# RESEARCH STATEMENT

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## INTRODUCTION

The subject of my research is geometric representation theory, and my chief interest is to study the geometry of algebraic varieties arising naturally from representation theory. My work brings together techniques from many areas of mathematics, including representation theory, algebraic and symplectic geometry, combinatorics, and differential equations.

Problems in representation theory can often be translated into algebraic geometry. However, many of these geometric objects are not well understood. My overall goal is to expand our understanding of the geometric side of representation-theoretic problems, and to discover new connections to representations of Lie algebras and associative algebras.

I summarize my current and recent work below. The remainder of this statement discusses more details and future directions.

**Calogero–Moser space and the Hilbert scheme of points (§ 1):** The Calogero–Moser space was first discovered in the theory of integrable systems, as a model for a one-dimensional $n$-body problem. It has since been found to intrinsically relate to many areas of algebra and geometry. For example, it parametrizes irreducible representations of an interesting non-commutative algebra (a rational Cherednik algebra). It is also a natural deformation of the Hilbert scheme of points on the plane.

In an ongoing project, I am working on constructing natural compactifications of the Calogero–Moser space, whose boundary points also have a description in terms of representation theory of algebras.

**Exotic Hessenberg varieties (§ 2):** The Springer resolution is an important resolution of singularities of the cone of nilpotent elements of a Lie algebra. Hessenberg varieties are certain subvarieties of the full flag variety of a reductive algebraic group, which generalize the fibers of the Springer resolution. The exotic representation of the symplectic group is a particular representation that encodes a well-behaved variant of the Springer resolution.

I am interested in generalizations of Hessenberg varieties constructed from the exotic representation (joint with William Graham). We expect to recover versions of results known for usual Hessenberg varieties.

**Perverse sheaves and torus actions (§ 3):** Perverse and intersection cohomology (IC) sheaves are fundamental tools used to study the topology of singular spaces. Additionally, they provide key geometric insight into problems of representation theory.

I have proved a Künneth-type formula to compute the cohomology of tensor products of IC (and equivariant IC) sheaves on varieties equipped with a torus action. Currently, I am working on generalizing this theorem to the case of toric varieties (joint with Sean Taylor).

**Singularity invariants and monodromy conjectures (§ 4):** Bernstein–Sato polynomials are invariants of singularities defined via differential operators. They are very difficult to compute explicitly, and are often studied through their connections to other invariants. The strong monodromy conjecture of Denef–Loeser predicts that the poles of a certain singularity invariant are also roots of the Bernstein–Sato polynomial.

I have proved this conjecture for singularities of reflection hyperplane arrangements of Weyl groups (joint with Robin Walters). We are trying to generalize our methods to prove the motivic version of this conjecture.

## 1. Calogero–Moser space and the Hilbert scheme of points

Let $V = \mathbb{C}^n$. Set $C_n$ to be the set of tuples $(X, Y, v, w) \in \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^*$ such that $[X, Y] + v \otimes w = I$. The group $GL(V)$ acts naturally on $C_n$. The *Calogero–Moser space* is defined to be $C_n/GL(V)$. It is a smooth, irreducible, affine, symplectic variety of dimension $2n$. My ongoing work is on a number of questions, each of which highlights a different feature of $\mathscr{C}_n$.

### 1.1. GIT compactifications.

Recall from [13] that points of $\mathscr{C}_n$ parametrize irreducible representations of the rational Cherednik algebra at special parameters. The projection $\pi \colon \mathscr{C}_n \to \mathbb{A}^n/S_n$ sends the class of $(X, Y, v, w)$ to the unordered collection of eigenvalues of $X$. A relative compactification of $\mathscr{C}_n$ over $\pi$ is known, such that the boundary has a description in terms of representation theory [14, 25]. Unfortunately, this compactification is not known to be algebraic. This motivates the following broad goal.

**Goal 1.1.** Use Geometric Invariant Theory (GIT) to construct relative compactifications of $\mathscr{C}_n$ whose boundary points have a description in terms of representation theory.

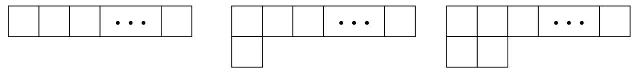
We discuss some initial progress. Let $\overline{C_n}$ be the simultaneous projective completion of $C_n$ along the coordinates of $Y$ and $w$. Explicitly, let $s$ be a new variable, and consider projective coordinates $[Y_{ij} : w_k : s]$ up to simultaneous scaling. The space $\overline{C_n}$ has the homogenized defining equation $[X, Y] + v \otimes w = sI$, and $C_n$ embeds into $\overline{C_n}$ as the points where $s = 1$. Taking a suitable GIT quotient of $\overline{C_n}$ by $GL(V)$ will yield a compactification of $\mathscr{C}_n$ over $\mathbb{A}^n/S_n$. The following proposition gives a complete classification of GIT-semistable points, where we fix the line bundle $\mathscr{O}(1)$ with the linearization given by the character $\det^{-1}$.

**Proposition 1.2** (Bapat, in preparation)**.** *A point $(X, v, [Y_{ij} : w_k : s])$ of $\overline{C_n}$ is $(\det^{-1})$-semistable if and only if both the following conditions hold.*

*(1) The vector $v$ is cyclic for $X$ and $Y$ together.*
*(2) The vector space spanned by $\{v, Xv, X^2v, \dots\}$ has codimension at most $1$.*

The work of G. Wilson [25] shows that over a special point of the form $(a, \dots, a) \in \mathbb{A}^n/S_n$, the fiber of $\pi$ is isomorphic to the disjoint union of the $n$-dimensional Schubert cells of an $(n, 2n)$ Grassmannian. Over any other point in $\mathbb{A}^n/S_n$, the fiber of $\pi$ can be recovered inductively by factorization, as the product of the special fibers for some smaller values of $n$. The next proposition explicitly describes the semistable locus from Proposition 1.2 in terms of the Schubert cell description of Wilson.

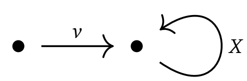
**Proposition 1.3** (Bapat, in preparation)**.** *Over a point $(a, \dots, a) \in \mathbb{A}^n/S_n$, the $(\det^{-1})$-semistable subset of the fiber of $\pi$ corresponds to exactly those $n$-dimensional Schubert cells of $\mathrm{Gr}(n, 2n)$ described by a partition of one of the following three types.*



*Over any other point in $\mathbb{A}^n/S_n$, the $(\det^{-1})$-semistable subset of the fiber of $\pi$ can be recovered by factorization.*

1.2. **Compactifications via quiver representations and the enhanced nilpotent cone.** The space $\mathscr{C}_n$ has a description as a Nakajima quiver variety for the one-loop quiver (the Jordan quiver). Its geometry is closely related to representations of the framed Jordan quiver, which have a rich combinatorial structure. Another point of view is provided by the study of the enhanced nilpotent cone (Achar–Henderson [2], Travkin [23]). We aim to use these insights to construct a compactification of $\mathscr{C}_n$.

We now restrict to the subset $C_n^0$ of $C_n$, consisting of points for which $X$ is a nilpotent matrix. Because of the factorization property explained above, results about a general point of $C_n$ can be recovered from this case. Let $\mathscr{N}(V)$ be the nilpotent cone of $\mathfrak{gl}(V)$. The product $V \times \mathscr{N}(V)$ is known as the enhanced nilpotent cone. Sending a tuple $(X, Y, v, w)$ to $(v, X)$ gives a projection $p\colon C_n^0 \to V \times \mathscr{N}(V)$. Moreover, a point $(v, X)$ of $V \times \mathscr{N}(V)$ can be interpreted as a representation of the framed Jordan quiver as follows.




It can be checked that the image of $p$ in $V \times \mathscr{N}(V)$ consists exclusively of *indecomposable* representations, which cover only some of the $GL(V)$-orbits on $V \times \mathscr{N}(V)$. This suggests that a compactification can be obtained by adding boundary points corresponding to the remaining orbits of $GL(V)$, which make up the decomposable representations of this quiver. The recent preprint [7] of Bellamy–Boos expands on the combinatorics of the $GL(V)$-orbits these representations, which is compatible with the Schubert cell description of Wilson [25]. It is also known to be compatible with the parametrization of $GL(V)$-orbits on $V \times \mathscr{N}(V)$ [2, 23]. Motivated by these observations, I am working towards the following goal.

**Goal 1.4.** Construct a compactification of $C_n^0$ over $V \times \mathscr{N}(V)$ that interacts well with the combinatorics of quiver representations and $GL(V)$-orbits in the enhanced nilpotent cone. Extend this compactification to all of $C_n$ by factorization. Finally, take a suitable GIT quotient to obtain a compactification of $\mathscr{C}_n$.

The orbits of $GL(V)$ on $V \times \mathscr{N}(V)$, as well as representations of the framed Jordan quiver, have equivalent combinatorial descriptions in terms of Young diagrams. The Schubert cell description of the fibers of $\pi$ precisely captures a large subset of these Young diagrams, but not all of it. Achieving the above goal would "complete" the Schubert cell description to give a unifying theory for each of these objects.

1.3. **Relationship with the Hilbert scheme of points on a plane.** The quiver variety interpretation of $\mathscr{C}_n$ gives rise to a flat one-parameter family $\mathfrak{X} \to \mathbb{A}^1$, with general fiber $\mathscr{C}_n$ and special fiber the Hilbert scheme of points on the plane (denoted $\mathrm{Hilb}^n(\mathbb{A}^2)$).

A two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathfrak{X}$, with the first factor acting "along the base" and the second acting "along the fibers". The special fiber is known to be invariant under the full torus, with finitely many fixed points indexed by partitions of $n$. The general fiber is invariant under one factor, with finitely many fixed points indexed by the same set [25], but in a non-trivial way. Therefore the action along the base flows a fixed point on $\mathscr{C}_n$ to a fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$.

In fact, the fixed point on $\mathscr{C}_n$ corresponding to a partition $\lambda$ flows to the fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$ corresponding to the same partition $\lambda$ (see, e.g. [8, Section 4]). The known proof is indirect, and an explicit degeneration is unknown.

**Goal 1.5.** Find an explicit degeneration of each torus-fixed point on $\mathscr{C}_n$ to the corresponding torus-fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Finding such a degeneration would aid in understanding the compactifications of $\mathscr{C}_n$ that we aim to construct. It would also be a good step towards solving a conjecture of Bellamy–Ginzburg about embedding the Grothendieck–Springer resolution of $\mathfrak{sl}_2$ into the degenerating family ([8, Conjecture 4.7]).

## 2. Exotic Hessenberg varieties

This is a joint project with William Graham.

2.1. **Preliminaries.** Let $G$ be a reductive algebraic group over $\mathbb{C}$. Let $B$ be a Borel subgroup. Let $\mathfrak{g}$ and $\mathfrak{b}$ be the Lie algebras of $G$ and $B$ respectively. Then $G/B$ is the flag variety, which can be thought of as the space of Borel subalgebras of $\mathfrak{g}$. For a fixed element $x \in \mathfrak{g}$, the Hessenberg variety corresponding to $H$ and $x$, denoted $\mathscr{B}(H,x)$, is defined as follows:

$$\mathscr{B}(H,x) = \{g \cdot \mathfrak{b} \in G/B \mid g^{-1} \cdot x \in H\}.$$

If $H$ is chosen to be $\mathfrak{b}$ and $x$ is chosen to be any element of the nilpotent cone of $\mathfrak{g}$, the space $\mathscr{B}(H,x)$ is naturally isomorphic to the corresponding Springer fiber.

Many results are known about Hessenberg varieties, especially with restrictions on $x$. For example, De Mari, Procesi, and Shayman [10] construct a Białynicki-Birula decomposition for $\mathscr{B}(H,x)$ when $x$ is a regular semisimple element. Explicit combinatorial formulas for cohomology classes were found in Anderson–Tymoczko [3]. Similarly, affine pavings of Hessenberg varieties have been studied in the cases where $x$ is regular nilpotent (Tymoczko [24]), regular (Precup [21]), minimal nilpotent (Abe–Crooks [1]), etc. In many of the above cases, the affine pavings found are compatible with the Schubert cell decomposition of $G/B$.

2.2. **Hessenberg varieties for the exotic nilpotent cone.** In this section, let $G$ be the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$. In [17], S. Kato defined a variety $\mathfrak{N}$, called the exotic nilpotent cone for $G$, via a representation called the exotic representation of $G$. The exotic Springer resolution is a version of the Springer resolution for $\mathfrak{N}$, which has better behavior than the usual Springer resolution for $G$. We have the following goal.

**Goal 2.1.** Understand exotic Hessenberg varieties, namely analogs of Hessenberg varieties for the exotic Springer resolution.

Let $V \cong \mathbb{C}^{2n}$ be the standard representation of $G$, where $\langle\cdot,\cdot\rangle$ is the chosen symplectic form. Set $S$ to be the space $\{x \in \mathfrak{gl}(V) \mid \langle xv, v\rangle = 0 \text{ for every } v \in V\}$, and set $\mathbb{V} = V \oplus S$. Then $\mathbb{V}$ is a $G$-representation, known as the exotic representation. The exotic nilpotent cone $\mathfrak{N}$ consists of all elements $(v,x)$ of $\mathbb{V}$ such that $x$ lies in $\mathscr{N}(V)$, the nilpotent cone of $\mathfrak{gl}(V)$. In an analogy with usual Hessenberg varieties, we give a provisional definition for exotic Hessenberg varieties.

**Definition 2.2.** Given $x \in \mathbb{V}$ and a $B$-stable vector subspace $H \subset \mathbb{V}$, we define the exotic Hessenberg variety $\mathfrak{B}(H,x)$ as follows:

$$\mathfrak{B}(H,x) = \{g \cdot \mathfrak{b} \in G/B \mid g^{-1} \cdot x \in H\}.$$

In the spirit of known results about usual Hessenberg varieties, we aim to investigate analogs for special classes of exotic Hessenberg varieties.

**Goal 2.3.** Define notions analogous to regular, semisimple, nilpotent, etc. for elements of $\mathbb{V}$. Understand the geometry and cohomology of special classes of exotic Hessenberg varieties for each of the above classes. In particular, determine whether they have affine pavings.

Achieving this goal would give us a handle on new subvarieties of flag varieties, that will also have interpretations in terms of the exotic Springer theory.

2.3. **Relationship with the enhanced nilpotent cone.** Given a vector space $W$ of dimension $n$, its enhanced nilpotent cone is the variety $W \times \mathcal{N}(W)$ (also discussed in § 1.2). As shown in [2, 23], the $GL(W)$-orbits on $W \times \mathcal{N}(W)$ are in bijection with the set of bi-partitions of $n$. It was shown by Kato [17] that the $\mathrm{Sp}(2n,\mathbb{C})$-orbits on $\mathfrak{N}$ are also in bijection with $BP_n$. Further, the following embeddings are known [2]:

$$W \times \mathcal{N}(W) \xhookrightarrow{\varphi_1} \mathfrak{N} \xhookrightarrow{\varphi_2} V \times \mathcal{N}(V).$$

These embeddings are compatible with the parametrization of orbits on each space (of $GL(W)$, $\mathrm{Sp}(2n,\mathbb{C})$, and $GL(V)$ respectively). Moreover, they respect closure orderings.

Since the enhanced nilpotent cone was originally introduced as an easier version of the exotic nilpotent cone, it is natural to extend our study of Hessenberg varieties to the enhanced nilpotent cone.

**Goal 2.4.** Understand Hessenberg varieties for particular $GL$-orbits in the enhanced nilpotent cone. Extend the embeddings $\varphi_1, \varphi_2$ to the corresponding Hessenberg varieties.

We expect this theory to be close to the classical theory of Hessenberg varieties. We also expect this theory to give useful information about exotic Hessenberg varieties.

## 3. Perverse sheaves and torus actions

Let $X$ be a smooth projective variety equipped with the action of an algebraic torus $T$. Fix a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to T$ such that the $\mathbb{G}_m$-fixed set is equal to the $T$-fixed set $X^T$. Let $F_1, \ldots, F_k$ be the irreducible components of the fixed set $X^{\mathbb{G}_m}$. In this situation, Białynicki-Birula [9] described two canonical decompositions of $X$ into $k$ locally-closed, $\mathbb{G}_m$-invariant subvarieties, called the *plus decomposition* ($\{S_1^+, \ldots, S_k^+\}$) and the *minus decomposition* ($\{S_1^-, \ldots, S_k^-\}$):

$$S_i^+ = \{x \in X \mid \lim_{t\to 0} \lambda(t) \cdot x \in F_i\}, \text{ and } S_i^- = \{x \in X \mid \lim_{t\to\infty} \lambda(t) \cdot x \in F_i\}.$$

Note that each $S_i^+$ and $S_i^-$ map to $F_i$ (by sending a point to its limiting point), and the fibers of these maps are affine spaces. In particular, we consider the case in which each $F_i$ is a point, and then each $S_i^{\pm}$ is itself an affine space.

Consider (say) the plus decomposition, and group the pieces $S_i^+$ by dimension. In general, the resulting configuration does not give a topological stratification, but we focus on the case when it does. Moreover, we assume that for each fixed point $w$, there is some one-parameter subgroup $\mu : \mathbb{G}_m \to T$ such that the attracting set of $\mu$ on $w$ has dimension $n$. The cell decomposition of $X$ so described is suitable for studying both the intersection cohomology sheaves and the topological (singular) cohomology.

I have proved the following theorem.

**Theorem 3.1** (Bapat, [4])**.** *Let $X$ be a smooth projective complex algebraic variety equipped with an action of an algebraic torus, with assumptions as before. Let $(\mathscr{L}_1, \dots, \mathscr{L}_m)$ be an $m$-tuple of IC sheaves on $X$ constant along the Białynicki-Birula strata, and let $\otimes$ denote the (derived) tensor product. Then the cup-product map induces the following isomorphism:*

$$H^\bullet(\mathscr{L}_1) \underset{H^\bullet(X)}{\otimes} \cdots \underset{H^\bullet(X)}{\otimes} H^\bullet(\mathscr{L}_m) \xrightarrow{\cong} H^\bullet(\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_m).$$

The same theorem holds for $T$-equivariant simple perverse sheaves on $X$ and their $T$-equivariant cohomology, as modules over the $T$-equivariant cohomology ring of $X$.

The most direct motivation for Theorem 3.1 comes from Ginzburg's paper [15], which itself is related to the work of Soergel [22] and Beilinson–Ginzburg–Soergel [6]. Under the hypotheses from the previous subsection, Ginzburg showed that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are two IC sheaves on $X$ constant along the Białynicki-Birula strata, then $\mathrm{Ext}^\bullet_{D^b_c(X)}(\mathscr{L}_1, \mathscr{L}_2) \cong \mathrm{Hom}^\bullet_{H^\bullet(X)}(H^\bullet(\mathscr{L}_1), H^\bullet(\mathscr{L}_2))$. My theorem is the analog of the theorem above (which discusses the derived Hom functor) for the derived tensor product functor.

Together with Sean Taylor, I am investigating a generalization of my theorem to all toric varieties.

## 4. Singularity invariants and monodromy conjectures

This is a joint project with Robin Walters.

**Overview.** In this section we describe several related invariants of hypersurface singularities. My main focus is Bernstein–Sato polynomials, also known as $b$-functions. Since $b$-functions are very hard to compute explicitly, it is fruitful to study connections to other invariants.

Let $f$ be a polynomial in $n$ variables, and let $\mathscr{D}$ be the ring of differential operators in $n$ variables. The *Bernstein–Sato polynomial* or *$b$-function* of $f$ is the minimal monic polynomial $b_f(s)$ that satisfies $L(s)f^{s+1} = b_f(s)f^s$ for some $L(s) \in \mathscr{D}[s]$. The $b$-function of $f$ is known to be an invariant of the singularities of the zero locus $V(f)$ of $f$.

The *Milnor fiber* is diffeomorphism class of spaces defined locally from a hypersurface. The cohomology groups of the Milnor fiber have a monodromy action, which encode information about singularities. Malgrange [18, 19] and Kashiwara [16] have shown that exponentials of the roots of the $b$-function are eigenvalues of the monodromy of the Milnor fiber.

The *local topological zeta function* is a singularity invariant first defined by Denef–Loeser [11]. It is a power series constructed from a resolution of singularities, using multiplicities of the exceptional divisors in the preimage of the hypersurface and in the relative canonical divisor.

**Conjecture** (Strong monodromy conjecture, Denef–Loeser [11])**.** *Every pole of the local topological zeta function of a hypersurface $V(f)$ is a root of the $b$-function of $f$.*

### 4.1. Hyperplane arrangements of Weyl type.

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank $r$. Let $\mathfrak{h}$ a Cartan subalgebra, and $W$ its Weyl group. Let $\xi$ be the product of all the positive roots of $\mathfrak{g}$. Then $\xi$ belongs to $\mathbb{C}[\mathfrak{h}]$, and cuts out a hyperplane arrangement that we call the Weyl arrangement. In a joint work with Robin Walters, I proved the strong monodromy conjecture (SMC) for all Weyl arrangements.

**Theorem 4.1** (Bapat–Walters, [5])**.** *The strong monodromy conjecture holds for any Weyl arrangement $\xi$. That is, each pole of the local topological zeta function of $\xi$ is also a root of $b_\xi(s)$.*

Denef and Loeser also defined a motivic version of the local topological zeta function [12], and there is a motivic version of the strong monodromy conjecture. We are working on using our methods to tackle the strong motivic monodromy conjecture in the case of Weyl arrangements.

Note that $\xi^2$ belongs to $\mathbb{C}[\mathfrak{h}]^W$, the space of $W$-invariants. Recall that $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring in $r$ variables, generated by homogeneous polynomials of degrees $d_1, \ldots, d_r$ respectively (the fundamental invariants). We will denote this polynomial ring by $\mathbb{C}[\mathfrak{h}/W]$. Let $\rho$ be the image of $\xi^2$ in $\mathbb{C}[\mathfrak{h}/W]$. In the paper [20], Opdam showed the $b$-function of $\rho$ to be

$$b_\rho(s) = \prod_{i=1}^{r} \prod_{j=1}^{d_i - 1} \left( s + \frac{1}{2} + \frac{j}{d_i} \right).$$

Opdam's method involved explicitly computing a suitable differential operator via hypergeometric shift operators. Although $\rho$ and $\xi$ are related to each other, their $b$-functions are not obviously related, because the differential operators on $\mathbb{C}[\mathfrak{h}]$ and on $\mathbb{C}[\mathfrak{h}/W]$ are quite different.

We proved the following theorem that relates the two.

**Theorem 4.2** (Bapat–Walters, [5])**.** *Let $\rho$ be the image of $\xi^2$ in $\mathbb{C}[\mathfrak{h}/W]$. Then $b_\rho(s) \mid b_\xi(2s+1)$.*

As a long-term goal, I am interested in computing the $b$-functions of $\xi$ and $\xi^2$ explicitly. Computer experiments indicate that they are much more complicated than the $b$-function of $\rho$. In the next part, we discuss the case of type $A$, for which we have an upper bound as well as a conjectured answer.

**The $b$-function of the Vandermonde determinant.** When $\mathfrak{g} = \mathfrak{sl}_n$, the function $\xi$ is simply the Vandermonde determinant on $n$ variables, defined as $\mathrm{VM}_n(x_1, \ldots, x_n) = \prod_{1 \le j < i \le n}(x_i - x_j)$. Surprisingly, its $b$-function is not known. We have proved the following (recursive) upper bound on $b_{\mathrm{VM}_n}(s)$.

**Theorem 4.3** (Bapat–Walters, in preparation)**.** *For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, denoted $\lambda \vdash n$, let $b_\lambda(s)$ denote the product of the $b$-functions of $\mathrm{VM}_{\lambda_i}$ for each $i$. Then we have that*

$$b_{\mathrm{VM}_n}(s) \,\Big|\, \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left( s + \frac{i}{\binom{n}{2}} \right).$$

Based on computer calculations, we conjecture that the above upper bound is strict.

**Conjecture 4.4.** *The $b$-function of $\mathrm{VM}_n$ is given by the following recursive formula.*

$$b_{\mathrm{VM}_n}(s) = \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left( s + \frac{i}{\binom{n}{2}} \right).$$

In the long run, we would like to prove this conjecture, as well as similar results for types other than type $A$.

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