

# STL Notes

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## 1 Autonomous Vehicles

### 1.1 Intersection Management

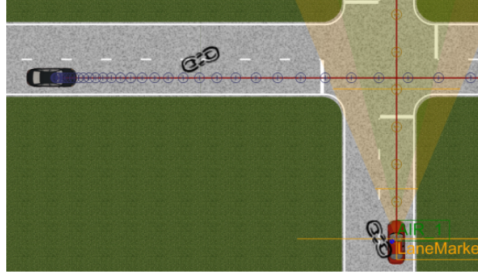


Figure 1: Intersection Scenario

In [1] the ego-vehicle's behavior at an intersection is defined using STL. The system has a state:

$$\mathbf{x}_t = [x_t^{ego} \ y_t^{ego} \ x_t^{adv} \ y_t^{adv} \ v_t^{ego} \ v_t^{adv}]^T \quad (1)$$

Figure 1 shows the ego-vehicle in blue and the adversary-vehicle in red. The authors of [1] use the following specification to describe the behavior:  $\phi_s$ : "When the ego-vehicle is within 2 meters of the adversary-vehicle the ego-vehicle will stop for two seconds."

$$\phi_s = \Box(|x_t^{ego} - x_t^{adv}| < 2) \implies \Box_{[0,2]}(|v_t^{ego}| < 0.1) \quad (2)$$

#### 1.1.1 Alternate Formulation

The above STL specification relies on the adversary-vehicle to stop at the intersection, but a car should always stop at an intersection. Inspired from [2] the following specification describes: "When the ego-vehicle reaches an intersection it will stop for one second and only cross when safe ( $\phi_{cross}$ ) otherwise it will stay stopped. ( $\phi_{stopped} \mathcal{U}_{[1,3]} \phi_{stopped}$ )."

$$\phi_{stopped} = (|v_t^{ego}| == 0) \quad (3)$$

$$\phi_{cross} = \Box_{[1,3]}(|y_t^{adv} - \mathbf{Y}^{INT}| > 0.1) \wedge (x_t^{ego} > 5) \quad (4)$$

$$\phi_{int} = \Box(|x_t^{ego} - \mathbf{X}^{INT}| < 0.1) \implies \Box[0.5, 1](\phi_{stopped}) \wedge (\phi_{stopped} \mathcal{U}_{[1,3]} \phi_{cross} \vee \phi_{stopped} \mathcal{U}_{[1,3]} \phi_{stopped}) \quad (5)$$

Where  $\mathbf{X}^{INT} = -5$  and  $\mathbf{Y}^{INT} = -5$  are the positional coordinates of the intersection.

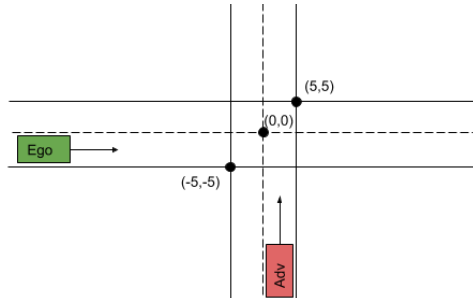


Figure 2: Alternate Scenario

## 2 STL for Hybrid Systems

### 2.1 Hybrid Time STL

#### 2.1.1 Hybrid System Solutions

Solutions to hybrid systems are defined on ordinary and discrete time:

1. Ordinary Time:  $t \in \mathbb{R}_{\geq 0} := [0, \infty)$
2. Discrete Time:  $j \in \mathbb{N} := \{0, 1, 2, \dots\}$

**Definition 2.1** (Hybrid Time Instance). A *hybrid time instance* is given by

$$(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$$

**Definition 2.2** (Compact Hybrid Domain). A set  $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$  is a *compact hybrid domain* if it can be written as

$$E = \bigcup_{j=0}^J ([t_j, t_{j+1}] \times \{j\}) \quad (6)$$

for a finite sequence of times  $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1}$ .

**Definition 2.3** (Hybrid Domain). A set  $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$  is a *hybrid domain* if for each  $(T, J) \in E$

$$E \cap ([0, T] \times \{0, 1, \dots, J\})$$

is a compact hybrid time domain.

**Definition 2.4** (Compact Hybrid Shift). Given a compact hybrid domain  $E$  and  $(t^*, j^*)$ , the forward compact hybrid shift of  $E$  by  $(t^*, j^*)$  is

$$(t^*, j^*) + E = \bigcup_{j=0}^J ([t_j + t^*, t_{j+1} + t^*] \times \{j + j^*\}) \quad (7)$$

and the backward hybrid shift of  $E$  by  $(t^*, j^*)$  is

$$E - (t^*, j^*) = \bigcup_{j=0}^J ([t_j - t^*, t_{j+1} - t^*] \times \{j - j^*\}) \quad (8)$$

for a finite sequence of times  $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1}$  and  $E$  satisfying (6).

**Definition 2.5** (Compact Hybrid Interval). A *compact hybrid interval* is defined by two hybrid time instances  $(t_A^*, j_A^*)$  and  $(t_B^*, j_B^*)$ , where  $t_A^* + j_A^* \leq t_B^* + j_B^*$ , and an hybrid domain  $E$ , over the range  $[t_A^*, t_B^*] \times \{j_A^*, \dots, j_B^*\}$ , by

$$\mathcal{I} := (t_A^*, j_A^*) + E = \bigcup_{j=0}^J ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\}) \quad (9)$$

- where  $J = j_B^* - j_A^*$  and  $t_0 = 0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1} = t_B^* - t_A^*$ .

**Definition 2.6** (Open Hybrid Interval). An open hybrid interval is defined by two hybrid time instances,  $(t_A^*, j_A^*)$  and  $(t_B^*, j_B^*)$ , on hybrid domain  $E$ , over the range  $[t_A^*, t_B^*] \times \{j_A^*, \dots, j_B^*\}$ , by the following cases:

1. Left open interval

$$\begin{aligned} \mathcal{I} &:= \left( (t_A^*, t_1 + t_A^*] \times \{j_A^*\} \right) \cup \left\{ (t_A^* + t_1, j_A^* + 1) + E \right\} \\ &= \left( (t_A^*, t_1 + t_A^*] \times \{j_A^*\} \right) \cup \left\{ \bigcup_{j=0}^J ([t_j + t_A^* + t_1, t_{j+1} + t_A^* + t_1] \times \{j + j_A^* + 1\}) \right\} \end{aligned} \quad (10)$$

- where  $J = j_B^* - j_A^* - 1$  and  $t_0 = 0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1} = t_B^* - t_A^* - t_1$ ;

2. Right open interval

$$\begin{aligned}\mathcal{I} &:= \left\{ (t_A^*, j_A^*) + E \right\} \cup \left( [t_{j_B}^*, t_B^*) \times \{j_B^*\} \right) \\ &= \left\{ \bigcup_{j=0}^J ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\}) \right\} \cup \left( [t_{j_B}^*, t_B^*) \times \{j_B^*\} \right)\end{aligned}\tag{11}$$

- where  $J = j_B^* - j_A^* - 1$  and  $t_0 = 0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1} = t_{j_B}^* - t_A^*$ ;

3. Open interval

$$\begin{aligned}\mathcal{I} &:= \left( (t_A^*, t_1 + t_A^*] \times \{j_A^*\} \right) \cup \left\{ (t_A^* + t_1, j_A^* + 1) + E \right\} \cup \left( [t_{j_B}^*, t_B^*) \times \{j_B^*\} \right) \\ &= \left( (t_A^*, t_1 + t_A^*] \times \{j_A^*\} \right) \cup \left\{ \bigcup_{j=0}^J ([t_j + t_A^* + t_1, t_{j+1} + t_A^*] \times \{j + j_A^* + 1\}) \right\} \cup \left( [t_{j_B}^*, t_B^*) \times \{j_B^*\} \right)\end{aligned}\tag{12}$$

- where  $J = j_B^* - j_A^* - 2$  and  $t_0 = 0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1} = t_{j_B}^* - t_A^*$ .

**Remark:** Authors of [3] denote a ordinary time interval  $\mathcal{I}$ , relative to ordinary time  $t$  as:

$$t + \mathcal{I} := \{t + t' : t' \in \mathcal{I}\}$$

The same notation can be used denoting a hybrid temporal interval,  $\mathcal{I}$  relative to hybrid time instance  $(t, j)$ :

$$(t, j) + I := \{(t + t', j + j') : (t', j') \in I\}$$

## 2.2 Continuous Time STL

A STL formula  $\phi$  is defined recursively as:

$$\psi ::= p^\mu \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_{\mathcal{I}}\psi \mid \psi_1 U_{\mathcal{I}}\psi_2 \quad (13)$$

Where  $\psi_1, \psi_2$  are STL formula and  $\mathcal{I}$  is an ordinary time interval of form  $\mathcal{I} = (t_a, t_b), (t_a, t_b], [t_a, t_b),$  or  $[t_a, t_b]$  with  $t_a < t_b$  and  $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ .

Inductive STL Definition

$$\begin{aligned} \phi &\models \psi && \Leftrightarrow (\phi, t) \models p \\ (\phi, t) &\models p^\mu && \Leftrightarrow \mu(\phi(t)) > 0 \\ (\phi, t) &\models \Box_{[a,b]}\psi && \Leftrightarrow \forall t' \in [t+a, t+b] \text{ s.t. } (\phi, t') \models \psi \end{aligned}$$

### 2.2.1 Possible Notion of Hybrid STL

Recursive STL Definition

$$\psi ::= p^\mu \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_I \psi \mid \psi_1 U_I \psi_2 \quad (14)$$

Where  $\psi_1, \psi_2$  are STL formula and  $I$  is a hybrid time interval.

Inductive STL Definition

$$\begin{aligned} \phi \models \psi & \Leftrightarrow (\phi, (t, j)) \models p \\ (\phi, (t, j)) \models p^\mu & \Leftrightarrow \mu(\phi(t)) > 0 \\ ((\phi, (t, j)) \models \Box_I \psi) & \Leftrightarrow \forall (t', j') \in (t, j) + I \text{ s.t. } (\phi, (t', j')) \models \psi \end{aligned}$$

### 2.2.2 Previous Notions of Hybrid STL

From Atreya - Modeling Hybrid Systems using Signal Temporal Logic

Inductive STL Definition

$$((\phi, (t, j)) \models \Box_{(t, j)} \psi) \Leftrightarrow \forall (t', j') \in \text{dom } \phi, t' + j' > t + j \text{ s.t. } (\phi, (t', j')) \models \psi \quad (15)$$

## .1 Lane Changing

The authors of [2] use STL and Hamilton Jacobi reachability to combine temporal specification with quantification of a reachable set: STL temporal specifications are related to reachability operators.

### .1.1 Example

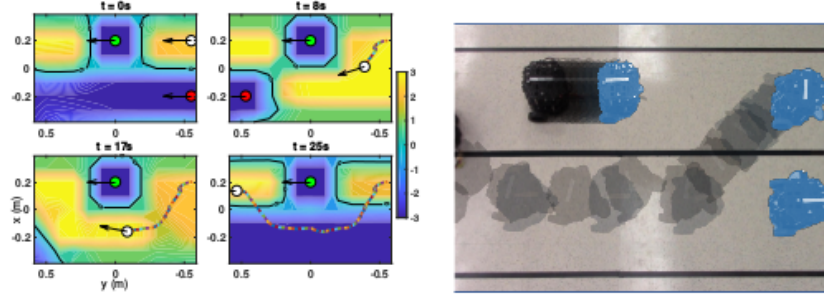


Figure 3: Lane Change Scenario

In Figure 3 white represents the autonomous car, green a slow car, and red an adjacent car. The white car's lane change specification is given as: "Within 25 seconds, satisfy general safety  $\phi$  until you can pass. Otherwise stay within the lane. Also ensure you are always within 5 seconds from re-entering the lane. ( $\psi^{lane}$ )."

$$\phi = \phi^{off-road} \wedge \phi^{on-road} \wedge \phi^{avoid} \quad (16)$$

$$\Box_{[0,25]} \Diamond_{[0,5]} \psi^{lane} \wedge (\phi \mathcal{U}_{[0,25]} \psi^{pass} \vee \phi \mathcal{U}_{[0,25]} \psi^{stay}) \quad (17)$$

## .2 Solutions for Discrete Systems

Consider a discrete system of the form:

$$x^+ \in G(x, \gamma) \quad (18)$$

Let  $x \in \mathbb{R}^n$  be states of the discrete system.

Let  $\gamma \in \mathbb{R}^m$  be input to the discrete system.

Discrete time is denoted:  $k \in \mathbb{N}$

Then, given an initial state,  $x_0$  and an input  $k \mapsto \gamma(k)$ , a state trajectory(solution) can be constructed. The domain of  $\gamma$  is given:  $dom \gamma = \{0, 1, 2, \dots, K\} \cap \mathbb{N}$  where  $K \in \mathbb{N} \cup \{\infty\}$ . Solutions are defined as:  $k \mapsto \phi(k)$  such that the following are satisfied:

1.  $\phi(0) = x_0$
2.  $dom \phi = dom \gamma$  if  $K = \infty$
3.  $dom \phi \setminus \{max \text{ dom } \phi\} = dom \gamma$  if  $K$  is finite
4.  $\forall k \in dom \gamma: \phi(k+1) = G(\phi(k), \gamma(k))$

### .2.1 Discrete Time STL

$$\psi ::= p^\mu \mid \neg \psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_I \psi \mid \psi_1 U_I \psi_2 \quad (19)$$

Where  $\phi_1, \phi_2$  are STL formula and  $I$  is an interval on  $dom \phi$ .

Inductive STL Definition

$$\begin{aligned} \sigma &\models \psi && \Leftrightarrow (\sigma, k) \models p \\ (\sigma, k) &\models p^\mu && \Leftrightarrow \mu(x_k, \gamma_k) > 0 \\ (\sigma, k) &\models \Box_{[n,m]} \psi && \Leftrightarrow \forall k' \in [k+n, k+m] \text{ s.t } (\sigma, k') \models \psi \end{aligned}$$

## References

- [1] V. Raman, A. Donzé, D. Sadigh, R. M. Murray, and S. A. Seshia, “Reactive synthesis from signal temporal logic specifications,” Seattle, Washington, 2015. [Online]. Available: <http://dl.acm.org/citation.cfm?doid=2728606.2728628>
- [2] M. Chen, Q. Tam, S. C. Livingston, and M. Pavone, “Signal Temporal Logic meets Hamilton-Jacobi Reachability: Connections and Applications,” 2018.
- [3] A. Donzé and O. Maler, “Robust Satisfaction of Temporal Logic over Real-Valued Signals.” Berlin, Heidelberg: Springer Berlin Heidelberg, 2010.