Project Two Template

MAT325: Calculus III: Multivariable Calculus

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Problem 1: Consider the iterated integral in cylindrical coordinates $\int_0^{\pi/3} \int_0^1 \int_{r^3}^{r} r\, dz\, dr\, d\theta.$

a) Use MATLAB and the trisurf() function to plot the solid region indicated by the integral limits. Choose an appropriate view to best visualize the region and label axes appropriately.

Solution:

```
syms r;

r = linspace(0, 1, 50);
theta = linspace(0, pi/3, 50);
[R, Theta] = meshgrid(r, theta);

xUpper = R .* cos(Theta);
yUpper = R .* sin(Theta);
zUpper = R;

xLower = R .* cos(Theta);
yLower = R .* sin(Theta);
zLower = R.^3;

vUpper = [xUpper(:), yUpper(:), zUpper(:)];
vLower = [xLower(:), yLower(:), zLower(:)];
dtUpper = delaunay(xUpper, yUpper);
```

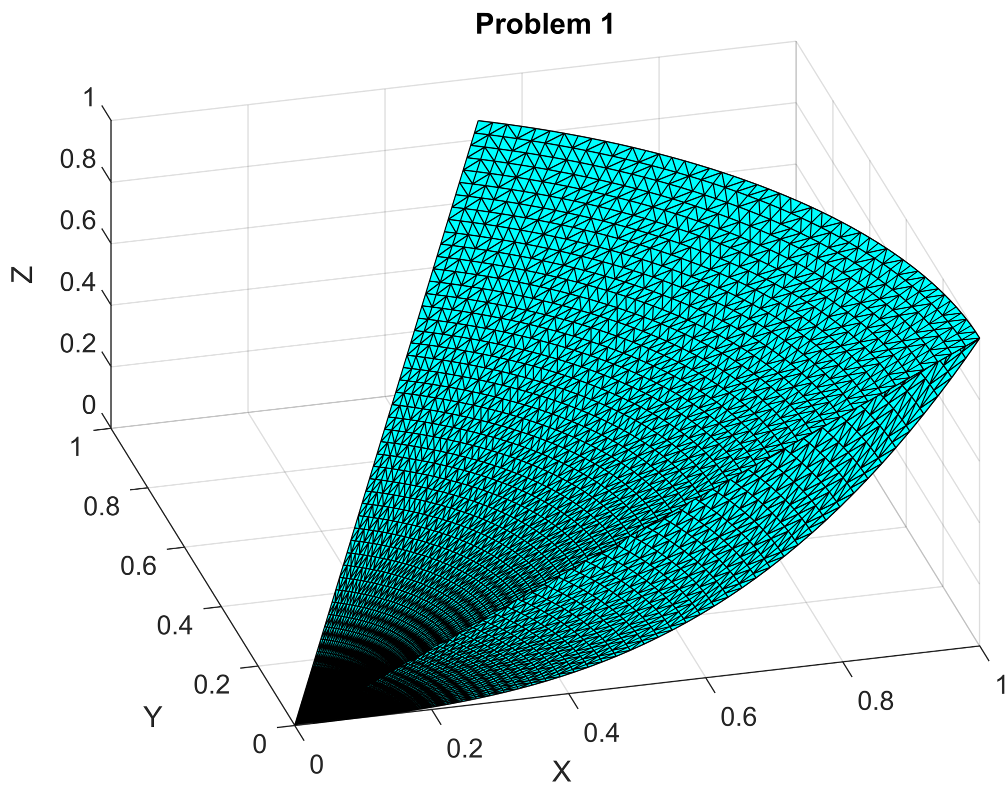
```
Warning: Duplicate data points have been detected and removed.
 Some point indices will not be referenced by the triangulation.
```

```
dtLower = delaunay(xLower, yLower);
```

```
Warning: Duplicate data points have been detected and removed.
 Some point indices will not be referenced by the triangulation.
```

```

figure;
trisurf(dtUpper,vUpper(:,1),vUpper(:,2),vUpper(:,3),'Facecolor','cyan');
hold on;
trisurf(dtUpper,vLower(:,1),vLower(:,2),vLower(:,3),'Facecolor','cyan');
```

```
xlabel('X');
ylabel('Y');
zlabel('Z');
title('Problem 1');
grid on;
view([-15 45]);
```



b) Solve the integral to find a value for the volume V of the solid.

Solution:

$$\int_{r^3}^{r} r\,\mathrm{dz} = r\int_{r^3}^{r} \mathrm{dz} = r(r-r^3) = r^2 - r^4$$

$$\int_{0}^{1} (r^2 - r^4)\,\mathrm{dr} = \left[\frac{r^3}{5} - \frac{r^5}{5}\right] = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

$$\int_{0}^{\frac{\mathrm{pi}}{3}} \frac{2}{15}\,d\theta = \frac{2}{15}\int_{0}^{\frac{\mathrm{pi}}{3}} d\theta = \frac{2}{15}\cdot\frac{\mathrm{pi}}{3} = \frac{2\pi}{45}$$

c) Use MATLAB and the integral3() function to numerical solve the iterated integral. Verify that your numerical value computed below matches the value computed in part (b).

Solution:

$2 \cdot \pi = 6.283185307 \div 45 = 0.1396263402$

```
f = @(x,y,z) (y);

xMin = 0;
xMax = pi/3;
yMin = 0;
yMax = 1;
zMin = @(x,y) y.^3;
zMax = @(x,y) y;

format long;
integral3(f,xMin,xMax,yMin,yMax,zMin,zMax,'Method','tiled')
```

```
ans =
   0.139626340157927
```

Problem 2: Consider the integral

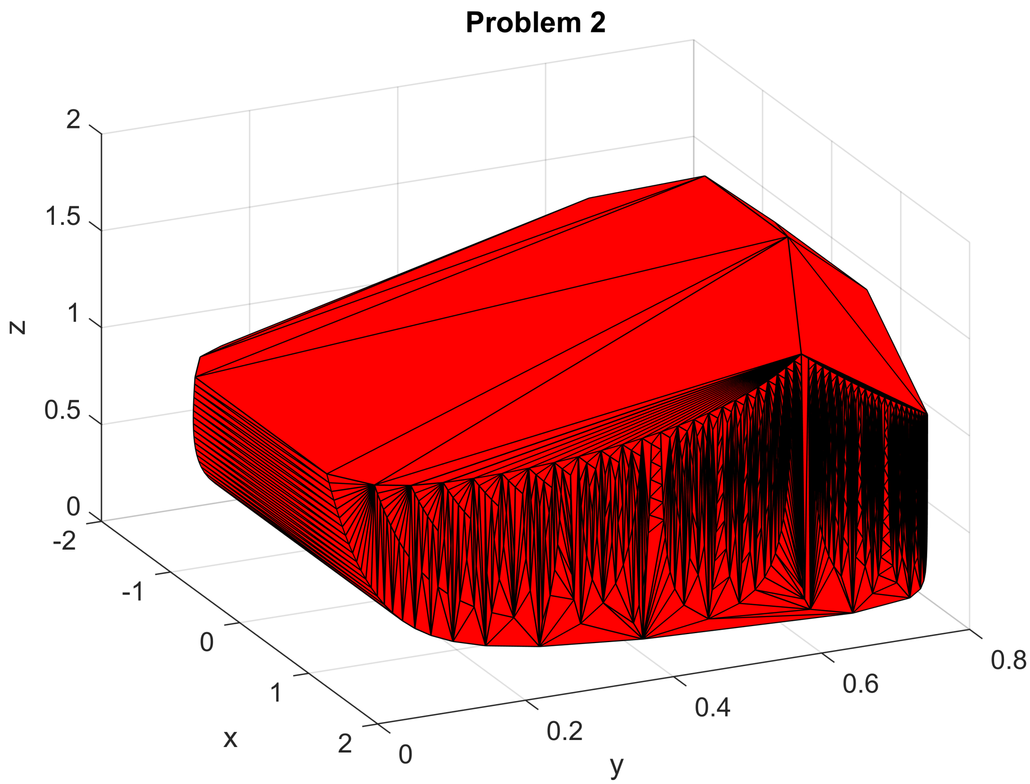
$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

a) Use MATLAB and the trisurf() function to plot the solid region given by the integral limits. Choose an appropriate view to best visualize the region and label axes appropriately.

Solution:

```
x = linspace(-1, 1, 50);
y = linspace(-1, 1, 50);
z = linspace(-1, 1, 50);
[X,Y,Z] = meshgrid(x,y,z);

Z = sqrt(X.^2 + Y.^2 + Z.^2);
Y = atan(sqrt(X.^2 + Y.^2)./Z);
X = atan(Y./X);

plotInd = (X >= -2*sqrt(2) & X <= 2*sqrt(2)) & (Y >= -sqrt(8 - x.^2) & Y <= sqrt(8
- x.^2)) & (Z >= sqrt(x.^2 + y.^2) & Z <= sqrt(16 - x.^2 - y.^2));

figure;
K1 = convhull(X(plotInd), Y(plotInd), Z(plotInd));
trisurf(K1,X(plotInd),Y(plotInd),Z(plotInd),'Facecolor','red');
xlabel('x');
ylabel('y');
zlabel('z');
```

```
title('Problem 2');
grid on;
view([65 30]);
```



b) Convert the integral into an integral in spherical coordinates.

$x = \rho \cdot \sin(\phi)\cos(\theta)$

$y = \rho \cdot \sin(\phi)\sin(\theta)$

$z = \rho \cdot \cos(\theta)$

$\rho^2 = x^2 + y^2 + z^2$

$\mathrm{dV} = \rho^2 \cdot \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^4 \rho \cdot \rho^2 \sin(\phi) d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^4 \rho^3 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

c) Solve the integral from part (b).

$$\int_0^4 \rho^3 \sin(\phi)\, d\rho = \sin(\phi)\int_0^4 \rho^3\, d\rho = \sin(\phi)\left(\frac{4^4}{4} - \frac{0^4}{4}\right) = \sin(\phi)\left(\frac{256}{4}\right) = 64\sin(\phi)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 64\sin(\phi)\, d\phi = 64\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(\phi)\, d\phi = 64\left(-\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right)\right) = 64\left(\frac{\sqrt{2}}{2}\right) = 32\sqrt{2}$$

$$\int_0^{2\pi} 32\sqrt{2}\, d\theta = 32\sqrt{3}\int_0^{2\pi} d\theta = 32\sqrt{2}\,(2\pi) = 64\pi\sqrt{2}$$

d) Use MATLAB and the integral3() function to numerically solve the integral. Verify that your numerical value computed below matches the value computed in part (c).

$64 \cdot \pi \cdot \sqrt{2} = 284.344508$

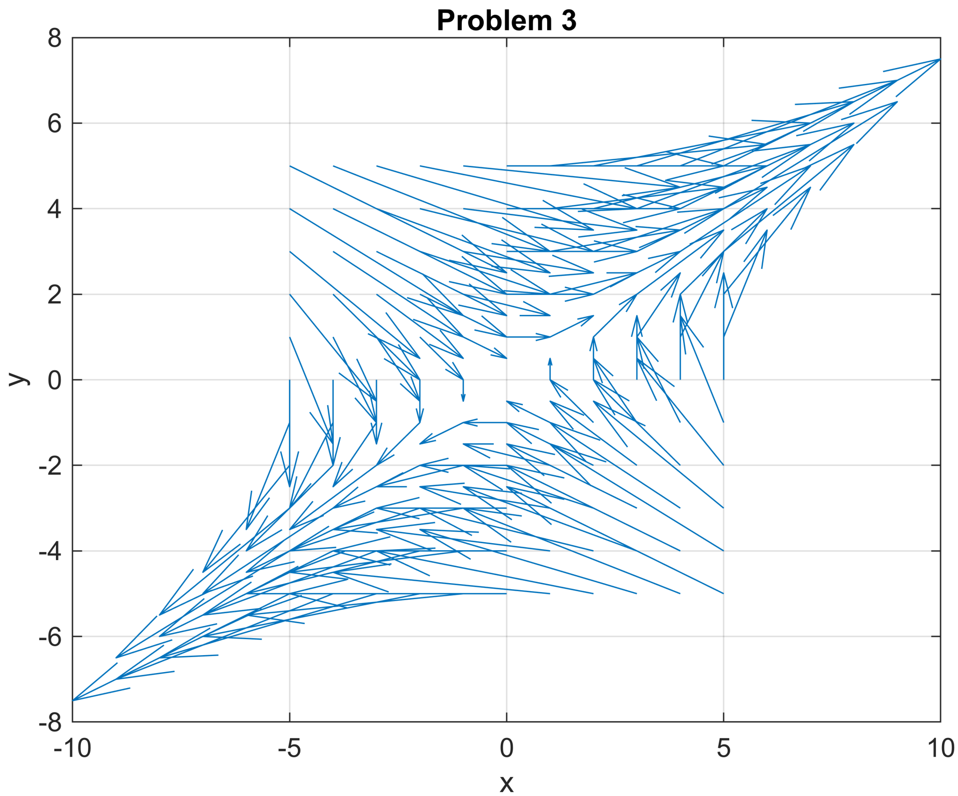
```
f = @(x,y,z) (z.^3.*sin(y));

xMin = 0;
xMax = 2*pi;
yMin = pi/4;
yMax = pi/2;
zMin = 0;
zMax = 4;

format long;
integral3(f,xMin,xMax,yMin,yMax,zMin,zMax,'Method','tiled')
```

```
ans =
    2.843445080422292e+02
```

Problem 3: Consider the vector field $\mathbf{F}(x, y) = y\mathbf{i} + 0.5x\mathbf{j}$.

a) Use MATLAB and the quiver() function plot the vector field.

```
[x, y] = meshgrid(-5:1:5, -5:1:5);
a = y;
b = 0.5*x;

figure;
quiver(x,y,a,b,'off');
xlabel('x');
ylabel('y');
title('Problem 3');
grid on;
```



b) Consider moving an object along a path C, with starting point $(1, 0)$ and end point $(0, 1)$. Find a parameteriziation $\mathbf{r}(t)$ for the curve.

$\vec{r}\,(t) = (1-t)(1,0) + t(0,1) = (1-t,0) + (0,t) = (1-t,t)$

$x(t) = 1 - t$

$y(t) = t$

$r(t) = (1-t)i + t\ j$

c) Compute the work done by the vector field in moving the object along the path C.

$\frac{\mathrm{dr}}{\mathrm{dt}} = \frac{d}{\mathrm{dt}}(1-t,t) = (-1,1)$

$F(x(t),y(t)) = F(1-t,t) = \mathrm{ti} + 0.5(1-t)j = \left(t, \frac{1}{2} - \frac{t}{2}\right)$

$F(r(t)) \cdot \frac{\mathrm{dr}}{\mathrm{dt}} = \left(t, \frac{1}{2} - \frac{t}{2}\right) \cdot (-1,1) = -t + \frac{1}{2} - \frac{t}{2} = \frac{1}{2} - \frac{3t}{2}$

$W = \left[\frac{t}{2} - \frac{3t^2}{4}\right]_0^1 = \frac{1}{2} - \frac{3}{4} - 0 - 0 = \frac{2}{4} - \frac{3}{4} = -\frac{1}{4}$

Problem 4: Consider the line integral $\int_C (6x - 5y)dx + (2x - 4y)dy$ where C is the ellipse given by $\frac{x^2}{4} + y^2 = 1$ in the counterclockwise direction.

a) Find a parameterization $\mathbf{r}(t)$ for the curve.

$\frac{x^2}{4} + y^2 = 1$ use trigonometric identity

$$\frac{(2\cos(t))^2}{4} + (\sin(t))^2 = \frac{4\cos^2(t)}{4} + \sin^2(t) = \cos^2 t + \sin^2 t = 1$$

$$r(t) = \langle 2\cos(t), \sin(t) \rangle$$

b) Use Green's Theorem to evaluate the integral.

$$\oint_C (6x - 5y)\mathrm{dx} + (2x - 4y)\mathrm{dy}$$

$$\frac{\delta Q}{\delta x} = \frac{\delta}{\delta x}(2x - 4y) = 2$$

$$\frac{\delta P}{\delta y} = \frac{\delta}{\delta y}(6x - 5y) = -5$$

$$\oint_C (6x - 5y)\mathrm{dx} + (2x - 4y)\mathrm{dy} = \int\int_D (2 - (-5))\mathrm{dA} = 7\int\int_D \mathrm{dA}$$

$$7\int\int_D \mathrm{dA} = 7(2\pi) = 14\pi$$

Problem 5: Consider the vector field

$\mathbf{F}(x, y, z) = 4y\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ and let the surface S be part of the sphere $x^2 + y^2 + z^2 = 9$ that is above the plane $z = 0$.

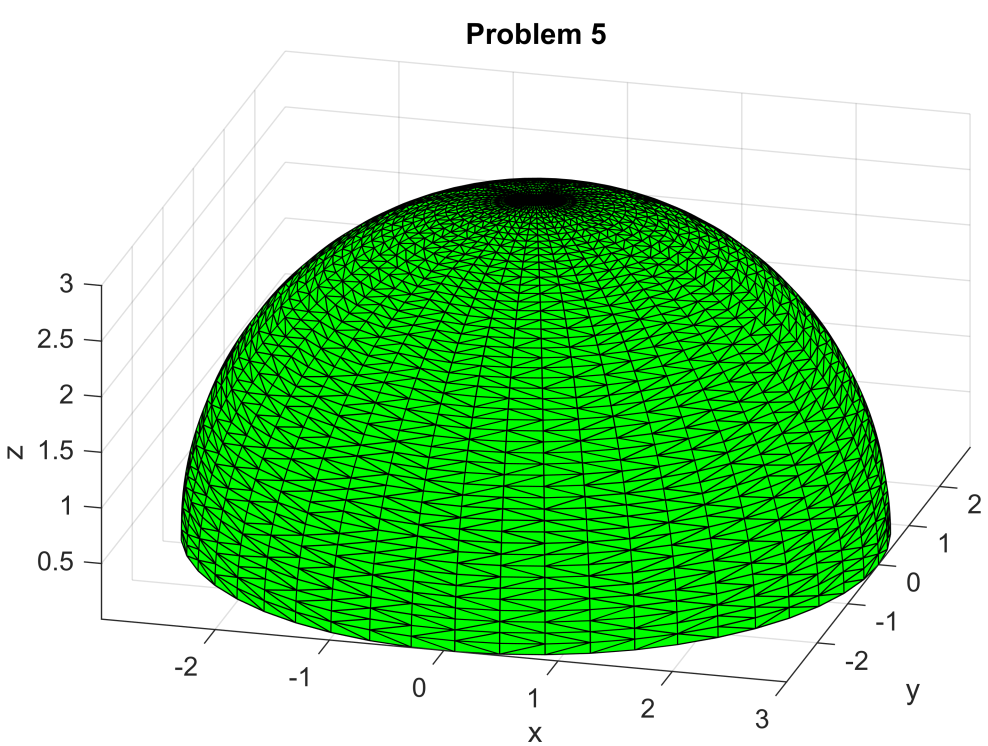
a) Use MATLAB and the trisurf() function to plot S. Choose an appropriate view to best visualize the region and label axes appropriately.

Solution:

```
clear all;

r = 3;
theta = linspace(0, 2*pi, 50);
phi = linspace(0, pi/2, 50);
[Theta, Phi] = meshgrid(theta, phi);

X = r * sin(Phi) .* cos(Theta);
Y = r * sin(Phi) .* sin(Theta);
Z = r * cos(Phi);
dt = delaunay(X(:), Y(:));
```

```
Warning: Duplicate data points have been detected and removed.
 Some point indices will not be referenced by the triangulation.
```

```

figure;
trisurf(dt, X(:), Y(:), Z(:),'Facecolor','green');
axis equal;
xlabel('x');
ylabel('y');
zlabel('z');
title('Problem 5');
grid on;
view([15 20]);
```



b) Use Stoke's Theorem to evalute the integral $\int\int_S (\text{curl } \mathbf{F} \cdot \mathbf{N}\, dS)$ for the given vector field $\mathbf{F}(x, y, z)$ and surface S.

Solution:

$r(t) = \langle 3\cos(t), 3\sin(t), 0\rangle$ where $0 \leq t \leq 2\pi$

$r'(t) = \langle -3\sin(t), 3\cos(t), 0\rangle$

$F(r(t)) = \langle 4(3\sin(t)), 0, 2(3\sin\text{I}(t))\rangle = \langle 12\sin(t), 0, 6\sin(t)\rangle$

$$\oint_C F \cdot \text{dr} = \int_0^{2\pi} \langle 12\sin(t), 0, 6\sin(t)\rangle \cdot \langle -3\sin(t), 3\cos(t), 0\rangle \text{ dt} = \int_0^{2\pi} -36\sin^2 t \text{ dt} = -18\int_0^{2\pi} (1 - \cos(2t)) \text{ dt}$$

$-18[(2\pi - 0) - (0 - 0)] = -36\pi$

Problem 6: Consider the vector field $\mathbf{F}(x, y, z) = (x^2 + 2y^2)\mathbf{i} + (x^2y - z)\mathbf{j} + (2y + 3z)\mathbf{k}$ and let the surface S be the surface of the cube $0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2$, excluding the $z = 0$ face.

a) Compute $\text{div } \mathbf{F}$.

Solution:

$P(x, y, z) = x^2 + 2y^2$

$Q(x, y, z) = x^2y - z$

$R(x, y, z) = 2y + 3z$

$$\frac{\delta P}{\delta x} = \frac{\delta}{\delta x}(x^2 + 2y^2) = 2x$$

$$\frac{\delta Q}{\delta y} = \frac{\delta}{\delta y}(x^2y - z) = x^2$$

$$\frac{\delta R}{\delta z} = \frac{\delta}{\delta z}(2y + 3z) = 3$$

$\text{div}(F) = x^2 + 2x + 3$

b) Use the Divergence theorem to compute the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{N}\, ds$ for the given vector field $\mathbf{F}$ and surface S.

Solution:

$$\int_0^2 \int_0^2 \int_0^2 x^2 + 2x + 3 \,\mathrm{dz}\,\mathrm{dy}\,\mathrm{dx}$$

$$\int_0^2 x^2 + 2x + 3 \,\mathrm{dz} = 2(x^2 + 2x + 3)$$

$$\int_0^2 x^2 + 2x + 3 \,\mathrm{dy} = 4(x^2 + 2x + 3)$$

$$\int_0^2 x^2 + 2x + 3 \,\mathrm{dx} = 4\left(\frac{8}{3} + 4 + 6\right) = 4\left(\frac{8}{3} + 10\right) = 4\left(\frac{38}{3}\right) = \frac{152}{3} = 50.6667$$

c) Use MATLAB and the integral3() function to numerically solve the integral. Verify that your numerical value computed below matches the value computed in part (b).

```
f = @(x,y,z) (x.^2 + 2*x + 3);

xMin = 0;
xMax = 2;
yMin = 0;
yMax = 2;
zMin = 0;
zMax = 2;

format long;
integral3(f,xMin,xMax,yMin,yMax,zMin,zMax,'Method','tiled')
```

```
ans =
  50.666666666666664
```