2 STL for Hybrid Systems

2.1 Hybrid Systems Preliminaries

Definition 2.1 (Hybrid System)**.** A hybrid system $\mathcal{H} = (C, F, D, G)$ is defined by a flow set C, flow map F, jump set D, and jump map G, following [?]. $x \in \mathcal{X}$ is the system's state where $\mathcal{X} \subset \mathbb{R}^n$ is the state space. Continuous dynamics are defined on flow set C and are captured by the set valued map $F : \mathcal{X} \rightrightarrows \mathcal{X}$. Discrete dynamics are defined on jump set D and are captured by by the set-valued map $G : \mathcal{X} \rightrightarrows \mathcal{X}$. Taken together, $\mathcal{H}$ can be written as:

$$\begin{aligned} \dot{x} \ &\in F(x) \qquad x \in C \\ x^+ &\in F(x) \qquad x \in D \end{aligned} \tag{6}$$

Solutions to a hybrid system $\mathcal{H}$ are defined on ordinary and discrete time:

1. Ordinary Time: $t \in \mathbb{R}_{\geq 0} := [0, \infty)$
2. Discrete Time: $j \in \mathbb{N} := \{0, 1, 2, ...\}$

Definition 2.2 (Hybrid Time Instance)**.** A *hybrid time instance* is given by

$$(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$$

Definition 2.3 (Compact Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid domain* if it can be written as

$$E := \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}) \tag{7}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$.

Definition 2.4 (Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid domain* if for each $(T, J) \in E$

$$E \cap ([0, T] \times \{0, 1, ..., J\})$$

is a compact hybrid time domain.

Definition 2.5 (Compact Hybrid Shift)**.** Given a compact hybrid domain E and (t^*, j^*), the forward compact hybrid shift of E by (t^*, j^*) is

$$(t^*, j^*) + E = \bigcup_{j=0}^{J} ([t_j + t^*, t_{j+1} + t^*] \times \{j + j^*\} \tag{8}$$

and the backward hybrid shift of E by (t^*, j^*) is

$$E - (t^*, j^*) = \bigcup_{j=0}^{J} ([t_j - t^*, t_{j+1} - t^*] \times \{j - j^*\} \tag{9}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$ and E satisfying (7).

Definition 2.6 (Compact Hybrid Interval)**.** A *compact hybrid interval* is defined by two hybrid time instances (t_A^*, j_A^*) and (t_B^*, j_B^*), where $t_A^* + j_A^* \leq t_B^* + j_B^*$, and an hybrid domain E, over the range $[t_A^*, t_B^*] \times \{j_A^*, ..., j_B^*\}$, by

$$\mathcal{I}_{t_A^*, j_A^*}^{t_B^*, j_B^*} := (t_A^*, j_A^*) + E = \bigcup_{j=0}^{J} ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\} \tag{10}$$

- where $J = j_B^* - j_A^*$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^*$.

Definition 2.7 (Open Hybrid Interval)**.** An open hybrid interval is defined by two hybrid time instances, (t_A^*, j_A^*) and (t_B^*, j_B^*), on hybrid domain E, over the range $[t_A^*, t_B^*] \times \{j_A^*, ..., j_B^*\}$, by the following cases:

1. Left open interval

$$\begin{aligned} \mathcal{I}_{t_A^*, j_A^*}^{t_B^*, j_B^*} : &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ (t_A^* + t_1, j_A^* + 1) + E \Big\} \\ &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ \bigcup_{j=0}^{J} ([t_j + t_A^* + t_1, t_{j+1} + t_A^* + t_1] \times \{j + j_A^* + 1\} \Big\} \end{aligned} \tag{11}$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^* - t_1$;

2. Right open interval

$$\mathcal{I}_{t_A^*,j_A^*}^{t_B^*,j_B^*} := \left\{(t_A^*, j_A^*) + E\right\} \cup \left([t_{j_B^*}, t_B^*) \times \{j_B^*\}\right) \tag{12}$$
$$= \left\{\bigcup_{j=0}^{J}([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\}\right\} \cup \left([t_{j_B^*}, t_B^*) \times \{j_B^*\}\right)$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_{j_B^*} - t_A^*$;

3. Open interval

$$\mathcal{I}_{t_A^*,j_A^*}^{t_B^*,j_B^*} := \left((t_A^*, t_1 + t_A^*] \times \{j_A^*\}\right) \cup \left\{(t_A^* + t_1, j_A^* + 1) + E\right\} \cup \left([t_{j_B^*}, t_B^*) \times \{j_B^*\}\right) \tag{13}$$
$$= \left((t_A^*, t_1 + t_A^*] \times \{j_A^*\}\right) \cup \left\{\bigcup_{j=0}^{J}([t_j + t_A^* + t_1, t_{j+1} + t_A^*] \times \{j + j_A^* + 1\}\right\} \cup \left([t_{j_B^*}, t_B^*) \times \{j_B^*\}\right)$$

- where $J = j_B^* - j_A^* - 2$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_{j_B^*} - t_A^*$.

Definition 2.8 (Hybrid Arc)**.** A function $\phi : E \mapsto \mathcal{X}$ is a *hybrid arc* if E is a hybrid time domain and for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $I_j := \{t : (t, j) \in E\}$. The interior of I_j is denoted int I_j.

Definition 2.9 (Solution of a Hybrid System)**.** A hybrid arc ϕ is a solution to hybrid system $\mathcal{H} = (C, F, D, G)$ if

1. $\phi(0, 0) \in C \cup D$
2. for all $j \in \mathbb{N}$ such that I^j has a nonempty interior
   - $\phi(t, j) \in C$ for all $t \in$ int I^j
   - $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I^k$
3. for all $(t, j) \in E$ such that $(t, j + 1) \in E$
   - $\phi(t, j) \in D$
   - $\phi(t, j + 1) \in G(\phi(t, j))$

2.2 STL Definitions

Definition 2.10. (Atomic Proposition)

An atomic proposition $p^\mu : \mathcal{X} \mapsto \mathbb{B}$ is a function that maps the state space of the system to a boolean value. The set of all atomic propositions is denoted by $\mathcal{P}$.

The function $\mu : \mathcal{X} \mapsto \mathbb{R}$ represents a robustness measure of the proposition and has following relation to an atomic proposition.

$$\begin{aligned} \mu > 0 &\implies p^\mu = 1 \\ \mu \leq 0 &\implies p^\mu = 0 \end{aligned}$$

Definition 2.11 (STL Grammar)**.** STL formulas are defined recursively by the following grammar:

$$\varphi ::= p^\mu \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_{\mathcal{I}}\, \varphi \mid \varphi_1\, \mathcal{U}_{\mathcal{I}}\, \varphi_2$$

Where $\mathcal{I}$ is a open or closed hybrid interval and φ, φ_1,φ_2 are STL formula.

The fact that a hybrid arc satisfies STL formula φ at hybrid time instance (t,j) is given by $\varphi(\phi(t, j)) = 1$. When a hybrid arc does not satisfy a proposition then $\varphi(\phi(t, j)) = 0$.

The validity of a formula φ with respect to hybrid arc ϕ at at time (t,j) is defined inductively as

$$\begin{aligned}
(\phi, (t, j)) &\models p^\mu &&\iff \mu(\phi(t, j)) > 0 \\
(\phi, (t, j)) &\models \varphi &&\iff \varphi(\phi(t, j)) = 1 \\
(\phi, (t, j)) &\models \neg\varphi &&\iff \neg((\phi, (t, j)) \models \varphi) \\
(\phi, (t, j)) &\models \varphi \wedge \psi &&\iff (\phi, (t, j)) \models \varphi \wedge (\phi, (t, j)) \models \psi \\
(\phi, (t, j)) &\models \Diamond_{\mathcal{I}_{t_a,j_a}^{t_b,j_b}}\, \varphi &&\iff \exists (t', j') \in (t, j) + \mathcal{I}_{t_a,j_a}^{t_b,j_b} \text{ s.t } (\phi, (t', j')) \models \varphi \\
(\phi, (t, j)) &\models \psi\, \mathcal{U}_{\mathcal{I}_{t_a,j_a}^{t_b,j_b}}\, \varphi &&\iff \exists (t', j') \in (t, j) + \mathcal{I}_{t_a,j_a}^{t_b,j_b} \text{ s.t } (\phi, (t', j')) \models \varphi \;\wedge\; \forall (t'', j'') \in \mathcal{I}_{t,j}^{t',j'},\; (\phi, (t'', j'')) \models \psi
\end{aligned}$$