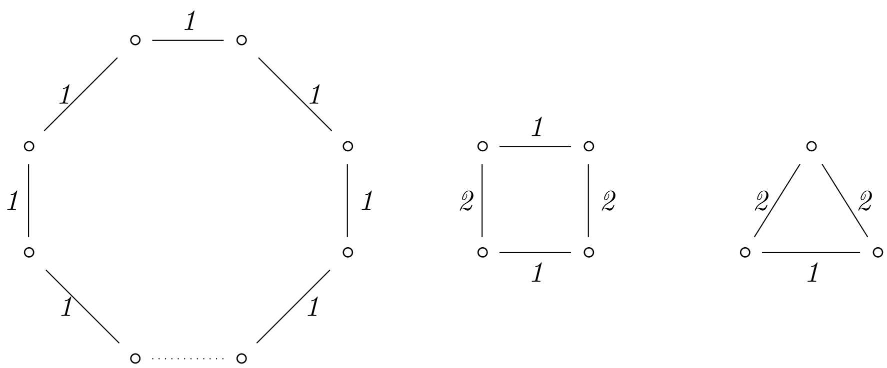
## List of Todos

## 1. Diagrams of Finite Type

In this section, we shall review the structure of diagrams of finite type, and how their cycles are effected by mutation. This section is simply a recap of [**?** , Section 2].

**Definition 1.1.** A *chordless cycle* of an unoriented graph $G$ is a connected subgraph $H \subset G$ such that the number of vertices in $H$ is equal to the number of edges in $H$, and the edges in $H$ form a single cycle.

**Proposition 1.2.** *Let $\Gamma$ be a diagram of finite type. Then, a chordless cycle in the unoriented graph of $\Gamma$ is cyclically oriented in $\Gamma$. Furthermore, the unoriented graph underlying the cycle must either be a cycle such that all edges have weight 1, a triangle with two edges of weight 2 and one of weight 1, or a square with two opposite edges of weight 2 and two opposite edges of weight 1, as pictured below.*




*Proof.* See [**?** , Proposition 2.1]. □

**Lemma 1.3.** *Let $\Gamma$ be a diagram of finite type with $\Gamma' = \mu_k(\Gamma)$ the mutation of $\Gamma$ at vertex $k$. Below, we list induced subdiagrams in $\Gamma$ on the left and the resulting induced subdiagrams in $\Gamma'$ with chordless cycles $C'$ on the right, after mutation at $k$. We draw the diagrams so that $C'$ always has a clockwise cycle. Furthermore, in case $(i)$, we assume $C'$ has at least three vertices, while in case $(j)$, we assume $C'$ has at least four vertices. Every chordless cycle in $\Gamma'$ is of one of these types.*

*(a)*


*(b)*

(c)


(d)

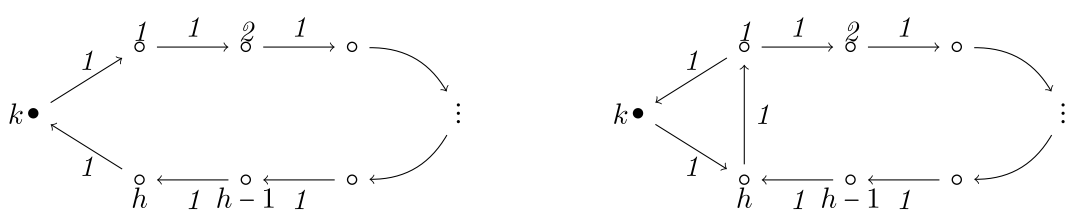

(e)


(f)

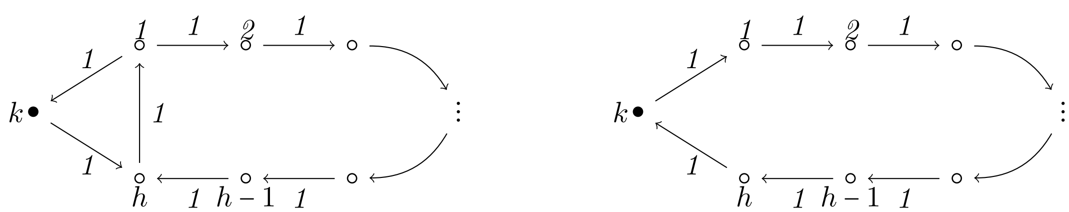

(g)


(h)


(i)




(j)




*(k) $C$ is an oriented cycle in $\Gamma$ not connected to $k$ and $C'$ is the corresponding cycle in $\Gamma'$.*

*(l) $C$ is an oriented cycle in $\Gamma$ with exactly one vertex in $C$ connected to $k$ by an edge of either weight 1 or 2. Then, $C'$ is the corresponding cycle in $\Gamma'$.*

*Proof.* See [**?** , Lemma 2.5]. ☐

## 2. The Group of a Diagram in an Artin Group

**Definition 2.1.** For $\Gamma$ a diagram of finite type, we define the associated Artin Group as follows. The associated artin group $W_\Gamma$ is generated by $s_i$, where there is one $s_i$ for each vertex $i$ in $\Gamma$. These generators are subject to the relations

(R2') For all $i \neq j$, we add the relations

$$\begin{cases} s_i s_j = s_j s_i, & \text{if there is no edge between } i \text{ and } j \\ s_i s_j s_i = s_j s_i s_j & \text{if there is an edge of weight 1 between } i \text{ and } j. \\ s_i s_j s_i s_j = s_j s_i s_j s_i & \text{if there is an edge of weight 2 between } i \text{ and } j. \\ s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i & \text{if there is an edge of weight 3 between } i \text{ and } j. \end{cases}$$

(R3')(a) For every chordless cycle of the form

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1} \longrightarrow i_0$$

such that all edges have weight 1, for all $i$, with $0 \leq a \leq d-1$, we include the relation

$$s_a s_{a+1}^{-1} s_{a+2}^{-1} \dots s_{a-2}^{-1} s_{a-1} s_{a-2} s_{a-3} \dots s_{a+1} = s_{a+1}^{-1} \dots s_{a-3}^{-1} s_{a-2}^{-1} s_{a-1} s_{a-2} \dots s_{a+1} s_a.$$

Where subscripts are taken $\pmod d$.

(R3')(b) For every chordless cycle of the form

$$\begin{tikzcd} & i \arrow[dr, "1"] & \\ k \arrow[ur, "2"] & & j \arrow[ll, "2"] \end{tikzcd}$$

we include the three relations

(1) $s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i$
(2) $s_j s_k^{-1} s_i s_k = s_k^{-1} s_i s_k s_j$
(3) $s_k^{-1} s_i^{-1} s_j s_i s_k s_i^{-1} s_j s_i s_k^{-1} s_i^{-1} s_j^{-1} s_i = e$.

(R3')(c) For every chordless cycle of the form

$$\begin{tikzcd} i \arrow[r, "1"] & j \arrow[d, "2"] \\ l \arrow[u, "2"] & k \arrow[l, "1"] \end{tikzcd}$$

we include the two relations

(1) $s_i s_j^{-1} s_k^{-1} s_l s_k s_j = s_j^{-1} s_k^{-1} s_l s_k s_j s_i$
(2) $s_k s_l^{-1} s_i^{-1} s_j s_i s_l = s_l^{-1} s_i^{-1} s_j s_i s_l s_k$

**Remark 2.2.** Note that if $\Gamma$ is the graph associated to a dynikin diagram, then $W_\Gamma$ as we have defined it is precisely the corresponding Artin group corresponding to that dynkin diagram. This is the case since in this case we have no cycles in $\Gamma$, and so we only have relation of the form $(R2')$, which define the Artin Group.

## 3. Symmetry among (R3') Relations

As in [**?** ], given the relations (R2'), many of the relations in (R3')(a) and (b) become redundant. For example,

**Lemma 3.1.** *Let $\Gamma$ be a diagram of finite type which contains a chordless cycle C:* $i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{d-1}$ *so that all edges have weight 1. Then if W is a group generated by $s_1, \ldots, s_n$ satifying the relations (R2') and $r(i_a, i_{a+1})$ for some $a \in \{1, \ldots, d\}$, all of the relations in (R3)(a) hold for C.*

*Proof.* As in Barot-Marsh, it suffices to prove that the relation r(0, 1) implies the relation r(d-1, 0). So suppose $W_\gamma$ satisfies the relation r(0, 1). Then we have

$$
\begin{aligned}
&s_{d-1}s_0^{-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_0\\
&= s_0^{-1}s_0s_{d-1}s_0^{-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_{d-1}^{-1}s_{d-1}s_0\\
&= s_0^{-1}s_{d-1}^{-1}s_0s_{d-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_{d-1}^{-1}s_{d-1}s_0\\
&= s_0^{-1}s_{d-1}^{-1}s_0s_1^{-1}\ldots s_{d-3}^{-1}s_{d-1}s_{d-2}s_{d-1}^{-1}s_{d-3}\ldots s_1s_{d-1}s_0\\
&= s_0^{-1}s_{d-1}^{-1}(s_0s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}^{-1}s_{d-1}s_{d-2}s_{d-3}\ldots s_1)s_{d-1}s_0\\
&= s_0^{-1}s_{d-1}^{-1}(s_1^{-1}\ldots s_{d-2}^{-1}s_{d-1}s_{d-2}s_{d-3}\ldots s_0)s_{d-1}s_0\\
&= s_0^{-1}s_{d-1}^{-1}(s_1^{-1}\ldots s_{d-3}^{-1}s_{d-1}s_{d-2}s_{d-1}^{-1}s_{d-3}\ldots s_0)s_{d-1}s_0\\
&= s_0^{-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_{d-1}^{-1}s_0s_{d-1}s_0\\
&= s_0^{-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_0s_{d-1}s_0^{-1}s_0\\
&= s_0^{-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_0s_{d-1}
\end{aligned}
$$

as required. Note that line 3 is equal to 4 and line 7 is equal to line 8 since the cycle is chordless, meaning that $s_{d-1}$ commutes with every element except $s_0$ and $s_{d-2}$. ☐

Furthermore, we obtain similar results for cycles containing edges of weight 2.

**Lemma 3.2.** *Let $\Gamma$ be a diagram of finite type containing a 3-cycle as in (R3')(b) of 2.1. Let W be the group with generators $s_1, \ldots, s_n$ defined by $\Gamma$. Then the relations (1) and (2) of (R3')(b) are equivalent, and they together imply the relation (3).*

*Proof.* The equivalence of (1) and (2) follows from the fact that

$$
\begin{aligned}
&s_1^{-1}s_3s_2s_3^{-1}s_1s_3s_2^{-1}s_3^{-1}s_1^{-1}s_3s_3^{-1}s_1\\
&= s_1^{-1}s_3s_2s_3^{-1}s_1s_3s_2^{-1}s_3^{-1}\\
&= s_1^{-1}s_2^{-1}s_3s_2s_1s_2^{-1}s_3^{-1}s_2
\end{aligned}
$$

In showing that (1) and (2) imply (3), we will underline the terms being manipulated in each line for emphasis.

$$
\begin{aligned}
&s_1s_3s_1(s_3^{-1}s_1^{-1}s_2s_1s_3s_1^{-1}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_1)s_1^{-1}s_3^{-1}s_1^{-1}\\
&= s_1\underline{s_3s_1s_3^{-1}s_1^{-1}}s_2s_1s_3s_1^{-1}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}\underline{s_1s_1^{-1}}s_3^{-1}s_1^{-1}\\
&= \underline{s_1s_1^{-1}s_3^{-1}s_1s_3s_2}s_1s_3s_1^{-1}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_2\underline{s_3^{-1}s_1s_3s_1}s_3s_1^{-1}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_2s_1s_3\underline{s_1s_3^{-1}s_3s_1^{-1}}s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_2\underline{e}s_1s_3s_2s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_2s_3\underline{s_3^{-1}s_1s_3s_2}s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= \underline{s_2s_3s_2}s_3^{-1}s_1s_3s_1s_3^{-1}s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_3s_2\underline{s_3s_3^{-1}}s_1\underline{s_3s_1s_3^{-1}s_1^{-1}}s_2^{-1}s_3^{-1}s_1^{-1}\\
&= s_3s_2\underline{s_1s_1^{-1}s_3^{-1}s_1s_3s_2^{-1}}s_3^{-1}s_1^{-1}\\
&= s_3\underline{s_2s_2^{-1}}s_3^{-1}s_1\underline{s_3s_3^{-1}}s_1^{-1}\\
&= \underline{s_3s_3^{-1}s_1s_1^{-1}}\\
&= e
\end{aligned}
$$

□

**Lemma 3.3.** *Let $\Gamma$ be a diagram of finite type containing a 4-cycle as in (R3')(c) of 2.1. Let $W$ be the group with generators $s_1, \ldots, s_n$ defined by $\Gamma$. Then the relations (1) and (2) of (R3')(c) are equivalent.*

*Proof.* We have that

$$
\begin{aligned}
&s_3^{-1}s_4^{-1}s_2(s_1s_2^{-1}s_3^{-1}s_4s_3s_2s_1^{-1}s_2^{-1}s_3^{-1}s_4^{-1}s_3s_2)s_2^{-1}s_4s_3\\
&= s_3^{-1}s_4^{-1}(s_2s_1s_2^{-1})(s_3^{-1}s_4s_3)(s_2s_1^{-1}s_2^{-1})s_3^{-1}s_4^{-1}s_3s_4s_3\\
&= s_3^{-1}s_4^{-1}s_1^{-1}s_2s_1s_4s_3s_4^{-1}s_1^{-1}s_2^{-1}s_1s_3^{-1}(s_4^{-1}s_3s_4)s_3\\
&= s_3^{-1}s_4^{-1}s_1^{-1}s_2s_1s_4s_3s_4^{-1}s_1^{-1}s_2^{-1}s_1s_4
\end{aligned}
$$

□

Finally, we conclude the section by establishing a relationship between the groups defined by $\Gamma$ and $\Gamma^{op}$.

**Lemma 3.4.** *(Analogue of 4.6) Let $W_\Gamma$ be generated by $s_1, \ldots, s_n$. Then $s_1^{-1}, \ldots, s_n^{-1}$ satisfy the relations (R2') and (R3') in $W_{\Gamma^{op}}$.*

*Proof.* One can see that the elements satisfy (R2') in $W_{\Gamma^{op}}$ by taking the inverse of both sides of the relation in $W_\Gamma$. To see that the elements satisfy (R3') in $W_{\Gamma^{op}}$, note that for a chordless cycle in $\Gamma$ with all weights equal to one, we have

$$s_0^{-1}\ldots s_{d-2}^{-1}s_{d-1}s_{d-2}\ldots s_0 = s_1^{-1}\ldots s_{d-2}^{-1}s_{d-1}s_{d-2}\ldots s_1$$

by the relation r(0,1) in (R3') in $W_\Gamma$. But then applying relations from (R2'), we have that

$$s_0^{-1}\ldots s_{d-1}s_{d-2}s_{d-1}^{-1}\ldots s_0 = s_1^{-1}\ldots s_{d-1}s_{d-2}s_{d-1}^{-1}\ldots s_1,$$

and since the cycle in chordless, we then have

$$s_0^{-1}s_{d-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_{d-1}^{-1}s_0 = s_{d-1}s_1^{-1}\ldots s_{d-3}^{-1}s_{d-2}s_{d-3}\ldots s_1s_{d-1}^{-1}.$$

Repeating this process, we find that

$$s_0^{-1}s_{d-1}s_{d-2}\dots s_2s_1s_2^{-1}\dots s_{d-2}^{-1}s_{d-1}^{-1}s_0 = s_{d-1}s_{d-2}\dots s_2s_1s_2^{-1}\dots s_{d-2}^{-1}s_{d-1}^{-1}.$$

But this can only occur if $s_1^{-1},\dots,s_n^{-1}$ satisfies the relation r'(0, d-1) in $\mathrm{W}_{\Gamma^{op}}$.

For a triangle as in (R3')(b) of 2.1, by the relation r'(1, 2) we have

$$s_1s_2^{-1}s_3s_2s_1^{-1} = s_2^{-1}s_3s_2.$$

Hence

$$s_1s_3s_2s_3^{-1}s_1^{-1} = s_3s_2s_3^{-1}.$$

But as before, this can occur if and only if $s_1^{-1}, s_2^{-1}, s_3^{-1}$ satisfy the relation r'(1, 3) in $\mathrm{W}_{\Gamma^{op}}$.

Finally, given a square labeled as in (R3')(c) and the relations r'(1, 2) and r'(3, 4), we have

$$\begin{aligned}
&s_2s_1s_4s_3s_4^{-1}s_1^{-1}\\
&= s_1s_1^{-1}s_2s_1s_4s_3s_4^{-1}s_1^{-1}\\
&= s_1s_2s_1s_2^{-1}s_3^{-1}s_4s_3s_1^{-1}\\
&= s_1s_2(s_1s_2^{-1}s_3^{-1}s_4s_3s_2)s_2^{-1}s_1^{-1}\\
&= s_1s_2(s_2^{-1}s_3^{-1}s_4s_3s_2s_1)s_2^{-1}s_1^{-1}\\
&= s_1s_4s_3s_4^{-1}s_2s_2^{-1}s_1^{-1}s_2\\
&= s_1s_4s_3s_4^{-1}s_1^{-1}s_2
\end{aligned}$$

But this relation holds if and only if $s_1^{-1},\dots,s_4^{-1}$ satisfy r'(2, 1) in $\mathrm{W}_{\Gamma^{Op}}$. Therefore, we are done. $\square$

## 4. Main Results

**Lemma 4.1.** *[Proposition 5.2 Analog] The elements $t_i$, for $i$ a vertex of $\Gamma$, satisfy the relations $(R2)$ and $(R3)$.*

After Lemma **??** we have left to check the relations $(R2)$ when both $i$ and $j$ are connected to $k$ and the relations $(R3)$. Beginning with the relations $(R2)$, and following cases a-f from Corollary 2.3 in Barot and Marsh:

a) i)

$$t_it_j = s_ks_is_k^{-1}s_ks_js_k^{-1} = s_ks_is_js_k^{-1} = s_ks_js_is_k^{-1} = t_jt_i$$

ii)

$$t_it_j = s_is_j = s_js_i = t_jt_i$$

b) i)

$$\begin{aligned}
t_it_jt_i &= s_ks_is_k^{-1}s_js_ks_is_k^{-1}\\
&= s_ks_is_js_ks_j^{-1}s_is_k^{-1}\\
&= s_ks_js_is_ks_is_j^{-1}s_k^{-1}\\
&= s_ks_js_ks_is_ks_j^{-1}s_k^{-1}\\
&= s_js_ks_js_is_j^{-1}s_k^{-1}s_j\\
&= s_js_ks_is_k^{-1}s_j\\
&= t_it_jt_i
\end{aligned}$$

$$
\begin{aligned}
t_j t_k^{-1} t_i t_k &= s_j s_k^{-1} s_k s_i s_k^{-1} s_k \\
&= s_j s_i \\
&= s_i s_j \\
&= t_k^{-1} t_i t_k t_j
\end{aligned}
$$

ii)

$$t_i t_j = s_i s_k s_j s_k^{-1} = s_i s_j^{-1} s_k s_j = s_j^{-1} s_k s_j s_i = s_k s_j s_k^{-1} s_i = t_j t_i$$

c) i)

$$t_i t_j = s_k s_i s_k^{-1} s_k s_j s_k^{-1} = s_k s_i s_j s_k^{-1} = s_k s_j s_i s_k^{-1} = t_j t_i$$

ii)

$$t_i t_j = s_i s_j = s_j s_i = t_j t_i$$

d) i)

$$
\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_j s_k s_i s_j s_k s_j^{-1} s_i^{-1} s_j s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_i s_k^{-1} s_k s_j s_k s_i s_k s_i^{-1} s_k^{-1} s_j^{-1} s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_k s_j s_i s_k s_i s_k s_i^{-1} s_k^{-1} s_i^{-1} s_k^{-1} s_j^{-1} s_k^{-1} \\
&= e
\end{aligned}
$$

We also have

$$t_j t_k^{-1} t_i t_k = s_j s_k^{-1} s_k s_i s_k^{-1} s_k = s_i s_j = t_k^{-1} t_i t_k t_j$$

ii)

$$
\begin{aligned}
t_i t_j &= s_i s_k s_j s_k^{-1} \\
&= s_i s_j^{-1} s_k s_j \\
&= s_j^{-1} s_k s_j s_i \\
&= s_k s_j s_k^{-1} s_i \\
&= t_j t_i
\end{aligned}
$$

e) i)

$$
\begin{aligned}
t_i t_j t_i t_j t_i^{-1} t_j^{-1} t_i^{-1} t_j^{-1} &= s_k s_i s_k^{-1} s_j s_k s_i s_k^{-1} s_j s_k s_i^{-1} s_k^{-1} s_j^{-1} s_k s_i^{-1} s_k^{-1} s_j^{-1} \\
&= s_i^{-1} s_k s_i s_j s_i^{-1} s_k s_i s_j s_i^{-1} s_k^{-1} s_i s_j^{-1} s_i^{-1} s_k^{-1} s_i s_j^{-1} \\
&= s_i^{-1} s_k s_j s_k s_j s_k^{-1} s_j^{-1} s_k^{-1} s_j^{-1} s_i \\
&= e
\end{aligned}
$$

We also have

$$t_j t_k^{-1} t_i t_k = s_j s_k^{-1} s_k s_i s_k^{-1} s_k = s_j s_i = s_i s_j = t_i t_j$$

ii)

$$
\begin{aligned}
s_k^{-1} t_i t_j t_i^{-1} t_j^{-1} s_k &= s_k^{-1} s_i s_k s_j s_k^{-1} s_i^{-1} s_k s_j^{-1} \\
&= s_i s_k s_i^{-1} s_j s_i s_k^{-1} s_i^{-1} s_j^{-1} \\
&= e
\end{aligned}
$$

f) i)

$$\begin{aligned} s_k^{-1}t_it_jt_it_j^{-1}t_i^{-1}t_j^{-1} &= s_is_k^{-1}s_js_ks_is_k^{-1}s_j^{-1}s_ks_i^{-1}s_k^{-1}s_j^{-1}s_k \\ &= e \end{aligned}$$

$$\begin{aligned} t_it_j^{-1}t_kt_jt_i^{-1}t_j^{-1}s_k^{-1}t_j &= s_ks_is_k^{-1}s_j^{-1}s_ks_js_ks_i^{-1}s_k^{-1}s_j^{-1}s_k^{-1}s_j \\ &= s_ks_is_js_ks_j^{-1}s_k^{-1}s_ks_i^{-1}s_k^{-1}s_j^{-1}s_k^{-1}s_j \\ &= s_ks_is_js_ks_j^{-1}s_i^{-1}s_k^{-1}s_j^{-1}s_k^{-1}s_j \\ &= s_ks_is_js_ks_j^{-1}s_i^{-1}s_js_k^{-1}s_j^{-1}s_k^{-1} \\ &= s_ks_js_ks_j^{-1}s_is_i^{-1}s_js_k^{-1}s_j^{-1}s_k \\ &= e \end{aligned}$$

ii) This follows from part (i) by symmetry