STL Notes

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1 Autonomous Vehicles

1.1 Intersection Management

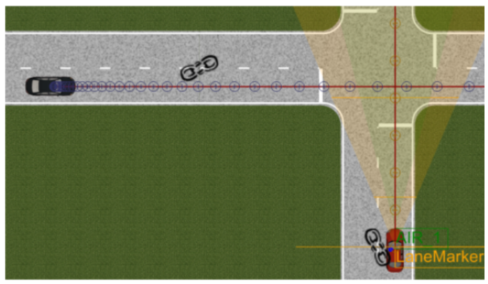

Figure 1: Intersection Scenario

In [1] the ego-vehicle's behavior at an intersection is defined using STL. The system has a state:

$$\mathbf{x_t} = [x_t^{ego}\ y_t^{ego}\ x_t^{adv}\ y_t^{adv}\ v_t^{ego}\ v_t^{adv}\ ]^T \tag{1}$$

Figure 1 shows the ego-vehicle in blue and the adversary-vehicle in red. The authors of [1] use the following specification to describe the behavior: ϕ_s: "When the ego-vehicle is within 2 meters of the adversary-vehicle the ego-vehicle will stop for two seconds."

$$\phi_s = \Box(|x_t^{ego} - x_t^{adv}| < 2) \implies \Box_{[0,2]}(|v_t^{ego}| < 0.1) \tag{2}$$

1.1.1 Alternate Formulation

The above STL specification relies on the adversary-vehicle to stop at the intersection, but a car should always stop at an intersection. Inspired from [2] the following specification describes: "When the ego-vehicle reaches an intersection it will stop for one second and only cross whens safe (ϕ_{cross}) otherwise it will stay stopped. ($\phi_{stopped}\mathcal{U}_{[1,3]}\phi_{stopped}$)."

$$\phi_{stopped} = (|v_t^{ego}| == 0) \tag{3}$$

$$\phi_{cross} = \Box_{[1,3]}(|y_t^{adv} - \mathbf{Y^{INT}}| > 0.1) \wedge (x_t^{ego} > 5) \tag{4}$$

$$\phi_{int} = \Box(|x_t^{ego} - \mathbf{X^{INT}}| < 0.1) \implies \Box[0.5,1](\phi_{stopped}) \wedge (\phi_{stopped}\mathcal{U}_{[1,3]}\phi_{cross} \vee \phi_{stopped}\mathcal{U}_{[1,3]}\phi_{stopped}) \tag{5}$$

Where $\mathbf{X^{INT}} = -5$ and $\mathbf{Y^{INT}} = -5$ are the positional coordinates of the intersection.

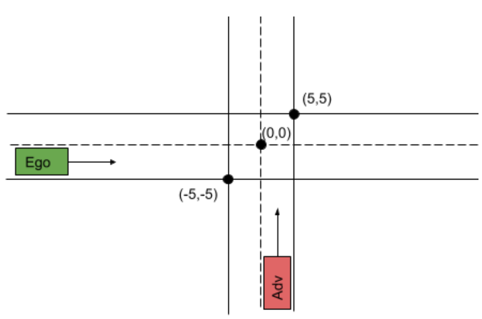

Figure 2: Alternate Scenario

2 STL for Hybrid Systems

2.1 Hybrid Systems Preliminaries

Definition 2.1 (Hybrid System)**.** A hybrid system $\mathcal{H} = (C, F, D, G)$ is defined by a flow set C, flow map F, jump set D, and jump map G, following [**?**]. $x \in \mathcal{X}$ is the system's state where $\mathcal{X} \subset \mathbb{R}^n$ is the state space. Continuous dynamics are defined on flow set C and are captured by the set valued map $F : \mathcal{X} \rightrightarrows \mathcal{X}$. Discrete dynamics are defined on jump set D and are captured by by the set-valued map $G : \mathcal{X} \rightrightarrows \mathcal{X}$. Taken together, $\mathcal{H}$ can be described as:

$$\begin{aligned} \dot{x} &\in F(x) \qquad x \in C \\ x^+ &\in F(x) \qquad x \in D \end{aligned} \tag{6}$$

Solutions to a hybrid system $\mathcal{H}$ are defined on ordinary and discrete time:

1. Ordinary Time: $t \in \mathbb{R}_{\geq 0} := [0, \infty)$
2. Discrete Time: $j \in \mathbb{N} := \{0, 1, 2, ...\}$

Definition 2.2 (Hybrid Time Instance)**.** A *hybrid time instance* is given by

$$(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$$

Definition 2.3 (Compact Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid domain* if it can be written as

$$E := \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}) \tag{7}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$.

Definition 2.4 (Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid domain* if for each $(T, J) \in E$

$$E \cap ([0, T] \times \{0, 1, ..., J\})$$

is a compact hybrid time domain.

Definition 2.5 (Compact Hybrid Shift)**.** Given a compact hybrid domain E and (t^*, j^*), the forward compact hybrid shift of E by (t^*, j^*) is

$$(t^*, j^*) + E = \bigcup_{j=0}^{J} ([t_j + t^*, t_{j+1} + t^*] \times \{j + j^*\} \tag{8}$$

and the backward hybrid shift of E by (t^*, j^*) is

$$E - (t^*, j^*) = \bigcup_{j=0}^{J} ([t_j - t^*, t_{j+1} - t^*] \times \{j - j^*\} \tag{9}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$ and E satisfying (7).

Definition 2.6 (Compact Hybrid Interval)**.** A *compact hybrid interval* is defined by two hybrid time instances (t_A^*, j_A^*) and (t_B^*, j_B^*), where $t_A^* + j_A^* \leq t_B^* + j_B^*$, and an hybrid domain E, over the range $[t_A^*, t_B^*] \times \{j_A^*, ..., j_B^*\}$, by

$$\mathcal{I} := (t_A^*, j_A^*) + E = \bigcup_{j=0}^{J} ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\} \tag{10}$$

- where $J = j_B^* - j_A^*$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^*$.

Definition 2.7 (Open Hybrid Interval)**.** An open hybrid interval is defined by two hybrid time instances, (t_A^*, j_A^*) and (t_B^*, j_B^*), on hybrid domain E, over the range $[t_A^*, t_B^*] \times \{j_A^*, ..., j_B^*\}$, by the following cases:

1. Left open interval

$$\begin{aligned} \mathcal{I} :&= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ (t_A^* + t_1, j_A^* + 1) + E \Big\} \\ &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ \bigcup_{j=0}^{J} ([t_j + t_A^* + t_1, t_{j+1} + t_A^* + t_1] \times \{j + j_A^* + 1\} \Big\} \end{aligned} \tag{11}$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^* - t_1$;

2. Right open interval

$$\begin{aligned}\mathcal{I} :&= \Big\{(t_A^*, j_A^*) + E\Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\}\Big) \\ &= \Big\{\bigcup_{j=0}^{J}([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\}\Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\}\Big)\end{aligned} \tag{12}$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \le t_1 \le t_2 \le ... \le t_{J+1} = t_{j_B^*} - t_A^*$;

3. Open interval

$$\begin{aligned}\mathcal{I} :&= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\}\Big) \cup \Big\{(t_A^* + t_1, j_A^* + 1) + E\Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\}\Big) \\ &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\}\Big) \cup \Big\{\bigcup_{j=0}^{J}([t_j + t_A^* + t_1, t_{j+1} + t_A^*] \times \{j + j_A^* + 1\}\Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\}\Big)\end{aligned} \tag{13}$$

- where $J = j_B^* - j_A^* - 2$ and $t_0 = 0 \le t_1 \le t_2 \le ... \le t_{J+1} = t_{j_B^*} - t_A^*$.

Definition 2.8 (Hybrid Arc)**.** A function $\phi : E \mapsto \mathcal{X}$ is a *hybrid arc* if E is a hybrid time domain and for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $I_j := \{t : (t, j) \in E\}$. The interior of I_j is denoted int I_j.

Definition 2.9 (Solution of a Hybrid System)**.** A hybrid arc ϕ is a solution to hybrid system $\mathcal{H} = (C, F, D, G)$ if

1. $\phi(0, 0) \in C \cup D$
2. for all $j \in \mathbb{N}$ such that I^j has a nonempty interior
   - $\phi(t, j) \in C$ for all $t \in \text{int } I^j$
   - $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I^k$
3. for all $(t, j) \in E$ such that $(t, j + 1) \in E$
   - $\phi(t, j) \in D$
   - $\phi(t, j + 1) \in G(\phi(t, j))$

2.2 STL Definitions

Definition 2.10. (Atomic Proposition)

An atomic proposition p^μ is a function that maps the state space of the system to a boolean value. The set of all atomic propositions is denoted by $\mathcal{P}$.

$$p^\mu \ : \ \mathcal{X} \mapsto \mathbb{B} \tag{14}$$

The function $\mu : \mathcal{X} \mapsto \mathbb{R}$ represents a robustness measure of the proposition such that. Where,

$$\begin{aligned}\mu > 0 &\implies p^\mu = 1 \\ \mu \le 0 &\implies p^\mu = 0\end{aligned}$$

Definition 2.11 (STL Grammar)**.** STL formulas are defined recursively by the following grammar:

$$\varphi ::= p^\mu \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_{\mathcal{I}}\, \varphi \mid \varphi_1\, \mathcal{U}_{\mathcal{I}}\, \varphi_2$$

Where $\mathcal{I}$ is a open or closed hybrid interval.

The fact that a hybrid arc satisfies STL formula φ at hybrid time instance (t,j) is given by $\varphi(\phi(t, j)) = 1$. When a hybrid arc does not satisfy a proposition then $\varphi(\phi(t, j)) = 0$.

The validity of a formula φ with respect to hybrid arc ϕ at at time (t,j) is defined inductively as

$$\begin{aligned}
(\phi, (t, j)) \models p^\mu \quad &\iff \mu(\phi(t, j)) > 0 \\
(\phi, (t, j)) \models \varphi \quad &\iff \varphi(\phi(t, j)) = 1 \\
(\phi, (t, j)) \models \neg\varphi \quad &\iff \neg((\phi, (t, j)) \models \varphi) \\
(\phi, (t, j)) \models \varphi \wedge \psi \quad &\iff (\phi, (t, j)) \models \varphi \wedge (\phi, (t, j)) \models \psi \\
(\phi, (t, j)) \models \Diamond_{\mathcal{I}}\, \varphi \quad &\iff \exists (t', j') \in (t, j) + \mathcal{I} \text{ s.t } (\phi, (t', j')) \models \varphi \\
(\phi, (t, j)) \models \psi\, \mathcal{U}_{\mathcal{I}}\, \varphi \quad &\iff \exists (t', j') \in (t, j) + \mathcal{I} \text{ s.t } (\phi, (t', j')) \models \varphi \ \wedge \ \forall (t'', j'') \in \mathcal{I} - (t, j) \text{ s.t } (\phi, (t'', j'')) \models \psi
\end{aligned}$$

The true and false conditions will be denoted by

$$\begin{aligned}\phi(t,j) \Vdash p^{\mu} &\implies p^{\mu}(\phi(t,j)) = 1 \\ \phi(t,j) \nVdash p^{\mu} &\implies p^{\mu}(\phi(t,j)) = 0\end{aligned}$$

Remark: Authors of [3] denote a ordinary time interval $\mathcal{I}$, relative to ordinary time t as:

$$t + \mathcal{I} := \{t + t' : t' \in \mathcal{I}\}$$

The same notation can be used denoting a hybrid temporal interval, $\mathcal{I}$ relative to hybrid time instance (t, j):

$$(t, j) + I := \{(t + t', j + j') : (t', j') \in I\}$$

2.3 Continuous Time STL

A STL formula ϕ is defined recursively as:

$$\psi ::= p^{\mu} \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_{\mathcal{I}}\psi \mid \psi_1 U_{\mathcal{I}} \psi_2 \tag{15}$$

Where ψ_1, ψ_2 are STL formula and $\mathcal{I}$ is an ordinary time interval of form $\mathcal{I} = (t_a, t_b), (t_a, t_b], [t_a, t_b)$, or $[t_a, t_b]$ with $t_a < t_b$ and $\mathcal{I} \subset \mathbb{R}_{\geq 0}$.

Inductive STL Definition

$$\begin{aligned}
&\phi \models \psi && \Leftrightarrow (\phi, t) \models p \\
&(\phi, t) \models p^{\mu} && \Leftrightarrow \mu(\phi(t)) > 0 \\
&(\phi, t) \models \Box_{[a,b]}\psi && \Leftrightarrow \forall t' \in [t + a, t + b] \; s.t \; (\phi, t') \models \psi
\end{aligned}$$

2.3.1 Possible Notion of Hybrid STL

Recursive STL Definition

$$\psi ::= p^{\mu} \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Diamond_I \psi \mid \psi_1 U_I \psi_2 \tag{16}$$

Where ψ_1, ψ_2 are STL formula and I is a hybrid time interval.

Inductive STL Definition

$$\begin{aligned}
&\phi \models \psi && \Leftrightarrow (\phi, (t, j)) \models p \\
&(\phi, (t, j) \models p^{\mu} && \Leftrightarrow \mu(\phi(t)) > 0 \\
&((\phi, (t, j)) \models \Box_I \psi && \Leftrightarrow \forall (t', j') \in (t, j) + I \ s.t. \ (\phi, (t', j')) \models \psi
\end{aligned}$$

2.3.2 Previous Notions of Hybrid STL

From Atreya - Modeling Hybrid Systems using Signal Temporal Logic

Inductive STL Definition

$$((\phi, (t, j)) \models \Box_{(t,j)} \psi \Leftrightarrow \forall (t', j') \in dom\ \phi, t' + j' > t + j \ s.t \ (\phi, (t', j')) \models \psi \tag{17}$$

.1 Lane Changing

The authors of [2] use STL and Hamilton Jacobi reachability to combine temporal specification with quantification of a reachable set: STL temporal specifications are related to reachability operators.

.1.1 Example

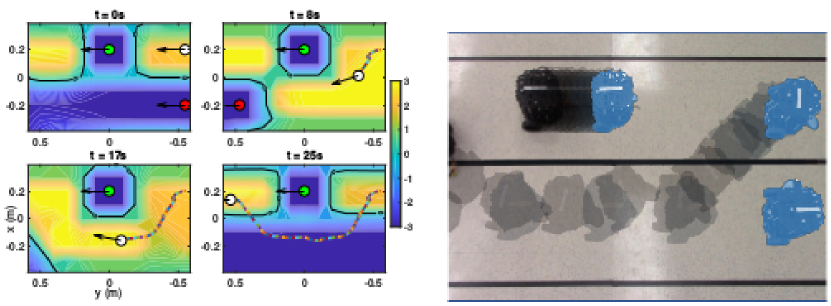

Figure 3: Lane Change Scenario

In Figure 3 white represents the autonomous car, green a slow car, and red an adjacent car. The white car's lane change specification is given as: "Within 25 seconds, satisfy general safety ϕ until you can pass. Otherwise stay within the lane. Also ensure you are always within 5 seconds from re-entering the lane.(ψ^{lane}.)"

$$\phi = \phi^{off-road} \wedge \phi^{on-road} \wedge \phi^{avoid} \tag{18}$$

$$\square_{[0,25]} \lozenge_{[0,5]} \psi^{lane} \wedge (\phi \mathcal{U}_{[0,25]} \psi^{pass} \vee \phi \mathcal{U}_{[0,25]} \psi^{stay}) \tag{19}$$

.2 Solutions for Discrete Systems

Consider a discrete system of the form:

$$x^+ \in G(x, \gamma) \tag{20}$$

Let $x \in \mathbb{R}^n$ be states of the discrete system.
Let $\gamma \in \mathbb{R}^m$ be input to the discrete system.
Discrete time is denoted: $k \in \mathbb{N}$
Then, given an initial state, x_0 and an input $k \mapsto \gamma(k)$, a state trajectory(solution) can be constructed. The domain of γ is given: $dom\ \gamma = \{0, 1, 2, ..., K\} \cap \mathbb{N}$ where $K \in \mathbb{N} \cup \{\infty\}$. Solutions are defined as: $k \mapsto \phi(k)$ such that the following are satisfied:

1. $\phi(0) = x_0$
2. $dom\ \phi = dom\ \gamma$ if $K = \infty$
3. $dom\ \phi \setminus \{max\ dom\ \phi\} = dom\ \gamma$ if K is finite
4. $\forall k \in dom\ \gamma$: $\phi(k+1) = G(\phi(k), \gamma(k))$

.2.1 Discrete Time STL

$$\psi ::= p^\mu \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \lozenge_I \psi \mid \psi_1 U_I \psi_2 \tag{21}$$

Where ϕ_1, ϕ_2 are STL formula and I is an interval on $dom\ \phi$.

Inductive STL Definition

$$\begin{aligned}
\sigma \models \psi &\Leftrightarrow (\sigma, k) \models p \\
(\sigma, k) \models p^\mu &\Leftrightarrow \mu(x_k, \gamma_k) > 0 \\
(\sigma, k) \models \square_{[n,m]} \psi &\Leftrightarrow \forall k' \in [k'+n, k'+m]\ s.t\ (\sigma, k') \models \psi
\end{aligned}$$

References

[1] V. Raman, A. Donzé, D. Sadigh, R. M. Murray, and S. A. Seshia, “Reactive synthesis from signal temporal logic specifications,” Seattle, Washington, 2015. [Online]. Available: http://dl.acm.org/citation.cfm?doid=2728606.2728628

[2] M. Chen, Q. Tam, S. C. Livingston, and M. Pavone, “Signal Temporal Logic meets Hamilton-Jacobi Reachability: Connections and Applications,” 2018.

[3] A. Donzé and O. Maler, “Robust Satisfaction of Temporal Logic over Real-Valued Signals.” Berlin, Heidelberg: Springer Berlin Heidelberg, 2010.