Modeling Self-Triggered Control Strategies Using the Hybrid System Framework

David Kooi

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University of California Santa Cruz

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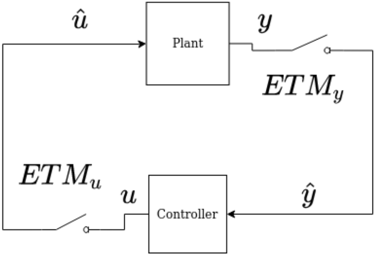
Motivation

Cyber-physical systems(CPSs) usually operate by

1. Taking sensor measurements,
2. Computing control,
3. Applying control.

In real systems each of these steps takes time.

Question: How to design event-trigger-mechanisms(ETMs) for the sensor and control such that system requirements(stability, safety, temporal requirements) are satisfied?



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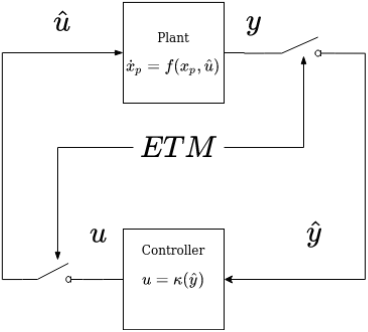
Outline

1. Event-Triggered CPS Model
2. Specialization for Self-Triggered Strategies
3. Review and Comparison of Two Self-Triggered Strategies

An Event Triggered CPS Model

Assumptions

1. The system is modeled as a differential equation
2. The control law κ is static
3. Plant and controller output operate under zero-order hold
4. Plant and controller output are triggered by synchronous ETMs.



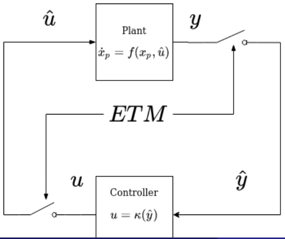
An Event Triggered CPS Model

- Developed from [Chai et al., 2017]

Hybrid Model

$$\mathcal{H}_{et} := (F, G, C, D)$$
$$z = (x_p, \hat{y}, \hat{u}, \chi)^\top$$

$$\dot{z} = F(z) = \begin{bmatrix} F_p(x_p, \hat{u}) \\ 0 \\ 0 \\ F_\chi(z) \end{bmatrix} \quad z^+ = G(z) = \begin{bmatrix} x_p \\ H_p(x_p) \\ \kappa(\hat{y}) \\ G_\chi(z) \end{bmatrix}$$
$$C = \{z \in Z : \gamma(\xi) = 0\} \quad D = \{z \in Z : \gamma(\xi) = 1\} \tag{1}$$



Plant State	$x_p \in \mathcal{X}_p \subset \mathbb{R}^{n_p}$
Plant Output	$y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$
Sampled Output	$\hat{y} \in \mathcal{Y}$
Controller Output	$u \in \mathcal{U} \subset \mathbb{R}^{n_u}$
Auxiliary Variables	$\chi \in X \subset \mathbb{R}^{n_\chi}$
Model State	$z = (x_p, \hat{y}, \hat{u}, \chi) \in Z$

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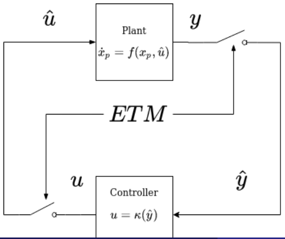

- Updates occur through the synchronous ETM
- ETM is modeled by a function $\gamma : \Xi \mapsto \{0, 1\}$
- $\Xi := \mathcal{Y} \times \mathcal{U} \times X$ is the domain of the data available to the ETM

An Event Triggered CPS Model

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Hybrid Model

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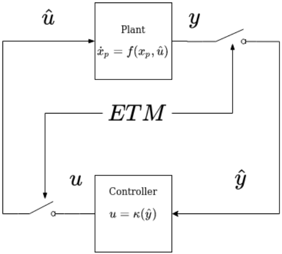


- Specialization to a self-triggered ETM requires defining:

Auxiliary State	χ
Auxiliary Dynamics	$F_\chi(\cdot)$
Auxiliary Update	$G_\chi(\cdot)$
ETM Function	$\gamma(\cdot)$

Periodic and Self-Triggered Sampling

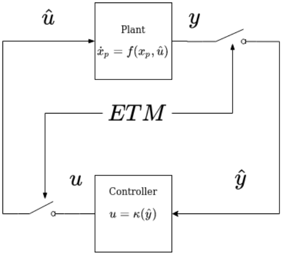
1. The Event-Triggered model define a sequence of times $\{t_i\}_0^\infty$ when the system is updated.
2. To illustrate the difference between periodic and self-triggered sampling consider the following

- **Periodic ETMs:** $t_{i+1} - t_i = T_s$
- **Self-Triggered ETMs:** $t_{i+1} - t_i = \Gamma(\xi)$,
  For $\xi \in \Xi$, $\Gamma : \Xi \mapsto \mathbb{R}_{\geq 0}$. (Recall that $\Xi := \mathcal{Y} \times \mathcal{U} \times X$.)

Self-Triggered Model

$$\mathcal{H}_{st} := (F, G, C, D)$$
$$z = (x_p, \hat{y}, \hat{u}, \chi)^\top$$

$$\dot{z} = F(z) = \begin{bmatrix} F_p(x_p, \hat{u}) \\ 0 \\ 0 \\ F_\chi(z) \end{bmatrix} \quad z^+ = G(z) = \begin{bmatrix} x_p \\ H_p(x_p) \\ \kappa(\hat{y}) \\ G_\chi(z) \end{bmatrix}$$
$$C = \{z \in Z : \gamma(\xi) = 0\} \quad D = \{z \in Z : \gamma(\xi) = 1\} \tag{4}$$



- Define:

Auxiliary State	$\chi := (\tau, T_s^*, \chi_3, ..., \chi_{n_\chi})^\top$
Auxiliary Dynamics	$F_\chi(\cdot) := (1, 0, \dot{\chi}_3, ...\dot{\chi}_n)^\top$
Auxiliary Update	$G_\chi(\cdot) := (0, \Gamma(\xi), \chi_3^+, ...\chi_n^+)^\top$
ETM Function	$\gamma(\xi) := \begin{cases} 0 \text{ if } \tau < T_s^* \\ 1 \text{ if } \tau \geq T_s^* \end{cases}$

Self Triggered Strategies [Tiberi and Johansson, 2013]

Assumptions

1. There exists a feedback law $\kappa : \mathcal{X} \mapsto \mathcal{U}$ such that the origin is asymptotically stable.
2. There is full state feedback. That is, at sample point t_i, the sampled state is $\hat{y} = H_p(x_p) = x_p := \hat{x}$

Self-Triggered Sampler

$$\Gamma_1(\hat{y}) = \frac{1}{2L} \ln(1 + 2\delta/||f(\hat{y}, \kappa(\hat{y}))||) \tag{5}$$

where $\delta > 0$, and $L = L_{f,u} L_{\kappa,x}$ is the product of the Lipschitz constants $L_{f,u}$ (Lipschitz constant of f with respect to u) and $L_{\kappa,x}$(Lipschitz constant of κ with respect to x. The triggering condition assumes that $||f(\hat{y}, \kappa(\hat{y}))|| > m$ for some $m > 0$.
$\implies$ Globally ultimately uniformly bounded(GUUB). I.e, For a arbitrarily large, $||x_o|| < a$,

$$||x_p|| \leq b > 0 \forall t \geq T$$

Self Triggered Strategies [Heemels et al., 2012]

Assumptions

1. F_p represents linear dynamics.
2. There exists a feedback law $\kappa : \mathcal{X} \mapsto \mathcal{U}$ such that ideal system F_p is exponentially asymptotically stable (E.A.S) to the origin.
3. There is full state feedback. That is, at sample point t_i, the sampled state is $\hat{y} = H_p(x_p) = x_p := \hat{x}$

Self Triggered Strategies [Heemels et al., 2012]

Since the linear system is E.A.S there exists a Lyapunov function $V(x_p)$ such that the following hold.

1. $\alpha_1||x_p||^2 \leq \quad V(x) \quad \leq \alpha_2||x_p||^2$,
2. $\dot{V}(x_p) \leq -\alpha_3||x_p||^2$,
3. $||\frac{\partial V}{\partial x_p}(x_p)|| \leq \alpha_4||x_p||$.

Consequently, the ideal system solutions starting at x_o evolve as

$$V(x_p) \leq V(x_o)e^{\lambda t} \quad \forall t \in \mathrm{dom} \;\; x_p, \;\; \forall x_o \in \mathcal{X},$$

where $\lambda \geq \alpha_3/(2\alpha_2)$.

Self Triggered Strategies [Heemels et al., 2012]

Strategy is to enforce

$$h_c(\hat{x}, t) = V(x(t)) - V(\hat{x})e^{-\lambda} \leq 0 \ \ \forall t \in [0, T_s]. \tag{6}$$

- If this is enforced the system remains (E.A.S) to the origin.
- The entire duration between samples must be exponentially decreasing as compared with the previous state $\hat{x}$.

Note: T_s is not known before hand.

$\implies$ Predict over a finite horizon $[\tau_{min,\tau_{max}}]$ to estimate the sampling period T_s such that (6) holds. Design parameters: τ_{min}, τ_{max}.

- $[\tau_{min,\tau_{max}}]$ is discretized into N_{max} pieces
- $N_{max} = \frac{\tau_{max}}{\Delta}$ and Δ is the discretization step.

The self-triggered sampler is then given by:

$$\Gamma_2(\hat{x}) := \max\{\tau_{min}, n(\hat{x})\Delta\} \tag{7}$$

$$n(\hat{x}) := \arg\max\{h_c(\hat{x}, n\Delta) : n \in [0, N_{max}]\} \tag{8}$$

Comparison of Methods

Double Integrator Model

$$\dot{x}p = F_p(x_p, u) = (x_2, u)^\top$$

- Full state feedback $\hat{y} = H_p(x_p) := \hat{x}$
- Stabilized with $u = \kappa(x_p) = -Kx$.

Comparison of Methods

Double Integrator Model

$$\dot{x}p = F_p(x_p, u) = (x_2, u)^\top$$

- Full state feedback $\hat{y} = H_p(x_p) := \hat{x}$
- Stabilized with $u = \kappa(x_p) = -Kx$.

[Tiberi and Johansson, 2013]

$$\Gamma_1(\hat{y}) = \frac{1}{2L} \ln(1 + 2\delta/||f(\hat{y}, \kappa(\hat{y}))||),$$

where $L = L_{f,u} L_{\kappa,x}$, and $\delta > 0.$, $L_{f,u} = 1$ and $L_{\kappa,x} = \max k_1, k_2$. Further, the norm of the dynamics is $||f(\hat{y}, \kappa(\hat{y}))|| = \sqrt{x_2^2 + u^2}$.

[Tiberi and Johansson, 2013]

k_1	1
k_2	1
$L_{f,u}$	1
$L_{\kappa,x}$	1
δ	0.5
$x(0)$	$(1,0)^\top$
$\min \|\|f(\hat{y}, \kappa(\hat{y}))\|\|$	0.0.1

Table 1: Simulation Parameters

Average Sample Time: 1.29s

Average Sample Time($t > 15$): 1.64s

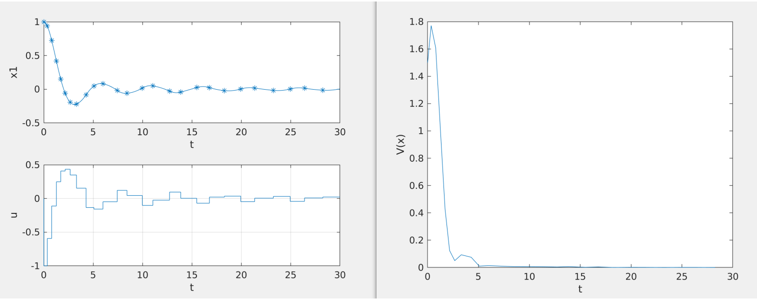


Figure 1: State Evolution, Control and V(x)

Comparison of Methods

[Heemels et al., 2012]

Recall the sampling method of

$$\Gamma_2(\hat{x}) := \max\{\tau_{min}, n(\hat{x})\Delta\}$$
$$n(\hat{x}) := \arg\max\{h_c(\hat{x}, n\Delta) : n \in [0, N_{max}]\}$$

with

$$h_c(\hat{x}, n\Delta) = V(x(n\Delta)) - V(\hat{x})e^{-\lambda} \leq 0,$$

where $x(n\Delta)$ is a prediction of the state evolution. In the case for linear systems an explicit solution is given by

$$\begin{aligned} x(t) &= e^{At}x(0) - \int_0^t e^{-At} BK\hat{x} d\tau \\ &= e^{At}x(0) - A^{-1}(e^{-At} - I)BK\hat{x} \end{aligned}$$

if A is invertable. In the simulation an approximation of A^{-1}.

Comparison of Methods

[Heemels et al., 2012]

k_1	1
k_2	1
λ	$1/(2\lambda_{max})$
Δ	0.1
τ_{max}	4
τ_{min}	0.1
$x(0)$	$(1, 0)^\top$

Table 2: Simulation Parameters

Average Sample Time: 1.15s
Average Sample Time($t > 15$): 1.01s

- Stable system has Lyapunov function $V(x) = x^\top Px$
- λ_{max} is largest eigenvalue of P

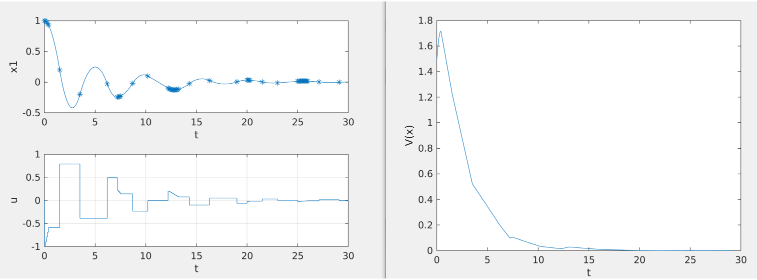


Figure 2: State Evolution, Control, and $V(x)$

Conclusion and Future Work

Conclusion

- Hybrid system framework can model general event-triggered CPSs
- Can be adapted to self-triggered strategies by adding a timer and predictive function Γ.
- [Tiberi and Johansson, 2013]: Larger sample periods at the cost of boundedness,
- [Heemels et al., 2012]: Smaller sample periods, but keeps asymptotic stability.

References

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