STL Notes

David Kooi

July 2, 2019

1 STL for Hybrid Systems

1.1 Hybrid Systems Preliminaries

Definition 1.1 (Hybrid System)**.** A hybrid system $\mathcal{H} = (C, F, D, G)$ is defined by a flow set C, flow map F, jump set D, and jump map G, following [3]. $x \in \mathcal{X}$ is the system's state where $\mathcal{X} \subset \mathbb{R}^n$ is the state space. Continuous dynamics are defined on flow set C and are captured by the set valued map $F : \mathcal{X} \rightrightarrows \mathcal{X}$. Discrete dynamics are defined on jump set D and are captured by by the set-valued map $G : \mathcal{X} \rightrightarrows \mathcal{X}$. Taken together, $\mathcal{H}$ can be written as:

$$\begin{aligned} \dot{x} &\in F(x) \qquad x \in C \\ x^+ &\in F(x) \qquad x \in D \end{aligned} \tag{1}$$

Solutions to a hybrid system $\mathcal{H}$ are defined on ordinary and discrete time:

1. Ordinary Time: $t \in \mathbb{R}_{\geq 0} := [0, \infty)$
2. Discrete Time: $j \in \mathbb{N} := \{0, 1, 2, ...\}$

Definition 1.2 (Hybrid Time Instance)**.** A *hybrid time instance* is given by

$$(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$$

Definition 1.3 (Compact Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid domain* if it can be written as

$$E := \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}) \tag{2}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$.

Definition 1.4 (Hybrid Domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid domain* if for each $(T, J) \in E$

$$E \cap ([0, T] \times \{0, 1, ..., J\})$$

is a compact hybrid time domain.

Definition 1.5 (Compact Hybrid Shift)**.** Given a compact hybrid domain E and (t^*, j^*), the forward compact hybrid shift of E by (t^*, j^*) is

$$(t^*, j^*) + E = \bigcup_{j=0}^{J} ([t_j + t^*, t_{j+1} + t^*] \times \{j + j^*\} \tag{3}$$

and the backward hybrid shift of E by (t^*, j^*) is

$$E - (t^*, j^*) = \bigcup_{j=0}^{J} ([t_j - t^*, t_{j+1} - t^*] \times \{j - j^*\} \tag{4}$$

for a finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$ and E satisfying (2).

Definition 1.6 (Hybrid Interval)**.** A *hybrid interval* is defined by two hybrid time instances (t_A^*, j_A^*) and (t_B^*, j_B^*), where $t_A^* + j_A^* \leq t_B^* + j_B^*$, and an hybrid domain E, over the range $[t_A^*, t_B^*] \times \{j_A^*, ..., j_B^*\}$.

Definition 1.7 (Compact Hybrid Interval)**.** A hybrid interval is a *compact hybrid interval* if it has the form

$$\mathcal{I} := (t_A^*, j_A^*) + E = \bigcup_{j=0}^{J} ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\} \tag{5}$$

- where $J = j_B^* - j_A^*$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^*$.

Definition 1.8 (Open Hybrid Interval)**.** A hybrid time interval is an *open hybrid interval* if it has the form of the following cases:

1. Left open interval

$$\begin{aligned}\mathcal{I} :&= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ (t_A^* + t_1, j_A^* + 1) + E \Big\} \\ &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ \bigcup_{j=0}^{J} ([t_j + t_A^* + t_1, t_{j+1} + t_A^* + t_1] \times \{j + j_A^* + 1\} \Big\}\end{aligned} \tag{6}$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_B^* - t_A^* - t_1$;

2. Right open interval

$$\begin{aligned}\mathcal{I} :&= \Big\{ (t_A^*, j_A^*) + E \Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\} \Big) \\ &= \Big\{ \bigcup_{j=0}^{J} ([t_j + t_A^*, t_{j+1} + t_A^*] \times \{j + j_A^*\} \Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\} \Big)\end{aligned} \tag{7}$$

- where $J = j_B^* - j_A^* - 1$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_{j_B^*} - t_A^*$;

3. Open interval

$$\begin{aligned}\mathcal{I} :&= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ (t_A^* + t_1, j_A^* + 1) + E \Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\} \Big) \\ &= \Big((t_A^*, t_1 + t_A^*] \times \{j_A^*\} \Big) \cup \Big\{ \bigcup_{j=0}^{J} ([t_j + t_A^* + t_1, t_{j+1} + t_A^*] \times \{j + j_A^* + 1\} \Big\} \cup \Big([t_{j_B^*}, t_B^*) \times \{j_B^*\} \Big)\end{aligned} \tag{8}$$

- where $J = j_B^* - j_A^* - 2$ and $t_0 = 0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1} = t_{j_B^*} - t_A^*$.

Definition 1.9 (Hybrid Arc)**.** A function $\phi : E \mapsto \mathcal{X}$ is a *hybrid arc* if E is a hybrid time domain and for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $I_j := \{t : (t, j) \in E\}$. The interior of I^j is denoted int I^j.

Definition 1.10 (Solution of a Hybrid System)**.** A hybrid arc ϕ is a solution to hybrid system $\mathcal{H} = (C, F, D, G)$ if

1. $\phi(0, 0) \in \overline{C} \cup D$
2. for all $j \in \mathbb{N}$ such that I^j has a nonempty interior
   - $\phi(t, j) \in C$ for all $t \in \text{int } I^j$
   - $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I^j$
3. for all $(t, j) \in \text{dom } \phi$ such that $(t, j + 1) \in \text{dom } \phi$
   - $\phi(t, j) \in D$
   - $\phi(t, j + 1) \in G(\phi(t, j))$

1.2 STL Definitions

Definition 1.11. (Atomic Proposition)

An atomic proposition $p^\mu : \mathcal{X} \mapsto \mathbb{B}$ is a function that maps the state space of the system to a boolean value. The set of all atomic propositions is denoted by $\mathcal{P}$.

The function $\mu : \mathcal{X} \mapsto \mathbb{R}$ represents a robustness measure of the proposition and has following relation to an atomic proposition.

$$\mu > 0 \Leftrightarrow p^{\mu} = 1$$
$$\mu \leq 0 \Leftrightarrow p^{\mu} = 0$$

Definition 1.12 (STL Grammar)**.** STL formulas are defined recursively by the following grammar:

$$\varphi ::= p^{\mu} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_{\mathcal{I}}\, \varphi \mid \varphi_1\, \mathcal{U}_{\mathcal{I}}\, \varphi_2$$

Where $\mathcal{I}$ is a open or closed hybrid interval and φ, φ_1,φ_2 are STL formulae.

The fact that a hybrid arc satisfies STL formula φ at hybrid time instance (t,j) is given by $\varphi(\phi(t,j)) = 1$. When a hybrid arc does not satisfy a proposition then $\varphi(\phi(t,j)) = 0$.

The validity of a formula φ with respect to hybrid arc ϕ at at time (t,j) is defined inductively as

$$\begin{aligned}
(\phi,(t,j)) &\models p^{\mu} && \iff \mu(\phi(t,j)) > 0\\
(\phi,(t,j)) &\models \varphi && \iff \varphi(\phi(t,j)) = 1\\
(\phi,(t,j)) &\models \neg\varphi && \iff \neg((\phi,(t,j)) \models \varphi)\\
(\phi,(t,j)) &\models \varphi \wedge \psi && \iff (\phi,(t,j)) \models \varphi \wedge (\phi,(t,j)) \models \psi\\
(\phi,(t,j)) &\models \Diamond_{\mathcal{I}}\, \varphi && \iff \exists (t',j') \in (t,j) + \mathcal{I} \ \text{ s.t } \ (\phi,(t',j')) \models \varphi\\
(\phi,(t,j)) &\models \psi\, \mathcal{U}_{\mathcal{I}}\, \varphi && \iff \exists (t',j') \in (t,j) + \mathcal{I} \ \text{ s.t } \ (\phi,(t',j')) \models \varphi \ \wedge \ \forall (t'',j'') \in ((t,j)+\mathcal{I}) \cap ([t,t'] \times [j,\ldots,j']), \ \ (\phi,(t'',j'')) \models \psi
\end{aligned}$$

References

[1] V. Raman, A. Donzé, D. Sadigh, R. M. Murray, and S. A. Seshia, "Reactive synthesis from signal temporal logic specifications," Seattle, Washington, 2015. [Online]. Available: http://dl.acm.org/citation.cfm?doid=2728606.2728628

[2] M. Chen, Q. Tam, S. C. Livingston, and M. Pavone, "Signal Temporal Logic meets Hamilton-Jacobi Reachability: Connections and Applications," 2018.

[3] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. New Jersey: Princeton University Press, 2012.

[4] A. Donzé and O. Maler, "Robust Satisfaction of Temporal Logic over Real-Valued Signals." Berlin, Heidelberg: Springer Berlin Heidelberg, 2010.