Homework_1

Math 742

1/19/2026

Casella/Berger

1.2d: Prove $A \cup B = A \cup (B \cap A^c)$

Solution Part 1

Let $x \in A \cup B$. This means that $x \in A$ or $x \in B$.

1. If $x \in A$, then $x \in A \cup (B \cap A^c)$ which implies that $A \cup B \subseteq A \cup (B \cap A^c)$

2. Suppose $x \notin A$ but $x \in B$. If $x \notin A \implies x \in A^c$. This means that $x \in B \cap A^c$ so $x \in A \cup (B \cap A^c)$ which implies that $A \cup B \subseteq A \cup (B \cap A^c)$.

3. Suppose $x \in A$ and $x \in B$. So again, $x \in A \cup (B \cap A^c)$ which implies that $A \cup B \subseteq A \cup (B \cap A^c)$.

$\boxed{A \cup B \subseteq A \cup (B \cap A^c)}$

Let $x \in A \cup (B \cap A^c)$. This means that $x \in A$ or $x \in B \cap A^c$.

If $x \in A$ then $x \in A \cup B$. Likewise, if $x \in B \cap A^c$, $x \in B$ so $x \in A \cup B$.

$\boxed{A \cup (B \cap A^c) \subseteq A \cup B}$

$\boxed{A \cup (B \cap A^c) = A \cup B}$

Part 2

After proving the above result, prove: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Solution

Notice that $A \cap (B \cap A^c) = \emptyset$.

From above, $A \cup B = A \cup (B \cap A^c)$, so $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c)$.

$\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A^c)$, so

$\Pr(B \cap A^c) = \Pr(B) - \Pr(B \cap A)$.

Putting it all together,

$\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c) = \Pr(A) + \Pr(B) - \Pr(B \cap A)$

1.3c DeMorgan's Laws. Prove this without Venn diagrams.

De Morgan's Laws

Let $A, B \subseteq \Omega$.

$$(A \cup B)^c = A^c \cap B^c$$

and,

$$(A \cap B)^c = A^c \cup B^c$$

Proof

Let $x \in \Omega$. Then

$$x \in (A \cup B)^c \iff x \notin A \cup B \iff (x \notin A) \text{ and } (x \notin B) \iff (x \in A^c) \text{ and } (x \in B^c) \iff x \in A^c \cap B^c.$$

Since this holds for all $x \in \Omega$, we conclude

$$\boxed{(A \cap B)^c = A^c \cup B^c}$$

Proof

Let $x \in \Omega$. Then

$$x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A \text{ or } x \notin B \iff x \in A^c \text{ or } x \in B^c \iff x \in A^c \cup B^c.$$

Since this holds for all $x \in \Omega$, we conclude that

$$\boxed{(A \cap B)^c = A^c \cup B^c}$$

1.47b: Prove that $F_X(x) = \frac{1}{1+e^{-x}}, x \in (-\infty, \infty)$ is a CDF.

The function $F_X(x) = \frac{1}{1+e^{-x}}, \qquad x \in \mathbb{R}$, is a cumulative distribution function.

Proof: A function $F : \mathbb{R} \to [0, 1]$ is a CDF if it satisfies:

- F is nondecreasing,
- F is right-continuous,
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Range For all $x \in \mathbb{R}$, we have $e^{-x} > 0$, hence $1 + e^{-x} > 1$ and therefore

$$0 < \frac{1}{1 + e^{-x}} < 1.$$

So $F_X(x) \in (0, 1)$ for all x.

Differentiate:

$$F_X(x) = \left(1 + e^{-x}\right)^{-1} \implies F_X'(x) = -(1 + e^{-x})^{-2} \cdot (-e^{-x}) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

Since $e^{-x} > 0$ and $(1 + e^{-x})^2 > 0$ for all x, we have $F_X'(x) > 0$ for all x. Thus F_X is strictly increasing (hence non-decreasing).

Right-continuity The function e^{-x} is continuous, and $1 + e^{-x} \neq 0$ for all x, so $F_X(x) = 1/(1 + e^{-x})$ is continuous on $\mathbb{R}$. Therefore it is right-continuous.

Limits at $\pm\infty$. As $x \to \infty$, $e^{-x} \to 0$, so

$$\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} \frac{1}{1 + e^{-x}} = \frac{1}{1 + 0} = 1.$$

As $x \to -\infty$, $e^{-x} \to \infty$, so

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} \frac{1}{1 + e^{-x}} = \frac{1}{1 + \infty} = 0.$$

Since F_X is nondecreasing, right-continuous, and has the correct limits at $-\infty$ and ∞, it is a CDF.

4.4 A PDF is defined by,

$$f(x, y) = \begin{cases} C(x + 2y) & : 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & : \text{ Otherwise} \end{cases}$$

a) Find the value of C.

$$1 = \int_0^1 \int_0^2 C(x + 2y) \, dx \, dy = C \int_0^1 \left[\int_0^2 (x + 2y) \, dx \right] dy.$$

Compute the inner integral:

$$\int_0^2 (x + 2y) \, dx = \left[\frac{x^2}{2} + 2yx \right]_0^2 = \frac{4}{2} + 4y = 2 + 4y.$$

Then

$$1 = C \int_0^1 (2 + 4y) \, dy = C \left[2y + 2y^2 \right]_0^1 = C(2 + 2) = 4C,$$

so,

$$\boxed{C = \frac{1}{4}.}$$

b) Find the marginal distribution of X.

For $0 < x < 2$,

$$f_X(x) = \int_0^1 f_{X,Y}(x, y) \, dy = \int_0^1 \frac{1}{4}(x + 2y) \, dy = \frac{1}{4} \left[x \int_0^1 dy + 2 \int_0^1 y \, dy \right].$$

Since $\int_0^1 dy = 1$ and $\int_0^1 y \, dy = \frac{1}{2}$,

$$f_X(x) = \frac{1}{4} \left(x + 2 \cdot \frac{1}{2} \right) = \frac{x + 1}{4}.$$

3

Thus,

$$f_X(x) = \begin{cases} \dfrac{x+1}{4}, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

c) Find the joint CDF of X and Y.

Because the support is the rectangle $0 < x < 2$, $0 < y < 1$, we integrate over

$$0 < x' \le \min(x, 2), \qquad 0 < y' \le \min(y, 1),$$

but the result must be written piecewise.

Case 1: $0 < x < 2$ and $0 < y < 1$.

$$F_{X,Y}(x, y) = \int_0^y \int_0^x \frac{1}{4}(x' + 2y')\, dx'\, dy'.$$

Inner integral:

$$\int_0^x (x' + 2y')\, dx' = \left[\frac{(x')^2}{2} + 2y'x' \right]_0^x = \frac{x^2}{2} + 2xy'.$$

Outer integral:

$$F_{X,Y}(x, y) = \frac{1}{4} \int_0^y \left(\frac{x^2}{2} + 2xy' \right) dy' = \frac{1}{4} \left(\frac{x^2}{2}y + xy^2 \right) = \frac{x^2 y}{8} + \frac{xy^2}{4}.$$

Case 2: $x \ge 2$ and $0 < y < 1$. Use Case 1 with $x = 2$:

$$F_{X,Y}(x, y) = \int_0^y \int_0^2 \frac{1}{4}(x' + 2y')\, dx'\, dy' = \frac{1}{4} \left(\frac{2^2}{2}y + 2y^2 \right) = \frac{2y + 2y^2}{4} = \frac{y + y^2}{2}.$$

Case 3: $0 < x < 2$ and $y \ge 1$. Use Case 1 with $y = 1$:

$$F_{X,Y}(x, y) = \int_0^1 \int_0^x \frac{1}{4}(x' + 2y')\, dx'\, dy' = \frac{x^2}{8} + \frac{x}{4}.$$

Case 4: $x \ge 2$ and $y \ge 1$.

$$F_{X,Y}(x, y) = 1.$$

Outside the support: If $x \le 0$ or $y \le 0$, then $F_{X,Y}(x, y) = 0$.

$$
F_{X,Y}(x, y) = \begin{cases} 0, & x \le 0 \text{ or } y \le 0, \\[2mm] \dfrac{x^2 y}{8} + \dfrac{xy^2}{4}, & 0 < x < 2,\ 0 < y < 1, \\[2mm] \dfrac{y + y^2}{2}, & x \ge 2,\ 0 < y < 1, \\[2mm] \dfrac{x^2}{8} + \dfrac{x}{4}, & 0 < x < 2,\ y \ge 1, \\[2mm] 1, & x \ge 2,\ y \ge 1. \end{cases}
$$

Think of the CDF as "how much of the probability rectangle have I covered?"

Inside the support $\to$ it grows in both x and y.

Past $x = 2 \to$ it can't grow in x anymore.

Past $y = 1 \to$ it can't grow in y anymore.

Those are exactly your Cases 2 and 3.

     d) Find the PDF of the random variable $Z = 9(X + 1)^2)$

From part (b), X has support $0 < x < 2$, hence

$$
1 < X + 1 < 3 \quad \implies \quad 1 < (X + 1)^2 < 9 \quad \implies \quad 9 < Z < 81.
$$

On $x > -1$ the map $z = 9(x + 1)^2$ is strictly increasing, and here $x \in (0, 2)$, so we can use the one-to-one change of variables:

$$
z = 9(x + 1)^2 \quad \implies \quad x = \frac{\sqrt{z}}{3} - 1, \qquad \frac{dx}{dz} = \frac{1}{6\sqrt{z}}.
$$

Therefore, for $9 < z < 81$,

$$
f_Z(z) = f_X\left(\frac{\sqrt{z}}{3} - 1\right)\left|\frac{dx}{dz}\right| = \left[\frac{\left(\frac{\sqrt{z}}{3} - 1\right) + 1}{4}\right]\left(\frac{1}{6\sqrt{z}}\right) = \left(\frac{\sqrt{z}}{12}\right)\left(\frac{1}{6\sqrt{z}}\right) = \frac{1}{72}.
$$

Thus

$$
f_Z(z) = \begin{cases} \dfrac{1}{72}, & 9 < z < 81, \\[2mm] 0, & \text{otherwise.} \end{cases}
$$

(So $Z \sim \text{Uniform}(9, 81)$.)

Note: This problem used the chain rule.

$$
\frac{d}{dz} F_X(g^{-1}(z)) = f_X(g^{-1}(z)) \cdot (g^{-1})'(z)
$$

4.10 The random pair (X, Y) has the distribution:

$f(x, y)$	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	1/12	1/6	1/12
$Y = 3$	1/6	0	1/6
$Y = 4$	0	1/3	0

a) Show that X and Y are dependent.

In order for X and Y to be independent, $f(x, y) = f_X(x)f_Y(y)$ for all (x, y).

First we find the marginal distributions:

X	$f_X(x)$
1	1/12+1/6=1/4
2	1/6+1/3=1/2
3	1/12+1/6=1/4

Y	$f_Y(= y)$
2	1/12+1/6+1/12=1/3
3	1/6+1/6=1/3
4	1/3

$f(1, 2) = f_X(1)f_Y(2) = (1/4)(1/3) = 1/12$

$f(2, 2) = f_X(2)f_Y(2) = (1/2)(1/3) = 1/6$

etc.

(All joint probabilities are equal to the product of the marginal probabilities)

b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

$f_{U,V}(u, v)$	$U = 1$	$U = 2$	$U = 3$
$V = 2$	1/12	1/6	1/12
$V = 3$	1/12	1/6	1/12
$V = 4$	1/12	1/6	1/12

Wasserman

3.10 $X \sim N(0,1)$, $Y \sim e^X$. Determine $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$

Solution

Let $X \sim N(0, 1)$ and define $Y = e^X$. The density of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad x \in \mathbb{R}.$$

By definition,

$$\mathbb{E}(Y) = \mathbb{E}(e^X) = \int_{-\infty}^{\infty} e^x f_X(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x e^{-x^2/2}\, dx.$$

$$e^x e^{-x^2/2} = e^{-(x^2 - 2x)/2}.$$

Complete the square:

$$x^2 - 2x = (x-1)^2 - 1,$$

so,

$$-(x^2 - 2x)/2 = -\frac{(x-1)^2}{2} + \frac{1}{2}.$$

Hence,

$$\mathbb{E}(Y) = \frac{e^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-1)^2/2}\, dx.$$

With the change of variables $u = x - 1$,

$$\int_{-\infty}^{\infty} e^{-(x-1)^2/2}\, dx = \int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}.$$

Therefore,

$$\boxed{\mathbb{E}(Y) = e^{1/2}}.$$

Second moment

Similarly,

$$\mathbb{E}(Y^2) = \mathbb{E}(e^{2X}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2x} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - 4x)/2}\, dx.$$

Complete the square:

$$x^2 - 4x = (x-2)^2 - 4,$$

so,

$$-(x^2 - 4x)/2 = -\frac{(x-2)^2}{2} + 2.$$

Thus,

$$\mathbb{E}(Y^2) = \frac{e^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-2)^2/2}\, dx = \frac{e^2}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^2.$$

Variance

Using $\mathbb{V}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$, we obtain

$$\boxed{\mathbb{V}(Y) = e^2 - e.}$$

Let X have a Gamma distribution with parameters α, β. Prove that $\mathbb{E}(X) = \alpha\beta$ and $\mathbb{V}(X) = \alpha\beta^2$ The Gamma PDF is defined as:

- $\alpha > 0$

- $\beta > 0$

$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$

Assume $X \sim \text{Gamma}(\alpha, \beta)$ with shape $\alpha > 0$ and scale $\beta > 0$,

(Notice the normalizing constant is $\frac{1}{\Gamma(\alpha)\beta^\alpha}$.)

Compute $\mathbb{E}(X)$.

By definition,

$$\mathbb{E}(X) = \int_0^\infty x f_X(x) \, dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^\alpha e^{-x/\beta} \, dx.$$

Make the substitution $t = x/\beta$ (so $x = \beta t$ and $dx = \beta \, dt$):

$$\int_0^\infty x^\alpha e^{-x/\beta} \, dx = \int_0^\infty (\beta t)^\alpha e^{-t} \beta \, dt = \beta^{\alpha+1} \int_0^\infty t^\alpha e^{-t} \, dt = \beta^{\alpha+1} \Gamma(\alpha+1).$$

Therefore,

$$\mathbb{E}(X) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \beta^{\alpha+1} \Gamma(\alpha+1) = \beta \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}.$$

Using the gamma recursion $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$, we get

$$\boxed{\mathbb{E}(X) = \alpha\beta}$$

Compute $\mathbb{V}(X)$

First compute $\mathbb{E}(X^2)$:

$$\mathbb{E}(X^2) = \int_0^\infty x^2 f_X(x) \, dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+1} e^{-x/\beta} \, dx.$$

With the same substitution $t = x/\beta$,

$$\int_0^\infty x^{\alpha+1} e^{-x/\beta} \, dx = \int_0^\infty (\beta t)^{\alpha+1} e^{-t} \beta \, dt = \beta^{\alpha+2} \int_0^\infty t^{\alpha+1} e^{-t} \, dt = \beta^{\alpha+2} \Gamma(\alpha+2).$$

Hence,

$$\mathbb{E}(X^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \beta^{\alpha+2} \Gamma(\alpha+2) = \beta^2 \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}.$$

Use $\Gamma(\alpha + 2) = (\alpha + 1)\Gamma(\alpha + 1) = (\alpha + 1)\alpha\Gamma(\alpha)$ to obtain

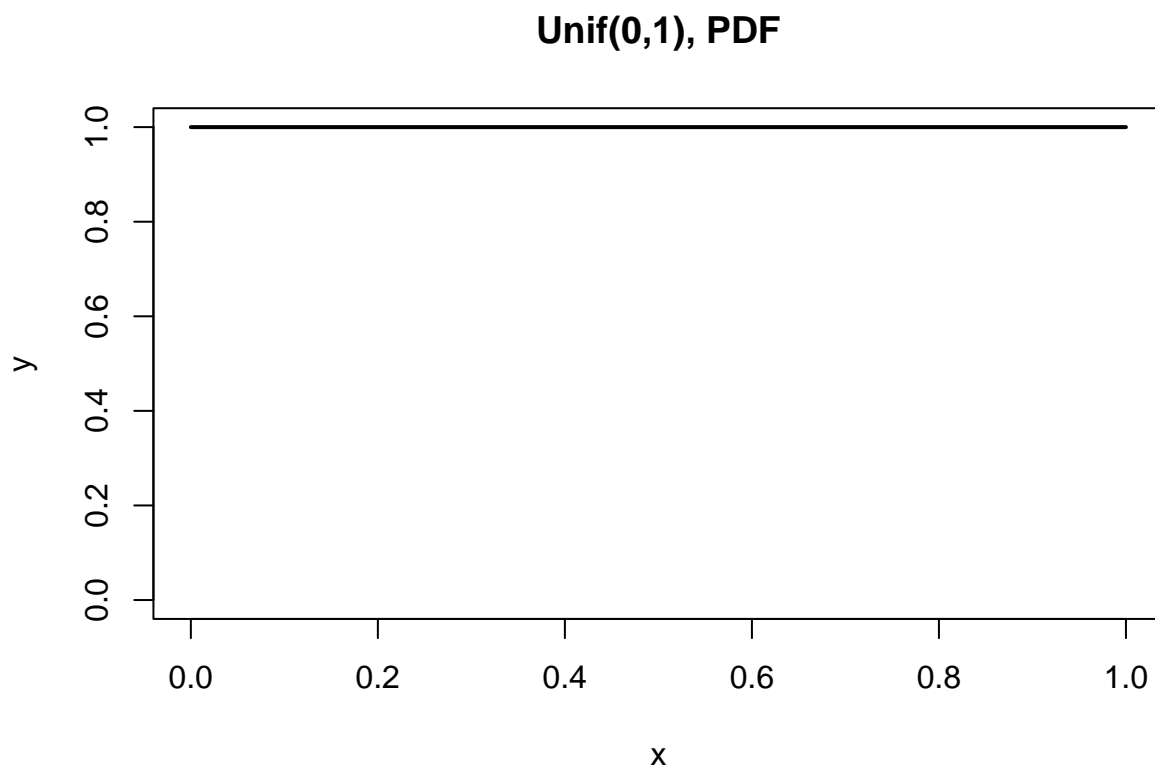$$\mathbb{E}(X^2) = \beta^2 \, \alpha(\alpha + 1).$$

Therefore,

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \beta^2 \, \alpha(\alpha + 1) - (\alpha\beta)^2 = \beta^2(\alpha^2 + \alpha - \alpha^2) = \alpha\beta^2.$$

3.19 This question is to help you understand the idea of a sampling distribution. Let $X_1, ..., X_n$ be iid with mean μ and variance 2. Let $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$. Then $\bar{X}_n$ is a statistic, that is, a function of the data. Since $\bar{X}_n$ is a random variable, it has a distribution. This distribution is called the sampling distribution of the statistic. Recall from Theorem 3.17 that $\mathbb{E}(\bar{X}_n) = \mu$ and $\mathbb{V}(\bar{X}_n) = \sigma^2/n$. Don't confuse the distribution of the data f_X and the distribution of the statistic $f_{\bar{X}_n}$. To make this clear, let $X_1, ..., X_n$ be Uniform(0,1). Let f_X be the density of the Uniform(0,1).

Plot f_X.

```
x <- seq(from=0,to=1,by=0.01)
y <- rep(1,length=length(x))
plot(x, y, xlim = c(0,1), ylim=c(0,1),type = "l", lwd = 2,
        main = "Unif(0,1), PDF")
```


Python Version

```
import numpy as np
import matplotlib.pyplot as plt

# Create grid
```

```
x = np.arange(0, 1.01, 0.01)
y = np.ones(len(x))

# Plot Uniform(0,1) PDF
plt.plot(x, y, linewidth=2)
plt.xlim(0, 1)
```

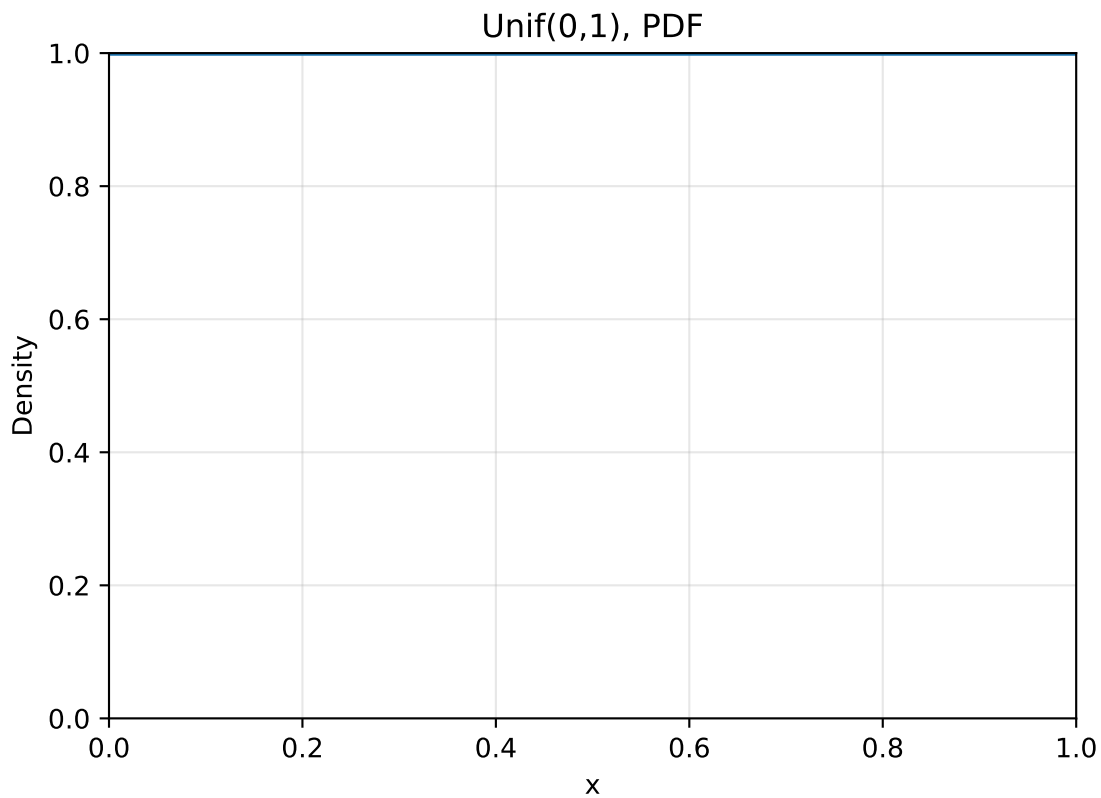## (0.0, 1.0)

```
plt.ylim(0, 1)
```

(0.0, 1.0)

```
plt.title("Unif(0,1), PDF")
plt.xlabel("x")
plt.ylabel("Density")

plt.grid(True, alpha=0.3)
plt.show()
```

Unif(0,1), PDF

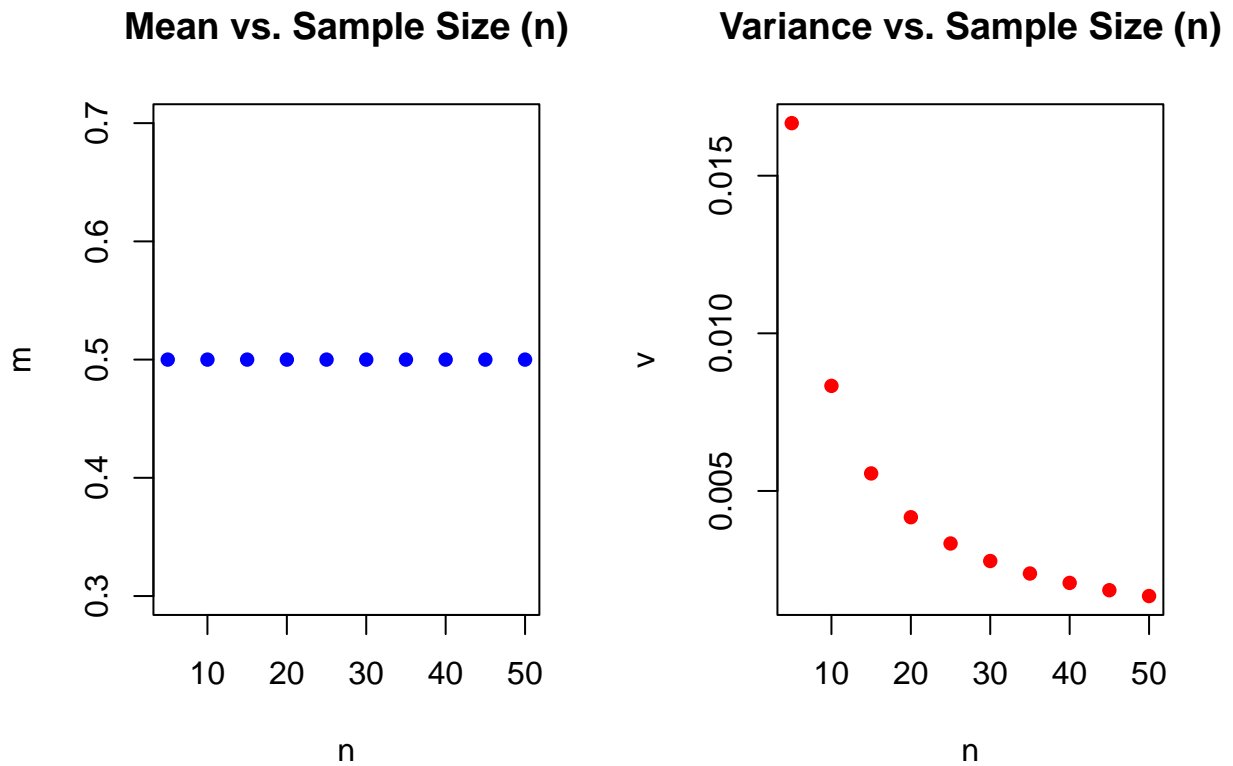Now let $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$. Find $\mathbb{E}(\bar{X}_n)$ and $\mathbb{V}(\bar{X}_n)$.

For the Unif(0,1) distribution, $\mu = (0+1)/2 = 1/2$ and $\sigma^2 = (1-0)^2/12 = 1/12$

$$\mathbb{E}(\bar{X}_n) = \mu = 1/2$$

$$\mathbb{V}(\bar{X}_n) = \sigma^2/n = (1/12)/n = 1/(12n)$$

Plot them as a function of n.

```r
n <- c(5,10,15,20,25,30,35,40,45,50)
m <- rep(1/2,10)
v <- 1/(12*n)
par(mfrow=c(1,2))
plot(n,m,pch=16,col="blue",main="Mean vs. Sample Size (n)")
plot(n,v,pch=16,col="red",main="Variance vs. Sample Size (n)")
```


Python Version

```python
import numpy as np
import matplotlib.pyplot as plt

# Data
n = np.array([5, 10, 15, 20, 25, 30, 35, 40, 45, 50])
m = np.repeat(1/2, len(n))
v = 1 / (12 * n)

# Set up side-by-side plots
plt.figure(figsize=(10, 4))

# Mean vs sample size
plt.subplot(1, 2, 1)
plt.scatter(n, m, color="blue")
plt.title("Mean vs. Sample Size (n)")
plt.xlabel("n")
plt.ylabel("Mean")
```
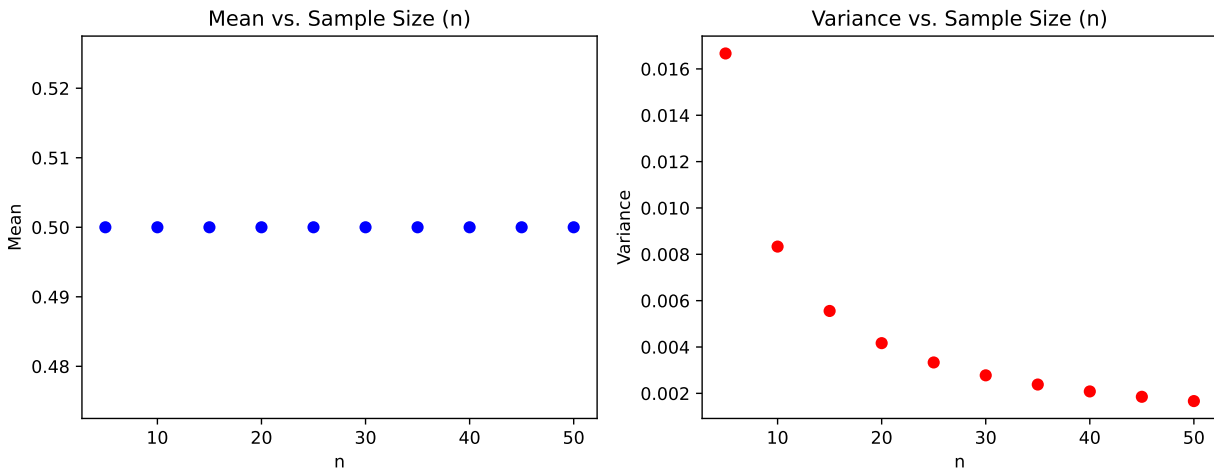
```python
# Variance vs sample size
plt.subplot(1, 2, 2)
plt.scatter(n, v, color="red")
plt.title("Variance vs. Sample Size (n)")
plt.xlabel("n")
plt.ylabel("Variance")

plt.tight_layout()
plt.show()
```


Interpret.

- The mean of the sampling distribution is the same, regardless of sample size.

- The variance has an exponential decay as n increases.

Now simulate the distribution of $\bar{X}_n$ for $n = 1, 5, 25, 100$. Check that the simulated values of $\mathbb{E}(\bar{X}_n)$ and $\mathbb{V}(\bar{X}_n)$ agree with your theoretical calculations. What do you notice about the sampling distribution of $\bar{X}_n$ as n increases?

```r
set.seed(742)

Nsim <- 50000              # number of repeated samples
n_vals <- c(1, 5, 25, 100) # sample sizes

# Theoretical values for Uniform(0,1)
mu <- 1/2
varX <- 1/12

# Storage for results
results <- data.frame(
  n = n_vals,
  mean_sim = NA_real_,
  var_sim = NA_real_,
  mean_theory = mu,
  var_theory = varX / n_vals
)

# Store xbar draws for plotting
```

```r
xbar_list <- vector("list", length(n_vals))
names(xbar_list) <- paste0("n=", n_vals)

# Simulation
for (i in seq_along(n_vals)) {
  n <- n_vals[i]
  # Each row is one simulated sample of size n
  X <- matrix(runif(Nsim * n), nrow = Nsim, ncol = n)
  xbar <- rowMeans(X)

  xbar_list[[i]] <- xbar

  results$mean_sim[i] <- mean(xbar)
  results$var_sim[i] <- var(xbar)
}

# Print comparison table
print(results)
```

```
##       n mean_sim      var_sim mean_theory   var_theory
## 1     1 0.498470 0.0832355708         0.5 0.0833333333
## 2     5 0.500344 0.0167308793         0.5 0.0166666667
## 3    25 0.500528 0.0032994555         0.5 0.0033333333
## 4   100 0.500262 0.0008399842         0.5 0.0008333333
```

```r
# ----------------------------------------------------------------
# Plots: sampling distributions
# ----------------------------------------------------------------
par(mfrow = c(2,2), mar = c(4,4,3,1))

for (i in seq_along(n_vals)) {
  n <- n_vals[i]
  xbar <- xbar_list[[i]]

  hist(
    xbar,
    breaks = 50,
    freq = FALSE,
    col = "gray85",
    border = "white",
    main = paste0("Sampling distribution of xbar (n = ", n, ")"),
    xlab = expression(bar(X))
  )

  # Overlay normal approximation from CLT
  xgrid <- seq(min(xbar), max(xbar), length.out = 500)
  lines(xgrid, dnorm(xgrid, mean = mu, sd = sqrt(varX / n)), col = "red", lwd = 2)

  abline(v = mu, col = "blue", lwd = 2, lty = 2)
  legend(
    "topright",
    legend = c("Simulated", "Normal approx", "True mean = 1/2"),
    col = c("black", "red", "blue"),
    lwd = c(8, 2, 2),
```
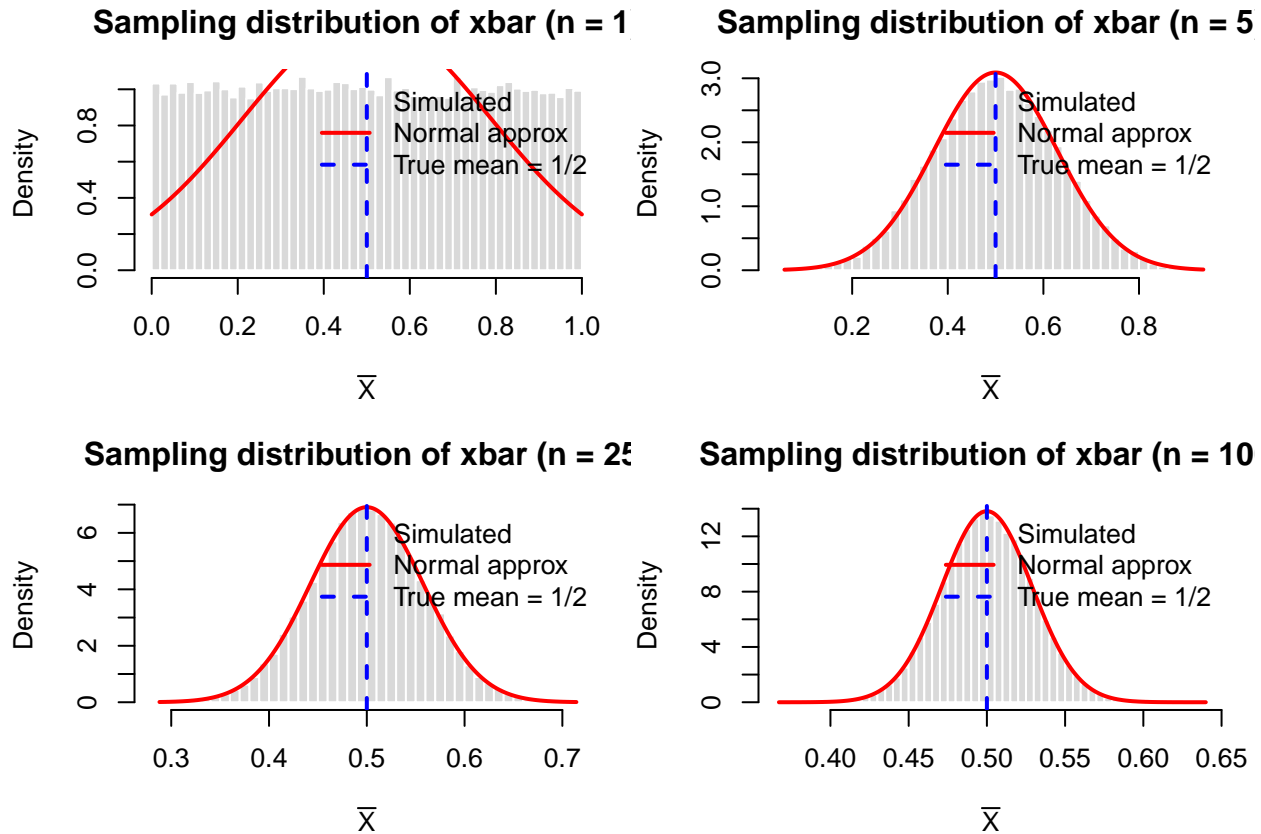
```
    lty = c(NA, 1, 2),
    bty = "n"
  )
}
```

Sampling distribution of xbar (n = 1

Sampling distribution of xbar (n = 5

Sampling distribution of xbar (n = 25

Sampling distribution of xbar (n = 10


```
# -----------------------------------------------------------
# What to notice (printed notes)
# -----------------------------------------------------------
cat("\nWhat to notice as n increases:\n")

##
## What to notice as n increases:
cat("- The sampling distribution of xbar becomes more concentrated around 0.5.\n")

## - The sampling distribution of xbar becomes more concentrated around 0.5.
cat("- The variance shrinks like 1/(12n).\n")

## - The variance shrinks like 1/(12n).
cat("- The shape becomes increasingly close to Normal by the CLT (red curve fits better as n grows).\n")

## - The shape becomes increasingly close to Normal by the CLT (red curve fits better as n grows).
cat("- For n=1, the distribution is Uniform(0,1); for moderate n it is already fairly bell-shaped.\n")

## - For n=1, the distribution is Uniform(0,1); for moderate n it is already fairly bell-shaped.
```

Python Version

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm


# ----------------------------------------------------------------
# Sampling distribution of the mean for X ~ Uniform(0,1)
# ASCII-only version (R Markdown safe)
# ----------------------------------------------------------------

np.random.seed(742)


Nsim = 50000                        # number of repeated samples
n_vals = np.array([1, 5, 25, 100])


# Theoretical values for Uniform(0,1)
mu = 0.5
varX = 1.0 / 12.0


# Storage for results
results = pd.DataFrame({
    "n": n_vals,
    "mean_sim": np.nan,
    "var_sim": np.nan,
    "mean_theory": mu,
    "var_theory": varX / n_vals
})


# Store sample mean draws for plotting
xbar_list = {}


# ----------------------------------------------------------------
# Simulation
# ----------------------------------------------------------------
for i, n in enumerate(n_vals):
    # Each row is one simulated sample of size n
    X = np.random.uniform(0.0, 1.0, size=(Nsim, n))
    xbar = X.mean(axis=1)

    xbar_list[f"n={n}"] = xbar

    results.loc[i, "mean_sim"] = xbar.mean()
    results.loc[i, "var_sim"] = xbar.var(ddof=1)


# Print comparison table
print(results)


##      n  mean_sim   var_sim  mean_theory  var_theory
## 0    1  0.499936  0.083346          0.5    0.083333
## 1    5  0.500105  0.016526          0.5    0.016667
## 2   25  0.500144  0.003335          0.5    0.003333
## 3  100  0.500136  0.000828          0.5    0.000833
```

```python
# ----------------------------------------------------------------
# Plots: sampling distributions
# ----------------------------------------------------------------
fig, axes = plt.subplots(2, 2, figsize=(10, 8))
axes = axes.flatten()

for ax, n in zip(axes, n_vals):
    xbar = xbar_list[f"n={n}"]

    ax.hist(
        xbar,
        bins=50,
        density=True,
        color="gray",
        edgecolor="white",
        alpha=0.8
    )

    # Overlay normal approximation from CLT
    xgrid = np.linspace(xbar.min(), xbar.max(), 500)
    ax.plot(
        xgrid,
        norm.pdf(xgrid, loc=mu, scale=np.sqrt(varX / n)),
        color="red",
        linewidth=2
    )

    ax.axvline(mu, color="blue", linestyle="--", linewidth=2)

    ax.set_title(f"Sampling distribution of sample mean (n = {n})")
    ax.set_xlabel("sample mean")
    ax.set_ylabel("density")

    ax.legend(
        ["Normal approximation", "True mean = 0.5"],
        loc="upper right",
        frameon=False
    )

plt.tight_layout()
plt.show()
```
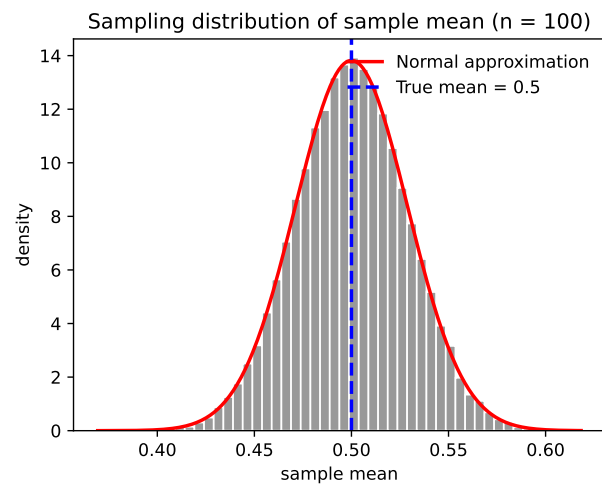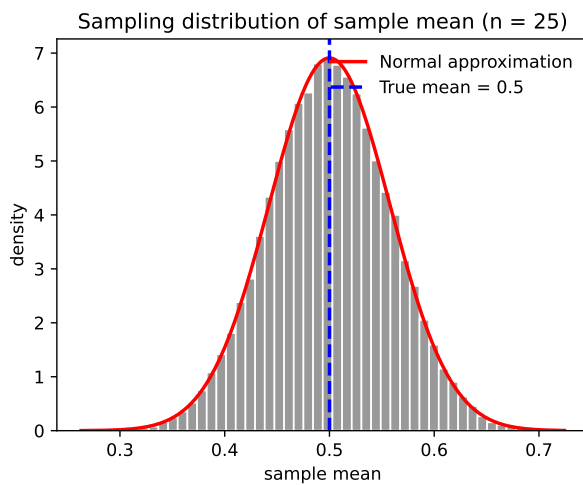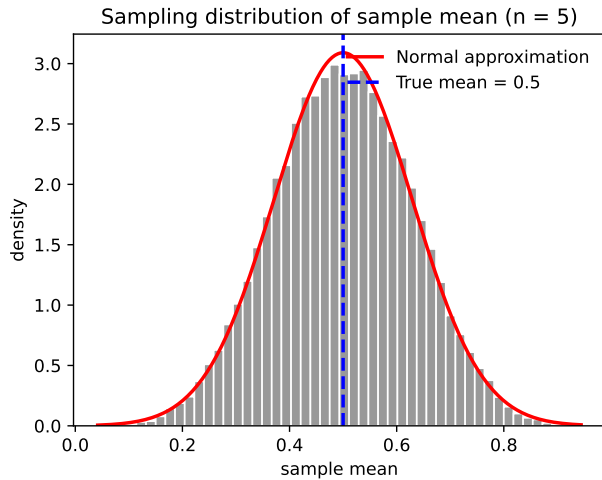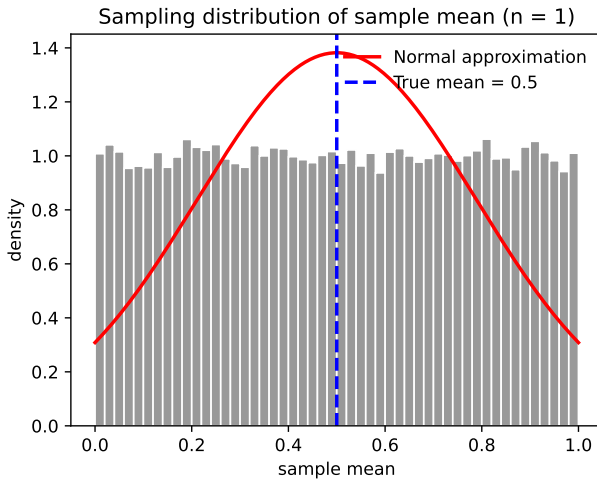
```
# -----------------------------------------------------------
# What to notice
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print("\nWhat to notice as n increases:")
```

```
##
## What to notice as n increases:
```

```
print("- The sampling distribution of the sample mean becomes more concentrated around 0.5.")
```

```
## - The sampling distribution of the sample mean becomes more concentrated around 0.5.
```

```
print("- The variance shrinks like 1/(12*n).")
```

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```

```
print("- The shape becomes increasingly close to Normal by the CLT.")
```

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```

```
print("- For n = 1, the distribution is Uniform(0,1); for moderate n it is already fairly bell-shaped.")
```

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```