Control Theory

Laplace Transforms

Fourier Transform Pair

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{j\omega t}\, d\omega$$

The problem with Fourier transforms is that they cannot handle initial conditions due to the infinite bounds on the integral. To handle initial conditions , consider the Laplace transform

$$\mathcal{L}[f(t)] = \tilde{F}(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

Laplace Transform Table

S.no	$f(t)$	$\mathcal{L}\{f(t)\}$	S.no	$f(t)$	$\mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$	11	$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
2	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
3	t^n	$\frac{n!}{s^{n+1}}$	13	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
4	$\sin at$	$\frac{a}{s^2+a^2}$	14	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
5	$\cos at$	$\frac{s}{s^2+a^2}$	15	$f'(t)$	$sF(s)-f(0)$
6	$\sinh at$	$\frac{a}{s^2-a^2}$	16	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
7	$\cosh at$	$\frac{s}{s^2-a^2}$	17	$\int_0^t f(u)du$	$\frac{1}{s}F(s)$
8	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	18	$t^n f(t)$ Where $n = 1,2,3,..$	$(-1)^n \frac{d^n}{ds^n}\{F(s)\}$
9	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	19	$\frac{1}{t}\{f(t)\}$	$\int_s^\infty F(s)ds$
10	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	20	$e^{at}f(t)$	$F(s-a)$

Inverse Laplace Transform

s is a complex number!

$$\boxed{s = \sigma + j\omega}$$

Since **s** is a complex number, the inverse transform is hard to compute directly and using the transform tables in reverse to find the inverse transform is the usual technique.

Initial and Final value Theorems

Initial value theorem

$$\boxed{\lim_{t \to 0^+} f(0^+) = \lim_{s \to \infty} s f(s)}$$

Final value theorem

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s f(s)}$$

Circuit Elements

Uncharged capacitor

$$\boxed{i(t) = C\frac{dv}{dt}}$$

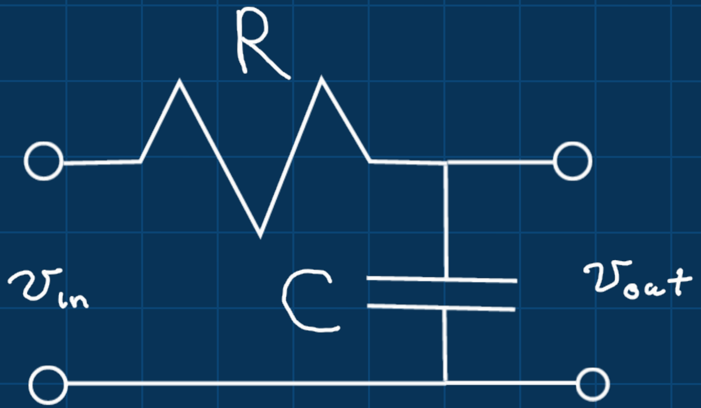
$$\boxed{\tilde{i}(s) = sC\,\tilde{v}(s) - C\,\cancelto{0}{v(0)}}$$

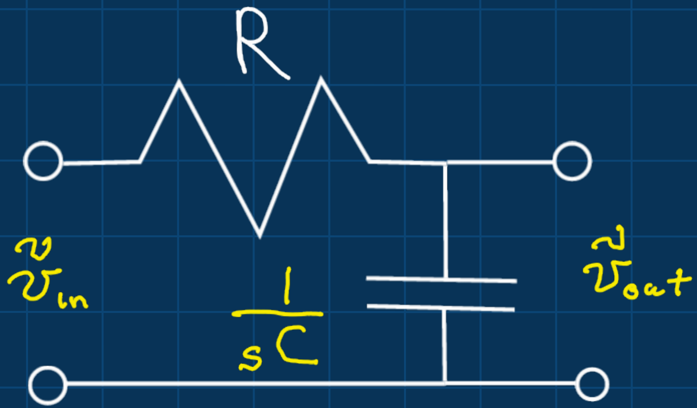
Uncharged inductor

$$\boxed{v(t) = L\frac{di}{dt}}$$

$$\boxed{\tilde{v}(s) = sL\,\tilde{i}(s)}$$

Low Pass Filter





$$\frac{\tilde{v}_{out}}{\tilde{v}_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC}$$

Frequency Response

$$s \Longrightarrow j\omega$$

$$\boxed{\frac{\tilde{v}_o}{\tilde{v}_i} = \frac{1}{1 + j\omega RC}}$$

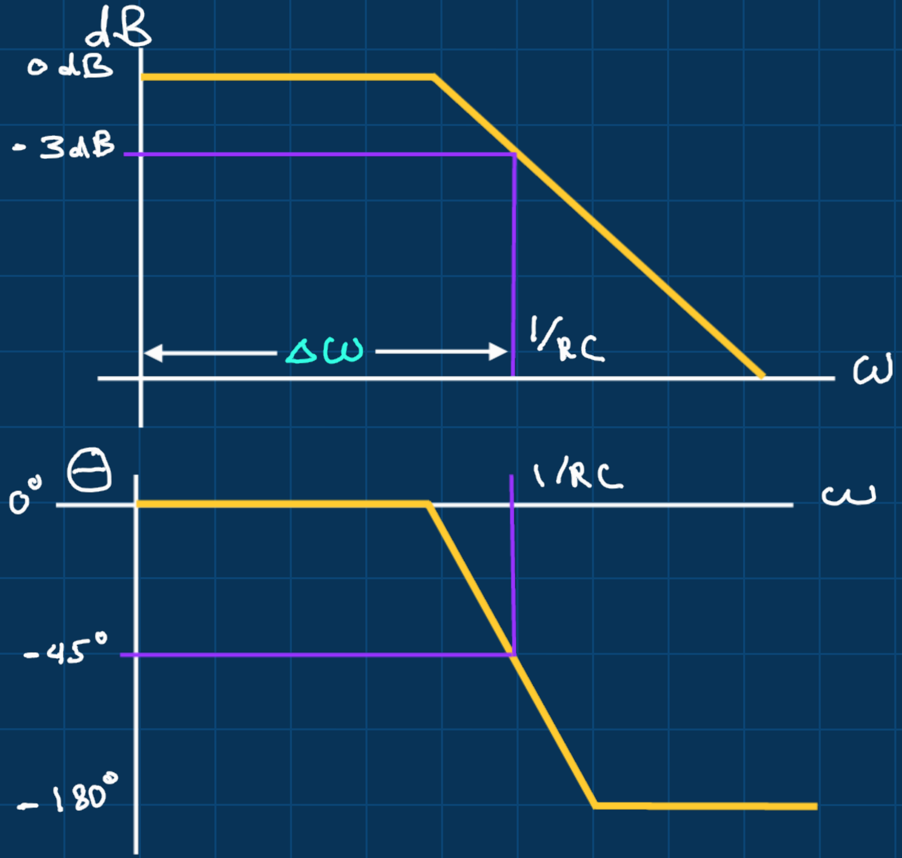
When $\omega = \frac{1}{RC}$

$$\boxed{\frac{V_o}{V_i} = \frac{1}{1+j} = \frac{1-j}{2}}$$

$$\boxed{\left|\frac{V_o}{V_i}\right| = \frac{1}{\sqrt{2}}}$$

$$\boxed{\angle\left(\frac{V_o}{V_i}\right) = -45^\circ}$$

Frequency Response



$$dB = 10\log_{10}\left(\frac{P_o}{P_1}\right)$$

$$dB = 10\log_{10}\left(\frac{|V_o|^2}{|V_i|^2}\right)$$

$$-3dB = 10\log_{10}\left(\frac{1}{2}\right)$$

$$\Theta = \angle \frac{V_o}{V_i}$$

Impulse Response

$$v_{in}(t) = V_i \tau_i \delta(t)$$

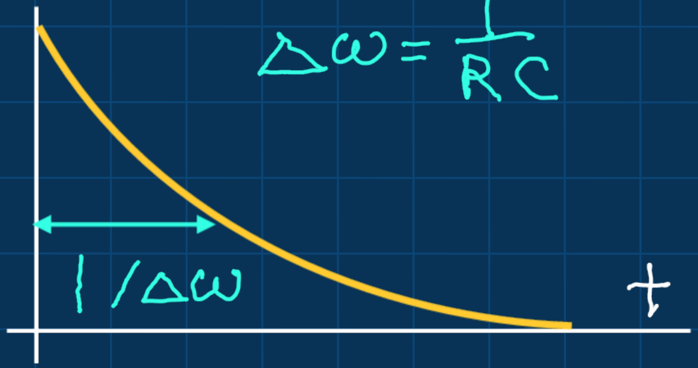
units of 1/time

$$V_{in}(s) = V_i \tau_i$$

$$V_o(s) = \frac{v_i \tau_i}{1 + sRC}$$

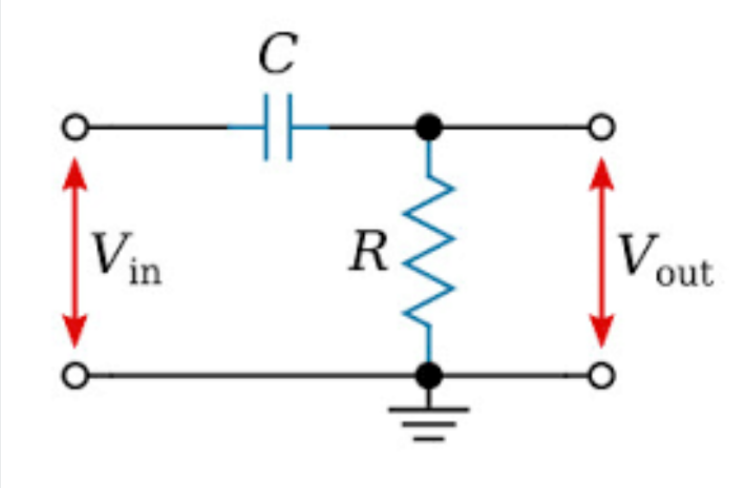
$$v_o(t) = v_i \Delta\omega \tau_i e^{-\Delta\omega t}$$

$$\Delta\omega = \frac{1}{RC}$$



The rms width of this exponential is $1/\Delta\omega$. Thus the bandwidth of a filter is inversely proportional to the time resolution

High Pass Filter



$$\frac{V_o}{V_i} = \frac{R}{\frac{1}{sC} + R}$$

$$\frac{v_o}{v_i} = \frac{sRC}{1 + sRC}$$

The response goes to zero at $s = 0$

The response goes to infinity at $s = -\frac{1}{RC}$

Poles and Zeros

We call $s = 0$ a zero of the filter

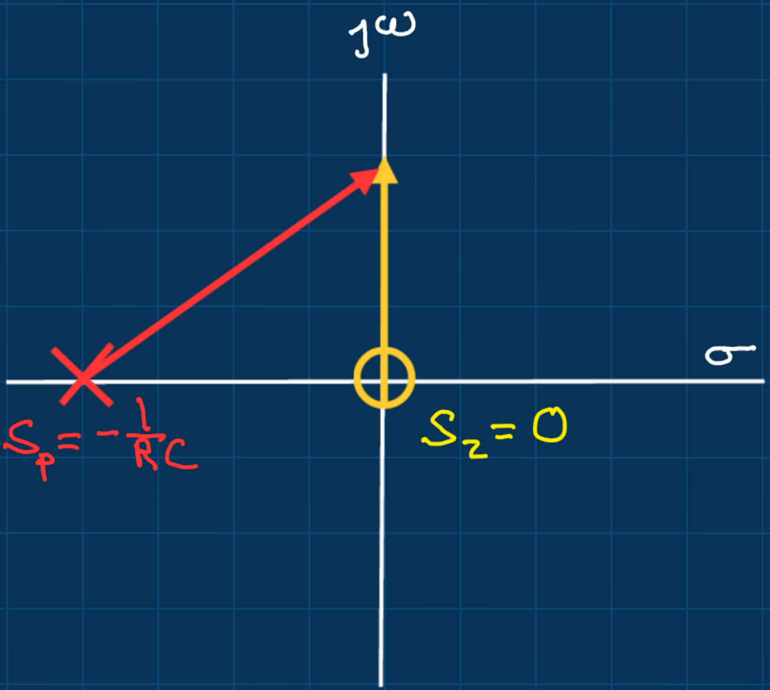
We call $s = -\frac{1}{RC}$ a pole of the filter

In general, any filter can be written as:

$$H(s) = \frac{(s - s_{z1})(s - s_{z2})\dots(s - s_{zn})}{(s - s_{p1})(s - s_{p2})\dots(s - s_{pn})}$$

Pole- Zero Constellation

We can plot the poles and zeros on the complex s plane



To compute the frequency response, we can draw arrows to any point on the imaginary $j\omega$ axis.

Bode Plots

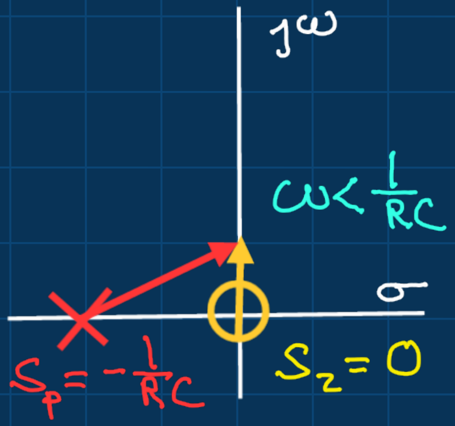
The magnitude of the response is the product of all the lengths of the zero arrows divided by the product of all the lengths of the pole arrows

The phase of the response is the sum of all the lengths of the zero arrows subtracte by the sum of all the lengths of the pole arrows

$$|H(j\omega)| = \frac{\prod_i |\vec{l}_{z_i}|}{\prod_i |\vec{l}_{p_i}|}$$

$$\angle H(j\omega) = \sum \angle \vec{l}_{z_i} - \sum \angle \vec{l}_{p_i}$$

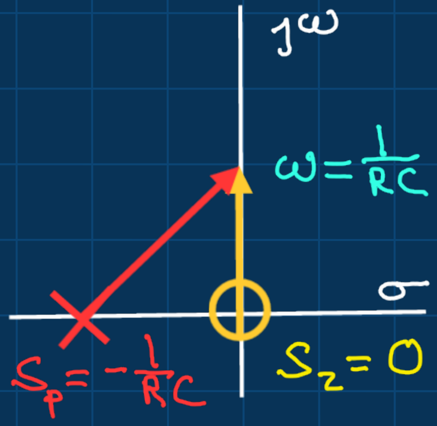
Bode Plots



$|l_z| < |l_p|$

$\angle l_z \simeq 90^\circ$

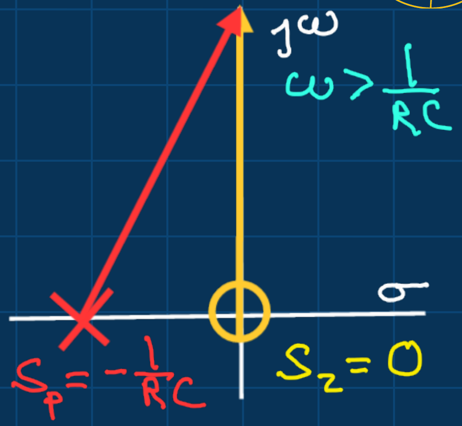
$\angle l_p \sim 0^\circ$



$\sqrt{2}\,|l_z| = |l_p|$

$\angle l_z \simeq 90^\circ$

$\angle l_p = 45^\circ$

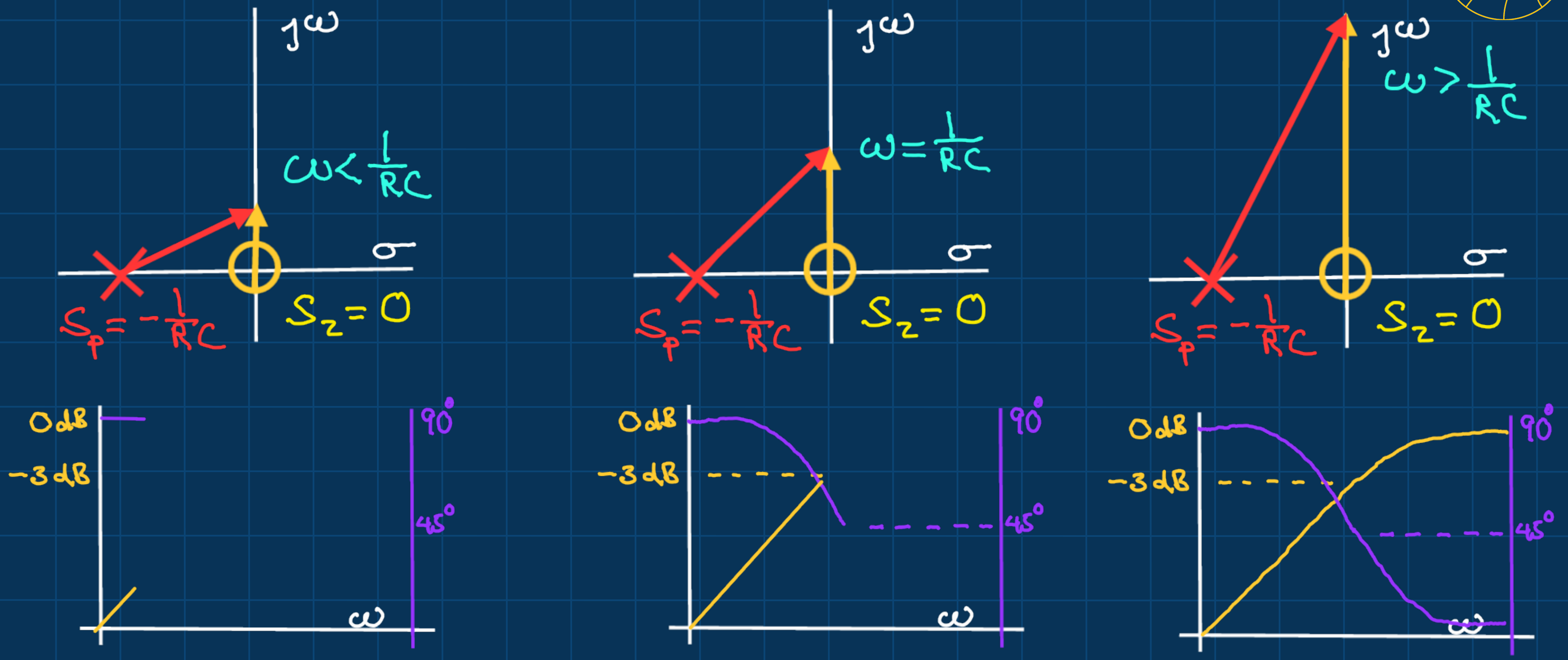


$|l_z| \sim |l_p|$

$\angle l_z \simeq 90^\circ$

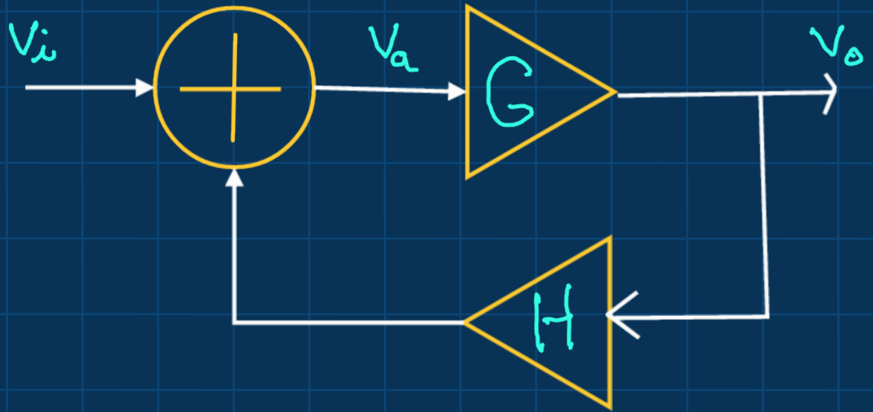
$\angle l_p \sim 90^\circ$

Bode Plots



jω
ω < 1/RC
σ
$S_p = -\frac{1}{RC}$
$S_z = 0$
0 dB
-3 dB
90°
45°
ω
jω
ω = 1/RC
σ
$S_p = -\frac{1}{RC}$
$S_z = 0$
0 dB
-3 dB
90°
45°
ω
jω
ω > 1/RC
σ
$S_p = -\frac{1}{RC}$
$S_z = 0$
0 dB
-3 dB
90°
45°
ω

Feedback



$$V_a = V_i + HV_o$$

$$V_o = GV_a$$

$$V_o = \frac{G}{1-GH} V_i$$

Open loop transfer function $= GH$

Closed loop transfer function $= \dfrac{G}{1-GH}$

Feedback

Let G be a low pass filter

$$G = \frac{G_o}{1 + s/\Delta\omega_o} \qquad H = H_o$$

$$\frac{V_o}{V_i} = \frac{\Delta\omega_o}{\Delta\omega_f} \frac{G_o}{1 - s/\Delta\omega_f} \qquad \frac{\Delta\omega_f}{\Delta\omega_o} = 1 - G_o H_o$$

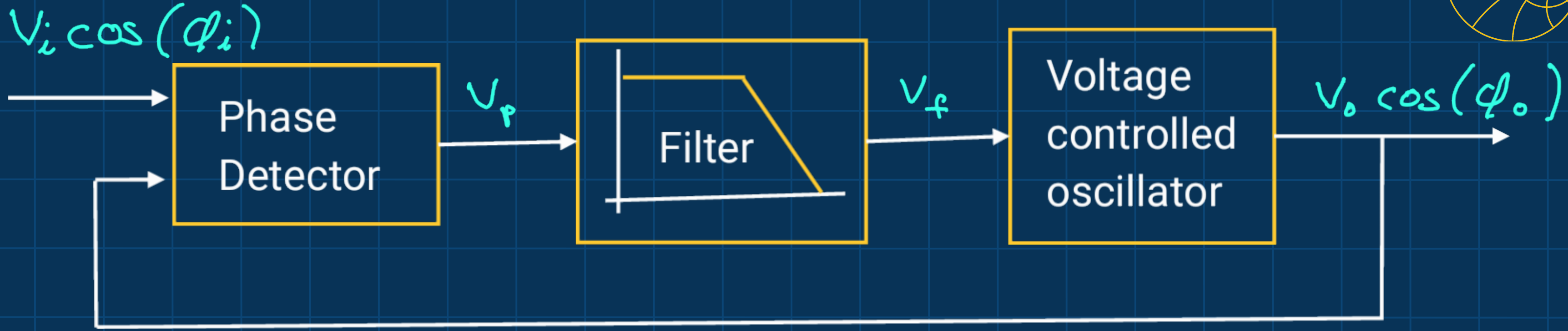
$$v_i(t) = V_i \tau_i \delta(t) \qquad v_o(t) = V_i \Delta\omega_o \tau_i e^{-\Delta\omega_f t}$$

Feedback

If $G_0 H_0 > 1$ then $\Delta\omega_f < 0$
and the system becomes unstable.
For stability, the poles of a system should be located on the left hand side of the s plane.

If $G_0 H_0 < 0$ the closed loop gain is reduced but the bandwidth is increased.

Phase- Locked Loop



If ω is a constant

$$\phi = \omega t + \psi$$

$$\boxed{V_p(s) = \phi_i(s) - \phi_o(s)}$$

$$\boxed{V_f(s) \quad \frac{1}{1+\frac{s}{\Delta\omega_f}} \quad V_p(s)}$$

Phase- Locked Loop

$$f_0(t) = K\, v_f(t) \qquad \varphi_0(t) = 2\pi \int f_0(t)\, dt$$

$$\varphi_0(s) = \frac{2\pi K}{s} V_f(s)$$

$$\varphi_0(s) = \frac{2\pi K \Delta\omega_f}{s^2 + s\Delta\omega_f + 2\pi K \Delta\omega_f}$$

10	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$

When $K = \dfrac{\Delta\omega_f}{8\pi}$

response is critically damped

$$\varphi_0(s) = \left[\frac{\Delta\omega_f}{2s + \Delta\omega_f}\right]^2$$