MAT 330: Differential Equations

Project Two Template

Complete this template by replacing the bracketed text with the relevant information.

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Problem 1: Consider the system of differential equations $\frac{dx_1}{dt} = x_1 + 3x_2$, $\frac{dx_2}{dt} = 4x_1 + 5x_2$ with initial conditions $x_1(0) = 2$ and $x_2(0) = -6$.

a) Compute the eigenvalues and eigenvectors of the system.

Solution:

$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$

$(A - 7I)v_1 = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}(x, y)$

$\det(A - \lambda I) = 0$

$-6x + 3y = 0$ and $4x - 2y = 0 \ \rightarrow y = 2x$

$\det\begin{bmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) - (3)(4) = 0$

Let $x = 1$

$\lambda^2 - 6\lambda - 7 = 0 \ \rightarrow (\lambda - 7)(\lambda + 1) = 0$

$(A - (-1)I)v_2 = \begin{bmatrix} 1-(-1) & 3 \\ 4 & 5-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}(x, y)$

Eigenvalues: $\lambda_1 = 7$ and $\lambda_2 = -1$

$2x + 3y = 0$ and

$4x + 6y = 0 \ \rightarrow x = -\frac{3}{2}y$

Let $y = 2$ and $x = -3$

Eigenvectors: $v_1 = \langle 1, 2 \rangle$ and $v_2 = \langle -3, 2 \rangle$

b) Compute the solution to the system of differential equations.

Solution:

$$(x_1(t)x_2(t)) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2$$

$$(x_1(t)x_2(t)) = c_1e^{7t}\langle 1, 2\rangle + c_2e^{-t}\langle -3, 2\rangle$$

$x_1 = c_1e^{7t} - 3c_2e^{-t}$ $\qquad$ $x_2 = 2c_1e^{7t} + 2c_2e^{-t}$

$x_1(0) = c_1 - 3c_2 = 2$ $\qquad$ $x_2(0) = 2c_1 + 2c_2 = -6$

Use the second equation to get c_1 then substitute into the first equation to get c_2

$c_1 = -3 - c_2$ $\qquad$ $c_2 = -\dfrac{5}{4}$

$$c_1 = -3 - \left(-\frac{5}{4}\right) = -\frac{12}{4} + \frac{5}{4} = -\frac{7}{4}$$

Substitute values for c_1 and c_2 into $x_1(t)$ and $x_2(t)$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{7}{4}e^{7t} + \frac{15}{4}e^{-t} \\ -\frac{7}{2}e^{7t} - \frac{5}{2}e^{-t} \end{bmatrix}$$

c) Use MATLAB to plot the direction field and eigenvectors of the system.

Solution:

```
A = [1 3; 4 5];
[V,D] = eig(A);
v1 = V(:,1)./min(V(:,1))
```

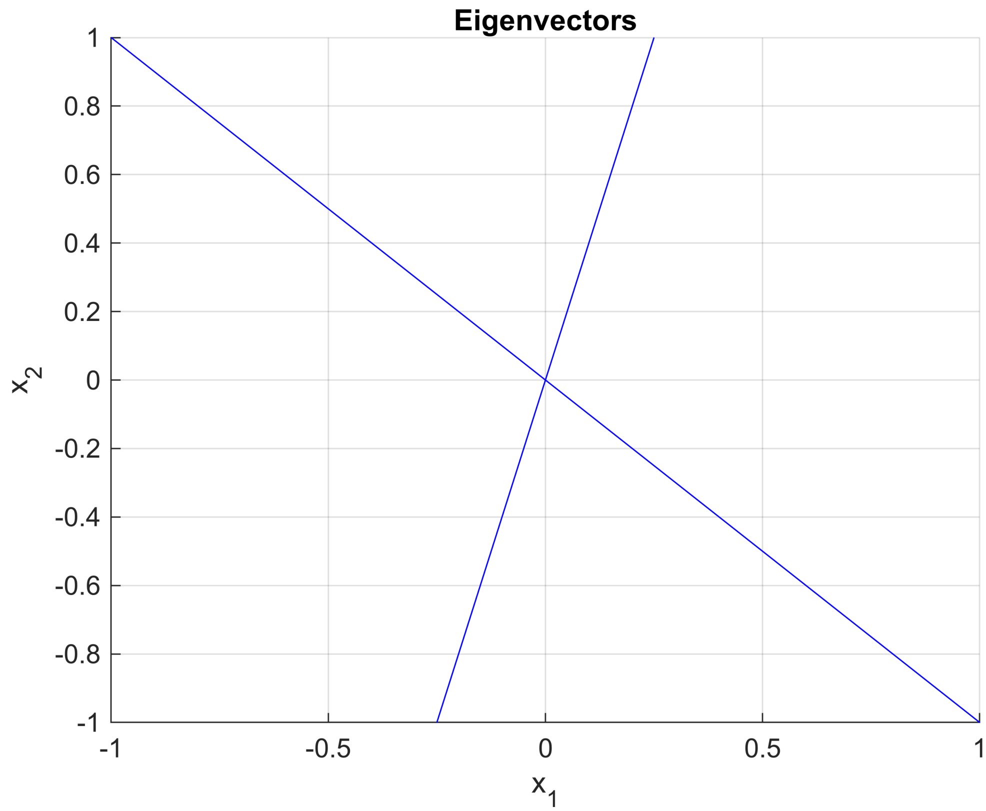
```
v1 = 2×1
    1.0000
   -0.6667
```

```
v2 = V(:,2)./min(V(:,2))
```

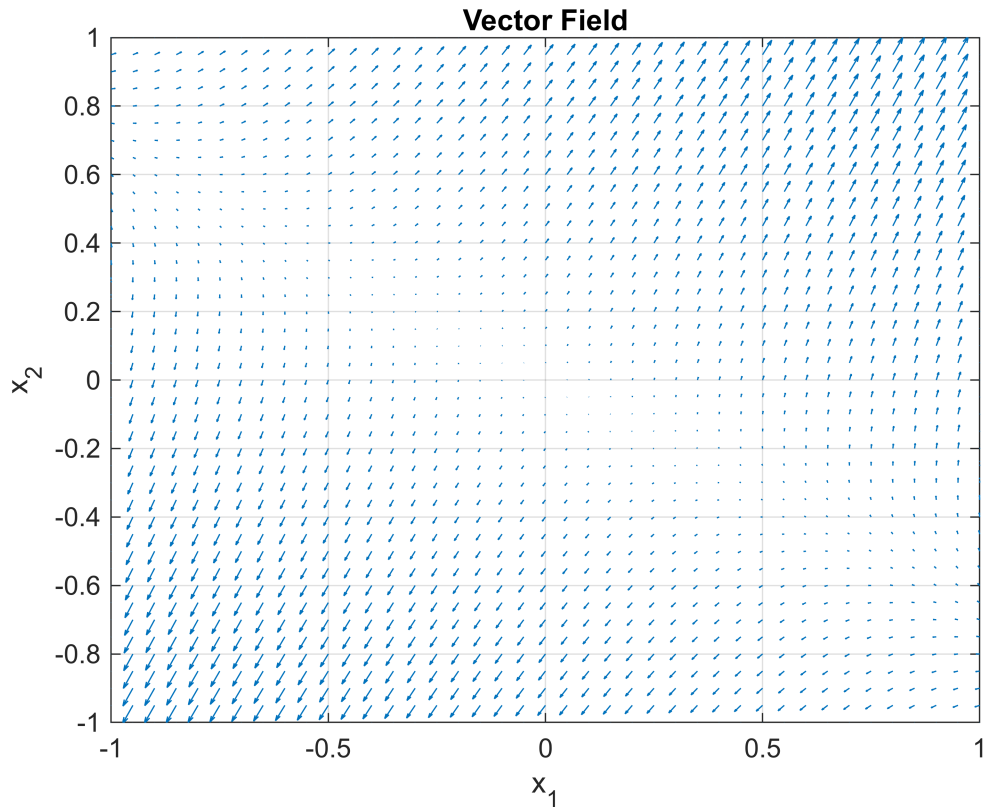
```
v2 = 2×1
    0.5000
    1.0000
```

```
x1Vec = -1:0.05:1;
x2Vec = -1:0.05:1;
vec1 = -1*x1Vec;
vec2 = 4*x2Vec;

figure;
hold on;
plot(x1Vec,vec1,'b');
plot(x2Vec,vec2,'b');
xlabel('x_{1}');
```

```
ylabel('x_{2}');
grid on;
title('Eigenvectors');
xlim([-1 1]);
ylim([-1 1]);
```



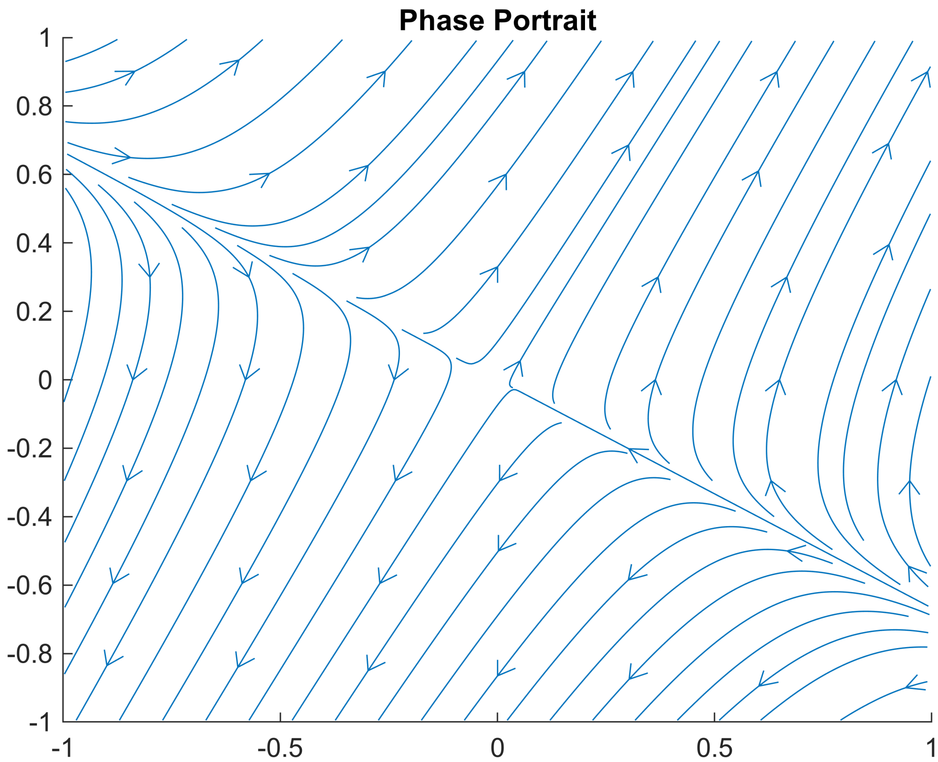

```
[x1, x2] = meshgrid(x1Vec,x2Vec);
x1dot = x1 + 3*x2;
x2dot = 4*x1 + 5*x2;

figure;
quiver(x1,x2,x1dot,x2dot);
xlabel('x_{1}');
ylabel('x_{2}');
grid on;
title('Vector Field');
xlim([-1 1]);
ylim([-1 1]);
```



d) Use MATLAB to plot the phase portrait of the system.

Solution:

```
figure;
streamslice(x1Vec,x2Vec,x1dot,x2dot);
title('Phase Portrait');
```



d) Use your phase portrait or numerical analysis to identify and classify any critical points.

Solution:

$x_1 + 3x_2 = 0 \qquad 4x_1 + 5x_2 = 0$

Use the first equation to find x_1 in terms of x_2 and substitute into the second equation

$4(-3x_2) + 5x_2 = 0 \ \rightarrow x_2 = 0$

Substitute into $x_1 \ \rightarrow x_1 + 3(0) = 0 \rightarrow x_1 = 0$

The critical point is $(0, 0)$.

Problem 2: Consider the system of differential equations

$$\frac{dx_1}{dt} = -3x_1 - 200x_2, \frac{dx_2}{dt} = 200x_1 - 3x_2.$$

a) Find the linearly independent solutions $x_1(t)$ and $x_2(t)$.

Solution:

$$A = \begin{bmatrix} -3 & -200 \\ 200 & -3 \end{bmatrix}$$

$$\det\begin{bmatrix} -3-\lambda & -200 \\ 200 & -3-\lambda \end{bmatrix} = (-3-\lambda)^2 - (-200)(200) = 0 \ \rightarrow (\lambda+3)^2 = 40000$$

$\lambda_1 = -3 + 200i$ and $\lambda_2 = -3 - 200i$

$-200\text{iv}_1 - 200v_2 = 0$ We can use $v_2 = -i\,v_1$ if we use $v_1 = 1$ we get $v_2 = -i$

$e^{(-3+200i)t}(1-i) = e^{-3t}(\cos(200t) + i\sin(200t))(1-i)$

Separate real and imaginary parts to get the values for $x(t)$

$$x_1(t) = \begin{bmatrix} e^{-3t}\cos(200t) \\ e^{-3t}\sin(200t) \end{bmatrix} \quad x_2(t) = \begin{bmatrix} e^{-3t}\sin(200t) \\ -e^{-3t}\cos(200t) \end{bmatrix}$$

$x_1(t) = c_1e^{-3t}\sin(200t) - c_2e^{-3t}\cos(200t) \quad x_2(t) = c_1e^{-3t}\cos(200t) + c_2e^{-3t}\sin(200t)$

b) Construct the fundamental matrix $\Phi(t)$.

Solution:

Using previous results $x_1(t) = e^{-3t}\cos(200t) \quad x_2(t) = e^{-3t}\sin(200t) \quad x_3 = e^{-3t}\sin(200t) \quad x_4 = -e^{-3t}\cos(200t)$

$$\phi(t) = \begin{bmatrix} e^{-3t}\cos(200t) & e^{-3t}\sin(200t) \\ e^{-3t}\sin(200t) & -e^{-3t}\cos(200t) \end{bmatrix}$$

c) Compute $\Phi(0)$ and $\Phi(0)^{-1}$.

Solution:

$$\phi(0) = \begin{bmatrix} e^{-3(0)}\cos(200(0)) & e^{-3(0)}\sin(200(0)) \\ e^{-3(0)}\sin(200(0)) & -e^{-3(0)}\cos(200(0)) \end{bmatrix} = \begin{bmatrix} 1\cdot 1 & 1\cdot 0 \\ 1\cdot 0 & -1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\phi(0)^{-1} = \begin{bmatrix} -e^{-3(0)}\cos(200(0)) & -e^{-3(0)}\sin(200(0)) \\ -e^{-3(0)}\sin(200(0)) & e^{-3(0)}\cos(200(0)) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

d) Use your results (b) and (c) to find the solution to the initial value problem $x_1(0) = 2$ and $x_2(0) = -10$.

Solution:

$x(0) = \begin{bmatrix} 2 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}(c_1, c_2)$ where $c_1 = 2$ and $c_2 = 10$

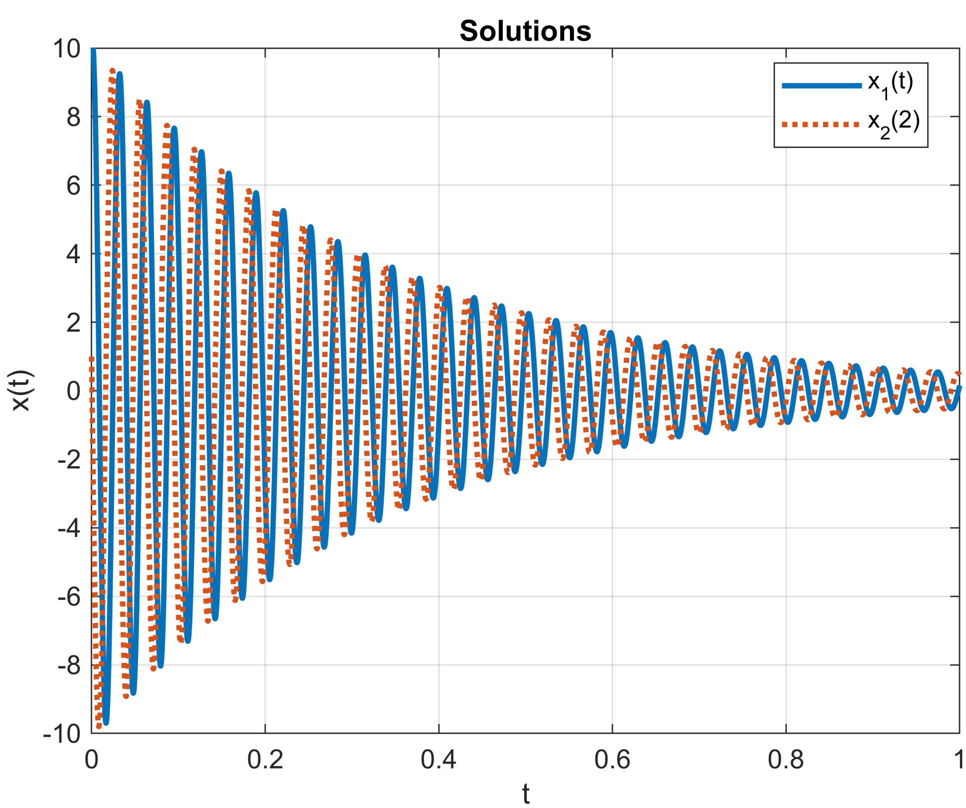
$x_1(t) = 2e^{-3t}\sin(200t) + 10e^{-3t}\cos(200t)$

$x_2(t) = 2e^{-3t}\cos(200t) - 10e^{-3t}\sin(200t)$

e) Use MATLAB to plot your solutions for $x_1(t)$ and $x_2(t)$ on a single figure for time $t = 0$ to $t = 1$.

Solution:

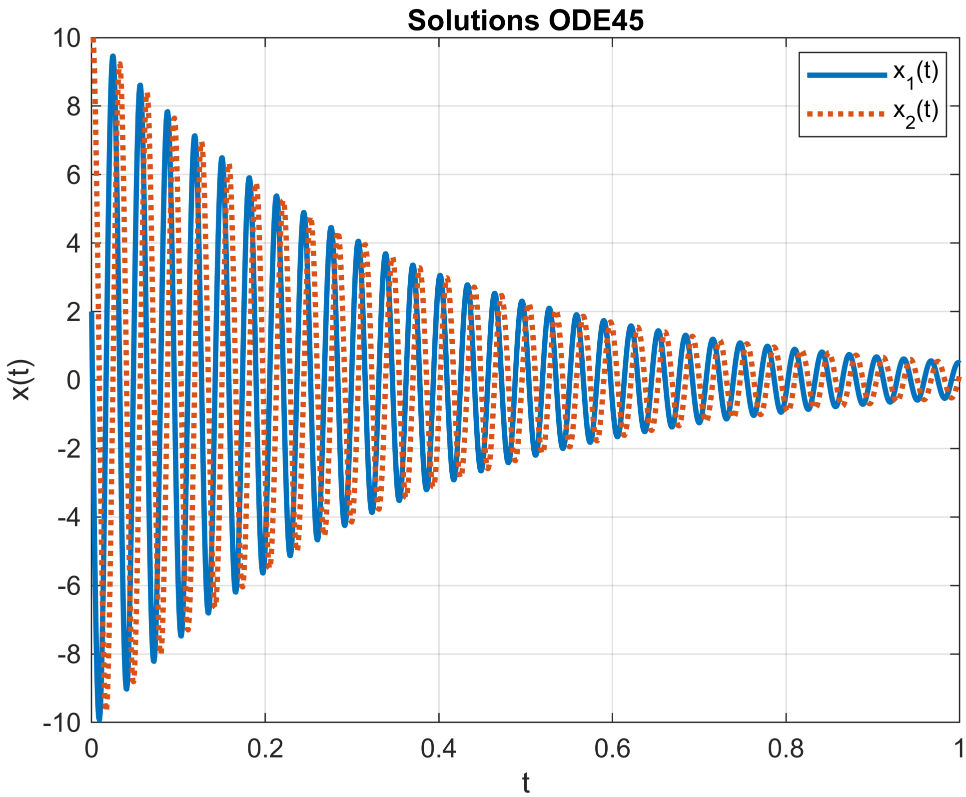
```
syms x1(t) x2(t) t;
C1 = 2;
C2 = 10;
x1(t) = exp(-3*t)*(C1*sin(200*t) + C2*cos(200*t));
x2(t) = exp(-3*t)*(cos(200*t) - C2*sin(200*t));

figure;
fplot(x1(t),[0 1],'-','linewidth',2);
hold on;
fplot(x2(t),[0 1],':','linewidth',2);
xlabel('t');
ylabel('x(t)');
title('Solutions');
grid on;
legend('x_{1}(t)','x_{2}(2)','location','best');
ylim([-10 10]);
```



f) Use the MATLAB ode45() function to solve the original system of equations and plot your solutions on a new figure. The results provided by ode45() and your solution from above should match.

Solution:

```
clear all;

function dXdt = problem02ODEFunction(t,X)
    x1 = X(1);
    x2 = X(2);

    dx1dt = -3*x1 - 200*x2;
    dx2dt = 200*x1 - 3*x2;

    dXdt = [dx1dt; dx2dt];
end

opts = odeset('RelTol',1e-6,'AbsTol',1e-8);
soln = ode45(@problem02ODEFunction,[0 1],[2 10],opts);

figure;
plot(soln.x,soln.y(1,:),'-','linewidth',2);
hold on;
plot(soln.x,soln.y(2,:),':','linewidth',2);
grid on;
title('Solutions ODE45');
xlabel('t');
ylabel('x(t)');
legend('x_{1}(t)','x_{2}(t)','location','best');
ylim([-10 10]);
```



Problem 3: Consider the differential equation

$$\frac{d^3y}{dt^3} + \frac{19}{12}\frac{d^2y}{dt^2} + \frac{19}{24}\frac{dy}{dt} + \frac{1}{8}y(t) = f(t)$$

that describes a system completely at rest, where the input applied is $f(t) = 1$ for $1 \le t < 10$ and zero elsewhere.

a) Write the input function $f(t)$ as a difference of time-shifted unit step functions.

Solution:

$t = 1$ and $t = 10$

$u(t-1)$ and $-u(t-10)$

Combine for $f(t)$

$f(t) = u(t-1) - u(t-10)$

b) Take the Laplace transform of the differential equation and solve for $Y(s)$. Use partial fraction expansion (PFE) to simplify your answer.

Solution:

$$s^3Y(s) + \frac{19}{12}s^2Y(s) + \frac{19}{24}s\,Y(s) + \frac{1}{8}Y(s) = L\,u(t-1) - L\,u(t-10)$$

$$s^3Y(s) + \frac{19}{12}s^2Y(s) + \frac{19}{24}s\,Y(s) + \frac{1}{8}Y(s) = \frac{e^{-s}}{s} - \frac{e^{-10s}}{s}$$

$$Y(s)\left(s^3 + \frac{19}{12}s^2 + \frac{19}{24}s + \frac{1}{8}\right) = \frac{(e^{-s} - e^{-10s})}{s}$$

$$Y(s) = \frac{(e^{-s} - e^{-10s})}{s\left(s^3 + \frac{19}{12}s^2 + \frac{19}{24}s + \frac{1}{8}\right)}$$

Factor the cubic polynomial in the denominator and find the roots of the equation.

$$\frac{24s^3 + 38s^2 + 19s + 3}{24}\ ,\ r_1 = -\frac{3}{4}\ , r_2 = -\frac{1}{2}\ , r_3 = -\frac{1}{3}$$

$$Y(s) = \frac{(e^{-s} - e^{-10s})}{s\left(s + \frac{3}{4}\right)\left(s + \frac{1}{2}\right)\left(s + \frac{1}{3}\right)}$$

$$F(s) = \frac{1}{s\left(s + \frac{3}{4}\right)\left(s + \frac{1}{2}\right)\left(s + \frac{1}{3}\right)} = \frac{A}{s} + \frac{B}{s + \frac{3}{4}} + \frac{C}{s + \frac{1}{2}} + \frac{D}{s + \frac{1}{3}}$$

Multiply by $s\left(s + \frac{3}{4}\right)\left(s + \frac{1}{2}\right)\left(s + \frac{1}{3}\right)$ and solve for A, B, C and D

$$F(s) = \frac{8}{s} - \frac{\left(\frac{64}{5}\right)}{s + \frac{3}{4}} + \frac{48}{s + \frac{1}{2}} - \frac{\left(\frac{216}{5}\right)}{s + \frac{1}{3}}$$

Apply partial fraction decomposition to $Y(s)$

$$Y(s) = (e^{-s} - e^{-10s})\left(\frac{8}{s} - \frac{\left(\frac{64}{5}\right)}{s + \frac{3}{4}} + \frac{48}{s + \frac{1}{2}} - \frac{\left(\frac{216}{5}\right)}{s + \frac{1}{3}}\right)$$

c) Find the solution of the differential equation by computing $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Solution:

$$L^{-1}\left\{\frac{8}{s}\right\} = 8 \qquad L^{-1}\left\{\frac{\left(\frac{64}{5}\right)}{s+\frac{3}{4}}\right\} = \frac{64}{5}e^{-\frac{3}{4}t} \qquad L^{-1}\left\{\frac{48}{s+\frac{1}{2}}\right\} = 48e^{-\frac{1}{2}t} \qquad L^{-1}\left\{\frac{\left(\frac{216}{5}\right)}{s\frac{+}{3}}\right\} = \frac{216}{5}e^{-\frac{1}{3}t}$$

$$L^{-1}\{e^{-s}F(s)\} = u_1(t)f(t-1)$$

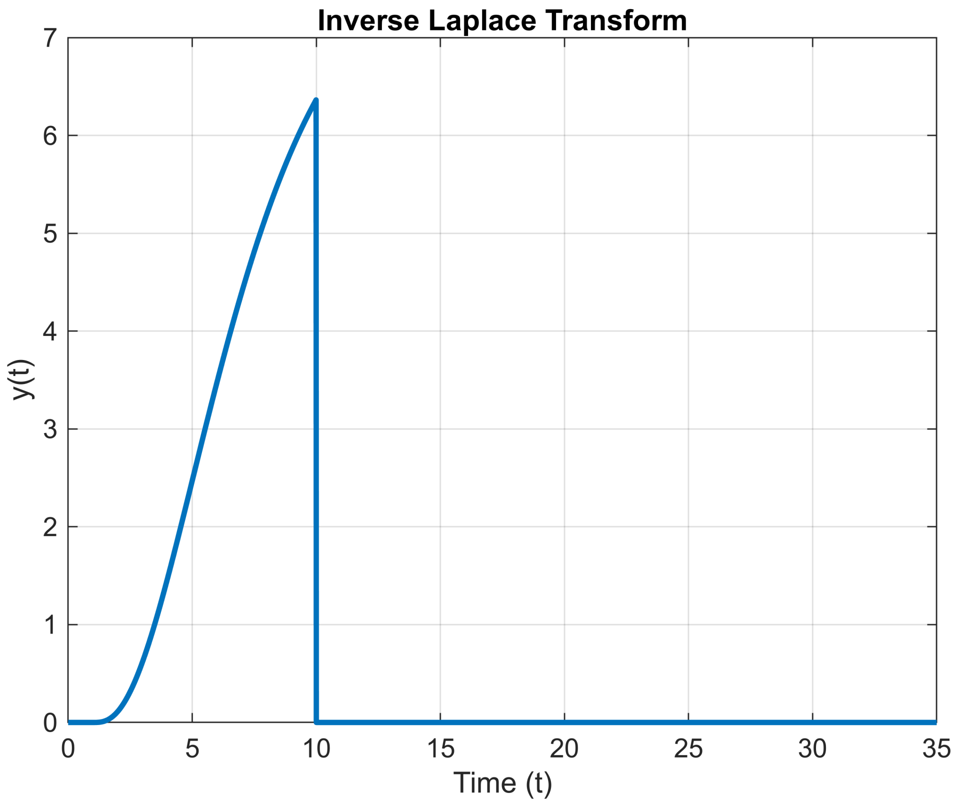
$$L^{-1}\{e^{-10s}F(s)\} = u_{10}(t)f(t-10)$$

$$y(t) = \begin{cases} 8 - \frac{64}{5}e^{-\frac{3}{4}(t-1)} + 48e^{-\frac{1}{2}(t-1)} - \frac{216}{5}e^{-\frac{1}{3}(t-1)} & 1 \le t < 10 \\ 0 & \text{Elsewhere} \end{cases}$$

d) Use MATLAB to plot your solution for time $t = 0$ to $t = 35$.

Solution:

```
clear all;
syms y(t) t;

y(t) = 8 - (64/5)*exp((-3./4)*(t-1)) + 48*exp((-1./2)*(t-1)) -
(216/5)*exp((-1./3)*(t-1));
t = 0:0.01:35;
ySolns = y(t);
ySolns(t < 1 | t >= 10) = 0;

figure;
plot(t,ySolns,'-','linewidth',2);
xlabel('Time (t)');
ylabel('y(t)');
title('Inverse Laplace Transform');
grid on;
```



e) Re-write the 3rd order differential equation as a system of three first-order differential equations.

Solution:

$$x_1 = y(t) = 8 - \frac{64}{5}e^{-\frac{3}{4}(t-1)} + 48e^{-\frac{1}{2}(t-1)} - \frac{216}{5}e^{-\frac{1}{3}(t-1)} \qquad \frac{\mathrm{dx}_1}{\mathrm{dt}} = x_2$$

$$x_2 = y'(t) \qquad \frac{\mathrm{dx}_2}{\mathrm{dt}} = x_3$$

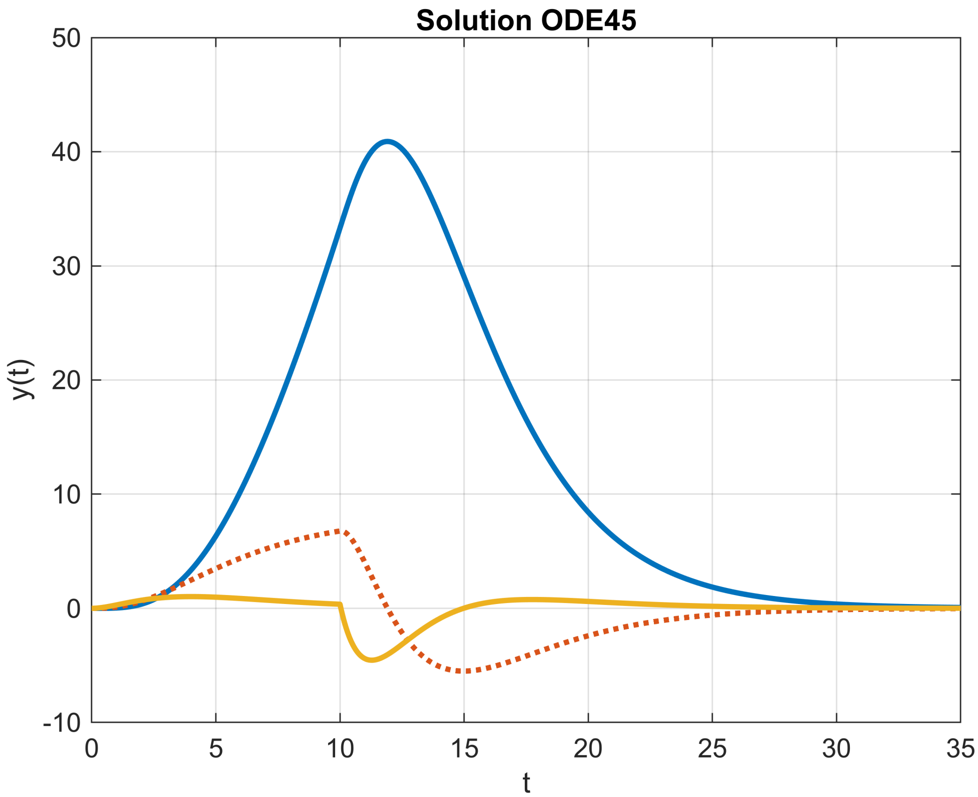
$$x_3 = y''(t) \qquad \frac{\mathrm{dx}_3}{\mathrm{dt}} = -\frac{1}{8}x_1 - \frac{19}{24}x_2 - \frac{19}{12}x_3 + f(t)$$

e) Use the MATLAB ode45() function to solve the system of equations and plot your solution on a new figure. The result provided by ode45() and your solution from above should match.

Solution:

```
clear all;

function dy = problem03ODEFunction(t,y)
    x1 = y(1);
    x2 = y(2);
    x3 = y(3);
```

```
    if t < 10
        f = t;
    else
        f = 0;
    end

    dx1dt = x2;
    dx2dt = x3;
    dx3dt = (-1/8)*x1 - (19/24)*x2 - (19/12)*x3 + f;

    dy = [dx1dt; dx2dt; dx3dt];
end

xRest = [0; 0; 0];
time = [0 35];

opts = odeset('RelTol',1e-6,'AbsTol',1e-8);
[t, y] = ode45(@problem03ODEFunction,time,xRest,opts);

figure;
plot(t,y(:,1),'-','linewidth',2);
hold on;
plot(t,y(:,2),':','linewidth',2);
plot(t,y(:,3),'-','linewidth',2);
grid on;
title('Solution ODE45');
xlabel('t');
ylabel('y(t)');
```



Problem 4: Consider the differential equation $y'' + 6y' + 5y = 3\delta(t-4)$ with initial conditions $y(0) = 5$ and $y'(0) = 2$.

a) Take the Laplace transform of the differential equation and solve for $Y(s)$. Use partial fraction expansion (PFE) to simplify your answer.

Solution:

Using Laplace transform properties we get:

$$L\delta(t-4) = e^{-4s}$$

Substitute initial conditions:

$$s^2Y(s) - 5s - 2 + 6s\,Y(s) - 30 + 5Y(s) = 3e^{-4s}$$

Combine like terms and solve for $Y(s)$:

$$(s^2 + 6s + 5)Y(s) = 5s + 32 + 3e^{-4s}$$

$$Y(s) = \frac{5s + 32 + 3e^{-4s}}{s^2 + 6s + 5}$$

Apply PFE:

$$\frac{5s+32}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5} \text{ where } A = \frac{27}{4} \text{ and } B = -\frac{7}{4}$$

$$\frac{3}{(s+1)(s+5)} = \frac{C}{s+1} + \frac{D}{s+5} \text{ where } C = \frac{3}{4} \text{ and } D = -\frac{3}{4}$$

$$Y(s) = \frac{\left(\frac{27}{4}\right)}{s+1} - \frac{\left(\frac{7}{4}\right)}{s+5} + e^{-4s}\left(\frac{\left(\frac{3}{4}\right)}{s+1} - \frac{\left(\frac{3}{4}\right)}{s+5}\right)$$

$$Y(s) = \frac{27}{4(s+1)} - \frac{7}{4(s+5)} + \frac{3e^{-4s}}{4(s+1)} - \frac{3e^{-4s}}{4(s+5)}$$

b) Find the solution of the differential equation by computing $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Solution:

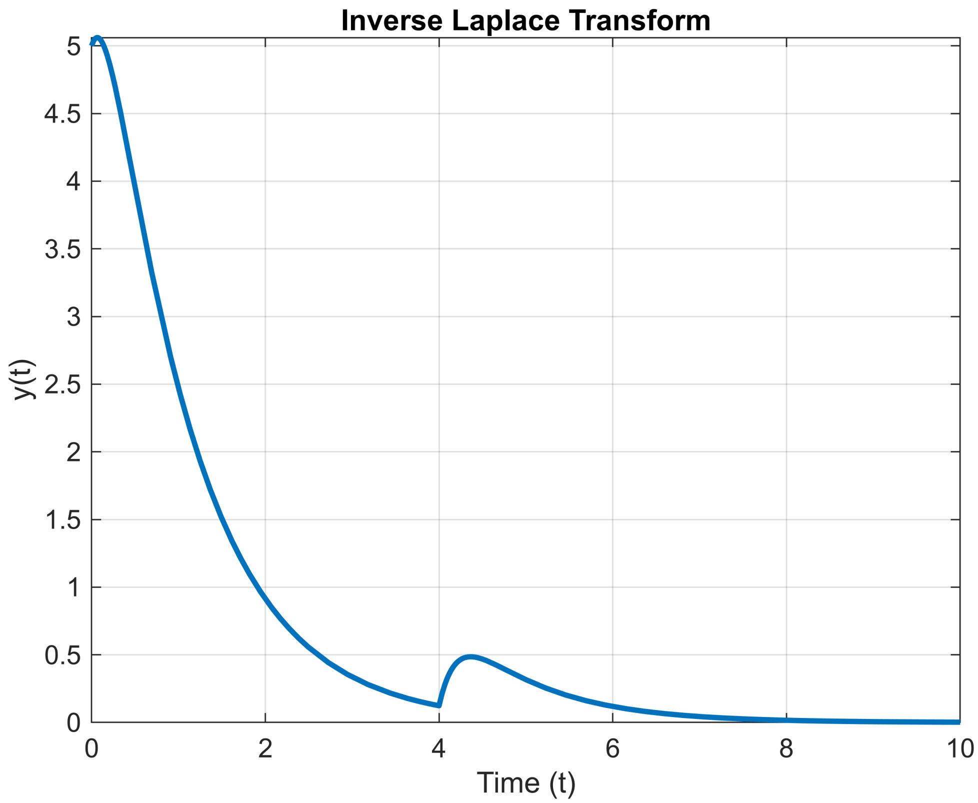
$$y(t) = \frac{27}{4}e^{-t} - \frac{7}{4}e^{-5t} + \frac{3}{4}e^{-(t-4)}u(t-4) - \frac{3}{4}e^{-5(t-4)}u(t-4)$$

$$y(t) = \frac{27}{4}e^{-t} - \frac{7}{4}e^{-5t} + \frac{3}{4}e^{-(t-4)}u(t-4)[e^{-(t-4)} - e^{-5(t-4)}]$$

c) Use MATLAB to plot your solution for time $t = 0$ to $t = 10$.

Solution:

```
clear all;
syms y(t) t;

y(t) = (27/4)*exp(-t) - (7/4)*exp(-5*t) + (3/4)*exp(-(t-4))*heaviside(t-4) -
(3/4)*exp(-5*(t-4))*heaviside(t-4);

figure;
fplot(y(t),[0 10],'-','linewidth',2);
xlabel('Time (t)');
ylabel('y(t)');
title('Inverse Laplace Transform');
grid on;
```



d) Use the MATLAB dsolve() function to solve the system of equations and plot your solution on a new figure. The result provided by dsolve() and your solution from above should match.

Solution:

```
clear all;
syms y(t) t;

Dy(t) = diff(y(t));
Dy2(t) = diff(y(t), 2);
ode = Dy2(t) + 6*Dy(t) + 5*y(t) == 3*dirac(t-4);
cond1 = y(0) == 5;
cond2 = Dy(0) == 2;
cond = [cond1; cond2;];

ySoln(t) = dsolve(ode, cond)
```

ySoln(t) =

$$e^{-t}\left(\frac{3\,e^{4}}{8}+\frac{27}{4}\right)-e^{-5\,t}\left(\frac{3\,e^{20}}{8}+\frac{7}{4}\right)+\frac{3\,\mathrm{sign}(t-4)\,e^{-t}\,e^{4}}{8}-\frac{3\,\mathrm{sign}(t-4)\,e^{-5\,t}\,e^{20}}{8}$$

```
figure;
fplot(ySoln(t),[0 10],'-','linewidth',2);
```

```
xlabel('Time (t)');
ylabel('y(t)');
title('dsolve() Solution');
grid on;
ylim([0 5.5]);
```