MAT 330: Differential Equations

Project One Template

Complete this template by replacing the bracketed text with the relevant information.

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Problem 1: Consider a time-varying population that follows the logistic population model. The population has a limiting population of 800, and at time $t = 0$, its population of 200 is growing at a rate of 60 per year.

a) Write a differential equation for the population $P(t)$ that models this scenario.

Solution:

$\frac{\mathrm{dP}}{\mathrm{dt}} = k \cdot P(M - P)$ where $P = 200$, $M = 800$, and $\frac{\mathrm{dP}}{\mathrm{dt}} = 60$

$$k = \frac{\frac{\mathrm{dP}}{\mathrm{dt}}}{P(M - P)}$$

$$k = \frac{\frac{\mathrm{dP}}{\mathrm{dt}}}{\mathrm{PM} - P^2}$$

$$k = \frac{60}{200(800) - 200^2}$$

$k = 0.0005$ therefore $60 = 0.0005 \cdot P(800 - P)$

b) Identify an appropriate solution technique and solve this differential equation.

Solution:

$$\frac{\mathrm{dP}}{\mathrm{dt}} = 0.0005 \cdot P(800 - P) = 0.4P - 0.0005P^2$$

$$\int \frac{\mathrm{dP}}{P(800 - P)} = \int 0.0005\,\mathrm{dt}$$

$$\ln|P| - \ln|800 - P| = 0.4t + C$$

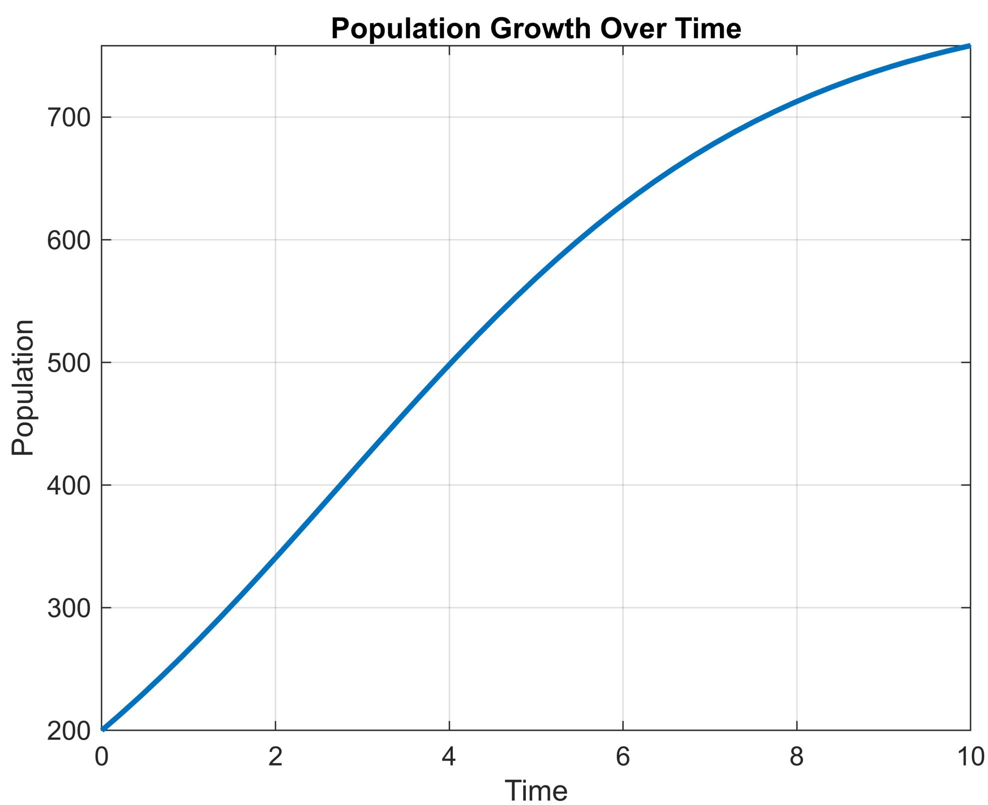
$$P(t) = \frac{800}{1 + e^{\ln(3) - 0.4t}}$$

c) Use MATLAB to plot your solution for $t = 0$ to $t = 10.$

Solution:

```
syms y(t) t;
y(t) = 800 ./ (1+exp(log(3)-0.4*t));


figure;
fplot(y(t),[0 10],'-','linewidth',2);
xlabel('Time');
ylabel('Population');
title('Population Growth Over Time');
grid on;
```



d) How long will it take for the population to achieve 95% of its maximum population?

Solution:

$$\frac{(0.95 \cdot 800)}{1 + e^{\ln(3) - 0.4t}} = 0$$

$$t = \frac{(\ln(759) - \ln(3))}{0.4}$$

$$t = 13.83347$$

It will take approximately 14 years to reach 95% of its maximum population.

e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.

Solution:

```
clear all;
syms P(t) t;

DP = diff(P(t));
ode = DP == 0.0005 * P * 800 - 0.0005 * P^2
```

```
ode(t) =
```

$$\frac{\partial}{\partial t} P(t) = \frac{2\,P(t)}{5} - \frac{P(t)^2}{2000}$$

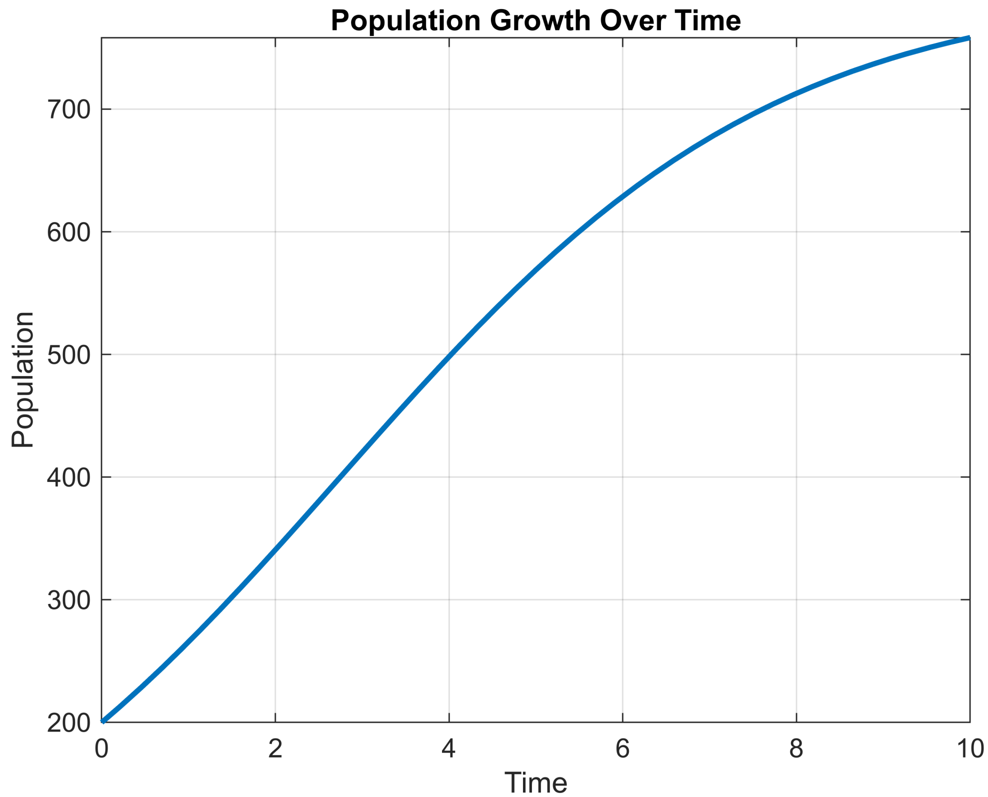
```
cond = P(0) == 200;
PSoln(t) = dsolve(ode, cond)
```

```
PSoln(t) =
```

$$\frac{800}{e^{\log(3) - \frac{2\,t}{5}} + 1}$$

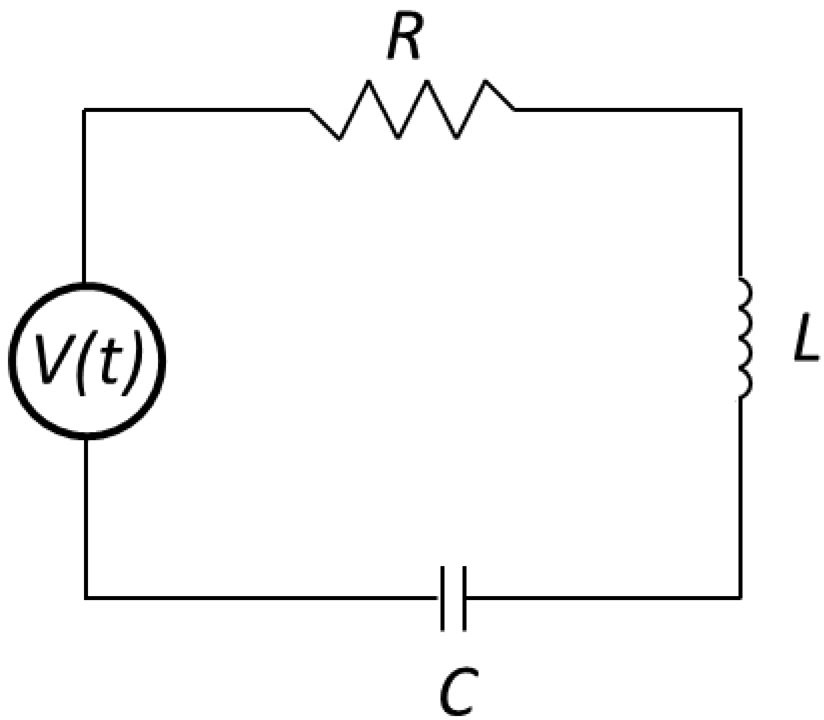
```
pSoln1 = PSoln(t);

figure;
fplot(pSoln1,[0 10],'-','linewidth',2);
hold on;
xlabel('Time');
ylabel('Population');
title('Population Growth Over Time');
grid on;
```



Problem 2: Consider the RLC circuit shown below, where the inital current flowing through the circuit at time $t = 0$ is $I_0 = 5$, and the initial charge on the capacitor at time $t = 0$ is $Q_0 = 2$. The components have values of

$$R = 100 \ \ \Omega, L = 5 \ \ \text{H}, \ \text{and} \ C = \frac{1}{450,500} \ \ \text{F}.$$



a) Write a differential equation for $Q(t)$, the charge across the capacitor, assuming the voltage source $V(t) = 0$.

Solution:

$$L\frac{d^2Q(t)}{\mathrm{dt}^2} + R\frac{\mathrm{dQ}(t)}{\mathrm{dt}} + \frac{Q(t)}{C} = 0$$

b) Classify the differential equation

Solution:

The differential equation above can be classified as a homogenous second-order linear differential equation.

c) Identify an appropriate solution technique, and solve the differential equation for $Q(t)$.

Solution:

$$5q'' + 100q' + 450500q = 0$$

$$5r^2 + 100r + 450500 = 0$$

$5(r^2 + 20r + 90100) = 0$ where $r_1 = -10 + 300i$ and $r_2 = -10 - 300i$

Therfore $Q(t) = e^{-10t}(C_1 \cdot \cos 300t + C_2 \cdot \sin 300t)$

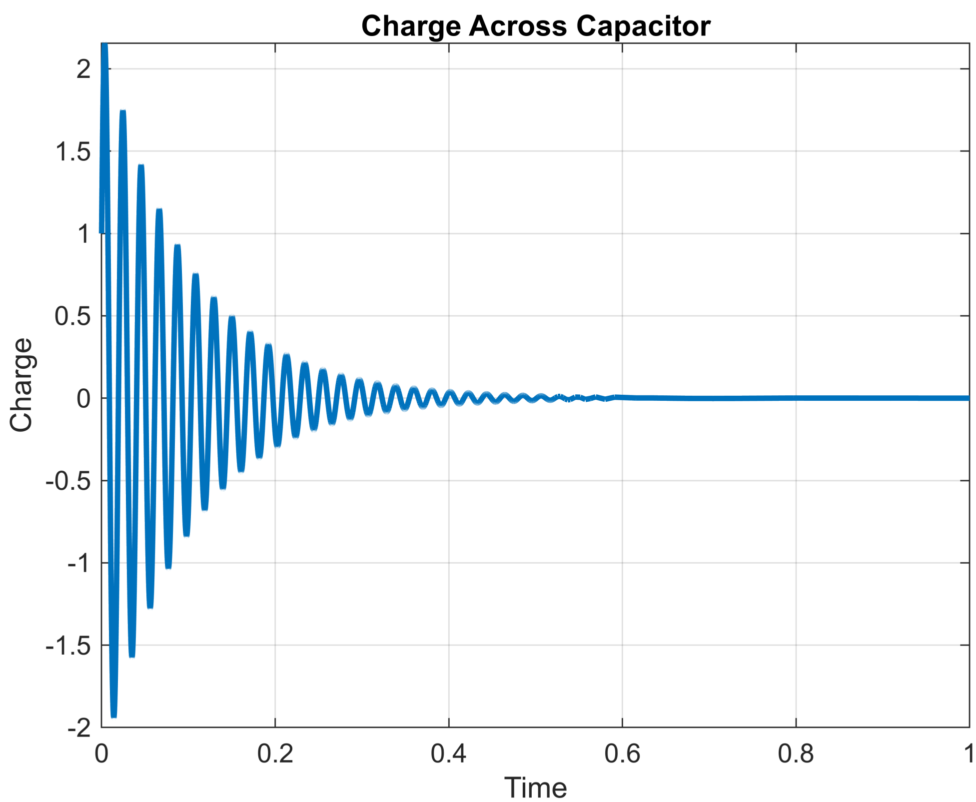
d) Use MATLAB to plot your solution for time t = 0 to 1.

Solution:

```
clear all;
syms Q(t) t;

C1 = 1;
C2 = 2;
Q(t) = exp(-10*t)*(C1*cos(300*t)+C2*sin(300*t));

figure;
fplot(Q(t),[0 1],'-','linewidth',2);
xlabel('Time');
ylabel('Charge');
title('Charge Across Capacitor');
grid on;
```



e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.

Solution:

```
clear all;
syms q(t) t;

Dq(t) = diff(q(t));
Dq2(t) = diff(q(t), 2);
ode = 5 * Dq2(t) + 100 * Dq(t) + 450500 * q(t) == 0
```

ode =

$$5\frac{\partial^2}{\partial t^2}\,q(t)+100\frac{\partial}{\partial t}\,q(t)+450500\,q(t)=0$$

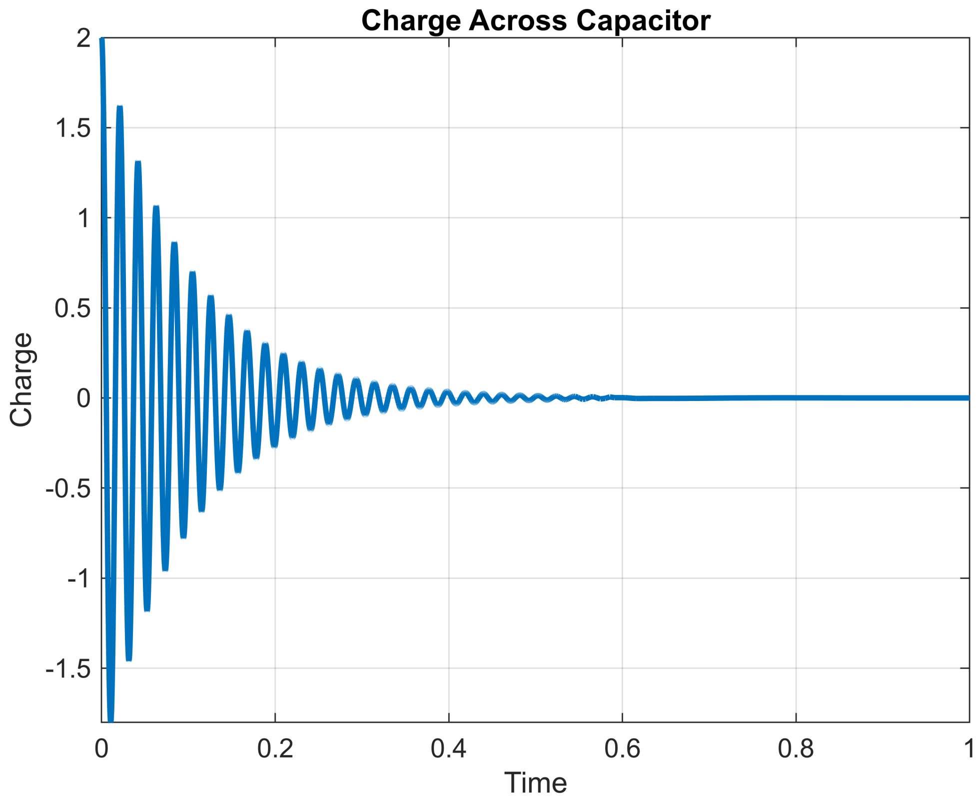
```
cond1 = q(0) == 2;
cond2 = Dq(0) == 5;
cond = [cond1; cond2;];

qSoln(t) = dsolve(ode, cond)
```

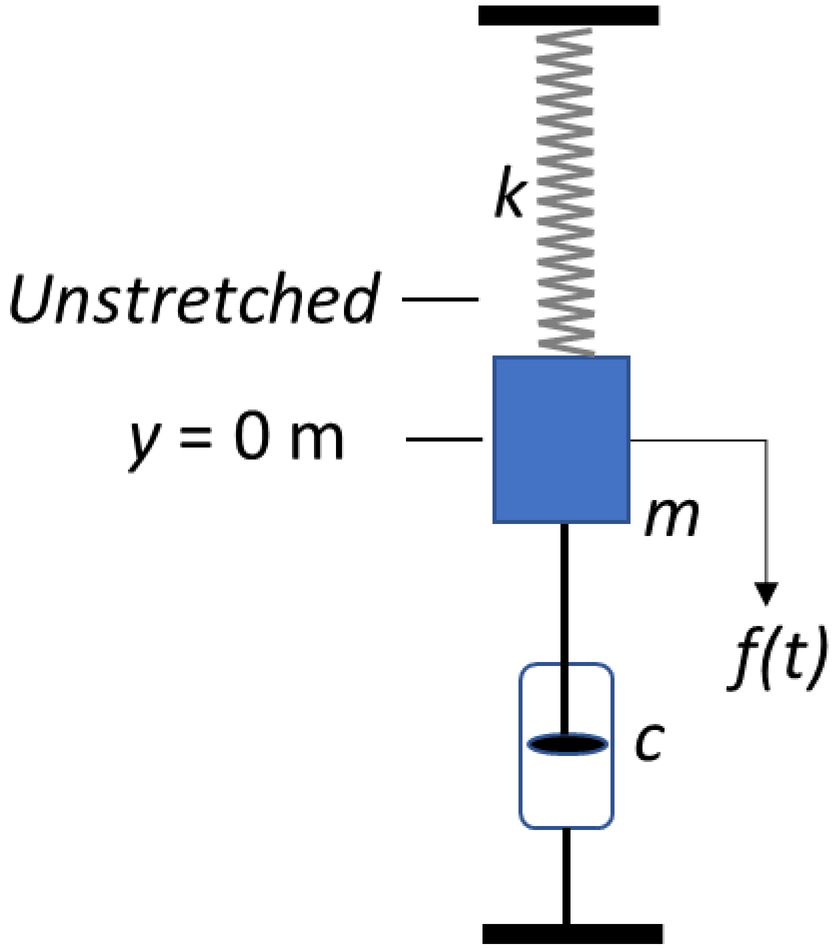
qSoln(t) =

$$\frac{e^{-10\,t}\,(24\cos(300\,t)+\sin(300\,t))}{12}$$

```
figure;
fplot(qSoln(t),[0 1],'-','linewidth',2);
xlabel('Time');
ylabel('Charge');
title('Charge Across Capacitor');
grid on;
```



Problem 3: Consider the spring-mass system shown below. A mass of $m = 5$ kg has stretched a spring by $s_0 = 1.5$ meters and is at rest. A damping system is connected that provides a damping force equal to 30 times the instantaneous velocity of the mass. At time $t = 0$, the mass at rest has an external force of $f(t) = 25\sin(10t)$ applied.



a) Write a differential equation that models the mass location $y(t)$ as a function of time.

Solution:

$\mathrm{my}'' + \mathrm{cy}' + \mathrm{ky} = F(t)$

Therfore $5y'' + 30y' + 32.667y = 25\sin(10t)$

b) Identify an appropriate solution technique, and solve the differential equation for $y(t)$.

Solution:

Non-Homogenous second order linear differntial equation can be solved with undetermined coefficients:

$5y'' + 30y' + 32.667y = 25\sin(10t)$

$y'' + 6y + 6.53y = 5\sin(10t)$

$\lambda^2 + 6\lambda^2 + 6.53 = 0$

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 6.53}}{2}$$

$$\lambda_1 = -1.43 \quad \lambda_2 = -4.57$$

$$y_c(t) = \mathrm{A}e^{-1.43t} + \mathrm{B}e^{-4.57t}$$

$$y_p(t) = \frac{5\sin(10t)}{-100 + 6D + 6.53}$$

$$y_p(t) = -0.03\cos(10t) - 0.05\sin(10t)$$

$$y(t) = -0.11e^{-1.43t} + 0.14e^{-4.57t} - 0.03\cos(10t) - 0.05\sin(10t)$$

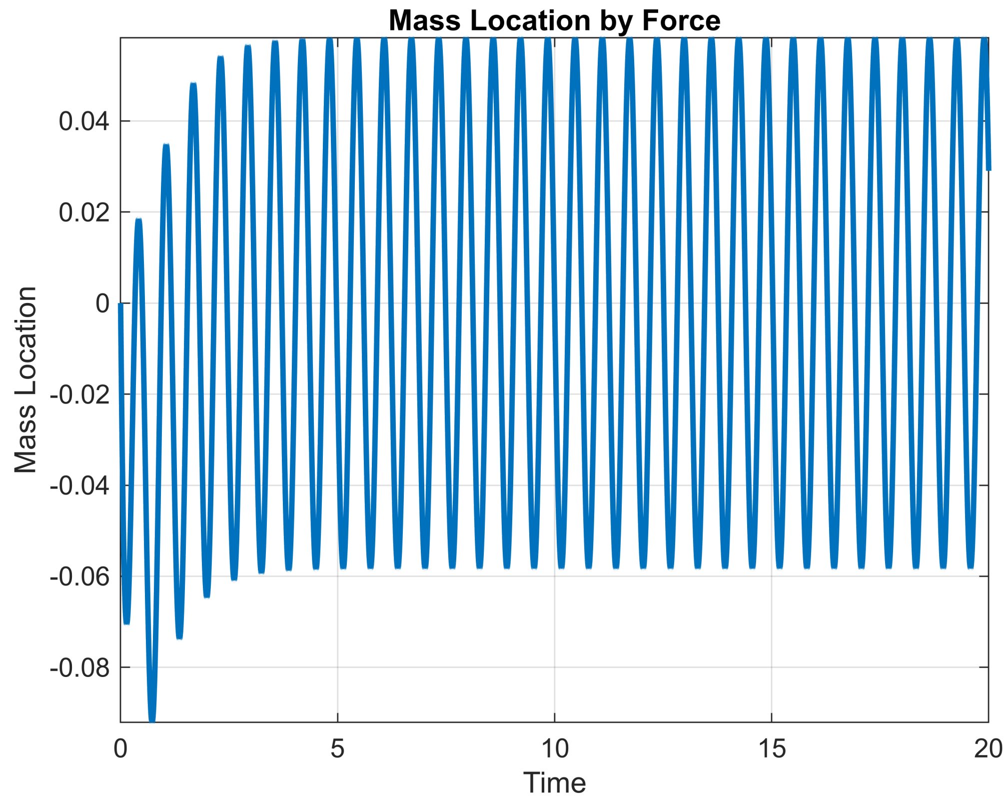
c) Use MATLAB to plot your solution for $t = 0$ to $t = 20$.

Solution:

```
clear all;
syms y(t) t;

y(t) = -0.11*exp(-1.43*t)+0.14*exp(-4.57*t)-0.03*cos(10*t)-0.05*sin(10*t);

figure;
fplot(y(t),[0 20],'-','linewidth',2);
xlabel('Time');
ylabel('Mass Location');
title('Mass Location by Force');
grid on;
```



d) What is the steady-state solution of $y(t)$?

Solution:

The steady-state solution is: $y_p(t) = -0.03\cos(10t) - 0.05\sin(10t)$

e) Use the MATLAB dsolve() function to solve the differential equation and plot the solution on a new figure. The results provided by dsolve() and your solution from part (b) should match.

Solution:

```
syms y(t) t;

Dy(t) = diff(y(t));
Dy2(t) = diff(y(t), 2);

ode = 5 * Dy2(t) + 30 * Dy(t) + 32.667 * y(t) == 25*sin(10*t)
```

ode =

$$5\,\frac{\partial^2}{\partial t^2}\,y(t) + 30\,\frac{\partial}{\partial t}\,y(t) + \frac{32667\,y(t)}{1000} = 25\sin(10\,t)$$

```
cond1 = y(0) == 1.5;
```

```
cond2 = Dy(0) == 0;
cond = [cond1; cond2;];
ySoln(t) = dsolve(ode, cond)
```

```
figure;
fplot(ySoln(t),[0 20],'-','linewidth',2);
xlabel('Time');
ylabel('Mass Location');
title('Mass Location by Force');
grid on;
```