HW1

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Math 742

Casella / Berger

1.2d : Prove $A \cup B = A \cup (B \cap C^c)$

I'll start with the Right-Hand Side and simplify it using the distribute Law, which is stated in theorem 1.1.4.

In set theory proofs, equality of sets is proven by double containment

ⓘ $A \cup B \subseteq A \cup (B \cap A^c) \quad \wedge \quad A \cup (B \cap A^c) \subseteq A \cup B$ ⓘⓘ

ⓘ Taking an arbitrary element $x \in A \cup B$. By definition of union:

$$x \in A \cup B \to x \in A \text{ or } x \in B$$

Now, lets consider the two cases.

case 1: $x \in A \longrightarrow x \in A \subseteq A \cup (B \cap A^c)$ Case 2: $x \in B \wedge x \notin A$ ⟨ $x \in B$, $x \in A^c$

So, $x \in B \cap A^c \longrightarrow x \in A \cup (B \cap A^c)$

In both cases, $x \in A \cup (B \cap A^c)$ thus: $A \cup B \subseteq (B \cap A^c)$

ⓘⓘ Taking an arbitrary $x \in A \cup (B \cap A^c) \longrightarrow x \in A \vee x \in (B \cap A^c)$

case 1: $x \in A$ then $x \in A \cup B$ case 2: $x \in B \cap A^c$ then $x \in B \to x \in A \cup B$

Again in both cases $x \in A \cup B$ Thus $A \cup (B \cap A^c) \subseteq A \cup B$

- After proving the above result, prove: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

from above $A \cup B = A \cup (B \cap A^c) \longrightarrow A$ and $B \cap A^c$ are disjoint $A \cap (B \cap A^c) = \phi$

why? $A \cap (B \cap A^c) \to$ using associative of intersection $A \cap (B \cap A^c) = (A \cap B) \cap A^c$. so any element x in this set must satisfy all 3 conditions: $x \in A$, $x \in B$, $x \in A^c$. Then by definition (A^c) $A \cap A^c = \phi$

Then $(A \cap B) \cap A^c \subseteq A \cap A^c = \phi \longrightarrow A \cap (B \cap A^c) = \emptyset$

from above, if $C \cap D = \phi$ then $P(C \cup D) = P(C) + P(D) \longrightarrow P(A \cup B) = P(A) + P(B \cap A^c)$

Rewritting $P(B \cap A^c)$, $B = (B \cap A) \cup (B \cap A^c)$ (decomposition)

$$P(B) = P(B \cap A) + P(B \cap A^c) \longrightarrow P(B \cap A^c) = P(B) - P(A \cap B)$$

Then $P(A \cup B) = P(A) + P(B \cap A^c)$

$$= P(A) + P(B) - P(A \cap B)$$

1.3c De Morgan's Laws. Prove this whitout Venn diagrams.

i $(A \cup B)^c = A^c \cap B^c \quad \wedge \quad$ ii $(A \cap B)^c = A^c \cup B^c$

i If an $x \in (A \cup B)^c \leftrightarrow x \notin A \vee x \notin B \leftrightarrow x \in A^c \wedge x \in B^c \leftrightarrow x \in A^c \cup B^c$

ii If an $x \in (A \cap B)^c \leftrightarrow x \notin A \cap B \leftrightarrow x \notin A \wedge x \notin B \leftrightarrow x \in A^c \vee x \in B^c \leftrightarrow x \in A^c \cup B^c$

1.4/10: Prove that $F_X(x) = \dfrac{1}{1+e^{-x}}$, $x \in (-\infty, \infty)$ is a CDF.

Our $F_X(x)$ satisfies the conditions of theorem 1.5.3

Theorem 1.5.3 The function $F(x)$ is a CDF $\leftrightarrow$ The following 3 conditions hold:

a. $\lim_{x \to -\infty} F(x) = 0 \wedge \lim_{x \to \infty} F(x) = 1$ (Monotonicity)

b. $F(x)$ is a NONDECREASING function of x.

c. $F(x)$ is RIGHT continuos; that is, for every number x_0, $\lim_{x \downarrow x_0} F(x) = F(x_0)$

outline of proof: To prove necessity, the three properties can be verified by writting F in term's of the probability function.

If $F_X(\cdot)$ is a CDF, $F_X(x) = P(X \le x)$. Hence $\lim_{x \to \infty} P(X \le x) = 0 \wedge \lim_{x \to -\infty} P(X \le x) = 1$

$F(x)$ is nondecreasing since the set $\{x : X \le x\}$ is nondecreasing in x.
Lastly, as $x \downarrow x_0$, $P(X \le x) \to P(X \le x_0)$, Then $F_X(\cdot)$ is RIGHT continuos.

4.4 A PDF is defined by,

$$f(x,y) = \begin{cases} c(x+2y) : 0 < y < 1 \wedge 0 < x < 2 \\ 0 \quad : \text{otherwise} \end{cases}$$

a) find the value of c.

$$\int_0^2 \int_0^1 c(x+2y)\,dy\,dx = 1 \text{ (The total probability)}$$

$$(y) \to \int_0^1 (x+2y)\,dy = (xy + y^2)\Big|_0^1 = \overset{(1)}{(x+1)} - \overset{(0)}{(0)} = x+1$$

$$(x) \to c\int_0^2 (x+1)\,dx = c\left(\frac{x^2}{2} + x\right)\Big|_0^2 = c\left(\overset{(2)}{(2+2)} - \overset{(0)}{(0)}\right) = c4$$

$$4c = 1 \longrightarrow c = 1/4$$

b) Find the marginal distribution of X.

To find $f_X(x)$, I should integrate the joint PDF over the range of y

$$f_X(x) = \begin{cases} \int_0^1 1/4\,(x+2y)\,dy \text{, we did it previously} = \frac{1}{4}(x+1), \quad 0<x<2 \\ 0 \qquad \text{otherwise} \end{cases}$$

c) Find the joint CDF of X and Y. $= F_{XY}(x,y) = P(X \le x, Y \le y)$

$$P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(v,u)\,dv\,du = \int_{0}^{x}\int_{0}^{y} 1/4\,(u+2v)\,dv\,du$$

Inner Integral (w.r.t. v) $\frac{1}{4}\left[uv + v^2\right]_0^y = \frac{1}{4}(uy + y^2)$

Outer Integral (w.r.t u) $\int_0^x \frac{1}{4}(uy + y^2)\,du = \frac{1}{4}\left[\frac{1}{2}u^2 y + uy^2\right]_0^x$

$F_{XY}(x,y) = \frac{x^2 y}{8} + \frac{xy^2}{4}$ for $0 < x < 2 \wedge 1 \le y$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du = \int_0^x\int_0^1 \frac{1}{4}(u+2v)\,dv\,du = \frac{x^2}{8} + \frac{x}{4}$$

Then, the full definition of F_{XY} is

$$F_{XY}(x,y) = \begin{cases} 0 & x \le 0 \ \vee \ y \le 0 \\ x^2y/8 + xy^2/4 & 0 < x < 2 \ \wedge \ 0 < y < 1 \\ y/2 + y^2/2 & 2 \le x \ \wedge \ 0 < y < 1 \\ x^2/8 + x/4 & 0 < x < 2 \ \wedge \ 1 \le y \\ 1 & 2 \le x \ \wedge \ 1 \le y \end{cases}$$

d) Find the PDF of the random variable $Z = 9(X+1)^2$

Apply Theorem 2.1.5 (Monotonicity) Computing derivative:

$g'(x) = -\frac{18}{(x+1)^3} < 0$ for $x > 0$. So $g(x)$ is strictly monotone decreasing on $0 < x < 2$

- Find the support of **Z**:

Evaluate endpoints at $x=0$: $z = \frac{9}{(1)^2} = 9$ at $x=2$: $z = \frac{9}{(3)^2} = 1$

Since g is decreasing

$$1 < Z < 9$$

- Invert the transformation

Start from

$z = \frac{9}{(x+1)^2}$ solve for x: $(x+1)^2 = \frac{9}{z} \rightarrow x+1 = \frac{3}{\sqrt{z}}$

$x = \frac{3}{\sqrt{z}} - 1$ (only + (positive) root is valid since $x+1>0$)

$\sqrt{z}$

- Compute the Jacobian. $\frac{dx}{dz} = -\frac{3}{2} z^{-3/2} \rightarrow \frac{dx}{dz} = \frac{3}{2z^{3/2}}$
- For a monotone transformation:

$$f_Z(z) = f_X(x(z)) \left| \frac{dx}{dz} \right|$$

- substitute

compute $f_X(x(z))$ $\qquad x+1 = \frac{3}{\sqrt{z}}$

$$f_X(x) = \frac{1}{4}(x+1) = \frac{3}{4\sqrt{z}}$$

- Multiply by Jacobian $f_Z(z) = \frac{3}{4\sqrt{z}} \cdot \frac{3}{2z^{3/2}} = \frac{9}{8z^2} \rightarrow f_Z(z) = \begin{cases} \frac{9}{8z^2}, & 1 < z < 9 \\ 0 & \text{otherwise} \end{cases}$

4.10 The random pair (X, Y) has the distribution

	$X=1$	$X=2$	$X=3$
$Y=2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$Y=3$	$\frac{1}{6}$	0	$\frac{1}{6}$
$Y=4$	0	$\frac{1}{3}$	0

a) Show that X and Y are dependent. Two discrete random variables $X \wedge Y$ are independent if: $P(X=x, Y=y) = P(X=x)P(Y=y) \ \forall x, y$

Once counter exemple $\qquad P(X=2, Y=3) = 0$

But if X and Y were independent:

$$P(X=2)\,P(Y=3) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$0 \neq \frac{1}{6} \rightarrow$ X and Y are dependent

b) Give a probability table for random variables U and V that have the same marginals as X as Y but are independent.

The distribution that satisfies $P(U=x, V=y) = P(U=x)P(V=y)$ where $U \sim X \wedge V \sim Y$ is

	U=1	U=2	U=3
V=2	1/12	1/6	1/12
V=3	1/12	1/6	1/12
V=4	1/12	1/6	1/12

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3.10 $X \sim N(0,1)$, $Y \sim e^{X}$. Determine $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$

$$\mathbb{E}(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{x(2-x)}{2}} dx \quad \text{let } u = x-1$$

$$\rightarrow \mathbb{E}(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1-u^2}{2}} du = e^{1/2} \int_{-\infty}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = e^{1/2}$$

$$\mathbb{E}(Y^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2x - \frac{x^2}{2}} dx \quad \text{lets } u = x-2$$

$$\rightarrow \mathbb{E}(Y^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2} e^{-u^2/2} du = e^{2}$$

$$\mathbb{V}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^2 - (e^{1/2})^2 = e^2 - e$$

NO TIME TO WORK ON IT

3.19

Extra Problem:

Let X have a gamma distribution with parameters α, β.

Prove that $\mathbb{E}(X) = \alpha\beta$ and $\mathbb{V}(X) = \alpha\beta^2$

The Gamma PDF is defined as:

- $\alpha > 0$
- $\beta > 0$

$$f_X(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \qquad \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$$

Show that: $\mathbb{E}(X) = \alpha\beta$ and $\mathbb{V}(X) = \alpha\beta^2$