Modeling Self Triggered Control Using Hybrid System Framework

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1 Introduction

Cyber-physical systems(CPS) consist of a physical plants and a discrete(cyber) controllers. The interaction between the plant and controller is naturally modelled as a hybrid system.

This report looks at a control technique known as self-triggered control. Self-triggered controllers compute the next sampling period when the plant state is measured. Control is then applied for that interval. This technique is part of a broader category of control techniques known as event-triggered control.

First, the report looks at general event-triggered control model build on the hybrid system's framework of [1]. An event-triggered model relies on event-triggering-mechanisms(ETMs) that update sensor data and apply control input. This model is specialized to demonstrate application of self-triggered ETMs.

Next, two self-triggered ETMs from the literature are reviewed and compared.

2 An Event-Triggered CPS Model

The following assumptions on the CPS model are made.

1. The system is modeled as a differential equation
2. The control law κ is static
3. Plant and controller output operate under zero-order hold
4. Plant and controller output are triggered by synchronous ETMs.

2.1 Closed Loop CPS with ETMs in the Loop.

This section uses the work from [1] to develop a simplified event-triggered CPS model.

Consider a CPS with a with the plant state $x_p \in \mathcal{X}_p \subset \mathbb{R}^{n_p}$, and the plant output under zero-order-hold $\hat{y} \in \mathcal{Y}$ and control under zero-order hold $\hat{u} \in \mathcal{U}$. By assumption $\hat{y}$ and $\hat{u}$ are updated at the same time.

This update occurs through an Event-Triggered Mechanism(ETM). An auxiliary state $\chi \in X \subset \mathbb{R}^{n_\chi}$ is defined to hold variables related to a system's ETMs. ETMs are triggered using an event triggered function $\gamma : \Xi \mapsto \{0, 1\}$, where $\Xi := \mathcal{Y} \times \mathcal{U} \times X$. The input to γ is given by $\xi := (\hat{y}, \hat{u}, \chi)$. When $\gamma(\xi) = 1$, $\hat{y}$ and $\hat{u}$ are updated. That is, the plant state is sampled and a new control is applied.

Consider the aggregate state $z = (x_p, \hat{y}, \hat{u}, \chi) \in Z \subset \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \times X$. Then, the event triggered CPS with the above assumptions can be modeled with the following hybrid system $\mathcal{H}_{et} := (F, G, C, D)$ defined as

$$\dot{z} = F(z) = \begin{bmatrix} F_p(x_p, \hat{u}) \\ 0 \\ 0 \\ F_\chi(z) \end{bmatrix} \quad z^+ = G(z) = \begin{bmatrix} x_p \\ H_p(x_p) \\ \kappa(\hat{y}) \\ G_\chi(z) \end{bmatrix}$$
$$C = \{z \in Z : \gamma(\xi) = 0\} \quad D = \{z \in Z : \gamma(\xi) = 1\} \tag{1}$$

where $F_p : \mathcal{X}_p \times \mathcal{U} \mapsto \mathcal{X}_p$ denotes the plant dynamics, $F_\chi : Z \mapsto X$ denotes the dynamics of the auxiliary variables, $H_p : \mathcal{X}_p \mapsto \mathcal{Y}$ denotes the sensor update function, $\kappa : \mathcal{Y} \mapsto \mathcal{U}$ denotes a static control law, and $G_\chi : Z \mapsto X$ denotes the auxiliary variable update function.

In order to define a hybrid model for a self-triggered control model the the auxiliary variable χ, it's update function, G_χ, and the ETM triggering fuction γ must be defined. In the next section the concept of self-triggered control is introduced and the identified variables and functions are defined.

3 Self-Triggered Control

Self-Triggered control is a aperiodic control method that predicts when next to sample and apply control. This contrasts periodic control where sampling period is constant. Regardless of the method, sampled data systems define a sequence of times $\{t_i\}_0^\infty$ where the system is updated. The difference between periodic ETMs and self-triggered ETMs are are follows:

- **Periodic ETMs:** $t_{i+1} - t_i = T_s \quad \forall i \in \mathbb{N}$
- **Self-Triggered ETMs:** $t_{i+1} - t_i = \Gamma(\xi)$,

where the self-triggered sampler is defined by a mapping $\Gamma : \Xi \mapsto \mathbb{R}_{\geq 0}$ that computes a sampling period from the current data available. A self-triggered control model therefore needs to define auxiliary variables to handle timers and their update functions.

Let $\chi := (\tau, T_s^*, \chi_3, ..., \chi_{n_\chi})^\top$ where $\chi_3, ...\chi_{n_\chi}$ are free auxiliary variables. The auxiliary dynamics and the update function G_χ defined as

$$G_\chi(z) := \begin{bmatrix} 0 \\ \Gamma(\xi) \\ \chi_3^+ \\ ... \\ \chi_{n_\chi}^+ \end{bmatrix}, \quad F_\chi(z) := \begin{bmatrix} 1 \\ 0 \\ \dot{\chi}_3^+ \\ ... \\ \dot{\chi}_{n_\chi}^+ \end{bmatrix} \tag{2}$$

Furthermore, the ETM triggering function can be defined as

$$\gamma(\xi) := \begin{cases} 0 \text{ if } \tau < T_s^* \\ 1 \text{ if } \tau \geq T_s^* \end{cases} \tag{3}$$

3.1 Model

A self triggered CPS can be modeled by a hybrid system $\mathcal{H}_{et} = (F, G, C, D)$ as in (1) with the auxiliary variables defined in (8) and the ETM triggering function in (3). It remains up to the self-triggered control designer to design the control law κ and a function Γ such that the system operates under desired requirements such as stability, safety, and non-zeno behavior.

3.2 Zeno Behavior in Self-Triggered Methods

An important consideration is avoiding Zeno behavior in the sample period. Zeno behavior can be avoided if it can be guaranteed that there exists a minimum-inter-event time that satisfies the control requirements.

4 Self-Triggered Strategies

This section provides an overview and comparision of two self-triggering ETMs.

4.1 Tiberi, Johansson [3]

Assumptions

1. There exists a feedback law $\kappa : \mathcal{X} \mapsto \mathcal{U}$ such that the origin is asymptotically stable.

The sampling sequence $\{t_i\}_0^\infty$ is inductively defined as

$$\begin{aligned} t_{i+1} &= t_i + \Gamma_1(\hat{y}) \quad \text{for} \\ \Gamma_1(\hat{y}) &= \frac{1}{2L}\ln(1 + 2\delta/||f(\hat{y}, \kappa(\hat{y}))||) \end{aligned} \tag{4}$$

where $\delta > 0$, and $L = L_{f,u}L_{\kappa,x}$ is the product of the Lipschitz constants $L_{f,u}$ (Lipschitz constant of f with respect to u) and $L_{\kappa,x}$(Lipschitz constant of κ with respect to x. The triggering condition assumes that $||f(\hat{y}, \kappa(\hat{y}))|| > m$ for some $m > 0$.

This strategy results in system solutions that are globally ultimately uniformly bounded(GUUB). That is, for an arbitrarily large a and with $||x_o|| < a$ it holds that $||x_p|| \leq b > 0 \forall t \geq T$.

4.2 Heemels, Johansson, Tabuada [2]

Assumptions

1. F_p represents linear dynamics.
2. There exists a feedback law $\kappa : \mathcal{X} \mapsto \mathcal{U}$ such that the system is exponentially asymptotically stable (E.A.S) to the origin.
3. There is full state feedback. That is, at sample point t_i, the sampled state is $\hat{x} := \hat{y} = x_p(t_i)$

Since the linear system is E.A.S there exists a Lyapunov function $V(x)$ such that the following hold.

1. $\alpha_1||x||^2 \leq \quad V(x) \quad \leq \alpha_2||x||^2$,

2. $\dot{V}(x) \leq -\alpha_3 ||x||^2$,

3. $||\frac{\partial V}{\partial x}(x)|| \leq \alpha_4 ||x||$.

Consequently, the ideal system solutions starting at x_o evolves as

$$V(x_p) \leq V(x_o)e^{\lambda t} \quad \forall t \in \text{dom} \;\; x_p, \;\; \forall x_o \in \mathcal{X},$$

where $\lambda \geq \alpha_3/(2\alpha_2)$.

[2] shows that enforcing the strategy

$$h_c(\hat{x}, t) = V(x(t)) - V(\hat{x})e^{-\lambda} \leq 0 \;\; \forall t \in [0, T_s], \tag{5}$$

keeps the system (E.A.S) to the origin. This condition states that the entire duration between samples must be exponentially decreasing as compared with the previous state $\hat{x}$.

Note that the sampling period T_s is not known before hand. Therefore, this self-triggered control method predicts solution over a finite horizon to estimate the sampling period T_s such that (5) holds. The range this finite horizon is $[\tau_{min,\tau_{max}}]$, where τ_{min} and τ_{max} are design parameters.

Next the range is discretized into N_{max} pieces, where

$$N_{max} = \frac{\tau_{max}}{\Delta},$$

and Δ is the discretization step.

The self-triggered sampler is then given by:

$$\Gamma_2(\hat{x}) := \max\{\tau_{min}, n(\hat{x})\Delta\} \tag{6}$$

$$n(\hat{x}) := \arg\max\{h_c(\hat{x}, n\Delta) : n \in [0, N_{max}]\} \tag{7}$$

Implementation of this method has two steps that occur at each sampling period: Computing the function h_c and evaluate $n(\hat{x})$ over the finite future horizon.

5 Comparison of Methods

This section provides a comparison between the self-triggered control methods given in Section 4. The comparison uses the hybrid self-triggered control model $\mathcal{H}_{et} = (F, G, C, D)$ as in (1) with auxiliary variables defined as $\chi = (\tau, T_s^*)$ with dynamics F_χ and update function G_χ defined as

$$G_\chi(z) := \begin{bmatrix} 0 \\ \Gamma(\xi) \end{bmatrix}, \quad F_\chi(z) := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and an ETM triggering condition given by (3).

5.1 Plant Model

A double integrator model is considered with a state vector $x_p = (x_1, x_2)^\top \in \mathbb{R}^2$ and dynamics

$$\dot{x}p = F_p(x_p) = (x_2, u)^\top. \tag{8}$$

Further, full state feedback is assumed. I.e, $\hat{y} = H_p(x_p) = x_p$. For this section we take $\hat{x} := \hat{y}$.

5.1.1 Continuous Time Stability

It can be shown that the plant dynamics F_p can be stabilized through the controller

$$u = \kappa(x) = -Kx_p = -[k_1 \quad k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{9}$$

Further, by the converse Lyapunov theorem, there exists a matrix P such that $V(x) = x^\top P x$ is a Lyapunov certificate of stability.

5.2 Tiberi, Johansson [3]

Recall that the sampling function given by this paper is

$$\Gamma_1(\hat{y}) = \frac{1}{2L} \ln(1 + 2\delta/||f(\hat{y}, \kappa(\hat{y}))||),$$

where $L = L_{f,u} L_{\kappa,x}$, and $\delta > 0$. Using the controller in (9) we have that $L_{f,u} = 1$ and $L_{\kappa,x} = \max k_1, k_2$. Further, the norm of the dynamics is $||f(\hat{y}, \kappa(\hat{y}))|| = \sqrt{x_2^2 + u^2}$.

k_1	1
k_2	1
$L_{f,u}$	1
$L_{\kappa,x}$	1
δ	0.5
$x(0)$	$(1,0)^\top$
$\min ||f(\hat{y}, \kappa(\hat{y}))||$	0.01

Table 1: Simulation Parameters

The norm $||f(\hat{y}, \kappa(\hat{y}))$ is artificially saturated above zero.

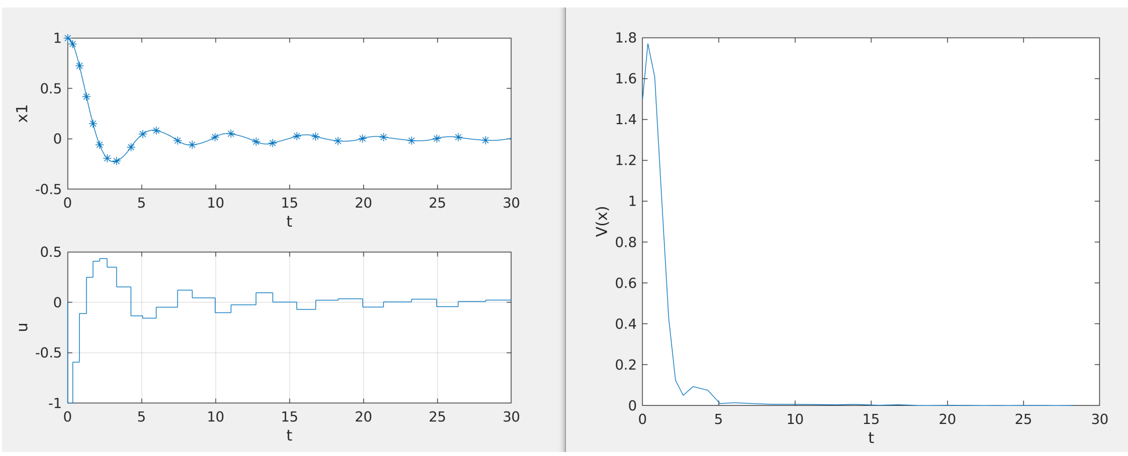

Figure 1: State Evolution, Control, and $V(x)$

Average Sample Time: 1.29s
Average Sample Time($t > 15$): 1.64s

5.3 Heemels, Johansson, Tabuada [2]

Recall that the sampling function given by this paper is

$$\begin{aligned} \Gamma_2(\hat{x}) &:= \max\{\tau_{min}, n(\hat{x})\Delta\} \\ n(\hat{x}) &:= \arg\max\{h_c(\hat{x}, n\Delta) : n \in [0, N_{max}]\} \end{aligned}$$

A Lyapunov function for this system is found by applying the feedback law (9) to the system (8). Thus we have

$$F_p(x_p) = Ax_p - BKx_p = \hat{A}x_p,$$

where $\hat{A} = A - BK$ is Hurwitz. Then, a Lyapunov function $V(x_p) = x_p^\top P x_p$ is a established by solving $\hat{A}P + P\hat{A}^\top + I = 0$ for P. The following parameters are derived from this choice of $V(x_p)$ and are used in simulations. Note, λ_{max} is the maximum eigenvalue of P.

k_1	1
k_2	1
λ	$1/(2\lambda_{max})$
Δ	0.1
τ_{max}	4
τ_{min}	0.1
$x(0)$	$(1, 0)^\top$

Table 2: Simulation Parameters

The function h_c is calculated before every jump. This is done by seeing that the linear system (8) evolves between updates as

$$\dot{x_p} = Ax_p + BK\hat{x},$$

where $\hat{x}$ is the last measurement taken. This solution is evaluated

The solution of this system can be written explicitly as

$$\begin{aligned} x(t) &= e^{At}x(0) - \int_0^t e^{-At}BK\hat{x}d\tau \\ &= e^{At}x(0) - A^{-1}(e^{-At} - I)BK\hat{x} \end{aligned}$$

In the simulation A^{-1} is taken as approximation.

The implementation creates the matrices $e^{A\Delta}$ and $e^{-A\Delta}$ and uses these to propagate the states through discrete steps as in (6).

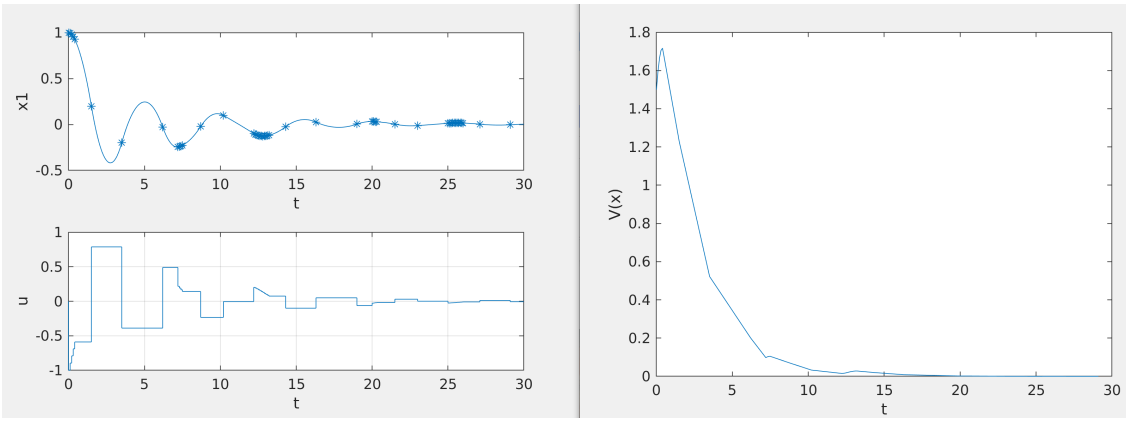

Figure 2: State Evolution, Control, and $V(x)$

Average Sample Time: 1.15s
Average Sample Time($t > 15$): 1.01s

References

[1] Jun Chai, Pedro Casau, and Ricardo G. Sanfelice. Analysis and design of event-triggered control algorithms using hybrid systems tools. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 6057–6062, Melbourne, Australia, December 2017. IEEE.

[2] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, Maui, HI, USA, December 2012. IEEE.

[3] U. Tiberi and K.H. Johansson. A simple self-triggered sampler for perturbed nonlinear systems. *Nonlinear Analysis: Hybrid Systems*, 10, November 2013.