Artin Group Presentations Arising from Cluster Algebras

Jacob Haley*, David Hemminger, Aaron Landesman, Hailee Peck*

University of Minnesota, Twin Cities REU

August 3, 2014

* denotes speaker

Outline

Brief History

Fomin and Zelevinsky introduced *cluster algebras*, generated from seed variables. These algebras are of finite type if they are generated from a finite number of seeds. Fomin and Zelevinsky also showed that cluster algebras of finite type can be classified by Dynkin diagrams.

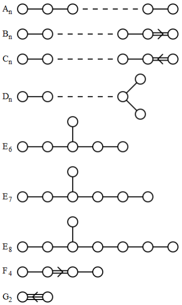

Figure 1 : Dynkin diagrams

Example

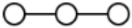

Figure 2 : Dynkin diagram of type A_3

Coxeter group relations:

- $s_1^2 = s_2^2 = s_3^2 = e$
- $(s_1 s_2)^3 = (s_2 s_3)^3 = e$
- $(s_1 s_3)^2 = e$

Artin group relations:

- $\langle s_1, s_2 \rangle^3 = \langle s_2, s_1 \rangle^3$
- $\langle s_2, s_3 \rangle^3 = \langle s_3, s_2 \rangle^3$
- $\langle s_1, s_3 \rangle^2 = \langle s_3, s_1 \rangle^2$

Alternatively, the above relations correspond to

- $s_1 s_2 s_1 = s_2 s_1 s_2$
- $s_2 s_3 s_2 = s_3 s_2 s_3$
- $s_1 s_3 = s_3 s_1$

Barot and Marsh extended the Coxeter group presentations to diagrams of finite type (making an allowance for chordless cycles), and proved that this group presentation is isomorphic to the Coxeter group associated to a Dynkin diagram. They also proved mutation invariance, up to isomorphism, for these Coxeter groups.

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Goal

Develop variations of relations for the Coxeter group associated to a diagram of finite type provided in Barot-Marsh to define the Artin group A_Γ corresponding to a diagram Γ. Prove that for $\Gamma' = \mu_k(\Gamma)$, we get $A_\Gamma \cong A_{\Gamma'}$.

Definition

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From the skew-symmetrisable matrix associated to a cluster algebra of finite type, we can associate an diagram Γ as follows: For $i, j \in V(\Gamma)$, $i \xrightarrow{w} j$ if and only if $B_{ij} > 0$ and $w = |B_{ij}B_{ji}|$ is the weight of the edge.

Mutation Rules

Proposition (Proposition 1.4, Barot-Marsh)

Let B be a 2-finite skew-symmetrisable matrix. Then $\Gamma(\mu_k(B))$ is uniquely determined by $\Gamma(B)$ as follows:

- *Reverse the orientations of all edges in $\Gamma(B)$ incident with k (leaving the weights unchanged)*
- *For any path in $\Gamma(B)$ of form $i \xrightarrow{a} k \xrightarrow{b} j$ (i.e. with a, b positive), let c be the weight on the edge $j \to i$, taken to be zero if there is no such arrow. Let c' be determined by $c' \geq 0$ and $c + c' = max(a, b)$. Then $\Gamma(B)$ changes in a predetermined way, taking the case $c' = 0$ to mean no arrow between i and j.*

Mutation Rules

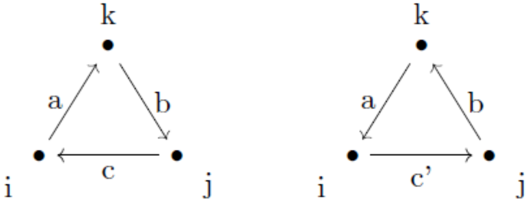

Figure 3 : Predetermined method for mutation at k

Unoriented structures underlying Γ

Proposition (Proposition 2.1, Barot-Marsh)

Any chordless cycle in Γ *must have an unoriented structure that is one of the following. Furthermore, it must be cyclically oriented.*

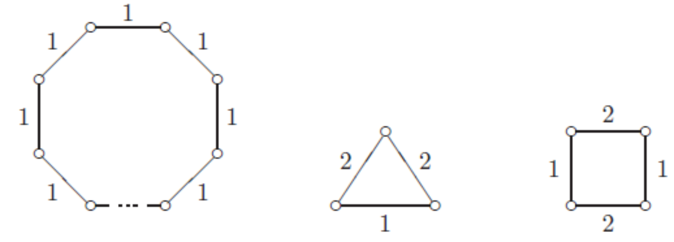

Figure 4 : Possible chordless cycles in a diagram

Definition

For a diagram Γ and $i, j \in V(\Gamma)$, define

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected;} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 1;} \\ 4 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 2;} \\ 6 & \text{if } i \text{ and } j \text{ are connected by an edge of weight 3.} \end{cases}$$

This definition will allow us to present generator relations in the group W_Γ.

Given a diagram Γ of finite type, Barot and Marsh define the Coxeter group W_Γ with generators $s_i, i = 1, 2, \ldots, n$, subject to the following relations:

- (R1) $s_i^2 = e$ for all i
- (R2) $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$

Furthermore, for a chordless cycle $C : i_0 \to i_1 \to \cdots \to i_{d-1} \to i_0$ and for $a = 0, 1, 2, \ldots, d-1$, define $r\left(i_a, i_{a+1}\right) = s_{i_a} s_{i_{a+1}} \cdots s_{i_{a+d-1}} s_{i_{a+d-2}} \cdots s_{i_{a+1}}$.
Then we have the following relations:

- (R3)(a) If all the weights in the edges of C are 1, then $r(i_a, i_{a+1})^2 = e$
- (R3)(b) If C has some edges of weight 2, then $r(i_a, i_{a+1})^k = e$ where $k = 4 - w_a$ and w_a is the weight of the edge $i_a - \ i_{a-1}$

Given a diagram Γ and the corresponding Coxeter group W_Γ, Barot and Marsh prove that this group is invariant (up to isomorphism) under mutation of Γ.

Theorem (Theorem 5.4, Barot-Marsh)

1. *Let Γ be a diagram of finite type and $\Gamma' = \mu_k(\Gamma)$ the mutation of Γ at vertex k. Then $W_\Gamma \cong W_{\Gamma'}$.*
2. *Let $\mathscr{A}$ be a cluster algebra of finite type. Then the groups W_Γ associated to the diagrams Γ arising from the seeds of $\mathscr{A}$ are all isomorphic (to the reflection group associated to the Dynkin diagram associated to $\mathscr{A}$).*

Let

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod 2 \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

That is, $\langle x_i, x_j \rangle$ is just an alternating sequence of x_i and x_j of length k. We also write $\langle x_i, x_j \rangle^{-k}$ to denote $\left(\langle x_i, x_j \rangle^k\right)^{-1}$.

Let $(i_0, \ldots, i_{d-1})$ be an ordered tuple such that the subgraph of Γ on the vertices $i_0, \ldots, i_{d-1}$ is a chordless cycle, with edges of nonzero weight from i_k to i_{k+1}, where subscripts are taken $\pmod d$. Then, denote

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \ldots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \ldots s_{i_{a+1}}.$$

Now we have

$$\langle x_i, x_j \rangle^k = \begin{cases} (x_i x_j)^{\frac{k}{2}}, & \text{if } k \equiv 0 \pmod 2 \\ (x_i x_j)^{\frac{k-1}{2}} x_i & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

And

$$p(i_a, i_{a+1}) = s_{i_{a+1}}^{-1} s_{i_{a+2}}^{-1} \dots s_{i_{a-2}}^{-1} s_{i_{a-1}} s_{i_{a-2}} s_{i_{a-3}} \dots s_{i_{a+1}}.$$

Additionally, let

$$t(i_a, i_{a+1}) = s_{i_a} p(i_a, i_{a+1}) s_{i_a}^{-1} p(i_a, i_{a+1})^{-1}.$$

These definitions will allow us to present generator relations in the group A_Γ.

Given a diagram Γ of finite type, we define the Artin group A_Γ with generators $s_i, i = 1, 2, \ldots, n$, subject to the following relations (noting that $t(i_a, i_{a+1}) = s_{i_a} p(i_a, i_{a+1}) s_{i_a}^{-1} p(i_a, i_{a+1})^{-1}$):

- (T2) With m_{ij} as defined previously, for all $i \neq j$, we add the relations $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$.
- (T3) Let $(i_0, i_1, \ldots, i_{d-1})$ be an ordered tuple as before. If additionally one of the following two conditions hold:
  1. All edges in the chordless cycle are of weight 1 or 2 and the edge $i_{d-1} \to i_0$ has weight 2.
  2. All edges in the chordless cycle have weight 1.

  then we include the relation $t(i_0, i_1) = e$.

Example

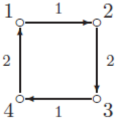

Figure 5 : A 4 cycle with edge weights of 1 and 2

Coxeter relations:

R1 $s_1^2 = s_2^2 = s_3^2 = s_4^2 = e$

R2 $(s_1 s_2)^3 = (s_3 s_4)^3 = e$

R2 $(s_2 s_3)^4 = (s_4 s_1)^4 = e$

R2 $(s_1 s_3)^2 = (s_2 s_4)^2 = e$

R3 $r(1,2)^2 = r(3,4)^2 = e$, or $(s_1 s_2 s_3 s_4 s_3 s_2)^2 = (s_3 s_4 s_1 s_2 s_1 s_4)^2 = e$

R3 $r(2,3)^3 = r(4,1)^3 = e$, or $(s_2 s_3 s_4 s_1 s_4 s_3)^3 = (s_4 s_1 s_2 s_3 s_2 s_1)^3 = e$

Example

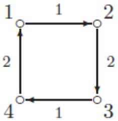

Figure 6 : A 4 cycle with edge weights of 1 and 2

Artin relations:

T2 $\langle s_1, s_2\rangle^3 = \langle s_2, s_1\rangle^3$, or $s_1 s_2 s_1 = s_2 s_1 s_2$

T2 $\langle s_2, s_3\rangle^4 = \langle s_3, s_2\rangle^4$, or $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$

T2 $\langle s_2, s_4\rangle^2 = \langle s_4, s_2\rangle^2$, or $s_2 s_4 = s_4 s_2$

The T2 relations $\langle s_3, s_4\rangle, \langle s_4, s_1\rangle, \langle s_1, s_3\rangle$ can be defined in a similar manner.

T3 $t(1,2) = s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 = e$, or
$s_1 s_2^{-1} s_3^{-1} s_4 s_3 s_2 = s_2^{-1} s_3^{-1} s_4 s_3 s_2 s_1$.

The T3 relation $t(3,4) = e$ can be defined in a similar manner.

Remark

Each Artin group has an associated Coxeter group defined by adding in the additional relations $s_i^2 = 1$ for all i. An Artin group is said to be of finite type if its associated Coxeter group is of finite type. To each Artin group of finite type we can assign it the same Dynkin diagram which is assigned to the Coxeter group associated to the Artin group.

Given a diagram Γ and the corresponding Artin group A_Γ, we prove this group is invariant (up to isomorphism) under mutation of Γ.

Theorem

Let Γ *be a diagram of finite type, and let* $\Gamma' = \mu_k(\Gamma)$ *be the mutation of* Γ *at vertex* k. *Then* $A_\Gamma \cong A_{\Gamma'}$.

Unoriented structures underlying Γ

Lemma (Lemma 2.2, Barot-Marsh)

For Γ, if we have a subdiagram of Γ with three connected vertices, then the unoriented graph underlying the subdiagram must be one of the following:

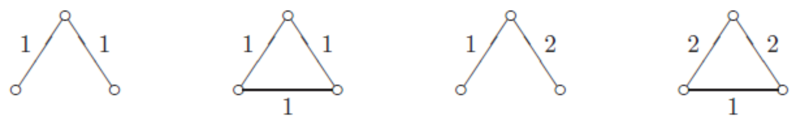

Figure 7 : Unoriented 3-vertex connected subdiagrams

Possible mutations for $i - k - j$

Corollary (Corollary 2.3, Barot-Marsh)

If in Γ we have $i, j, k \in V(\Gamma), i \neq k \neq j$, and we have $i - k - j$, then the only possible mutations for this connected path between the three vertices are the following:

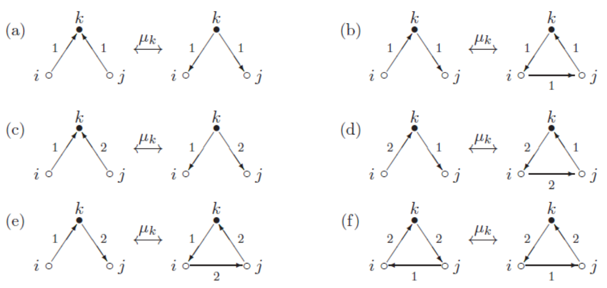

Figure 8 : Mutation of three connected vertices

[1] Barot, M. and Marsh, R., Reflection Group Presentations Arising from Cluster Algebras, *Preprint*, arXiv:1112.2300 (2011)