Project One Template

MAT325: Calculus III: Multivariable Calculus

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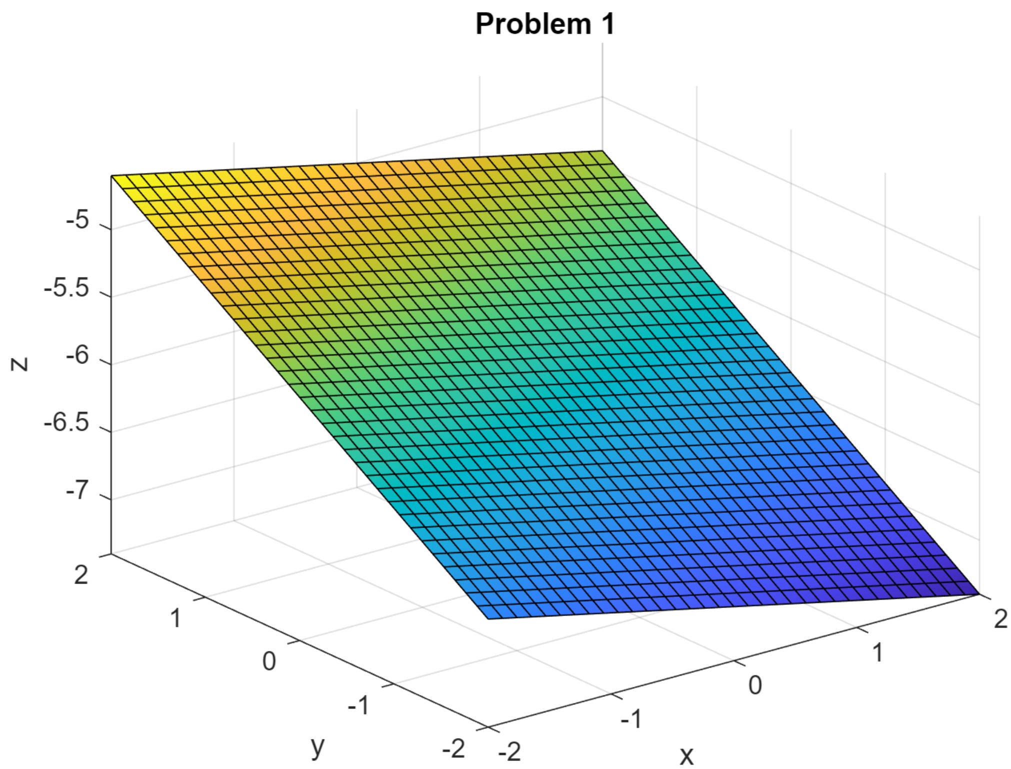
Problem 1: Consider the plane defined by the equation $2x - 5y + 10z + 60 = 0$.

a) Use MATLAB and the surf() function to plot the plane over the interval $x \in [-2, 2]$, $y \in [-2, 2]$. Choose an appropriate view to best visualize the plane and label axes appropriately.

Solution:

```
syms x y;

z(x,y) = (-2*x+5*y-60) ./ 10;

figure;
fsurf(z,[-2,2,-2,2]);
xlabel('x');
ylabel('y');
zlabel('z');
title('Problem 1');
grid on;
```



b) Where does the plane intersect the x-axis, y-axis, and z-axis? Provide a coordinate (x, y, z) for each intersection.

Solution:

x-axis: $2x - 5(0) + 10(0) + 60 = 0$

$2x = -60$ therefore: $(-30, 0, 0)$

y-axis: $2(0) - 5y + 10(0) + 60 = 0$

$-5y = -60$ therfore: $(0, 12, 0)$

z-axis: $2(0) - 5(0) + 10z + 60 = 0$

$10z = -60$ therefore: $(0, 0, -6)$

c) Compute the normal vector $\mathbf{n}$ to the plane. Express $\mathbf{n}$ in terms of the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution:

$2x - 5y + 10z + 60 = 0$

$2x - 5y + 10z$

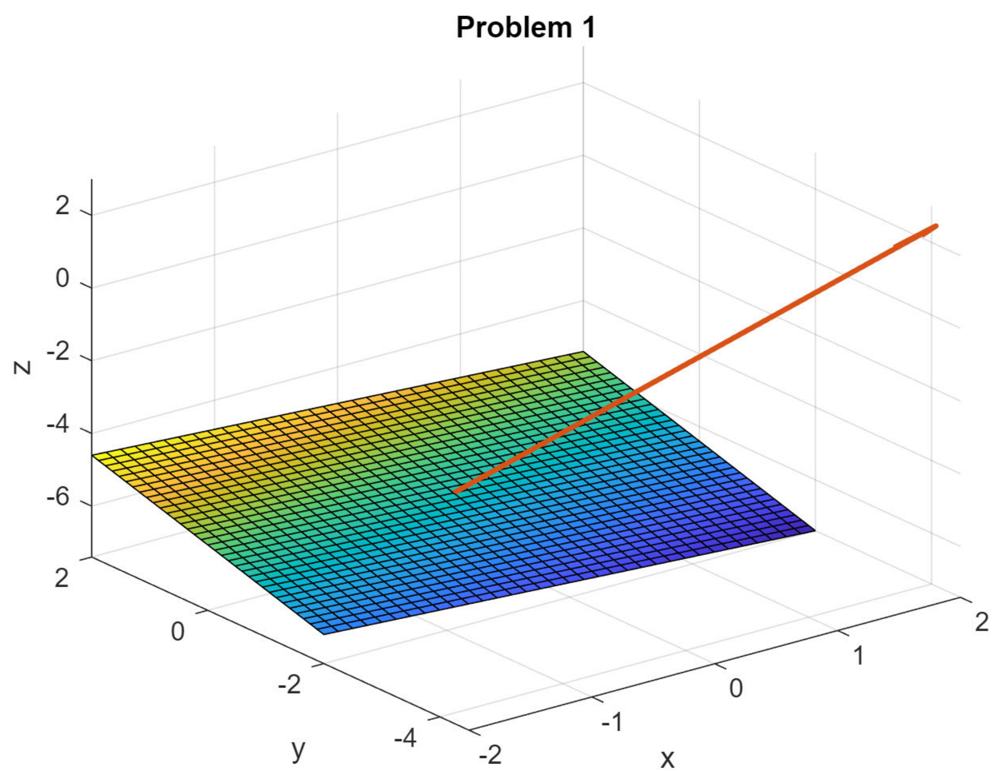
$\langle 2, -5, 10 \rangle$

$2i - 5j + 10k$

d) Use MATLAB to plot the plane (as in part (a)) and a normal vector to the plane using the surfnorm() function at the point $(0, 0, -6)$. Normalize and scale the surfnorm()-computed vector to show it equals the value found in part (c). Make sure to use the axis equal command so the plane and vector look orthogonal as desired.

Solution:

```
syms x y;

z(x,y) = (-2*x+5*y-60) ./ 10;
normV = [2 -5 10];
Z = [0 0 -6];

figure;
fsurf(z,[-2,2,-2,2]);
hold on;
quiver3(Z(1),Z(2),Z(3),normV(1),normV(2),normV(3),'linewidth',2);
xlabel('x');
ylabel('y');
zlabel('z');
title('Problem 1');
grid on;
```

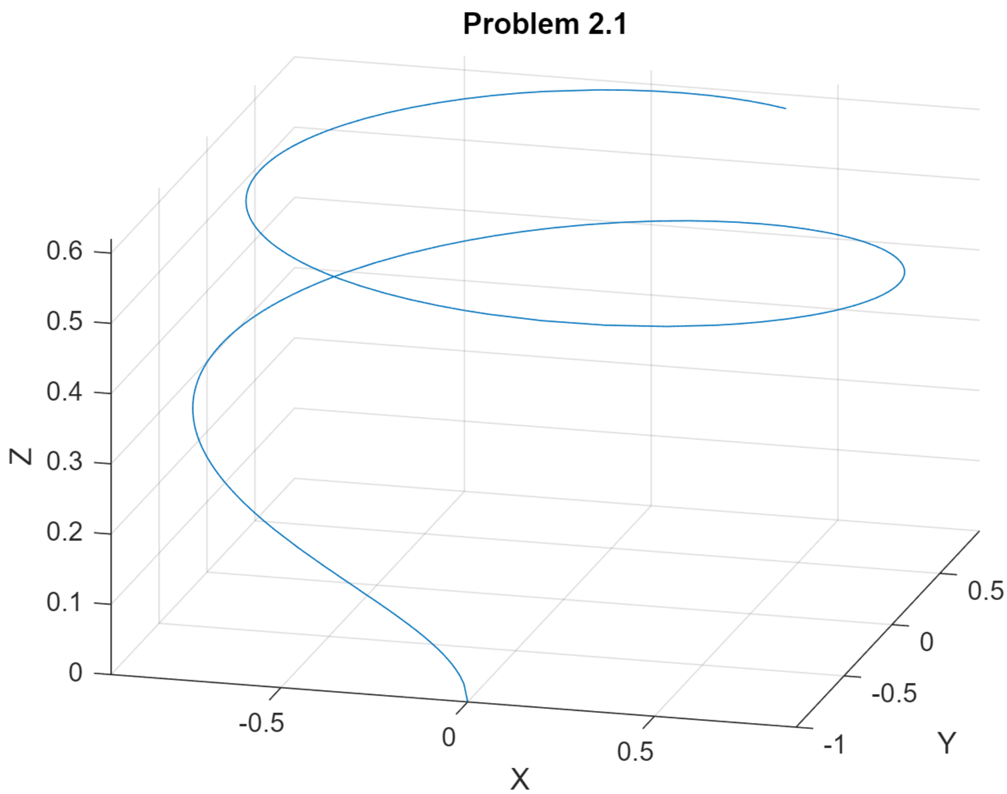


Problem 2: Consider a moving projectile with position vector given as $\mathbf{r}(t) = 2\cos(t)\mathbf{i} - 2\sin(t)\mathbf{j} + \sqrt{t}\,\mathbf{k}$.

a) Use MATLAB and the plot3() function to plot the projectile location for time $t \in [0, 10]$. Choose an appropriate view to best visualize the curve and label axes appropriately.

Solution:

```
clear all;
syms t;

rPrimeX(t) = -2*sin(t);
rPrimeY(t) = -2*cos(t);
rPrimeZ(t) = 1 ./ 2*sqrt(t);

rNorm(t) = sqrt(rPrimeX(t).^2+rPrimeY(t).^2+rPrimeZ(t).^2);

TX(t) = rPrimeX(t)./rNorm(t);
TY(t) = rPrimeY(t)./rNorm(t);
TZ(t) = rPrimeZ(t)./rNorm(t);

figure;
fplot3(TX(t),TY(t),TZ(t),[0,10]);
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 2.1');
view([15 25]);
```



b) Compute the velocity vector $\mathbf{v}(t)$ and acceleration vector $\mathbf{a}(t)$.

Solution:

$$v(t) = \frac{d}{\mathrm{dt}}\left(2\cos(t)i - 2\sin(t)j + \sqrt{t}\,k\right)$$

$$v(t) = -2\sin(t)i - 2\cos(t)j + \frac{1}{2\sqrt{t}}k$$

$$a(t) = \frac{d}{\mathrm{dt}}\left(v(t) = -2\sin(t)i - 2\cos(t)j + \frac{1}{2\sqrt{t}}k\right)$$

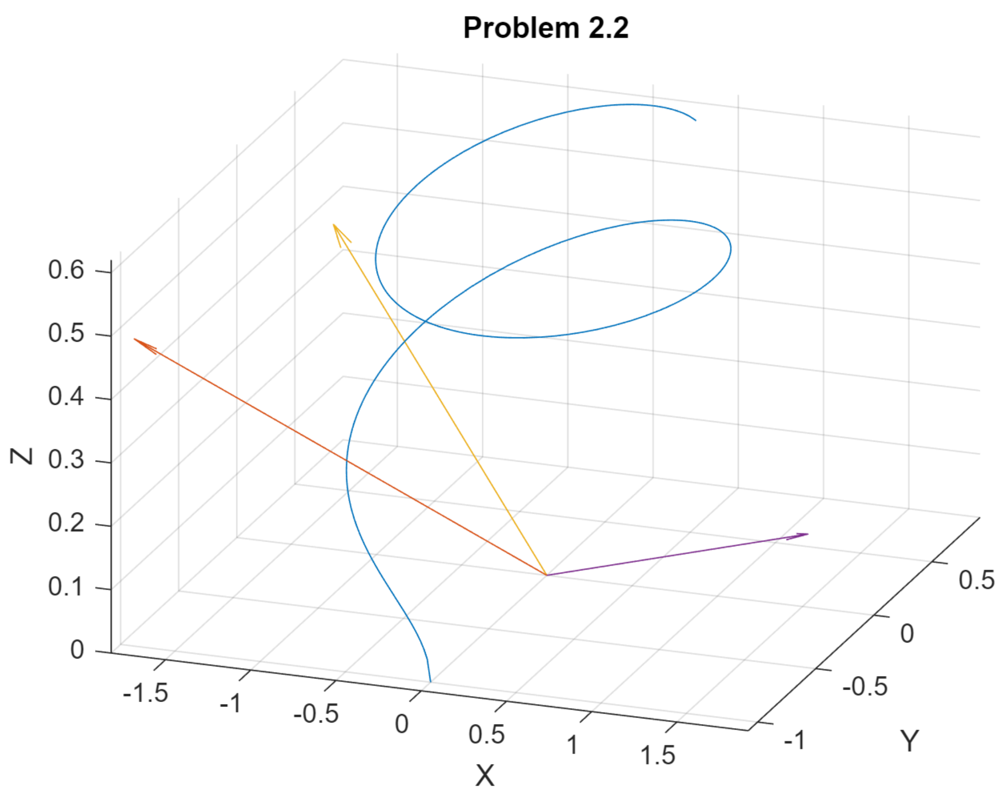
$$a(t) = -2\cos(t)i + 2\sin(t)j - \frac{1}{4\sqrt{t^3}}k$$

c) Use MATLAB to replicate your plot from part (a) and then use the quiver3() function to plot the velocity vector of the particle at times $t = 1$, $t = 2$, and $t = 5$. Choose an appropriate view to best visualize the curve and label axes appropriately.

Solution:

```
clear all;
syms t;

rPrimeX(t) = -2*sin(t);
```

```
rPrimeY(t) = -2*cos(t);
rPrimeZ(t) = 1 ./ 2*sqrt(t);

rNorm(t) = sqrt(rPrimeX(t).^2+rPrimeY(t).^2+rPrimeZ(t).^2);

TX(t) = rPrimeX(t)./rNorm(t);
TY(t) = rPrimeY(t)./rNorm(t);
TZ(t) = rPrimeZ(t)./rNorm(t);

t1 = [-2*sin(1) -2*cos(1) 0.5];
t2 = [-2*sin(2) -2*cos(2) (sqrt(2))./4];
t3 = [-2*sin(5) -2*cos(5) (sqrt(5))./10];

figure;
fplot3(TX(t),TY(t),TZ(t),[0,10]);
hold on;
quiver3(0,0,0,t1(1),t1(2),t1(3),'off');
quiver3(0,0,0,t2(1),t2(2),t2(3),'off');
quiver3(0,0,0,t3(1),t3(2),t3(3),'off');
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 2.2');
view([20 30]);
```



d) Compute unit normal vector $\mathbf{N}(t)$.

Solution:

$$v(t) = -2\sin(t)i - 2\cos(t)j + \frac{1}{2\sqrt{t}}k$$

$$||v(t)|| = \sqrt{(-2\sin(t))^2 + (-2\cos(t))^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}$$

$$\sqrt{4\sin^2 t + 4\cos^2 t + \frac{1}{4t}}$$

$$\sqrt{4 + \frac{1}{4t}}$$

$$T(t) = \frac{-2\sin(t)i - 2\cos(t)j + \frac{1}{2\sqrt{t}}k}{\sqrt{4 + \frac{1}{4t}}}$$

$$T'(t) = \frac{-2\cos(t)i + 2\sin(t)j - \frac{1}{4 \cdot t^{\frac{3}{2}}}k}{-\frac{1}{4 \cdot t^{\frac{3}{2}}\sqrt{16t+1}}} \;=\; -4t^{\frac{3}{2}}\sqrt{16t+1}\left(2\sin(t)j - \frac{k}{4t^{\frac{3}{2}}}\right) + 4t^{\frac{3}{2}}\sqrt{16t+1}\,(2\cos(t)i)$$

$$T'(t) = 8t^{\frac{3}{2}}\sqrt{16t+1}\cos(t)i - 8t^{\frac{3}{2}}\sqrt{16t+1}\sin(t)j + \sqrt{16t+1}\,k$$

$$||T'(t)|| = \sqrt{\left(8t^{\frac{3}{2}}\sqrt{16t+1}\cos(t)\right)^2 - \left(8t^{\frac{3}{2}}\sqrt{16t+1}\sin(t)\right)^2 + \left(\sqrt{16t+1}\right)^2}$$

$$\sqrt{64t^3(16t+1)\cos^2 t + 64t^3(16t+1)\sin^2 t + 16t + 1}$$

$$||T'(t)|| = \sqrt{64t^3(16t+1) + 16t + 1}$$

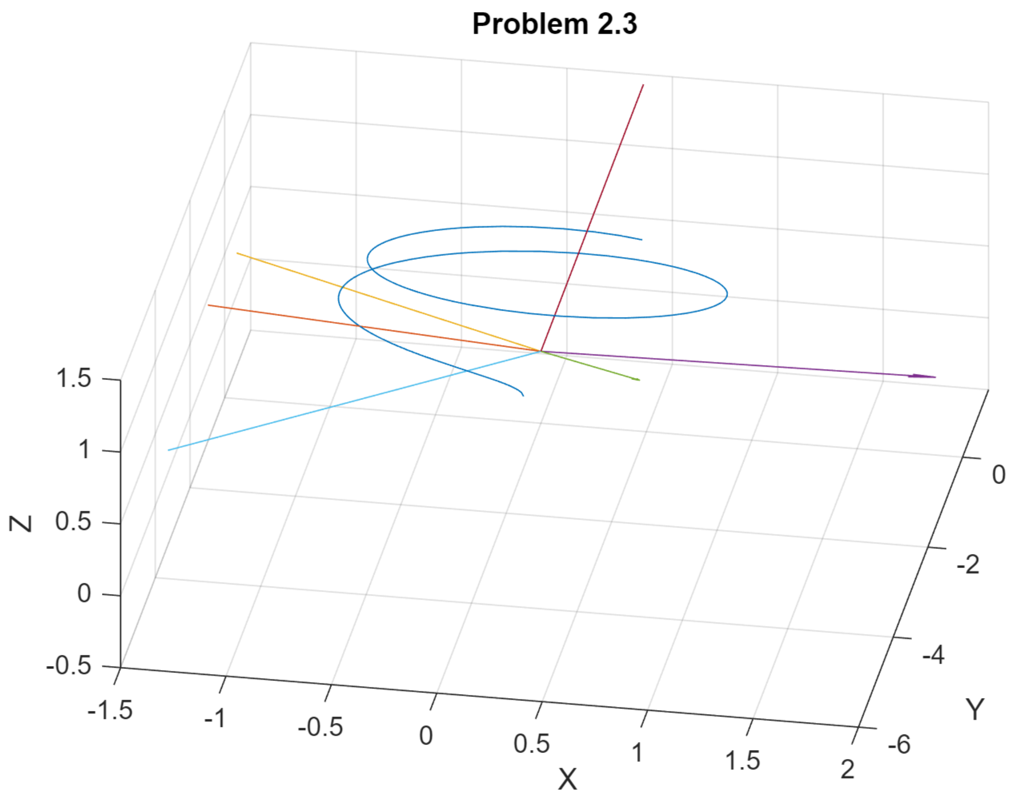
$$N(t) = \frac{8t^{\frac{3}{2}}\sqrt{16t+1}\cos(t)i - 8t^{\frac{3}{2}}\sqrt{16t+1}\sin(t)j + \sqrt{16t+1}\,k}{\sqrt{64t^3(16t+1) + 16t + 1}}$$

e) Use MATLAB to replicate your plot from part (c) and then use the quiver3() function to plot the unit normal vector of the particle at times $t = 1$, $t = 2$, and $t = 5$. Your figure should now contain the original curve, the velocity vectors from part (c), and the unit normal vectors from part (d). Choose an appropriate view to best visualize the curve and label axes appropriately.

Solution:

```
clear all;
```

```
syms t;

rPrimeX(t) = -2*sin(t);
rPrimeY(t) = -2*cos(t);
rPrimeZ(t) = 1 ./ 2*sqrt(t);

rNorm(t) = sqrt(rPrimeX(t).^2+rPrimeY(t).^2+rPrimeZ(t).^2);

TX(t) = rPrimeX(t)./rNorm(t);
TY(t) = rPrimeY(t)./rNorm(t);
TZ(t) = rPrimeZ(t)./rNorm(t);

t1 = [-1.683 -1.081 0.5];
t2 = [-1.819 0.832 0.354];
t3 = [1.918 -0.567 0.224];

n1 = [0.536 -0.835 0.124];
n2 = [-23.697 -51.779 2.517];
n3 = [2.522 8.524 9.939];

figure;
fplot3(TX(t),TY(t),TZ(t),[0 10]);
hold on;
quiver3(0,0,0,t1(1),t1(2),t1(3),'off');
quiver3(0,0,0,t2(1),t2(2),t2(3),'off');
quiver3(0,0,0,t3(1),t3(2),t3(3),'off');
quiver3(0,0,0,n1(1),n1(2),n1(3),'off');
quiver3(0,0,0,n2(1),n2(2),n2(3),'off');
quiver3(0,0,0,n3(1),n3(2),n3(3),'off');
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 2.3');
view([10 50]);

xlim([-1.5 2])
ylim([-6 1.5])
zlim([-0.5 1.5])
```




```
Vt = [-1.683 -1.081 0.5; -1.819 0.832 0.354; 1.918 -0.567 0.224];
Nt = [0.536 -0.835 0.124; -23.697 -51.779 2.517; 2.522 8.524 9.939];
dot(Vt,Nt)
```

```
ans = 1×3
   47.0400   -47.0106     3.1794
```

f) Compute an expression for the dot product between the velocity vector $\mathbf{v}(t)$ and unit normal vector $\mathbf{N}(t)$, i.e. compute $\mathbf{v}(t) \cdot \mathbf{N}(t)$. For values of t computed for the figure, does your expression for the dot product seem reasonable based on your plots from part (e)? Yes or no? Explain.

Solution:

$$v(t) = \begin{bmatrix} -1.683 & -1.081 & 0.5 \\ -1.819 & 0.832 & 0.354 \\ 1.918 & -0.567 & 0.224 \end{bmatrix}$$

$$N(t) = \begin{bmatrix} 0.536 & -0.835 & 0.124 \\ -23.697 & -51.779 & 2.517 \\ 2.522 & 8.524 & 9.939 \end{bmatrix}$$

$$v(t) \cdot N(t) = [47.04 \quad -47.012 \quad 3.179]$$

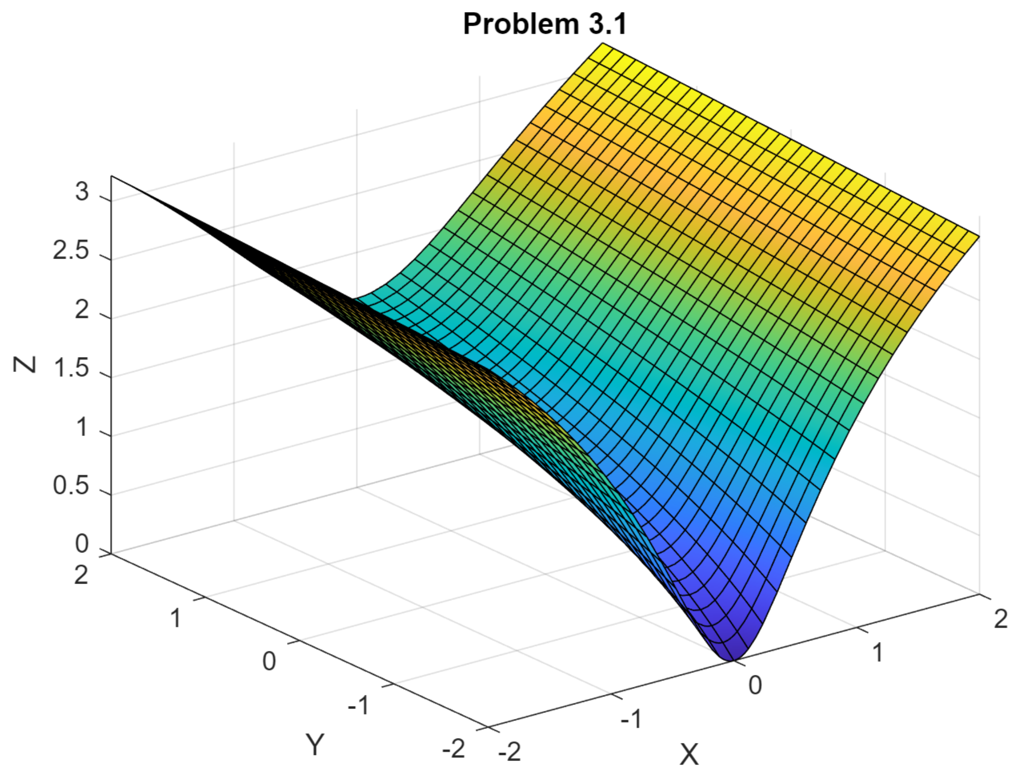
Honestly, I have no idea if the expression seems reasonable. I am not 100% sure that I computed the unit normal vector correctly. Based off of the given equations, the values in the matrices are rounded to the third decimal place for simplicity. The computation was done by MATLAB (as seen above).

Problem 3: Consider the surface defined by the equation $z = \ln(5x^2 + y + 3)$.

a) Use MATLAB and the surf() function to plot the plane over the interval $x \in [-2, 2]$, $y \in [-2, 2]$. Choose an appropriate view to best visualize the plane and label axes appropriately.

Solution:

```
clear all;
syms x y;

z(x,y) = log(5*x^2+y+3);

figure;
fsurf(z(x,y),[-2,2,-2,2]);
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 3.1');
grid on;
```



b) Find an equation for a tangent plane to the surface at the point $P = (1, 1)$.

Solution:

$z = \ln(5x^2 + y + 3)$

$z_0 = \ln(5(1)^1 + 1 + 3) = \ln(9)$

$$\frac{\partial z}{\partial x} = \frac{d}{\mathrm{dx}} \ln(5x^2 + y + 3) = \frac{10x}{5x^2 + y + 3} = \frac{10}{9}$$

$$\frac{\partial z}{\partial y} = \frac{d}{\mathrm{dy}} \ln(5x^2 + y + 3) = \frac{1}{5x^2 + y + 3} = \frac{1}{9}$$

$$z - \ln(9) = \frac{10}{9}(x - 1) + \frac{1}{9}(y - 1)$$

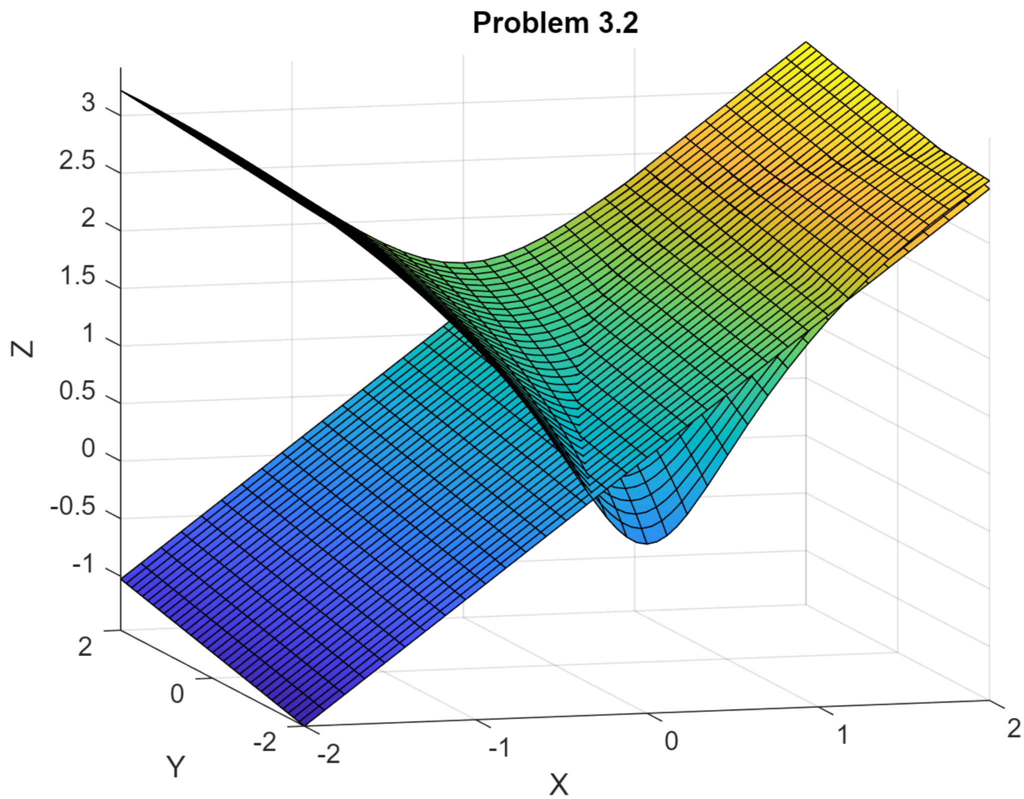
$$z = \frac{10}{9}(x - 1) + \frac{1}{9}(y - 1) + \ln(9)$$

c) Use MATLAB to replicate your plot from part (a) but now also plot the tangent surface computed in part (b). Also, use the plot3() function to plot the point where the tangent plane was computed at. Set the view so it's clear the tangent plane is indeed tangent to the surface at the given point.

Solution:

```
clear all;
```

```
syms x y;

z(x,y) = log(5*x^2+y+3);
z0(x,y) = (10./9)*(x-1)+(1./9)*(y-1)+log(9);

figure;
fsurf(z(x,y),[-2,2,-2,2]);
hold on;
fsurf(z0(x,y),[-2,2,-2,2]);
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 3.2');
view([-15 10]);
grid on;
```

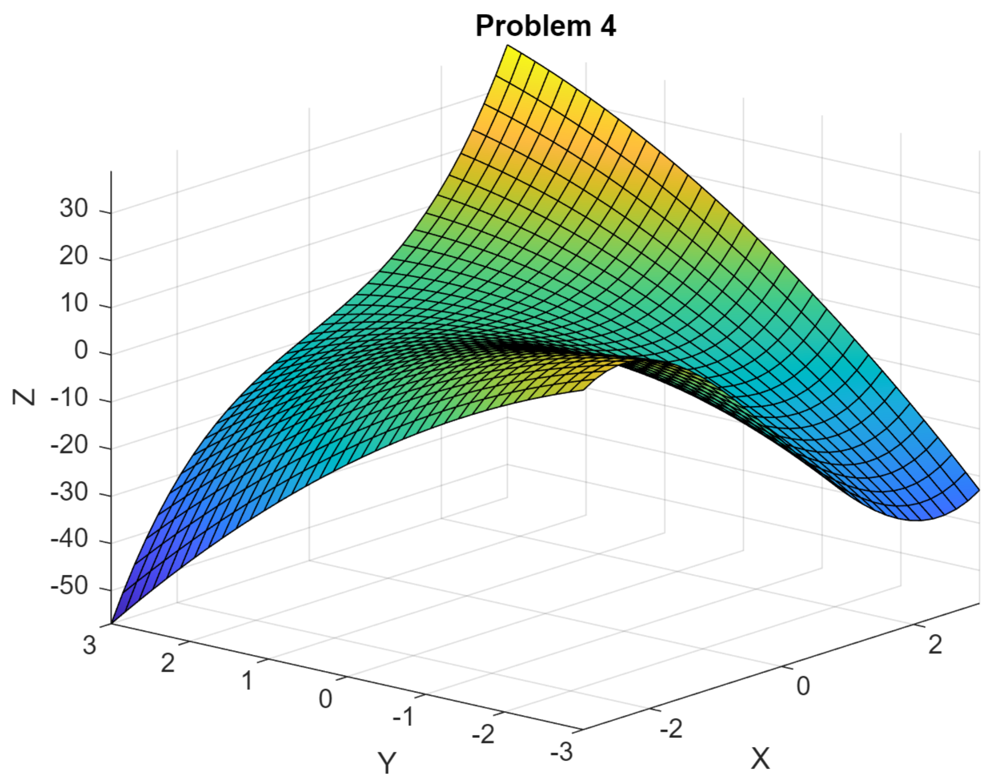


Problem 4: Consider the function $f(x, y) = x^3 - 5x + 4xy - y^2$ for $x \in [-3, 3]$, $y \in [-3, 3]$.

a) Use MATLAB and the surf() function to plot the surface over the given closed and bounded region. Choose an appropriate view to best visualize the plane and label axes appropriately.

Solution:

```
clear all;
syms x y;

f(x,y) = x^3-5*x+4*x*y-y^2;

figure;
fsurf(f,[-3,3,-3,3]);
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 4');
view([-50 20]);
grid on;
```



b) Find the critical points of the function on the bounded region and use the MATLAB plot3() function to plot the critical points on the surface from part (a).

Solution:

$f(x,y) = x^3 - 5x + 4xy - y^2$

$\frac{\partial f}{\partial x} = 3x^2 - 5 + 4y$

$\frac{\partial f}{\partial y} = 4x - 2y$

$3x^2 - 5 + 4y = 0$ and $4x - 2y = 0$ where $y = 2x$

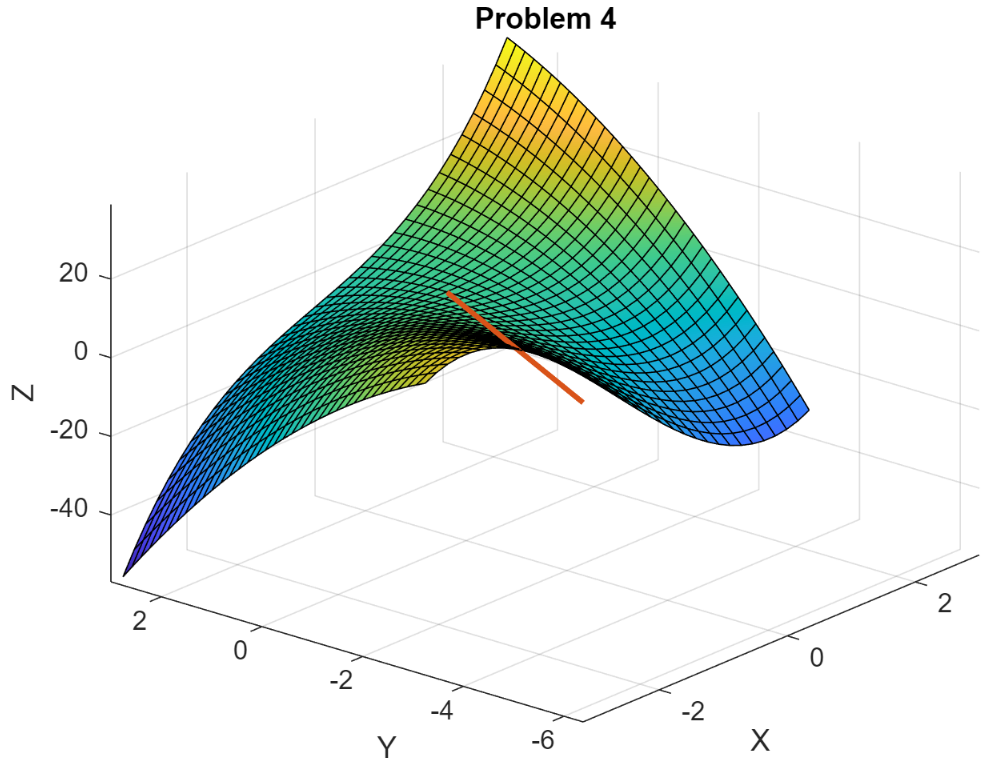
$3x^2 - 5 + 4(2x) = 3x^2 + 8x - 5 = 0$

$x = \frac{-8 \pm \sqrt{8^2 - 4(3)(-5)}}{2(3)}$

$x = \frac{-4 \pm \sqrt{31}}{3}$ therefore $y = 2 \cdot \frac{-4 \pm \sqrt{31}}{3}$

$\left(\frac{-4 + \sqrt{31}}{3}, \frac{-8 + 2\sqrt{31}}{3}\right)$ and $\left(\frac{-4 - \sqrt{31}}{3}, \frac{-8 - 2\sqrt{31}}{3}\right)$ are the critical points.

```
clear all;
syms x y;

f(x,y) = x^3-5*x+4*x*y-y^2;

x1 = (-4+sqrt(31))./3;
x2 = (-4-sqrt(31))./3;
y1 = (-8+2*sqrt(31))./3;
y2 = (-8-2*sqrt(31))./3;
z1 = f(x1, y1);
z2 = f(x2, y2);

X = [x1 x2];
Y = [y1 y2];
Z = [z1 z2];

figure;
fsurf(f,[-3,3,-3,3]);
hold on;
plot3(X,Y,Z,'linewidth',2);
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 4');
view([-50 30]);
grid on;
```



c) Use the second derivative test to classify each critical point as either a local maximum, local minimum, or saddle point.

Solution:

$\dfrac{\partial f}{\partial x} = 3x^2 - 5 + 4y$ and $\dfrac{\partial f}{\partial y} = 4x - 2y$

$\dfrac{\partial^2 f}{\partial^2 x} = 6x$ and $\dfrac{\partial^2 f}{\partial^2 y} = -2$

$\dfrac{\partial^2 f}{\partial^2 x \partial^2 y} = 4$

$D(x, y) = 6x(-2) - 4^2 = -12x - 16$

$D_1 = -12\left(\dfrac{-4 + \sqrt{31}}{3}\right) + 16$ and $D_2 = -12\left(\dfrac{-4 - \sqrt{31}}{3}\right) + 16$

$D_1 = -4\sqrt{31}$ and $D_2 = 4\sqrt{31}$

Since $D_1 < 0$ then $\left(\dfrac{-4 + \sqrt{31}}{3}, \dfrac{-8 + 2\sqrt{31}}{3}\right)$ is a saddle point.

Since $D_2 > 0$ we check $D = 6\left(\frac{-4-\sqrt{31}}{3}\right)$ which is $-8-2\sqrt{31}$. Since $D < 0$ then $\left(\frac{-4-\sqrt{31}}{3}, \frac{-8-2\sqrt{31}}{3}\right)$ is a local maximum.