

Stochastic Gradient Descent with Momentum and Line Searches

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Abstract

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1 Introduction

1.1 Optimization problem

$$\min_{w \in \mathbb{R}^p} f(w) = L(w) + \lambda \Omega(w)$$

$$\min \sum_{i=1}^N \log(1 + \exp(-y^{(i)} w^T x^{(i)})) + \lambda \|w\|^2 \quad (1)$$

where $i = \dots, N$ are the dataset indices, $y^i \in \{-1, 1\}$ is the response variable corresponding to the negative or positive class, $x^i \in \mathbb{R}^p$ are dataset examples.

$$\nabla f(w) = X^T r + 2\lambda w, \quad r = -y^{(i)} \sigma(-y^{(i)} w^T x^{(i)})$$

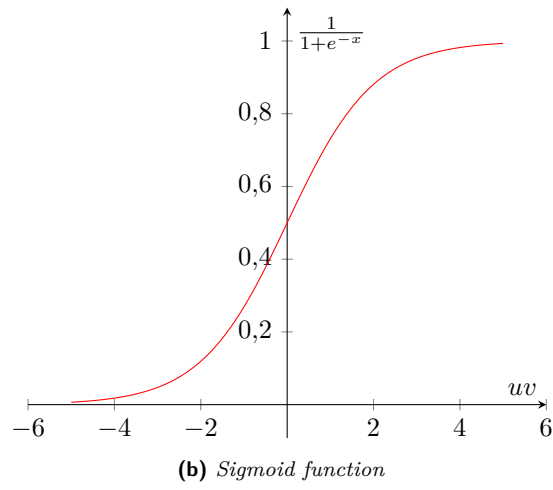
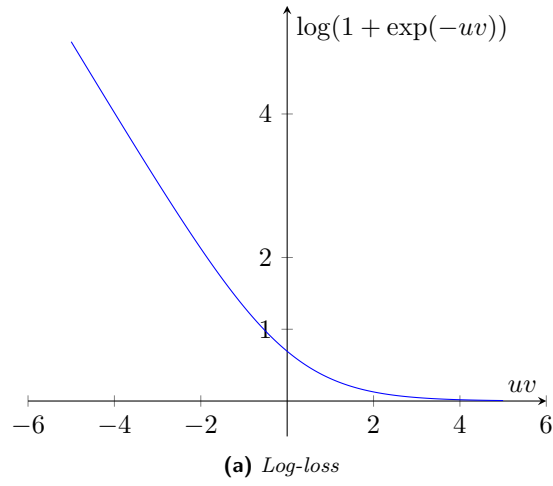
$$\nabla^2 f(w) = X^T D X + 2\lambda I, \quad d_{ii} = \sigma(y^{(i)} w^T x^{(i)}) \sigma(-y^{(i)} w^T x^{(i)})$$

Proposition 1. *Problem (1) admits a unique optimal solution.*

$$X^T = \begin{pmatrix} 1 & x_1^{(1)} & x_2^{(1)} & \dots & x_p^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & \dots & x_p^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(N)} & x_2^{(N)} & \dots & x_p^{(N)} \end{pmatrix} \in \mathbb{R}^{N \times (p+1)}$$

$$w = \begin{pmatrix} b \\ w_1 \\ w_2 \\ \vdots \\ w_p \end{pmatrix} \in \mathbb{R}^{p+1}$$

$$y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(N)} \end{pmatrix} \in \{-1, 1\}$$

**Figure 1:** Log-loss and sigmoid function plots

- $uv \gg 0$: the example is labelled correctly
- $uv \ll 0$: the class assigned to the example is the wrong one
- the hessian matrix is positive defined $\forall w$, this means that the objective function, which is quadratic, is coercive and for the continuity that function admits global minimum, so $f(w)$ has finite inferior limit
- the hessian matrix being positive defined implies also that the objective function is strictly convex (on the other hand the loss function is just convex, due to its hessian matrix being positive semi-defined), this implies that if the global minimum exists, that solution is unique
- a global minimum is a point that satisfy $\nabla f(w^*) = 0$, which is a sufficient condition implied by the convexity of the problem, see figure 1a on the preceding page
- the ℓ_2 regularization implies that the objective function is strongly convex, this speeds up the convergence
- further more we can assume that $\nabla f(w)$ is Lipschitz-continuous with constant L

2 Mini-batch gradient descent variants

$$f(w) = \sum_{i=1}^N f_i(w)$$

2.1 Fixed step-size

Mini-batch Gradient Descent with fixed or decreasing step-size

```

1 given  $w^0 \in \mathbb{R}^n$ ,  $k = 0$  e  $\{\alpha_k\} \mid \alpha_k = \alpha \vee \alpha_k = \frac{\alpha_0}{k+1}$ 
2 while  $(\|\nabla f(w^k)\| > \varepsilon(1 + |f(w)|))$ 
3   shuffle  $\{1, \dots, N\}$  and split  $B_1, \dots, B_{N/M}$  such that  $1 < |B_t| = M \ll N$ 
4   set  $y_0 = w^k$ 
5   for  $t = 1, \dots, N/M$ 
6     get mini-batch indices from  $B_t$ 
7     approximate true gradient  $\nabla f_{i_t}(w) = \frac{1}{M} \sum_{j \in B_t} \nabla f_j(y_{t-1})$ 
8     compute direction  $d_t = -\nabla f_{i_t}(y_{t-1})$ 
9     make a (internal) step  $y_t = y_{t-1} + \alpha_k d_t$ 
10  end for
11  update weights  $w^{k+1} = y_{N/M}$ 
12  epoch ends  $k = k + 1$ 
13 end while

```

2.2 Stochastic line search

Mini-batch Gradient Descent with Armijo line search

```

1 given  $w^0 \in \mathbb{R}^{p+1}$ ,  $\gamma \in (0, 1)$ ,  $\delta \in (0, 1)$ ,  $\alpha_0 \in \mathbb{R}^+$ 
2  $k = 0$ 
3 while  $(\|\nabla f(w^k)\| > \varepsilon(1 + |f(w)|))$ 
4   shuffle  $\{1, \dots, N\}$  and split  $B_1, \dots, B_{N/M}$  such that  $1 < |B_t| = M \ll N$ 
5   set  $y_0 = w^k$ 
6   for  $t = 1, \dots, N/M$ 
7     get mini-batch indices  $i_t$  from  $B_t$ 
8     approximate true gradient  $\nabla f_{i_t}(w) = \frac{1}{M} \sum_{j \in B_t} \nabla f_j(y_{t-1})$ 
9     compute direction  $d_t = -\nabla f_{i_t}(y_{t-1})$ 
10     $\alpha = \text{reset}()$ ,  $q = 0$ 
11    compute potential next step  $y_t = y_{t-1} + \alpha d_t$ 
12    while  $(f_{i_t}(y_t) > f_{i_t}(y_{t-1}) + \gamma \alpha \nabla f_{i_t}(y_{t-1})^T d_t)$ 
13      reduce step-size  $\alpha = \delta \alpha$ 
14      rejections counter  $q = q + 1$ 
15    end while
16    set optimal mini-batch step-size  $\alpha_t = \alpha$ 
17    make a (internal) step  $y_t = y_{t-1} + \alpha_t d_t$ 
18  end for
19  update weights  $w^{k+1} = y_{N/M}$ 
20  epoch ends  $k = k + 1$ 
21 end while

```

2.3 Fixed momentum term

Mini-batch Gradient Descent with fixed Momentum term and fixed step-size

```

1 given  $w^0 \in \mathbb{R}^{p+1}$ ,  $\{\alpha_k\} = \alpha$ ,  $\{\beta_k\} = \beta \in (0, 1)$ 
2  $k = 0$ 
3 while  $(\|\nabla f(w^k)\| > \varepsilon(1 + |f(w)|))$ 
4   shuffle  $\{1, \dots, N\}$  and split  $B_1, \dots, B_{N/M}$  such that  $1 < |B_t| = M \ll N$ 
5   set  $y_0 = w^k$ ,  $d_0 = 0$ 
6   for  $t = 1, \dots, N/M$ 
7     get mini-batch indices  $i_t$  from  $B_t$ 
8     approximate true gradient  $\nabla f_{i_t}(w) = \frac{1}{M} \sum_{j \in B_t} \nabla f_j(y_{t-1})$ 
9     compute direction  $d_t = -((1 - \beta)\nabla f_{i_t}(y_{t-1}) + \beta d_{t-1})$ 
10    make a (internal) step  $y_t = y_{t-1} + \alpha_k d_t$ 
11  end for
12  update weights  $w^{k+1} = y_{N/M}$ 
13  epoch ends  $k = k + 1$ 
14 end while

```

Mini-batch Gradient Descent with Armijo line search for step-size and Momentum correction

```

1 given  $w^0 \in \mathbb{R}^{p+1}$ ,  $\gamma \in (0, 1)$ ,  $\delta_a \in (0, 1)$ ,  $\alpha_0 \in \mathbb{R}^+$ ,  $\delta_m \in (0, 1)$ ,  $\beta_0 \in (0, 1)$ 
2  $k = 0$ 
3 while  $(\|\nabla f(w^k)\| > \varepsilon(1 + |f(w)|))$ 
4   shuffle  $\{1, \dots, N\}$  and split  $B_1, \dots, B_{N/M}$  such that  $1 < |B_t| = M \ll N$ 
5   set  $y_0 = w^k$ ,  $d_0 = 0$ 
6   for  $t = 1, \dots, N/M$ 
7     get mini-batch indices  $i_t$  from  $B_t$ 
8     approximate true gradient  $\nabla f_{i_t}(w) = \frac{1}{M} \sum_{j \in B_t} \nabla f_j(y_{t-1})$ 
9     compute potential next direction  $d_t = -((1 - \beta)\nabla f_{i_t}(y_{t-1}) + \beta d_{t-1})$ 
10     $q_m = 0$ 
11    while  $(\nabla f_{i_t}(y_{t-1})^T d_t \geq 0)$ 
12      reduce momentum term  $\beta = \delta_m \beta$ 
13      rejections counter  $q_m = q_m + 1$ 
14    end while
15    set optimal mini-batch momentum term  $\beta_t = \beta$ 
16     $\alpha = \text{reset}()$ ,  $q_a = 0$ 
17    compute potential next step  $y_t = y_{t-1} + \alpha d_{t-1}$ 
18    while  $(f_{i_t}(y_t) > f_{i_t}(y_{t-1}) + \gamma \alpha \nabla f_{i_t}(y_{t-1})^T d_t)$ 
19      reduce step-size  $\alpha = \delta_a \alpha$ 
20      rejections counter  $q_a = q_a + 1$ 
21    end while
22    set optimal mini-batch step-size  $\alpha_t = \alpha$ 
23    make a (internal) step  $y_t = y_{t-1} + \alpha_t d_t$ 
24  end for
25  update weights  $w^{k+1} = y_{N/M}$ 
26  epoch ends  $k = k + 1$ 
27 end while

```

Mini-batch Gradient Descent with Armijo line search for step-size and Momentum restart

```

1 given  $w^0 \in \mathbb{R}^{p+1}$ ,  $\gamma \in (0, 1)$ ,  $\delta_a \in (0, 1)$ ,  $\alpha_0 \in \mathbb{R}^+$ ,  $\delta_m \in (0, 1)$ ,  $\{\beta_k\} = \beta \in (0, 1)$ 
2  $k = 0$ 
3 while  $(\|\nabla f(w^k)\| > \varepsilon(1 + |f(w)|))$ 
4   shuffle  $\{1, \dots, N\}$  and split  $B_1, \dots, B_{N/M}$  such that  $1 < |B_t| = M \ll N$ 
5   set  $y_0 = w^k$ ,  $d_0 = 0$ 
6   for  $t = 1, \dots, N/M$ 
7     get mini-batch indices  $i_t$  from  $B_t$ 
8     approximate true gradient  $\nabla f_{i_t}(w) = \frac{1}{M} \sum_{j \in B_t} \nabla f_j(y_{t-1})$ 
9     compute potential next direction  $d_t = -((1 - \beta)\nabla f_{i_t}(y_{t-1}) + \beta d_{t-1})$ 
10    if  $(\nabla f_{i_t}(y_{t-1})^T d_t \geq 0)$ 
11      restart direction  $d_t = d_0$ 
12    end if
13    set mini-batch momentum term  $\beta_t = \beta_0$ 
14     $\alpha = \text{reset}()$ ,  $q_a = 0$ 
15    compute potential next step  $y_t = y_{t-1} + \alpha d_{t-1}$ 
16    while  $(f_{i_t}(y_t) > f_{i_t}(y_{t-1}) + \gamma \alpha \nabla f_{i_t}(y_{t-1})^T d_t)$ 
17      reduce step-size  $\alpha = \delta_a \alpha$ 
18      rejections counter  $q_a = q_a + 1$ 
19    end while
20    set optimal mini-batch step-size  $\alpha_t = \alpha$ 
21    make a (internal) step  $y_t = y_{t-1} + \alpha_t d_t$ 
22  end for
23  update weights  $w^{k+1} = y_{N/M}$ 
24  epoch ends  $k = k + 1$ 
25 end while

```

3 Experiments