Lecture 2: Maxwells Equations

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10:47 AM

Maxwells Eq.

$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial B}{\partial t} = 0$

$\vec{\nabla} \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J}$

$\vec{B}$: magnetic flux density   webers/m^2

$\vec{D}$: electric flux density   coloumbs/m^2

$\vec{E}$: electric field strength   V/m

$\vec{H}$: magnetic field strength   A/m

$\vec{J}$: electric current density   A/m^2

ρ : charge density

starting assumptions

1) medium is constant w/ respect to time

2) source free zone i.e. $\rho = 0$ $\vec{J}_s = 0$

To simplify Maxwell we can relate
D w/ E and B w/ H
in general

(1) $D_i / \varepsilon_0 = \sum_j \varepsilon_{ij} E_j + \sum_{jk} \chi_{ijk} E_j E_k + O(E^3)$

i, j, k are unit vectors

$\varepsilon = \varepsilon_0 \varepsilon_{ijk}$ permittivity ε_0 : perm free space
8.854×10^{-12} Farad/m

ε_{ijk} is a vector where each component relates
E in a direction, ε is a function of

ω
χ : electric susceptibility
relates ε & E via higher order terms

second set of assumptions

1) ε is isotropic

$\varepsilon_i = \varepsilon_j = \varepsilon_k = \varepsilon_r$

2) medium is linear w/ respect to field strength

i.e. behaves the same for $E = 1$ V/m as 1 kV/m

$\chi = 0$

(1) $\Rightarrow$ $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

final assumptions

1) ε_r is a function of frequency (material dispersion)
assume we are in a regime where
ε_r is constant over freq. band of interest

2) deal only w/ transparent materials
i.e. no loss or gain as light
propagates

ε_r is purely real

Now

$\vec{D}(\vec{r}) = \varepsilon_0 \varepsilon_r \vec{E}(\vec{r})$ as w/ most cases

$\vec{B}(\vec{r}) = \mu_0 \mu_r \vec{H}(\vec{r})$ $\mu_r = 1$

$$\vec{B}(\vec{r}) = \mu_0 \mu_r \vec{H}(\vec{r}) \qquad \mu_r = 1, \quad \mu_0 = 4\pi \times 10^{-7} \text{ Henry/m}$$

rewrite Maxwell's equations

$$\nabla \cdot H(r,t) = 0 \qquad \vec{\nabla} \times \vec{E}(\vec{r},t) + \mu_0 \frac{\partial \vec{H}(\vec{r},t)}{\partial t} = 0$$

$$\nabla \cdot [\epsilon(\vec{r}) \vec{E}(r,t)] = 0 \qquad \vec{\nabla} \times \vec{H}(\vec{r},t) - \epsilon_0 \epsilon_r(r) \frac{\partial E(r,t)}{\partial t} = 0$$

H & E are functions of both time & space, have equations that depend on variations of both.

Break down $\vec{E}$ & $\vec{H}$ for a second

$$\vec{E}(r) = \hat{i} E_i(r) + \hat{j} E_j(r) + \hat{k} E_k(r)$$

$$\vec{H}(r) = \hat{i} H_i(r) + \hat{j} H_j(r) + \hat{k} E_k(r)$$

$\vec{E}, \vec{H}$: vectors in space $E_{i,j,k}, H_{i,j,k}$ scalars that rep. quantity of $\vec{E}$ & $\vec{H}$ pointing in $\hat{i}, \hat{j}, \hat{k}$ respectively

Because we have assumed linearity we can separate spatial & temporal dependence w/ <u>NO</u> loss of generality

w/o loss of generality can assume time variational component is sinusoidal (Fourier decomp)

complex exponential $e^{-j\omega t}$

$j = \sqrt{-1}$, ω: angular freq $= 2\pi f$

t: time

now $\vec{H}(r,t) = \vec{H}(r)e^{-j\omega t}$

$\vec{E}(r,t) = \vec{E}(r)e^{-j\omega t}$

$H(r), E(r)$ spatial field profiles

"mode profiles"

Plug into Maxwell

$$\nabla\cdot H(r,t) = \nabla\cdot\vec{H}(r)\cancel{e^{-j\omega t}} = 0$$

$$\nabla\cdot E(r,t) = \nabla\cdot\varepsilon_r E(r)\cancel{e^{-j\omega t}} = 0$$

again utilize Fourier to create spacial modes of plane waves

$$\vec{H}(r) = \vec{a}\,e^{j\vec{k}\cdot\vec{r}}$$

$\vec{a}$: vector of field values in directions

$\vec{k}$: wavevector ; unit vector in direction of propogation

$\vec{r}$: position vector i.e. "where am I"

$$\nabla\cdot\vec{a}\,e^{j\vec{k}\cdot\vec{r}} = 0$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(a_x\hat{x}+a_y\hat{y}+a_z\hat{z}\right)e^{jk\cdot r} = 0$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(a_x\hat{x}e^{j\vec{k}\cdot\vec{r}}+a_y\hat{y}e^{j\vec{k}\cdot\vec{r}}+a_z\hat{z}e^{j\vec{k}\cdot\vec{r}}\right) = 0$$

$$\frac{\partial}{\partial x}a_x\hat{x}e^{j(k_x\hat{x}+k_y\hat{y}+k_z\hat{z})\cdot r} + \quad +$$

$$\frac{\partial}{\partial x}a_x\hat{x}e^{j(k_x r_x\hat{x}+k_y r_y\hat{y}+k_z r_z\hat{z})} + \quad +$$

$$j\,a_x k_x r_x\hat{x}e^{j(k_x r_x\hat{x}+k_y r_y\hat{y}+k_z r_z\hat{z})}$$

$j a_x k_x r_x \hat{x} e \ldots +$

$j a_y k_y r_y \hat{y} e^{j k_x r_x \hat{x} + k_y r_y \hat{y} + k_z r_z \hat{z}} +$

$j a_z k_z r_z \hat{z} e^{j k_x r_x \hat{x} + k_y r_y \hat{y} + k_z r_z \hat{z}} = 0$

only way for this to be true is if all terms are true. Only way for all terms $= 0$ is if either a_i or $k_i = 0$

trivial cases $a_x = a_y = a_z = 0$ i.e. no field

$k_x = k_y = k_z = 0$ no movement

or if $a_i \neq 0$ $k_i = 0$

no field strength in the direction of propagation!

known as Transverse Waves

mathematically stated $\vec{a} \cdot \vec{k} = 0$

as we move forward we will use this to enforce boundary conditions

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now move on to curl equations

$$\nabla \times \vec{E}(r) e^{j\omega t} + \mu_0 \frac{\partial H(r) e^{j\omega t}}{\partial t} = 0$$

$$\nabla \times \vec{H}(r) e^{j\omega t} - \epsilon_0 \epsilon_r \frac{\partial E(r) e^{j\omega t}}{\partial t} = 0$$

lets pick on the first one

$$\nabla \times \vec{E}(r) \cancel{e^{j\omega t}} + \mu_0 j\omega \vec{H}(r) \cancel{e^{j\omega t}} = 0$$

① $\nabla \times \vec{E}(r) - j\mu_0 \omega \vec{H}(r) = 0$

② $\nabla \times \vec{H}(r) + j\epsilon_0 \epsilon_r \omega \vec{E}(r) = 0$

decouple $H(r)$ & $E(r)$ dependence

②/ϵ_r $\quad \nabla \times \frac{1}{\epsilon_r} H(r) + j\omega \epsilon_0 \vec{E}(r) = 0$

take curl

$$\nabla \times \left(\nabla \times \frac{1}{\epsilon_r} H(r) \right) + j\omega\epsilon_0 \nabla \times E(r) = 0$$

① $\nabla \times \vec{E}(r) = j\mu_0 \omega \vec{H}(r)$

$$\nabla \times \left(\nabla \times \frac{1}{\epsilon_r} H(r) \right) - \omega^2 \epsilon_0 \mu_0 H(r) = 0$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \therefore \quad \epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\boxed{\nabla \times \left(\frac{1}{\epsilon(r)} \nabla \times \vec{H}(r) \right) = \left(\frac{\omega}{c} \right)^2 \vec{H}(r)}$$

the strategy for crystal analysis will be to find solutions to this equation that satisfy transversality

w/ $H(\vec{r})$ it is easy to find $E(r)$

use Maxwell

for shorthand rewrite M.E. as a Hermetian operator

$$\nabla \times \left(\frac{1}{\epsilon(r)} \nabla \times H(r) \right) = \hat{\Theta} \quad : \text{eigenfunction}$$

$$\hat{\Theta} H(r) = \left(\frac{\omega}{c}\right)^2 H(r)$$

eigenvector eigenvalue

eigenproblems: any operator that when it acts on a function returns the function times a constant

$\hat{\Theta}$: is Hermetian which formally mean $(\vec{F}, \hat{\Theta}\vec{G}) = (\hat{\Theta}F, G)$

but we won't go through the proof

Hermetian operators are well studied & have compelling properties

1) $\hat{\Theta}$ is positive semi-definite

$\omega \geq 0$ & purely real

2) 2 modes of $H(r)$ $H_1(r)$ $H_2(r)$ that have different ω are orthogonal

These will come in handy later

Scaling Properties of Maxwell

Maxwells equation are <u>scale invariant</u>

no dependency on length anywhere

M.E. also is scale invariant. This mean

that if we solve a system, as long as we scale the dimensions of interest the same as ω the solution is still valid

there is also no set value for $\epsilon(r)$ if determine solution for $\epsilon(r)$ then increase all $\epsilon(x, y, z)$ by constant then $\omega \rightarrow s\omega$