+1}) s_{i_a}^{-1} p(i_a, i_{a+1})^{-1}$):

- (T2) With m_{ij} as defined previously, for all $i \neq j$, we add the relations $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$.
- (T3) Let $(i_0, i_1, \ldots, i_{d-1})$ be an ordered tuple as before. If additionally one of the following two conditions hold:
  1. All edges in the chordless cycle are of weight 1 or 2 and the edge $i_{d-1} \to i_0$ has weight 2.
  2. All edges in the chordless cycle have weight 1.

  then we include the relation $t(i_0, i_1) = e$.

Example

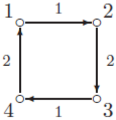


Figure 5: A 4 cycle with edge weights of 1 and 2

Coxeter relations:

R1 $s_1^2 = s_2^2 = s_3^2 = s_4^2 = e$

R2 $(s_1 s_2)^3 = (s_3 s_4)^3 = e$

R2 $(s_2 s_3)^4 = (s_4 s_1)^4 = e$

R2 $(s_1 s_3)^2 = (s_2 s_4)^2 = e$

R3 $r(1,2)^2 = r(3,4)^2 = e$, or $(s_1 s_2 s_3 s_4 s_3 s_2)^2 = (s_3 s_4 s_1 s_2 s_1 s_4)^2 = e$

R3 $r(2,3)^3 = r(4,1)^3 = e$, or $(s_2 s_3 s_4 s_1 s_4 s_3)^3 = (s_4 s_1 s_2 s_3 s_2 s_1)^3 = e$

Example

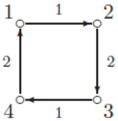


Figure 6: A 4 cycle with edge weights of 1 and 2

Artin relations:

T2 $\langle s_1, s_2 \rangle^3 = \langle s_2, s_1 \rangle^3$, or $s_1 s_2 s_1 = s_2 s_1 s_2$

T2 $\langle s_2, s_3 \rangle^4 = \langle s_3, s_2 \rangle^4$, or $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$

T2 $\langle s_2, s_4 \rangle^2 = \langle s_4, s_2 \rangle^2$, or $s_2 s_4 = s_4 s_2$

The T2 relations $\langle s_3, s_4 \rangle, \langle s_4, s_1 \rangle, \langle s_1, s_3 \rangle$ can be defined in a similar manner.

T3 $t(1,2) = s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 = e$, or
$s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 = s_2^{-1} s_3^{-1} s_4 s_3 s_2 s_1$.

The T3 relation $t(3,4) = e$ can be defined in a similar manner.

Artin group invariance

Given a diagram Γ and the corresponding Artin group A_Γ, we prove this group is invariant (up to isomorphism) under mutation of Γ.

Theorem

(HHLP, 14) Let Γ be a diagram of finite type, and let $\Gamma' = \mu_k(\Gamma)$ be the mutation of Γ at vertex k. Then $A_\Gamma \cong A_{\Gamma'}$.

Constructing the Isomorphism

To prove this theorem, our strategy will be to construct homomorphisms so that the following diagram commutes.

$$\begin{tikzcd} A_{\Gamma^{op}} \arrow[r, "\phi_{op}"] & A_{(\Gamma')^{op}} \arrow[d, "\Delta'"] \\ A_{\Gamma} \arrow[u, "\Delta"] & A_{\Gamma'} \arrow[l, "\phi"] \end{tikzcd}$$

Then the map ϕ will give us the desired isomorphism.

Constructing the Isomorphism

We first prove that $A_\Gamma \cong A_{\Gamma^{op}}$, where Γ^{op} is the diagram obtained by reversing all arrows in Γ.

Lemma

Let A_Γ be generated by $s_1, \ldots, s_n$, and let $A_{\Gamma^{op}}$ be generated by $r_1, \ldots, r_n$. Then the map

$$\Delta : s_i \to r_i^{-1}$$

defines an isomorphism between A_Γ and $A_{\Gamma^{op}}$.

Constructing the Isomorphism

Now let A_Γ be generated by $s_1, \dots, s_n$ and let $A_{\Gamma'}$ be generated by $u_1, \dots, u_n$. Consider the map $\phi : A_{\Gamma'} \to A_\Gamma$ defined as follows:

$$\phi(u_i) = \begin{cases} s_k s_i s_k^{-1} & \text{if there is an arrow from i to k in } \Gamma \\ s_i & \text{otherwise} \end{cases}$$

Lemma

The map ϕ is a well-defined homomomorphism.

Unoriented structures underlying Γ

In proving that the map is well-defined, we make use of the fact that diagrams of finite-type have a nice underlying structure. For example, when dealing with the (T2) relations, we have

Lemma (Lemma 2.2, Barot-Marsh)

For Γ of finite-type, if we have a subdiagram of Γ with three connected vertices, then the unoriented graph underlying the subdiagram must be one of the following:

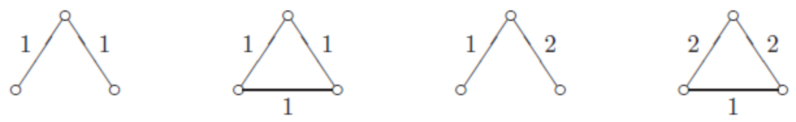


Figure 7: Unoriented 3-vertex connected subdiagrams

Possible mutations for $i - k - j$

Corollary (Corollary 2.3, Barot-Marsh)

If in Γ *we have* $i, j, k \in V(\Gamma), i \neq k \neq j$, *and we have* $i - k - j$, *then the only possible mutations for this connected path between the three vertices are the following:*

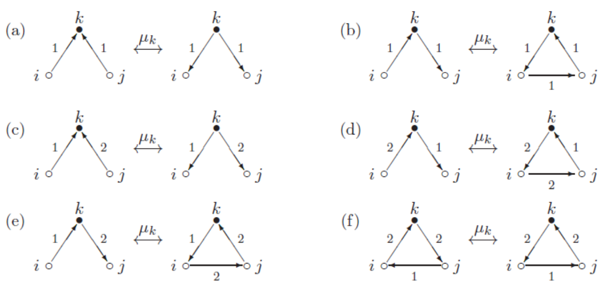


Figure 8: Mutation of three connected vertices

Sketch of Proof of the Main Theorem

Now consider the diagram:

$$\begin{tikzcd}
A_{\Gamma^{op}} \arrow[r, "\phi_{op}"] & A_{(\Gamma')^{op}} \arrow[d, "\Delta'"] \\
A_{\Gamma} \arrow[u, "\Delta"] & A_{\Gamma'} \arrow[l, "\phi"]
\end{tikzcd}$$

One can check that this diagram commutes, and so we find that the map ϕ is actually an isomorphism, which proves our theorem.

Extending Result to Other Diagrams

It is natural to ask whether a similar result holds for diagrams that are not of finite type. In particular, we will examine diagrams which are mutation equivalent to affine Dynkin diagrams.

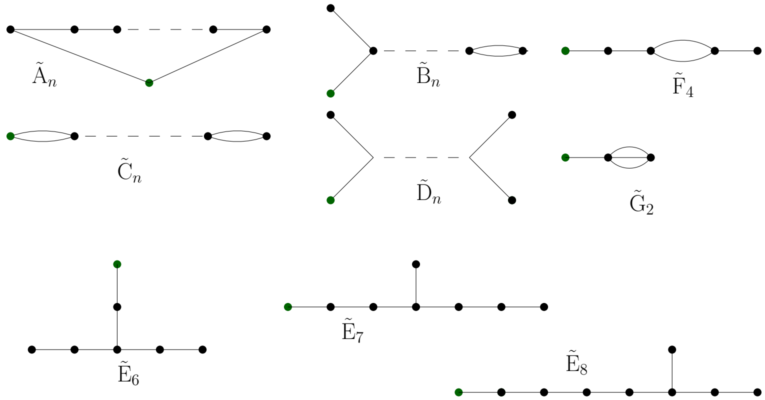


Figure 9: Affine Dynkin Diagrams

Extending the Barot-Marsh Presentation

In a paper by Felickson and Tumarkin, the authors obtain a similar result for coxeter groups. For a diagram Γ of affine type with n+1 nodes, they define a group W_Γ generated by elements $s_1, \dots, s_{n+1}$ and subject to the following relations:
(R1) $s_i^2 = e$ for $i \in \{1, \dots, n\}$.
(R2) $(s_i s_j)^{m_{ij}} = e$ where

$$m_{ij} = \begin{cases} 2 & \text{if there is no arrow between i and j in } \Gamma \\ 3 & \text{if there is an arrow of weight 1 between i and j in } \Gamma \\ 4 & \text{if there is an arrow of weight 2 between i and j in } \Gamma \\ 6 & \text{if there is an arrow of weight 3 between i and j in } \Gamma \\ \infty & \text{otherwise} \end{cases}$$

Extending the Barot-Marsh Presentation

(R3) For every chordless oriented cycle:

$$i_0 \xrightarrow{w_{i_0}} i_1 \xrightarrow{w_{i_1}} \cdots \xrightarrow{w_{i_{d-2}}} i_{d-1} \xrightarrow{w_{i_{d-1}}} i_0,$$

define for l $\in \{0, \ldots, d-1\}$,

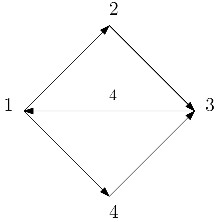
$$t(l) = (\prod_{j=l}^{l+d-2} \sqrt{w_{i_j}} - \sqrt{w_{i_{l+d-1}}})^2.$$

Then take the relation $(s_{i_l} p(i_l, i_{l+1}))^{m(l)} = e$ where

$$m(l) = \begin{cases} 2 & \text{if } t(l) = 0 \\ 3 & \text{if } t(l) = 1 \\ 4 & \text{if } t(l) = 2 \\ 6 & \text{if } t(l) = 3 \end{cases}$$

Extending the Barot-Marsh Presentation

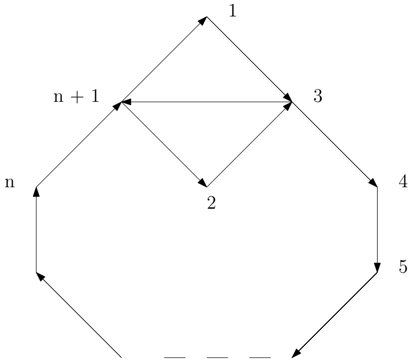
(R4) To a subdiagram of the form



we add the relation $(s_1 s_2 s_3 s_4 s_3 s_2)^2 = e$.

Extending the Barot-Marsh Presentation

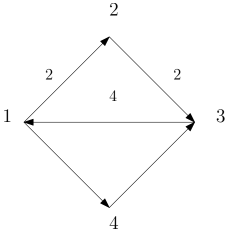
To a subdiagram of the form



we add the relation $(s_1 s_2 s_3 s_2 s_1 s_4 s_5 \dots s_n s_{n+1} s_n \dots s_5 s_4)^2 = e$

Extending the Barot-Marsh Presentation

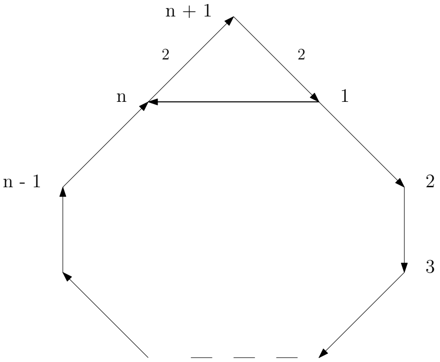
To a subdiagram of the form



we add the relations $(s_2 s_3 s_4 s_1 s_4 s_3)^2 = e$ and $(s_2 s_1 s_4 s_3 s_4 s_1)^2 = e$.

Extending the Barot-Marsh Presentation

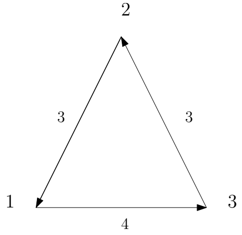
To a subdiagram of the form



we add the relation $(s_{n+1}s_1s_{n+1}s_2s_3\ldots s_{n-1}s_ns_{n-1}\ldots s_3s_2)^2 = e$.

Extending the Barot-Marsh Presentation

To a subdiagram of the form



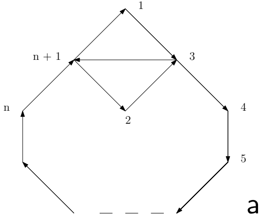
we add the relations $(s_2 s_1 s_2 s_1 s_2 s_3)^2 = e$ and $(s_2 s_3 s_2 s_3 s_2 s_1)^2 = e$

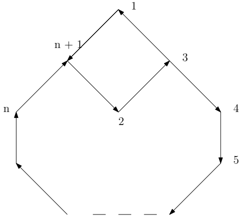
Theorem (Felickson-Tumarkin)

The group W_Γ is invariant up to isomorphism under diagram mutation.

Purpose of (R4) Relations

In their paper, Felickson and Tumarkin show that all of the (R4) relations are necessary for this theorem to be true. For example,

consider the mutation of  at the vertex 1, which is the diagram



Extending our Artin Group Presentations

Let Γ be a diagram of affine type on n+1 nodes. Then we define A_Γ to be the group generated by $s_1, \ldots, s_{n+1}$ satisfying the relations
(T2) $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$ where

$$m_{ij} = \begin{cases} 2 & \text{if there is no arrow between i and j in } \Gamma \\ 3 & \text{if there is an arrow of weight 1 between i and j in } \Gamma \\ 4 & \text{if there is an arrow of weight 2 between i and j in } \Gamma \\ 6 & \text{if there is an arrow of weight 3 between i and j in } \Gamma \\ \infty & \text{otherwise} \end{cases}$$

Extending our Artin Group Presentations

(T3) For every chordless oriented cycle:

$$i_0 \xrightarrow{w_{i_0}} i_1 \xrightarrow{w_{i_1}} \cdots \xrightarrow{w_{i_{d-2}}} i_{d-1} \xrightarrow{w_{i_{d-1}}} i_0,$$

define for l $\in \{0, \ldots, d-1\}$,

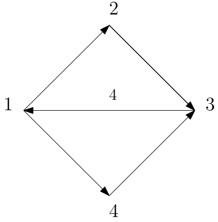
$$t(l) = \big(\prod_{j=l}^{l+d-2} \sqrt{w_{i_j}} - \sqrt{w_{i_{l+d-1}}}\big)^2.$$

Then take the relation $\langle s_{i_l} p(i_l, i_{l+1}) \rangle^{m(l)} = \langle p(i_l, i_{l+1}) s_{i_l} \rangle^{m(l)}$ where

$$m(l) = \begin{cases} 2 & \text{if } t(l) = 0 \\ 3 & \text{if } t(l) = 1 \\ 4 & \text{if } t(l) = 2 \\ 6 & \text{if } t(l) = 3 \end{cases}$$

Extending our Artin Group Presentations

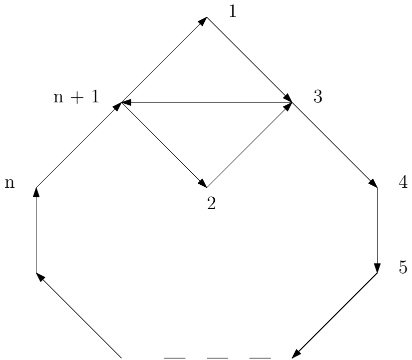
(T4) To a subdiagram of the form



we add the relation $(s_2 s_1 s_2^{-1}) s_4^{-1} s_3 s_4 = s_4^{-1} s_3 s_4 (s_2 s_1 s_2^{-1})$.

Extending our Artin Group Presentations

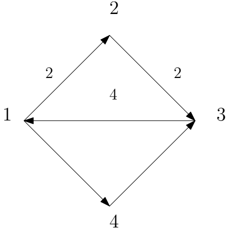
To a subdiagram of the form



we add the relation $s_2(s_3^{-1}s_4^{-1}\dots s_n^{-1}(s_1 s_{n+1} s_1^{-1})s_n \dots s_4 s_3)$
$= (s_3^{-1}s_4^{-1}\dots s_n^{-1}(s_1 s_{n+1} s_1^{-1})s_n \dots s_4 s_3)s_2$

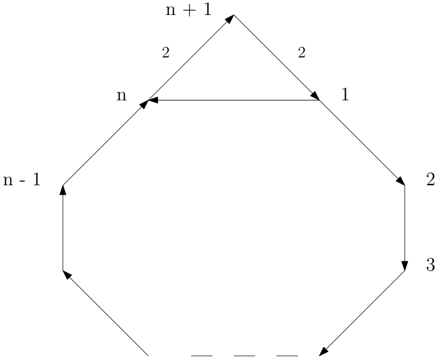
Extending our Artin Group Presentations

To a subdiagram of the form



we add the relations $s_2 s_3^{-1}(s_4 s_1 s_4^{-1})s_3 = s_3^{-1}(s_4 s_1 s_4^{-1})s_3 s_2$ and $s_2 s_1 s_4^{-1} s_3 s_4 s_1^{-1} = s_1 s_4^{-1} s_3 s_4 s_1^{-1} s_2$.

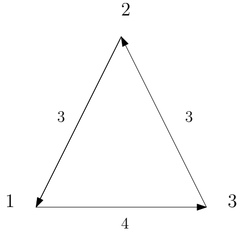
To a subdiagram of the form



we add the relation $s_1(s_2^{-1}\dots s_{n-1}^{-1}(s_{n+1}s_ns_{n+1}^{-1})s_{n-1}\dots s_2) = (s_2^{-1}\dots s_{n-1}^{-1}(s_{n+1}s_ns_{n+1}^{-1})s_{n-1}\dots s_2)s_1$.

Extending our Artin Group Presentations

To a subdiagram of the form



we add the relations $s_2 s_1^{-1}(s_2 s_3 s_2^{-1}) s_1 = s_1^{-1}(s_2 s_3 s_2^{-1}) s_1 s_2$ and $s_1 s_2 s_3 s_2 s_3^{-1} s_2^{-1} = s_2 s_3 s_2 s_3^{-1} s_2^{-1} s_1$.

Conjecture

A_Γ is invariant up to isomorphism under diagram mutation.

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